BASIC
REINFORCED CONCRETE
DESIGN
A TEXT-BOOK FOR STUDENTS AND ENGINEERS

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VOLUME I
ELEMENTARY

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PUBLISHER'S NOTE

This book is published in two volumes each of which is sold separately. It is, however, necessary for Volume II to be read in conjunction with Volume I.

This book deals mainly with the principles of design and to some extent is complementary to the author's other books on reinforced concrete design and construction:

"EXAMPLES OF THE DESIGN OF REINFORCED CONCRETE BUILDINGS IN ACCORDANCE WITH THE BRITISH STANDARD CODES OF PRACTICE"

"REINFORCED CONCRETE DESIGNER'S HANDBOOK"

"CONCRETE CONSTRUCTION"

For particulars of these and other books in the "CONCRETE SERIES"

see the page facing page 264.
PREFACE

The purpose of this book is to present the principles of the design of reinforced concrete structures in accordance with British practice. It is assumed that the reader has some knowledge of the theory of structures and the strength of homogeneous materials, but has no knowledge of their application to reinforced concrete construction. The subject is dealt with from the elements of design to a stage at which the reader should be able to undertake the design of simple structures and, under competent supervision, to design more advanced works. The book is in three parts*, namely (i) the fundamental principles and derivation of basic formulae and other design data; (ii) the design of simple structural members; and (iii) the consideration of more complex members and the design of complete structures.

The principles dealt with in Part I include working stresses, and resistance of simple members to concentric and eccentric axial compressive or tensile forces, to bending and shearing, and to axial forces combined with bending. Elementary consideration of the bond between concrete and steel is also given. The basic theoretical analyses derived in Part I are applied in Part II to the design of simple members such as solid slabs, rectangular and flanged beams, columns, and the more common types of foundation, in this order as it is the sequence in which the detailed design of a building proceeds.

Loads and bending moments, and especially the bending moments on continuous beams are dealt with at the beginning of Part II but only summarily since it is presupposed that the reader is reasonably well versed in methods of calculating ordinary bending moments and shearing forces. More detailed attention is given in Part III to bending moments and forces relating to parts of structures peculiar to reinforced concrete, such as slabs spanning in two directions, flat slabs, frames, special forms of roof, walls of containers, retaining structures, and the like. The resistances of members of less regular cross-section than those in Parts I and II are dealt with in Part III, in which the general case of a member of any cross-section is analysed as from this case the formulae for members of regular cross-sections are derived. The method adopted in Part III of proceeding from the complex general case to the more simple special cases is advantageous because the entire conception of the interaction of the various factors is presented at the beginning, and successive simplifications lead to the practical working formulae for the more common simple cases. Among other matters dealt with in Part III are bending in two planes, shearing force acting simultaneously with direct compression and other

* Parts I and II are in Volume I; part III is in Volume II.
cases of combined stresses, torsion, and the effects of change of temperature and shrinking of concrete.

Readers who have some knowledge of the theory of reinforced concrete design, and who may therefore think that they need not read this part, are advised to read these chapters at least once before proceeding to apply the principles and formulæ as is done in Part II and in the more complex studies in Part III.

It is impracticable in two volumes to consider in detail the design of the many structures commonly constructed in reinforced concrete; therefore Part III concludes with a survey of structures including buildings, bridges, tanks, industrial structures, and the like; a brief indication is given regarding design consideration and procedure. The reader is advised to consult articles in technical journals and books dealing with particular structures. Short bibliographies are given at the end of some chapters and elsewhere in the text. References to books and documents that apply to several parts of the book are given on page 8.

The symbols used in the formulæ in each part of this book are given in summaries as well as in relevant places in the text. For convenience of reference, the enumeration of the formulæ is such that the first figures in the number of a formula indicate the number of the chapter in which the formula is given.

In addition to the designs of complete structural members in Part II and complete parts of structures in Part III, numerous examples of more simple calculations are given throughout the book and especially in Part I. The examples should not be considered merely as arithmetical exercises in substituting numerical values in formulæ. Simple examples are an essential preliminary step in the study of design because they serve to acquaint the reader with the various terms, symbols, and formulæ used in the more complex designs. Where possible, each example is selected so that it illustrates some point in design. In many examples it is required to select suitable reinforcement bars to resist the forces calculated. To avoid unnecessary arithmetic, which may distract from the main purpose of an example, the cross-sectional areas of round bars and factors of moments of resistance are tabulated on pages 123 and 124. With these exceptions, tables and charts of design data, which are of great value in practice but which may conceal the underlying principles of design, are excluded from this book, the purpose of which is to be instructive.

LONDON, 1962

C. E. R.
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INTRODUCTION

REINFORCED concrete is a combination of two dissimilar materials, namely concrete, which has considerable crushing strength but offers little resistance to tensile forces, and steel which has great tensile strength and in the form of bars is embedded in those parts of a structural member of concrete which are likely to be subjected to tension, thereby reinforcing the member. Steel reinforcement is also provided to augment the compressive resistance of some members. A reinforced concrete member of typical beam-slab-and-column construction is illustrated on page 2 and the arrangement of the reinforcement in the several members is shown on page 3.

The properties of the concrete and of the steel reinforcement must be considered before proceeding to the analysis of the structural action of the combined materials because it is essential to realise from the beginning that theoretical design calculations are related to actual materials which, however high their qualities, suffer from the variabilities of all natural or manufactured substances. Safe working stresses and the allied consideration of factors of safety must be considered in relation to the properties of concrete and steel on which properties these stresses depend. The assessment of safe working stresses is based mainly on experience, although the recommendations given in codes of practice are a primary guide.

Principles of Design.—The apparent mathematical nature of design, which has been emphasised since the inception of reinforced concrete at the end of the nineteenth century, should not be allowed to obscure its practical nature. Design is not merely a matter of substituting numerical factors in formulae, but includes the proper selection of the numerical values to be so substituted and the acceptance or otherwise of the results of the calculation. A thorough understanding of the basic principles is therefore essential before applying formulae to the design of actual structures. Although much of the application can be learned from books, it should be realised that knowledge obtained by working with a competent designer is equally, if not more, valuable.

Consideration of the principles of design of reinforced concrete should commence with the most simple case, that is the resistance of a member of rectangular cross-section to concentric thrust and to bending. The next stage is to consider this simple member to be subjected to these two effects simultaneously. The opposite condition, that is the resistance to a direct pull either with or without bending, should then be considered. Members of more complex shape, which are common in reinforced concrete and subjected to more complex bending and twisting actions, can then be dealt with.

Resistance to bending may be based on either the ordinary “elastic”
Typical Beam, Slab and Column Construction.
(See also facing page.)
Arrangement of Reinforcement Bars in Typical Beam, Slab and Column Construction.

(See facing page.)
or modular-ratio method in which the elastic moduli of the concrete and steel and conditions at working load are taken into account, or the load-factor or ultimate-load method in which the conditions at near failure are considered. Resistance to shearing is an important aspect of reinforced concrete design, and a simple empirical method of determining the amount of reinforcement required to resist the tensile stresses resulting from shearing is generally adopted.

The effectiveness of reinforced concrete as a structural material depends on the combined action of the concrete and reinforcement, which in turn depends on the adhesion of the concrete to the surface of the steel bars. It is useless to determine from elaborate calculations the amount of reinforcement required to resist bending and other effects if the bars are arranged in such a way that the assumed working stresses in the steel cannot be developed and are not fully effective because they are not properly secured in the concrete.

An important preliminary is the consideration of the loads and pressures which have to be or are likely to be supported and the determination of the bending moments and forces produced by these effects. Other effects, such as change of temperature and shrinking of concrete, also have to be taken into account.

Economical Design.—The determination of the sizes of the members and the amount of reinforcement required to enable them to withstand the forces or other effects to which they will be subjected is the object of detailed design. Detailed design is, however, only one of the two main parts of structural design, the other being the primary design, that is the initial planning or arranging of the members so that the external forces or loads on the structure are transmitted to the foundation in the most economical manner consistent with the purpose of the structure; knowledge of how to do this is derived from experience, from a study of existing structures, and from comparison of alternative designs. Much economy can be effected if the shapes of concrete members are as plain as practicable so that the shuttering or temporary moulds in which the concrete is cast is simplified, thereby saving time and cost of construction. Equally important is the arrangement of the reinforcement, especially the positions of junctions or splices which should conform to the stages into which the operations of placing the concrete is divided. The detailed designs given in this book should be studied from this point of view. Structural design is only one stage in the building of a structure; and building is essentially a practical operation. Therefore works in progress should be examined so that constructional procedure and the problems encountered at each stage of the work may be realised. Designs can then be made that take into account the sequence of operations and alleviate rather than add to the constructional difficulties and expense.

Derivation of Formulae.—The theoretical formulæ for the resistance
of reinforced concrete members to forces and bending appear to be the result of rigid mathematical analyses, and are so to the extent that they are derived strictly from certain assumptions. Since these assumptions are not always correct formulae based on them cannot be accurate, but fortunately in most cases the numerical results obtained from the formulae seem to be true enough to enable the formulae to be adopted almost as "rule of thumb" bases for sound design. It is generally assumed in the design of beams that the concrete in the tension zone offers no resistance because its tensile strength is small and it is likely to crack; but under ordinary working conditions, and especially in solid slabs, the concrete may not be cracked completely and then offers some tensile resistance to bending. When concrete hardens it shrinks and tends to contract; this tendency is resisted by the reinforcement, which has no similar tendency to shrink, and sometimes by the ends of the member being fixed in position. The concrete may therefore be in tension, a condition which is generally neglected in theoretical analyses. Again, the elastic modulus of concrete, which is taken into account in the modular-ratio theory, is a variable property, but a constant modulus is assumed in most analyses. It is probably because an unreal value of the modulus is assumed that defects due to other omissions and incorrect assumptions are offset. These matters should be borne in mind from the outset of the study of design so that the designer will not fall into the error of thinking that the formulae are infallible. It should, however, be realised that the results obtained from using rational "accepted" formulae lead to sound design although the theoretical basis may, to some extent, be deficient.

Glossary.—In the following are given brief explanations of a few of the terms used in reinforced concrete. (See also B.S. No. 2787: "Glossary of Terms for Concrete and Reinforced Concrete ".)

MATERIALS.

Reinforcement.—Steel in the form of bars or wires, or similar, embedded in the concrete to resist mainly tensile forces.

Aggregate.—The inert material, such as gravel or crushed rock and sand, which is mixed with cement and water to form concrete.

Admixture.—A material other than aggregate, cement or water which is mixed in the concrete in small quantities to produce a desired modification of the properties of the concrete, such as improved workability, acceleration of setting or hardening, waterproofing, etc.

Portland Cement.—A fine powder resulting from grinding the clinker produced by burning a mixture of calcareous and argillaceous materials. When mixed with a suitable amount of water, the material sets and hardens. Rapid-hardening Portland cement (or high-early-strength cement) hardens more rapidly at early ages than ordinary Portland cement.

High-alumina Cement.—A cement resulting from grinding the clinker
produced by melting a mixture of calcareous materials and materials rich in alumina.

Water-Cement Ratio.—The ratio of the weight of water to the weight of cement in freshly-mixed concrete.

Consistency.—The fluidity of freshly-mixed concrete; also termed (wrongly) Consistency.

Workability.—The property of freshly-mixed concrete which determines the ease with which it can be manipulated and consolidated in the shuttering (or similar) and around the reinforcement.

Setting.—The initial solidifying of the mixture of cement, water, and aggregates.

Hardening of Concrete.—The progressive strengthening of the mixture of cement, water, and aggregates subsequent to setting.

Shrinkage.—The reduction in volume of a mass of freshly-mixed concrete which occurs as it sets and hardens.

Creep.—The slow inelastic deformation or movement of concrete in a stressed member; also termed Plastic Flow.

Design.

Effective Depth.—The distance from the compressed edge of a member subjected to bending to the centroid of the reinforcement in tension.

Overall Depth.—The distance from the compressed edge to the tensioned edge of a member subjected to bending.

Lever Arm.—The distance from the centre of action of the compressive resistance of a member subjected to bending to the centroid of the reinforcement.

Neutral Plane.—The plane in the depth of a member subjected to bending at which there is theoretically no stress longitudinally; also called Neutral Axis.

Modular Ratio.—The ratio of the elastic moduli of steel and concrete.

Modular-Ratio Method.—A method of design based on the assumption that the steel and concrete are elastic within the range of the permissible stresses, and that the modular ratio is constant.

Load-Factor Method.—A method of design based on the strength of a member (at near-failure) and incorporating a factor to determine the safe working load; also called Ultimate-Load Method.

Modulus of Rupture.—The calculated maximum tensile stress in a beam of plain concrete at failure.

Bond.—The adhesion of concrete to the reinforcement.

Flanged Beam.—A beam comprising a compression flange, and a rib, such as a beam having cross-section like a T or an inverted L; also may have a flange at the tensile edge, thereby having a cross-section like an I.
INTRODUCTION

SHORT COLUMN.—A column (or strut) which is short in relation to its length and therefore not liable to buckling.

SLENDER COLUMN.—A column (or strut) which is long in relation to its length and therefore more liable to buckling than a short column.

FLAT SLAB.—Planar floor or roof construction comprising only a slab supported on columns; also called Beamless or Mushroom Floor.

PRISMATIC STRUCTURES.—A roof, or similar construction, comprising planar slabs constructed monolithically but at various angles; also called Hipped-plate or Folded-Plate Construction.

BARREL VAULT.—A thin slab in the form of part of a cylinder and generally forming a roof; see "Shell Construction".

SHELL CONSTRUCTION.—Roofs and similar construction comprising thin curved slabs which resist direct stresses but do not theoretically resist bending.

CONSTRUCTION.

CAST-INSITU CONCRETE.—Concrete that while still plastic is deposited in its final position in the work under construction; concrete cast in place.

PRECAST CONCRETE.—Concrete which is cast in moulds and, when hardened, the member so cast is removed to be incorporated in a structure or other works.

PLAIN CONCRETE.—Concrete without reinforcement; also Non-Reinforced Concrete or, if in bulky construction, Mass Concrete.

REINFORCED CONCRETE (see page 1).

MONOLITHIC CONSTRUCTION.—A structure the parts of which are of cast-insitu concrete and is without permanent joints.

CONSTRUCTION JOINT.—The joint between concrete already placed (and hardened) and freshly-mixed concrete.

CURING.—The process of preventing excessive evaporation of the moisture from the surface of newly cast concrete.

COVER OF CONCRETE.—The least distance between the face of a concrete member and a reinforcement bar or wire.

SHUTTERING (also FORMWORK, MOULDS, CENTERING).—The temporary supports provided for the concrete when first deposited and until it hardens.

Notation.—The clarity of analyses and design calculations depends largely on the symbols adopted for the factors in the formulae. To avoid confusion, a symbol should always have the same meaning, and it is desirable that the notation should conform to an accepted standard. The notation in this book is in accordance with British Standards; where additional symbols are used, they are mnemonic where practicable and as like a standard symbol as possible. The definitions of the symbols used in the formulae are summarised at the beginning of each part. Other symbols which are used in the derivation of the formulae but do not appear
in the final formulæ are excluded but are explained in the relevant text, as are all other symbols.

Bibliography.—The books and other publications in the following are of general interest to all students and others concerned with reinforced concrete design and supplement the subject matter in this book. Publications relating specifically to the subject matter of a particular chapter are noticed in the chapter concerned. British and foreign periodicals dealing with civil engineering should also be perused for information on the latest advances in design and construction of reinforced concrete works.

Note.—The construction "B.S. " denotes a publication of the British Standards Institution.

(1).—"The Structural Use of Normal Reinforced Concrete in Buildings." B.S. Code No. 114.
(3).—"London Building By-laws."
(4).—"Memorandum on Bridge Design and Construction." Ministry of Transport.
(5).—"Concrete Year Book." (Published annually.)
PART I

PRINCIPLES OF DESIGN

The fundamental principles on which the calculation of the resistance of structural members of reinforced concrete are based are described in this part and the basic design formulae are derived. Although the contents of this part are mainly theoretical, it is important that it should not be thought that design is merely a matter of juggling formulae. It must be realised throughout that materials, namely concrete and steel, which have different properties and are of variable qualities, are being dealt with and that the ordinary theories of their resistances do not take into account all the vagaries of these materials. The theoretical considerations in this part are therefore preceded by a brief description of these materials and the stresses that may be permitted with safety.

NOTATION FOR PART I

Dimensions and Properties of Sections.—"Area" means "cross-sectional area" unless stated otherwise.

$A$: Gross area of a member. $A_B$: Half cross-sectional area of helical binding in vertical cross-section of unit length of column. $A_h$: Equivalent area of helical binding (volume of binding in unit length of column). $A_o$: Net area of concrete excluding reinforcement. $A_e$: Equivalent area in concrete units. $A_c$: Area of concrete in core of column excluding longitudinal reinforcement. $A_s$: Area of reinforcement ($A_{st} + A_{st}$ or $A_{st} + A_{st}$). $A_{nc}$, $A_{nl}$: Area of reinforcement in compression and tension respectively. $A_{nb}$: Area of "balanced" amount of reinforcement. $A_{s1}$, $A_{s2}$: Area of reinforcement at opposite faces of member. $A_w$: Area of both arms of a binder. $a_1$: Lever-arm factor ($l_a/d_1$).

$b$: Breadth of rectangular beam or column or flange of tee-beam or ell-beam. $b_r$: Breadth of rib of tee-beam or ell-beam.

$D$: Diameter of reinforcement bar. $D_B$: Diameter of the core of a column with helical binding. $d$: Overall depth of a beam; thickness of a slab; width or least lateral dimension of a column. $d_n$: Depth to the neutral plane ($n_d d_1$). $d_s$: Thickness of flange of tee-beam or ell-beam ($s d_1$). $d_l$: Distance of reinforcement in tension above bottom edge of beam ($f_1 d_1$). $d_w$: Rise of inclined bar. $d_1$: Effective depth of beam or slab. $d_b$: Depth to reinforcement in compression ($f_2 d_1$).

$e$, $s$: Eccentricity of thrust or pull measured about the centroid of the stressed area and the centre of the reinforcement in tension respectively. $e_o$: Limiting eccentricity for load $P_b$. $e_1$: Eccentricity factor ($e/d_1$).

$f_1$, $f_2$: Cover ratios of reinforcement in tension and compression respectively ($f_1 = d/t_1$; $f_2 = d/s_1$). $g$: Position of eccentric thrusts.

$h$, $h_o$, $h_c$: Distances from compressed edge to centre of total compression, to centre of compression of concrete, and to centre of compression of reinforcement respectively.
BASIC REINFORCED CONCRETE DESIGN

I: Moment of inertia.  
k: Least radius of gyration of a column.
L: Actual length of a column.  
i: Effective height of column.  
la: Lever arm of moment of resistance (a1d1).  
lac, las: Lever arm of compressive resistance of the concrete and reinforcement in compression respectively.
lo: Bond-length without end anchorage.
N: Number of "bar diameters" in bond-length.  

n1: Neutral-plane factor (d1/dd).

\( s \): Sum of perimeter of reinforcement bars.

\( r_b \): Ratio of "balanced" amount of reinforcement.  
\( r_c, r_s \): Ratio of reinforcement in compression and tension respectively.  
\( r_s \): Ratio of total reinforcement \( \left( \frac{A_{st} + A_{sc}}{bd} \right) \).

\( s \): Spacing or pitch of binders and of helical binding.  
\( s_1 \): Ratio of thickness of flange to effective depth of a tee-beam or ell-beam (d1/dd).

\( \bar{X}, \bar{Y} \): Co-ordinates of centroid of a cross-section if concrete is effective in tension.  
\( \bar{X}' \): Distance to centroid if concrete is ineffective in tension.

\( x_1 \): Spacing factor of inclined bars (spacing = x1d1).

\( Z, Z_1 \): Section modulus.  

\( Z_0, Z_1 \): Section modulus for edge in greatest compression and for opposite edge of a beam respectively.

\( \alpha \): Angle of inclination of inclined bars.  

\( \delta A_{st} \): Area of one bar.

FORCES, STRESSES, AND MOMENTS.

\( E_c, E_s \): Elastic moduli of concrete and steel (or secant modulus of steel).

\( F \): Concentric thrust.  

\( F_a, F_t \): Compressive and tensile forces respectively.  

\( F_{ca} \): Compressive force in concrete.  

\( F_{sc}, F_{st} \): Compressive and tensile forces respectively in reinforcement.  

\( F_{st}, F_{sc} \): Forces in reinforcement \( A_{st} \) and \( A_{sc} \) respectively.  

\( F_T \): Axial pull.  

\( f_{cb} \): Compressive stress in concrete due to bending.  

\( f_{ct}, f_{st} \): Compressive and tensile direct stresses respectively in concrete.  

\( f_{ct}, f_{st} \): Compressive and tensile stresses respectively in reinforcement.

\( J, J_s \): Compression factors for concrete and reinforcement respectively (combined stress).

\( K_s \): Factor in formula for combined stresses.

\( M \): Bending moment.  

\( M_c, M_s \): Bending moments resisted by concrete and reinforcement in compression respectively.  

\( M_T \): Moment of resistance to bending.  

\( M_{tr}, M_{st} \): Moments of compressive and tensile resistance respectively.  

\( M_1 \): Bending moment at section where effective depth is \( d_1 \).

\( m \): Modular ratio \( (E_s/E_c) \).

\( N \): Eccentric thrust.

\( P \): Safe eccentric thrust on a short column; safe load on a short column subjected to bending.  

\( P_B, P_c, P_s \): Compressive resistances due to binding, concrete in core, and longitudinal reinforcement respectively in a column with helical binding.  

\( P_b \): "Limiting" eccentric load on a column.  

\( P_a \): Concentric thrust.  

\( P_0 \): Safe concentric load on a short column.  

\( P_1 \) to \( P_n \): Eccentric thrusts.  

\( P_{cb}, P_{ct} \): Permissible compressive stresses in concrete in bending and in direct compression respectively.  

\( P_{ct} \): Permissible tensile stress in concrete.  

\( P_{st}, P_{st} \): Permissible compressive and tensile stresses respectively in reinforcement.  

\( P_{sy} \): Yield-point stress or equivalent yield stress in steel.

\( Q \): Shearing force.  

\( Q_{cb}, Q_{CL} \): Moment-of-compressive-resistance factor for modular-ratio method and load-factor method.  

\( q \): Shearing stress.

\( R_l \): Load-reduction factor for a slender column.

\( s_{th}, s_{th} \): Average and local bond stresses respectively.

\( U \): Factor in formula for \( P \) (load-factor method).  

\( u \): Crushing strength of concrete cubes.  

\( X, Y \): Factors in formula for \( P_b \) and \( P \) respectively.

\( \mu \) (\( \mu \)): Poisson’s ratio.
CHAPTER I
MATERIALS AND PERMISSIBLE STRESSES

Ordinary reinforced concrete is concrete in which reinforcement in the form of steel bars is embedded. There are many types of concrete and reinforcement, but the consideration of the materials in this chapter and the examples in subsequent chapters deal with Portland-cement concrete of density and strength suitable for structures and reinforcement in the form of plain round bars of mild steel. Some other types of steel reinforcement are described but reference should be made elsewhere\(^5\) * for descriptions of concretes which have special properties to suit particular purposes, such as low or high density, or resistance to heat.

**Structural Concrete.**

Concrete comprises a chemically-inert strong mineral aggregate in a matrix consisting of cement and water. The quality of these materials, the physical and chemical properties of which are established by standard tests, are described briefly in the following.

**Aggregates\(^{1,1}\).**—The aggregates in structural concrete are generally separated into coarse and fine materials. The coarse aggregate is generally natural gravel or crushed stone, the sizes of the pieces of stone being not smaller than \(\frac{3}{4}\) in. nor, in ordinary structures, greater than \(\frac{3}{4}\) in. In concrete in large masses, larger stones may be used. The fine aggregate is natural sand or, less commonly, the finer material from crushed stone, and is graded from particles \(\frac{3}{8}\) in. in size down to the finest particles but excluding dust. The grading, or granulometry, of the sand between these limits has more effect on the quality of the concrete than has the grading of the coarse aggregate: for example, there should not be a high proportion of the finest particles. The aggregates must be free of dust or adhering materials so that the cement can adhere to the surfaces of the pieces of stone and particles of sand. There must be no organic or inorganic materials mixed with the aggregates which may affect adversely the setting of the cement.

**Cement\(^{1,2}\).**—The cement is generally normal-setting Portland cement of either ordinary or rapid-hardening type or Portland-blastfurnace cement. Special cements are used for particular purposes. If extra rapid-hardening, chemical-resistant, or refractory properties are required, high-alumina cement is commonly used. Sulphate-resistant cement may be used for

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* References thus \(^{(1)}\) apply to the general Bibliography on page 8. References thus \(^{(1,1)}\) apply to the Bibliography for Chapter I which is given on page 39.
Concrete that is likely to be exposed to attack by sulphates in weak to medium concentrations. The adverse effects of the heat generated by the chemical reactions during the setting of cement in concrete in large masses are reduced if low-heat Portland cement is used.

**Water**\(^{(1,3)}\)._—The quality of the water used in mixing the concrete must be such that the chemical reactions which take place during the setting of the cement are not impaired. Water that is potable is, in general, suitable for making concrete.

**Proportions of Materials.**—The proportions in which the materials are mixed for structural concrete should be such that, when it has hardened, a dense concrete is obtained. The greater the density of concrete the greater its strength and durability. The ideal concrete is one in which the spaces between the coarse aggregate are filled with fine aggregate, and the voids in the combined aggregate are filled completely with cement. It is rarely that the voids are just completely filled. If they are not filled the concrete will contain small voids and will be less dense. If the combined volumes of fine aggregate and cement exceed slightly the volume of voids in the coarse aggregate the number and size of the voids are likely to be less and a more dense concrete is obtained; on the other hand, if the volume of the fine aggregate and cement greatly exceeds the volume of the voids in the coarse aggregate, the concrete is likely to be less dense. Therefore there are limited proportions in which the constituents can be mixed to give the densest concrete. These proportions can be determined almost exactly for materials of constant quality, as in a laboratory, but on a construction site the vagaries of natural materials make the practicable determination one of trial and error.

The proportions of the materials in concrete are described by the relative amounts of cement and aggregate. The description may be the weight of cement in a cubic yard of finished concrete, but the usual method is to state the ratios of the volumes of fine and coarse aggregates to unit volume of cement. A common mixture is one containing dry materials in the proportions of 1 cwt. of cement to 2\(\frac{1}{2}\) cu. ft. of fine aggregate to 5 cu. ft. of coarse aggregate. Assuming that loose cement weighs 90 lb. per cubic foot, these proportions can be expressed volumetrically as 1 : 2 : 4. Other proportions, such as 1 : 1\(\frac{1}{2} : 3\) and 1 : 1 : 2, are sometimes used for concrete in which stresses greater than normal are to be resisted, and in the case of concrete in large masses subjected to low stresses, the proportions may be 1 : 3 : 6. The most favourable ratio of fine to coarse aggregate depends on the grading of the materials, and the proportions are varied to give the densest mixture of suitable workability but with the same ratio of cement to total aggregate. For example, a mixture specified to be nominally 1 : 2 : 4 may be between 1 : 1\(\frac{1}{2} : 4\frac{1}{2}\) and 1 : 2\(\frac{1}{2} : 3\frac{1}{2}\).

There are many publications\(^{(1,4)}\) dealing with proportioning and making concrete, but, as previously stated, trial-and-error methods using the actual
materials to be used for the structure give in general the most satisfactory results.

**Strength.**—For the purpose of design the most important property of concrete is its crushing strength, on which the permissible working stresses depend. It is necessary to define when and how the crushing strength shall be determined, because the strength of concrete increases with age and the apparent crushing strength depends on the shape of the test specimen and the conditions under which the specimens are made, stored, and tested. Practice in Great Britain is to define the crushing strength as the least intensity of pressure (lb. per square inch) required to crush a 6-in. cube of concrete twenty-eight days after casting the cube, which is made and tested in accordance with standard methods\(^{(1-5)}\). Cubes may be made on the site of construction (and are then called "works cubes") or in a laboratory. The strength of a laboratory-made cube may be as much as 50 per cent. greater than that of a works cube of the same concrete. In some foreign standards, notably those of the U.S.A., the strength of concrete is defined by the results of crushing cylinders of concrete, the apparent crushing strength of a cylinder being from 65 to 85 per cent. of the strength of a cube. It is therefore necessary to consider the shape of the test specimen when comparing results from different sources.

The strength of concrete depends on several factors other than age and the foregoing conditions affecting apparent strength. Apart from the obvious need to have strong aggregates and sound cement, the amount of water in relation to the quantity of cement is the factor which has the greatest influence on the strength of concrete.

**Water-cement Ratio and Workability.**—The smaller the ratio of water to cement (above a certain limit) the greater the strength. If the quantity of water is too small there may be insufficient to ensure that the chemical action of the setting of cement is complete; also the materials may be too dry to be properly mixed or the resulting mixture may be too stiff to be satisfactorily compacted. If too much water is used there may be a tendency for the stones to separate from the finer materials after mixing. Evaporation of the excess water produces a less dense, more porous, and friable concrete which is liable to crack due to shrinking while hardening. An average amount of water in reinforced concrete is that which results in a water-cement ratio of about \(\frac{1}{4}\) (by weight). To ensure that concrete can be placed properly and compacted around the reinforcement bars, it is necessary for it to be workable. Workability is difficult to assess quantitatively as the degree of workability required depends on the size of the member being cast, the congestion or otherwise of the reinforcement, the means used to consolidate or compact the concrete and, to some extent, on the method of transporting the concrete. For example, if the concrete is to be consolidated by mechanical vibration, as is most common, it should be drier (that is have a lower water-cement
ratio) than if it is to be compacted by hand tools. Concrete that is conveyed from the mixer to the site of placing by pumping usually requires to be more fluid than if other means are used, and a water-cement ratio of 0.7 may be necessary. Increasing workability by increasing the amount of water results in a decrease of strength. Increasing the proportion of cement, and to a limited extent the proportion of sand, increases the workability with little or no change in the amount of water required, and consequently there may be an increase of strength due to the reduction in the water-cement ratio. Thus a 1:1\frac{1}{4}:3 concrete having the same degree of workability as a 1:2:4 concrete, which contains less cement, is likely to be stronger than the 1:2:4 concrete, mainly because of the lower water-cement ratio.

Reinforcement.

The most common reinforcement for concrete comprises plain round hot-rolled bars of mild steel, but cold-worked bars and wires of greater tensile strength are also used. The properties and strengths of various types of reinforcement obtainable in Great Britain are described in British Standards(1.6). The tensile strength of mild steel bars for reinforcing concrete must be not less than 63,000 lb. per square inch. Other hot-rolled bars include bars of medium-tensile steel, which are a superior mild-steel bar having a tensile strength not less than 74,000 lb. per square inch, and bars of high-tensile steel having a tensile strength of not less than 83,000 lb. per square inch. Mild-steel bars are, however, the most common.

Cold-worked reinforcement bars and wire are generally proprietary materials and may be mild-steel bars which have been twisted or otherwise deformed without being heated, or mild-steel wire that has been drawn cold. One effect of twisting or drawing steel while it is cold is to increase its tensile strength. Cold-worked bars are obtainable as twisted square bars and twisted ribbed bars. Twisted bars of not less than \(\frac{3}{8}\) in. diameter should have a tensile strength not less than 70,000 lb. per square inch. Cold-drawn wire, which should have a tensile strength of not less than 83,000 lb. per square inch, is generally provided in the form of a welded mesh suitable for the reinforcement of slabs. Other steel meshes may comprise woven twisted bars or may be in the form of expanded metal.

Permissible Working Stresses.

Before structural members can be designed, the safe working stresses in the concrete and reinforcement must be determined. These stresses depend on the crushing strength of the concrete and the tensile strength of the steel. It is generally assumed that the materials are not inferior to a certain quality and that normal methods of design are adopted. Basic
working stresses permissible in ordinary design are recommended in British standard codes of practice, local building by-laws, memoranda issued by Government departments, such as those for the design of road bridges, and the like\(^{(1-4)}\). In each document the permissible stresses are related to minimum specified strengths of the materials and generally to recommended methods of design. If more accurate, or more approximate, methods of design are adopted the permissible stresses may be greater or should be smaller respectively. In all cases, however, the permissible stresses must be only a fraction of the strengths of the materials so as to provide a margin for such indeterminate factors as differences between the assumed and actual magnitudes of the loads, variations in the bending moments and forces to which the member may be subjected, and variations in the strengths of the materials, and to allow for the numerous secondary stresses due to such effects as the shrinking of the concrete upon setting, changes of temperature and humidity, unequal settlement of foundations, accidental overloading, and the like effects which are commonly excluded from the design calculations. Obviously the more factors included in the stress analysis the smaller need be the margin between the assumed working stress and the strength of the material. This margin is incorporated in a factor of safety.

**Factor of Safety.**

A structure is safe from collapse under normal working conditions if the stresses in all its parts are less than the strength (expressed as resistances to stress) or other critical property of the materials of which the structure is composed. The greater the ratio of strength to working stress, the greater is the margin or factor of safety. The need for a factor of safety is obvious, as some margin must be allowed for deficiencies of the theory used in the design, defects in materials, indifferent workmanship, and other effects mentioned in the preceding paragraph. In its simplest form the factor of safety would appear to be the ratio of the apparent strength to the greatest working stress. This relationship does not apply directly to reinforced concrete for the reasons, among others, given in the following.

The apparent crushing strength of concrete depends on the nature of the compression, that is whether it is due to a concentric thrust or to bending or to thrust combined with bending. The crushing strength of a non-reinforced concrete cube is not an absolute measure of the apparent strength of the same concrete in a structural member of reinforced concrete, although it is related thereto. The stresses in concrete as ordinarily calculated do not take into account variations due to shrinking (see page 29), creep (see page 29), and similar phenomena. The distribution of stress near failure is very different from that assumed in the design to occur
at working load. The tensile strength of steel is not the criterion of the value of reinforcement, since the permanent extension which occurs at a smaller stress may cause a member to be useless before the bars are stressed to an intensity equal to their tensile strength. A reinforced concrete member may collapse, or become useless, because of excessive distortion due to a breakdown of compressive resistance of the concrete or tensile resistance of the reinforcement or to failure of bond or in shearing; in members as designed ordinarily it is not easy to predict which of these factors is the weakest link and therefore which stress or combination of stresses establishes the factor of safety. Overstressing of one part of a member may be relieved by understressig elsewhere, and therefore investigation of the stresses at one section only do not establish the probable margin of safety.

In the narrowest sense the factor of safety of a member should be defined therefore as the ratio of the load which causes collapse, or uselessness, to the design load. Strictly these loads should exclude the permanent dead load, since the useful load is the imposed or live load. For example, the factor of safety of a floor designed to support a specified imposed load is the imposed load causing collapse, or uselessness, of the floor divided by the specified imposed load. Although this is a rational conception of the factor of safety, it is not the general interpretation because it is common to consider the resistance of the separate members comprising a structure. Therefore, the factor of safety of a beam is generally considered to be the ratio of the moment of resistance just before failure to the calculated moment of resistance at the permissible safe working stresses. Neither of these moments of resistance is easy to determine; that at failure can be equated only to the apparent bending moment causing failure and, although under laboratory conditions this bending moment can be assessed accurately for a simply-supported beam, it cannot be so determined in practice because of the indeterminate degree of restraint at the supports. The safe moment of resistance also is indeterminate, and the calculated resistance depends on the theory applied in the calculation. Further, failure of a beam may be failure in shearing and not in resistance to bending. Similarly the load causing the collapse of a column subjected to a concentric load can be determined under laboratory conditions, but in practice the load may not be truly concentric, or the column may be subjected to bending due to indeterminate restraint at the ends which may cause failure at a less load than in the laboratory tests. It is therefore impossible to be certain about the factor of safety of a reinforced concrete member. It appears that, in the case of members designed in accordance with the ordinary methods, factors between two and four are likely for beams and columns, but ordinary solid slabs have considerably greater factors of safety which may be as much as ten.

The factors of safety of such structures as retaining walls, and such
parts of a structure as the foundations or those subjected to wind, are in a different category from the factors of safety of separate members. For example, a retaining wall may fail not only as a result of the materials of which it is constructed being overstressed, but by overturning or tilting or sliding; the pressure of soils is a factor known only very approximately, and therefore a margin must be allowed in the calculation of stability to allow for possible pressures in excess of that assumed and to ensure that under the most likely adverse conditions the wall is not just on the point of moving. The overturning effects of wind, the forces due to which are likewise very indefinite, must be considered similarly. The factor of safety of a foundation structure is dependent not only on the strength of the materials of which it is constructed but also on the margin between the pressure imposed on the ground under the foundation structure and the pressure which causes excessive settlement or other movement of the ground. The safe resistance of soils is indefinite, not only because their physical properties are variable and are not exactly determinable but because the criterion of what constitutes excessive settlement or other movement differs with each type of structure and, to a certain extent, in the opinion of different engineers.

The foregoing observations emphasise the fact that structural engineering design is by no means confined to substituting numerical values in formulæ, but requires the application of knowledge gained from experience.

Basic Stresses Permissible in Concrete.

The safe working stresses permissible in concrete are based on the crushing strength as defined by the strength of 6-in. works cubes at twenty-eight days. If ordinary care be exercised in selecting and mixing the materials, the crushing strength of Portland-cement concrete is not likely to be less than 3000 lb. per square inch for 1:2:4 concrete, 3750 lb. per square inch for 1:1½:3 concrete, and 4500 lb. per square inch for 1:1:2 concrete. It is not unusual for greater strengths to be obtained, and at least 4000 lb. per square inch should be expected for 1:2:4 concrete if its making and placing are well controlled. In factories making precast concrete products, 6000 lb. per square inch at seven days is not an uncommon strength for 1:1½:3 concrete, and it may be greater at early ages if the concrete is cured in steam. Special concrete for prestressed construction but made with ordinary materials may attain strengths of 8000 lb. per square inch or more at twenty-eight days.

Compressive Stress.—The basic compressive stress permissible in concrete subjected to bending is generally one-third of the crushing strength. Thus the permissible compressive stress in bending in ordinary concrete is generally 1000 lb. per square inch in 1:2:4 mixtures and 1250 lb. per square inch in 1:1½:3 mixtures. Concrete richer in cement
than $1 : 1 \frac{1}{2} : 3$ is rarely used for members subjected to bending only, but may be used for members in compression such as columns or arches subjected to bending, in which case the stress permissible in $1 : 1 : 2$ concrete is 1500 lb. per square inch. For columns and other members subjected to concentric thrust without bending, the permissible compressive stress is about three-quarters (generally 76 per cent.) of the stress permissible in bending, and the basic stresses generally recommended are 760 lb. per square inch for $1 : 2 : 4$ concrete, 950 lb. for $1 : 1 \frac{1}{2} : 3$ concrete, and 1140 lb. for $1 : 1 : 2$ concrete. The compressive stresses permissible in concrete of proportions between $1 : 2 : 4$ and $1 : 1 : 2$ are inversely proportional to the ratio of the amount of total aggregate to the amount of cement; for example, this ratio for $1 : 1 \frac{1}{2} : 3 \frac{1}{2}$ concrete is $1 : 5$, which is intermediate between $1 : 6$ for $1 : 2 : 4$ concrete and $1 : 4 \frac{1}{2}$ for $1 : 1 \frac{1}{2} : 3$ concrete; and the permissible stress in bending is $1000 + \frac{(1250 - 1000)(6 - 5)}{(6 - 4 \frac{1}{2})}$, that is 1167 lb. per square inch.

**Tensile Stress and Modulus of Rupture.**—The resistance of concrete to direct tensile force is not easy to determine, so the tensile resistance in bending is commonly used as the criterion. This resistance, called the modulus of rupture, is determined by breaking a small non-reinforced concrete beam of standard dimensions by applying a load at the midpoint. The bending moment at failure divided by the modulus of the section gives theoretically the maximum tensile stress in bending, which is greater than the direct tensile stress. In general, the greater the crushing strength the greater the tensile strength. For concrete of medium quality the tensile strength is nearly proportional to, and equal to, about one-tenth of the crushing strength. For concrete of higher quality, the tensile strength is less than that given by direct proportionality to the crushing strength. Weak concrete may have practically no tensile strength. The modulus of rupture of ordinary structural Portland-cement concrete at seven days would be probably not less than 350 lb. per square inch for $1 : 2 : 4$ concrete, 400 lb. for $1 : 1 \frac{1}{2} : 3$ concrete, and 450 lb. for $1 : 1 : 2$ concrete. The tensile strength is important in structures containing liquids, and it is general to adopt a tensile stress of 190 lb. per square inch in members of $1 : 1 : 6 : 3 : 2$ concrete in direct tension and 270 lb. per square inch in members subjected to bending. The corresponding stresses in $1 : 2 : 4$ concrete are 175 lb. and 245 lb. per square inch respectively. Tensile stresses in the concrete should also be limited to these values in structures in which cracking may permit corrosive fumes to attack the reinforcement, examples being structures over steam-operated railways and in gas-works and chemical and similar industrial works.

**Shearing Stress.**—Although the tensile strength of concrete is not generally considered to be effective in reinforced concrete members subjected to bending, it is important in the resistance to shearing forces which
induce tensile stresses. The tensile stresses due to shearing forces are limited by restricting the shearing stress in the concrete to about one-tenth of the numerical value of the compressive stress permissible in bending, if reinforcement is not provided to resist the shearing forces. The shearing stress permissible in $1:2:4$ ordinary structural concrete is therefore one-tenth of 1000 lb. per square inch, that is 100 lb. per square inch. In concrete stronger in compression the permissible shearing stress is slightly less than one-tenth of the compressive stress; for example, in $1:1\frac{1}{4}:3$ concrete it is generally $115$ lb. per square inch. If reinforcement be provided to resist the shearing forces it is still advisable to restrict the tensile stress by limiting the shearing stress to four times the stress permissible if reinforcement be not provided, that is to 400 lb. per square inch in ordinary $1:2:4$ concrete and 460 lb. per square inch in $1:1\frac{1}{4}:3$ concrete. It is advisable to adopt as low a shearing stress as practicable in beams and, although the foregoing stresses may be suitable for primary beams, shearing stresses of, say, two-thirds of these stresses should be the upper limit for secondary beams in buildings. A smaller shearing stress is also recommended in structures containing liquids, say, not more than $280$ lb. and $250$ lb. per square inch in $1:1.6:3.2$ and $1:2:4$ concretes respectively.

Basic Stresses Permissible in Reinforcement.

Properties of Steel.—When a mild steel bar is subjected to a small tensile stress the elongation of the bar is elastic and is proportional to the stress. At a stress equal to about half the tensile stress that would break the bar, several phenomena occur. The extension is no longer proportional to the stress but increases at a greater rate than the stress; at a slightly greater stress the bar ceases to be entirely elastic, that is only part of the extension disappears when the force on the bar is entirely relaxed. At a slightly greater stress yielding takes place, that is the bar continues to increase in length although the tensile force remains constant. These three stresses, which are indicated in the stress-strain diagrams in Fig. 1, do not differ greatly from one another and are called the limit of proportionality, the elastic limit, and the yield point respectively. It is obvious that a bar stressed beyond the elastic-limit stress becomes valueless in a structure, and therefore this stress is the criterion of the usefulness of the bar. Since it is easier to determine the yield-point stress than either of the other stresses, the yield-point stress is generally considered to be the critical stress. Twisted bars and cold-drawn wire do not generally exhibit a definite yield point as do mild-steel bars, as is seen by the typical stress-strain curves for cold-worked reinforcement in Fig. 1. The critical stress in a cold-worked bar or wire is therefore considered to be an arbitrary stress called the equivalent yield stress, which is the stress at which the permanent elongation of the steel is not more than $\frac{1}{4}$ per cent. (0.0025).
An alternative definition is that the equivalent yield stress is the stress at which a total elongation of \( \frac{1}{8} \) per cent. (0.0050) is produced. There is little difference between the stresses according to these two definitions, as is shown by the stress-strain curves for a typical twisted bar in Fig. 1.

![Stress-strain Diagram](image)

**Fig. 1.—Properties of Reinforcing Steels: Stress-strain Diagrams.**

The diagram gives also the corresponding curve for typical cold-drawn wire.

**Tensile Stresses.**—The basic permissible or safe working tensile stress is a fraction, usually one-half, of the yield-point or equivalent yield stress. Therefore, whereas the basic compressive stress permissible in concrete is based directly on the crushing strength of the concrete, the basic tensile
stress permissible in reinforcement is based on a condition of stress considerably less than the tensile strength of the steel. Whatever may be the yield-point or equivalent yield stress, the basic tensile stress permissible in a bar should not exceed 30,000 lb. per square inch, the object of this restriction being to avoid excessive cracking of the concrete surrounding the bar. Since even moderate cracking may be deleterious in a structure in a corrosive atmosphere it is advisable to limit the tensile stress to, say, 20,000 lb. per square inch. Similarly, wide cracks are undesirable in structures containing liquids, and the tensile stress in the reinforcement in such structures should not exceed, say, 12,000 lb. per square inch in order to prevent the concrete cracking; limiting the tensile stress in the concrete as described reduces the risk of cracking.

**Mild Steel.**—In ordinary mild-steel bars for which no yield-point stress is specified the basic permissible tensile stress is generally 20,000 lb. per square inch, that is half an assumed yield-point stress of 40,000 lb. per square inch. A stress of 20,000 lb. per square inch is satisfactory in all mild-steel bars except bars of large diameter, because the extra rolling to which bars of smaller diameter are subjected to reduce them in size has the effect of increasing the yield-point stress. The permissible tensile stress in bars exceeding $\frac{1}{4}$ in. in diameter, which are not subjected to so much rolling and therefore have a lower yield-point stress, should not be more than 18,000 lb. per square inch. The largest mild-steel bar in general use is $1\frac{1}{2}$ in. in diameter, but bars up to 2 in. in diameter are used in very large members. The smallest bars are generally $\frac{1}{4}$ in. in diameter, but $\frac{3}{8}$-in. or even $\frac{1}{2}$-in. bars may be used in very small members or thin slabs.

**Medium-tensile Steel Bars.**—The minimum specified yield-point stress of bars of medium tensile steel is 44,000 lb. per square inch for bars not greater than 1 in. in diameter and 41,500 lb. per square inch for bars up to $1\frac{1}{2}$ in. in diameter. On the basis that the safe tensile stress does not exceed half the yield-point stress, basic permissible stresses are 22,000 lb. per square inch for bars not greater than 1 in. and 20,750 lb. for bars up to $1\frac{1}{2}$ in. in diameter.

**High-tensile Steel Bars.**—For hot-rolled high-tensile steel bars having a specified yield-point stress of not less than 51,500 lb. per square inch for bars not exceeding 1 in. and 49,500 lb. for bars up to $1\frac{1}{2}$ in. in diameter, the basic permissible tensile stresses are 25,750 lb. and 24,750 lb. per square inch respectively, or for practical purposes, say, 25,000 lb. per square inch for bars of all sizes up to and including $1\frac{1}{2}$ in. in diameter.

**Cold-worked Steel.**—The effect of cold working to form twisted bars is to increase considerably the yield-point stress or, since most bars of this type do not exhibit a definite yield point, to produce a higher equivalent yield stress. The minimum specified equivalent yield stress of twisted square bars smaller than $\frac{3}{8}$ in. is 70,000 lb. per square inch, and
for larger bars it is 60,000 lb. per square inch. The basic tensile stress permissible in all twisted square bars is therefore 30,000 lb. per square inch. Round twisted ribbed bars generally have an equivalent yield stress exceeding 60,000 lb. per square inch, and the basic permissible tensile stress is therefore 30,000 lb. per square inch.

**Cold-drawn Wire.**—Cold-drawn wire as commonly provided in the form of welded meshes has a specified equivalent yield stress of not less than 70,000 lb. per square inch and the basic tensile stress permissible therein under normal conditions is 30,000 lb. per square inch.

**Compressive Stresses.**—The basic compressive stress permissible in a reinforcement bar is also related to the yield-point stress or equivalent yield stress, but the permissible stresses are generally lower. For example, the permissible compressive stress in mild-steel bars is 18,000 lb. per square inch for bars not exceeding \( \frac{1}{4} \) in. in diameter and 16,000 lb. per square inch for larger bars. The corresponding permissible stress in other bars is half the yield-point or equivalent yield stress but in no case greater than 23,000 lb. per square inch.

**Bond Stresses.**

The adhesion between the reinforcement and the surrounding concrete (see page 111) depends on the quality of the concrete and the character of the surface of the bar. The basic permissible average bond stress between plain round bars and ordinary structural concrete is generally about 20 lb. per square inch in excess of the permissible shearing stress, that is 120 lb. per square inch of area of the surface of the bar embedded in \( 1:2:4 \) concrete, 135 lb. per square inch if in \( 1:1\frac{1}{2}:3 \) concrete, and 150 lb. per square inch if in \( 1:1:2 \) concrete. The permissible local bond stress is greater than the permissible average bond stress, and acceptable stresses for plain round bars embedded in ordinary structural concrete are 180 lb. per square inch for \( 1:2:4 \) concrete, 200 lb. for \( 1:1\frac{1}{2}:3 \) concrete, and 220 lb. for \( 1:1:2 \) concrete.

Deformed bars, such as twisted square and twisted ribbed bars, provide mechanical anchorage in addition to the adhesion between the concrete and the surface of the bar, and the basic permissible average and local bond stresses may be increased by up to 25 per cent. if standard tests demonstrate a sufficient increase in the value of the bond.

**Variations of Basic Permissible Stresses.**

The numerical values of the basic stresses permissible in concrete and reinforcement given in the foregoing apply to materials of ordinary qualities and strengths as described and to structures designed in accordance with common methods for normal working conditions. In some circumstances
basic stresses may be increased, and in some circumstances must be decreased, to take into account conditions differing from those to which the basic working stresses apply. The factors affecting the permissible stresses are considered in the following.

**Strength of the Concrete.**—An important variation is that due to the strength of the concrete. If the ordinary crushing strength (for example 3000 lb. per square inch at twenty-eight days of works cubes of 1:2:4 concrete) is not obtained, the compressive stresses permissible in bending and direct compression must be reduced in proportion to the actual strength, which should be not less than three-quarters of the ordinary strength. For example, the least crushing strength of works cubes of 1:2:4 concrete at twenty-eight days may be 75 per cent. of 3000 lb. per square inch, that is 2250 lb. per square inch, in which case the compressive stress permissible in bending must not exceed \( \frac{2250}{3000} \) of the ordinary permissible stress of 1000 lb. per square inch, that is 750 lb. per square inch; the permissible direct compressive stress is 76 per cent. of 750 lb., that is 570 lb. per square inch. If, however, the crushing strength of ordinary structural concrete is consistently greater than the minimum strength required, the permissible compressive stresses may be increased in proportion to the crushing strength up to 25 per cent. in excess of the basic stresses; that is for 1:2:4 concrete, if the crushing strength is consistently not less than 3750 lb. per square inch, the stresses permissible in bending and in direct compression may be 1250 lb. and 950 lb. per square inch respectively. If the qualities and proportions of the materials and the methods of measuring and mixing are such that a superior concrete is consistently produced and there is a high standard of supervision to ensure maintenance of this superiority, the permissible working compressive stresses may be increased accordingly. For such special concrete the mixture should have not less than 1 part of cement to 8 parts of total aggregate by weight and the crushing strength of works cubes at twenty-eight days should be not less than 2250 lb. per square inch and may be up to 6000 lb. per square inch. Within these limits the compressive stress permissible in bending is one-third of the crushing strength, that is if the crushing strength is 6000 lb. per square inch the permissible stress is 2000 lb. per square inch. The permissible stress in direct compression is 76 per cent. of 2000 lb., that is 1520 lb. per square inch.

**Quality of Aggregates.**—The permissible stresses recommended in the foregoing relate to concrete made with aggregates that conform to the standard requirements, but they may apply also to concrete made with other aggregates if the crushing strength of the concrete is not less than that of concrete made with standard aggregates. If a lower strength only can be obtained for concrete with non-standard aggregates, the compressive
stress permissible in bending should not exceed a quarter of the crushing strength, and the corresponding permissible direct compressive stress should be 76 per cent. of this lower stress. For example, if the crushing strength is 2250 lb. per square inch the permissible stresses are only 562 lb. per square inch in bending and 427 lb. in direct compression.

**Shearing and Bond Stresses.**—If the permissible compressive stress differs from the basic stress due to any of the foregoing causes, the permissible shearing stress and the average and local bond stresses must be altered and should be based on the compressive stress permissible in bending in accordance with the following rules. The permissible shearing stress and the average and local bond stresses are 10 per cent., 12 per cent., and 18 per cent. respectively of the compressive stress permissible in bending if the compressive stress does not exceed 1000 lb. per square inch. Otherwise the permissible shearing stress without reinforcement is 40 lb. per square inch plus 6 per cent. of the compressive stress permissible in bending; the average bond stress is 20 lb. per square inch in excess of the shearing stress; and the local bond stress is 100 lb. per square inch plus 8 per cent. of the compressive stress permissible in bending. Therefore, if the compressive stress permissible in bending is 750 lb. per square inch, the permissible shearing stress is 75 lb. per square inch, the permissible bond stress is 90 lb. per square inch, and the permissible local bond stress is 135 lb. per square inch, compared with 100 lb., 120 lb., and 180 lb. per square inch respectively if the permissible compressive stress is 1000 lb. per square inch. If the permissible compressive stress is 1200 lb. per square inch, the permissible shearing stress and the average and local bond stresses are, in accordance with the foregoing rules, 112 lb., 132 lb., and 196 lb. per square inch respectively. The foregoing bond stresses are for plain round bars and may be increased by 25 per cent. if deformed bars are used.

**High-alumina Cement Concrete.**—If $1:2:4$ concrete is made with standard aggregates and high-alumina cement, greater strengths are likely than for Portland-cement concrete, and the working stresses may be increased accordingly. If the minimum crushing strength at two days is not less than 5000 lb. per square inch, the permissible stresses are the same as those for $1:1:2$ Portland-cement concrete, namely 1500 lb. per square inch in bending and 1140 lb. per square inch in direct compression; the permissible shearing stress and the average and local bond stresses are 130 lb., 150 lb., and 220 lb. per square inch respectively. High-alumina cement concrete not leaner than $1:8$ and not richer than $1:5$ by weight may have a crushing strength at two days up to 6000 lb. per square inch, in which case the working stresses may be increased in the same proportion as for special Portland-cement concrete.

**Age at Loading.**—The basic permissible stresses in concrete are generally considered to be safe if the member is fully loaded not earlier
than one month after casting. Since concrete increases in strength with age; higher stresses are permissible if the total load assumed in the design is not applied until later. For example, if the age at loading is two months the permissible compressive stresses may be 10 per cent. in excess of the basic stress; if three months, 16 per cent.; if six months, 20 per cent.; and if twelve months, 24 per cent. These increases apply to the basic permissible compressive stresses in Portland-cement concrete or these stresses as varied to allow for crushing strengths smaller or greater than those to which the basic stresses are related. To allow for the age at loading, the shearing stress and bond stresses may be increased in compliance with the rules given in the foregoing. The permissible stress must be reduced if the working load is likely to be imposed earlier than a month after casting unless the member is supported in such a manner that it is not fully stressed. Such lower stresses should be related to the crushing strength of the concrete at the age of loading. The compressive stress permissible in bending should be not more than one-third of this crushing strength, which may generally be interpolated from the strength at seven days and twenty-eight days of cubes made of concrete of the same quality as it is proposed to use in the work.

**Stresses due to Wind.**—It is common to allow greater permissible stresses if the effect of wind on the structure is included in the calculation of the bending moments and forces. Generally, the increase permitted is 25 per cent. in excess of the basic permissible compressive, shearing, and bond stresses in Portland-cement or high-alumina cement concrete; if these stresses are varied to allow for the actual crushing strength or for age at loading, the increase is 25 per cent. of the permissible stress as so varied. A similar increase may be made to the basic stresses in the reinforcement, but the increased stress should not exceed 30,000 lb. per square inch. It is, however, essential that the stresses induced by the design load, excluding the effects of wind, should not exceed the basic or varied permissible stresses. The increase should not be made in the case of chimneys, towers and other structures where the wind forces are the major effects.

**Stresses in Slender Members.**—Reduction in permissible compressive stresses must be made for slender members. The reduced stresses permissible in the concrete and reinforcement in long columns (as defined on page 37) and similar members subjected to direct compression are described on page 37. The reduced stresses in narrow beams are dealt with on page 174.

**Poor Workmanship and Constructional Difficulties.**—If the structure is likely to be constructed with indifferent workmanship or under incompetent supervision the quality of the concrete may be poor, and it may be lower than that of the test cubes. In such circumstances, a compressive stress of, say, 600 lb. per square inch in $1:2:4$ concrete and a tensile stress of, say, 16,000 lb. per square inch in mild steel
reinforcement should not be exceeded; all bars in tension should have hooks or similar anchorages (as described on page 113), and shearing forces should be resisted by reinforcement, no reliance being placed on the concrete in this respect. Similar reductions in working stresses should be adopted if the work is difficult to construct, for example, in restricted working space (as in underpinning) or where concrete is placed under water.

**Relationship between Stresses and Data used in Design.**—Stresses smaller or greater than the basic permissible stresses should be adopted depending on the adequacy of the design data. For example, if the greatest working load is not known exactly, or if the calculations do not take into account the effects of changes of temperature, torsion, shrinking of concrete, and similar secondary effects as described in Chapter XIV, the stresses must be reduced. Such effects may be omitted when preparing a preliminary design, but should so far as possible be included in the final design. For example, in the design of a column of a building frame, the basic stresses permissible in bending apply to the condition of bending combined with axial load, but if bending is excluded from the preliminary calculation smaller stresses must be adopted. The obvious rule is that the fewer the factors included in the calculations the smaller must be the permissible working stresses.

On the other hand, the stresses may be greater than the basic permissible stresses if all factors that are known or can be reasonably foreseen are taken into account. For example, in a railway bridge the basic permissible stresses should not be exceeded if impact, lurching, centrifugal force and, in the case of an arch, temperature, shrinking, and elastic contraction only are taken into account, but if in addition the effects of wind, braking, tractive effort, and friction of girder bearings are taken into account then stresses up to 20 per cent. greater are permissible.

**Effect of Vibration.**—If a member is to be subjected to vibration, the working stresses should be reduced to allow for possible fatigue unless this effect has been taken into account in assessing the load. Reinforcement bars in tension in such members should be provided with hooks or other equally effective end anchorages.

**Effect of Fatigue.**—Fatigue causes a reduction of the strength of materials, including concrete, and occurs when they are subjected to loads or other forces which cause stresses to be repeatedly increased and relaxed or which may cause stresses to alternate between compression and tension. Fluctuating stresses which do not exceed about half the strength of the concrete do not generally produce fatigue, but above this limit repetition of stress causes a deterioration of strength that may eventually cause failure at working loads. Since in most designs the greatest working stresses are below this limit there may be no danger from this source, but, since rapidly-fluctuating loads are often accompanied by vibration, the actual maximum stresses produced are difficult to assess, and it is
MATERIALS AND PERMISSIBLE STRESSES

important that reduced working stresses, or a design load in excess of the actual value of the dynamic load, should be adopted. Steel is also subject to fatigue but this phenomenon is unimportant in reinforced concrete.

**Occasional Exceptional Loads.**—If the working load will be mainly permanent, it is advisable to adopt lower stresses in concrete and reinforcement unless precautions are taken to support the member until it has obtained at least the age assumed in assessing the basic permissible stresses. On the other hand, it may be permissible to increase the stresses for possible occasional exceptional loads, that is if the greatest live load is likely to act infrequently; examples of such loads are a heavy vehicle crossing a bridge at intervals of several years; possible but unlikely waterlogged conditions behind a wall retaining dry or well-drained ground; an accidental wheel load on the footpath or part of a bridge other than the roadway (it is generally permissible under this loading to adopt working stresses up to 50 per cent, more than the basic stress). In these and similar cases, the stresses due to the dead load and the ordinary live load must not exceed the normal basic permissible stresses.

**Properties of Reinforced Concrete.**

**Modular Ratio.**—An important property of reinforced concrete is the modular ratio, that is the ratio of the elastic modulus of steel to that of concrete, because the numerical value of this ratio is required in design in accordance with the ordinary "elastic" theory. Since the elastic modulus of a material is the theoretical stress that produces unit elastic strain, and since strain is the increase (or decrease) of unit length of material when it is subjected to compression or tension, the relation between stress, strain, and elastic modulus is

\[
\text{Elastic modulus} = \frac{\text{stress}}{\text{strain}} = \frac{\text{load}}{\text{area}} \times \frac{\text{original length}}{\text{increase in length}}
\]

The elastic strain of structural materials at working stresses is exceedingly small, being of the order of 0.001; the elastic modulus is therefore numerically greater than a thousand times the working stress. For example, the modulus of elasticity of ordinary mild steel at stresses less than the limit of proportionality is about 30,000,000 lb. per square inch, but for cold-worked bars it is generally lower, say, 27,000,000 lb. per square inch. The elastic modulus of concrete is much more variable, and increases with increase in cement content. It also varies with age and with the number of times stresses are successively induced and relaxed, and is different in compression and tension. Generally, the elastic modulus of concrete is not less than 1,600,000 lb. per square inch, but may exceed 5,000,000 lb. per square inch. The nominal value commonly assumed in ordinary calculations is 2,000,000 lb. per square inch. The modular ratio
of mild steel and concrete of ordinary quality is therefore $30 \times 10^6$, that is $15$, which is the nominal value generally recommended for the design of common structures. Tests show that when deformation is considered or high-strength concrete is used a smaller modular ratio should be adopted, and a reasonable ratio seems to be $40,000$ divided by the crushing strength. The common ratio of $15$ should not be thought of as the exact numerical value of the modular ratio, but rather as a coefficient introduced into the calculation of the resistance of a reinforced concrete member to allow for the assumptions made and the factors neglected in the derivation of working formulae, in addition to taking into account the difference between the moduli of concrete and steel.

**Poisson's Ratio.**—A primary stress in one direction in a material produces a secondary stress in a direction at right-angles to that of the primary stress. The ratio of the secondary stress to the primary stress, called Poisson's ratio, is variable for concrete. An average value of $0.15$ is sometimes recommended. In many cases of ordinary calculations this effect can be neglected, although the inclusion or omission of Poisson's ratio can make a substantial difference to the numerical results obtained, for example, when considering panels of slabs spanning in two directions as discussed in Chapter XV, Vol. II.

**Changes of Temperature.**—The thermal properties of concrete are of importance in the design of chimneys, tanks containing hot liquids, arches, long buildings, and other structures in which provision is made to resist the stresses due to changes of temperature (see Chapter XIV, Vol. II), unless the strains due to such changes are limited by dividing the structure by expansion gaps (generally miscalled joints). The coefficient of linear expansion and contraction of concrete may vary between $5 \times 10^{-6}$ and $6.5 \times 10^{-6}$ per degree Fahrenheit, the lower value being for $1:2:4$ concrete and the higher for concrete rich in cement. An average of $0.000055$ per deg. Fahr. is commonly assumed and is about the same as that for mild steel.

**Thermal Conductivity.**—The thermal conductivity of concrete, on which its insulation value depends, varies with the density of the material. Lightweight cellular or porous concrete has a much lower conductivity than dense structural concrete and therefore has greater insulation value. The conductivity of ordinary structural concrete, expressed as the number of British thermal units which pass in one hour through 1 sq. ft. of concrete of 1 in. thickness is about 7 for each degree Fahrenheit difference of temperature between the two faces.

**Heat of Hydration of Cement.**—Heat is generated by the chemical process which occurs as water acts upon cement. This heat of hydration may be unable to dissipate from the interior of a large mass of concrete quickly enough to avoid a considerable increase of internal temperature,
and consequently strains due to differential thermal conditions are produced. These strains, and the stresses they produce, may be avoided or reduced by placing the concrete in batches at such intervals that each batch may cool before adjacent batches are deposited, by using a cement which produces less heat than ordinary Portland cement, by inserting artificial cooling devices in the concrete, or by two or more of these methods in combination.

**Shrinking of Concrete.**—Concrete tends to shrink as it dries out during hardening (Chapter XIV, Vol. II). If a reinforced concrete member is restrained in such a way that it cannot contract freely, the tendency to shrink sets up tensile stresses in the concrete and corresponding compressive stresses in the reinforcement. The linear coefficient of contraction due to setting of concrete, if the member is entirely unrestrained either externally or by the presence of reinforcement, is about 0.0005 if the concrete can dry out completely. About half this contraction may take place during the first month after casting. Maintaining concrete damp during its early age reduces, or may even prevent, contraction due to drying out until the concrete has obtained sufficient strength to resist the tensile forces induced by external or internal restraint. If a restrained member cannot become completely dry, as in reservoirs and the like, a maximum shrinkage coefficient of 0.0002 is generally adopted in design. Since this shrinkage may correspond to a stress of about 500 lb. per square inch, it is important that measures be taken to resist shrinkage stresses by providing a strong and well-cured concrete, or to neutralise them entirely or in part by providing contraction gaps or by constructing the member in short lengths. In some structures, such as arches, the effect of shrinking is taken into account in the stress analysis, but generally this effect is omitted from the calculations.

**Creep and Plastic Yielding of Concrete.**—In addition to elastic strain, other deformations of strained concrete occur in the course of time. Creep is the slow permanent contraction, or extension, that occurs when concrete is subjected to compression, or tension. Creep proceeds more slowly in the course of time and appears to be proportional to the stress in the concrete, and to be greater the earlier the age of the concrete when it is loaded. The magnitude of this deformation at ordinary working stresses is about the same as the strains due to shrinking. The ultimate effect of creep is to reduce the stress in the concrete and therefore require more resistance from the reinforcement. The allied phenomenon of plastic yield, which occurs when concrete is overstrained, is of some assistance to the designer since yielding of over-stressed parts of a monolithic structure may be offset by the inherent strength of adjacent and less highly-stressed parts.

**Fire Resistance.**—Concrete has a high degree of fire resistance, a property which was an important factor in the development of reinforced
concrete in its early days. The resistance to fire depends primarily on
the type of aggregate, but the thickness of the member and the thickness
of concrete covering the reinforcement also have an effect. Hard limestone
aggregate is more resistant to fire than other stones such as flint gravel.
The effect of the thickness of the member is such that a reinforced concrete
wall 9 in. thick has a resistance of three times that of a wall 4 in. thick.
Similarly a solid slab 6 in. thick, with 1-in. cover of concrete over the
bars, has a resistance four times that of a 4-in. slab with \( \frac{1}{16} \)-in. cover.

It is seen from the foregoing that the properties of concrete and its
reinforcement are more complex than those of some other structural
materials, and that consideration of these properties and the assessment
of reasonable working stresses make the design of reinforced concrete
structures an interesting engineering operation and not merely a matter
of unconsidered adoption of the recommendations of codes of practice and
building regulations.

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CHAPTER II

RESISTANCE TO CONCENTRIC THRUST

The simplest structural member is a short strut subjected to a concentric thrust, that is a compressive load the line of action of which is parallel to the longitudinal axis of the strut and passes through the centroid of its cross-section. For this condition the compressive strain is uniform over any section at right-angles to the longitudinal axis of the strut. If the strain is uniform, the stress in a strut of homogeneous material is likewise uniform. Because the elastic properties of concrete and steel differ, reinforced concrete is not homogeneous; therefore the stresses in the concrete and steel differ although the strain is the same in each material. If the line of action of the thrust does not pass through the centroid the load is eccentric, the strains and stresses are not uniform, and the complex condition considered in Chapter IV is produced. The effect of a concentric thrust on a simple reinforced concrete strut is considered in the following by alternative methods of analysis, namely, the equal-strain or modular-ratio method, and the load-factor or ultimate-load method. The different bases of these two fundamental methods is seen clearly in the consideration of members subjected to concentric thrust. The most common example of a reinforced concrete member that can be designed as a short strut with concentric load is an interior column of a building; details of a typical column of this type are shown in the diagram on page 3.

Equal-strain or Modular-ratio Method.

If a block of reinforced concrete (Fig. 2) is subjected to a concentric compressive force it contracts, and if the force is much less than that which would cause failure the contraction is proportional to the applied force. The contraction is the same for the concrete and the embedded reinforcement so long as the bond between the concrete and steel bars is not broken. For equal compressive strain, which is the amount of contraction in unit length, the compressive stresses in the concrete and steel are proportional respectively to the elastic moduli of the two materials. If the elastic modulus $E_s$ of steel is $m$ times greater than the elastic modulus $E_c$ of concrete, that is the modular-ratio $\frac{E_s}{E_c}$ is $m$, the stress in the reinforcement is $m$ times the stress in the surrounding concrete. Therefore unit area of reinforcement is $m$ times as effective in resisting compression as unit area of concrete. The equivalent cross-sectional area $A_s$ of a reinforced concrete member can therefore be expressed as the sum of the net area $A_c$ of the concrete, excluding the reinforcement, and $m$ times the
area $A_{re}$ of the reinforcement, that is $A_e + m A_{re}$. It is more convenient to express the equivalent area in terms of the gross cross-sectional area $A$ of the member, which is $A_e + A_{re}$. Therefore

$$A_e = A + (m - 1) A_{re}. \quad (2.1)$$

This formula can be readily applied to members of regular cross section such as the common sections in Fig. 2, namely,

- Rectangular member: $A_e = b d + (m - 1) A_{re}$
- Square member: $A_e = d^2 + (m - 1) A_{re}$
- Octagonal member: $A_e = 0.828d^2 + (m - 1) A_{re}$ \quad (2.1a)

Fig. 2.—Reinforced Concrete Member Subjected to Concentric Thrust.

The centroid of such regular cross-sections, the position of which is defined by the co-ordinates $\bar{X}$ and $\bar{Y}$ in Fig. 2, coincides with the geometrical centre if the reinforcement is symmetrically disposed; that is $\bar{X} = \frac{1}{2} d$ and $\bar{Y} = \frac{1}{2} b$. The equivalent cross-sectional areas of reinforced concrete members of other regular and irregular shapes and the positions of the centroids are given in Chapter XIII, Vol. II. Rectangular sections with unsymmetrical reinforcement are considered in Chapter III. Only if the line of action of the applied thrust passes through the centroid as established by $\bar{X}$ and $\bar{Y}$ is the thrust concentric and the strain uniform.
For equilibrium, the total compressive force in the concrete and steel must be equal to the applied force. The stress in the concrete is therefore the concentric force divided by the equivalent cross-sectional area, that is if a concentric thrust $F$ is applied to a strut of effective cross-sectional area $A_e$, the compressive stress $f_{ce}$ in the concrete is $\frac{F}{A_e}$ and the compressive stress $f_{se}$ in the steel is $mf_{ce}$. Conversely, if the compressive stress permissible in the concrete is $p_{ce}$, the safe concentric thrust $P_0$ which the member can withstand safely is $p_{ce}A_e$. The safe thrust generally depends on the stress in the concrete and not on the stress in the steel because, although $f_{se}$ is $m$ times $p_{ce}$, and $m$ may be from 10 to 18, $mp_{ce}$ is much less than the safe stress $p_{se}$ permissible in the steel. For example, if $p_{ce}$ is 760 lb. per square inch and $m$ is 15, $f_{se}$ is 11,400 lb. per square inch, whereas the compressive stress $p_{se}$ generally permissible in mild steel is 18,000 lb. per square inch as explained on page 22.

By substituting $A + (m - 1)A_{se}$ from formula (2.1) for $A_e$ in $\frac{F}{A_e}$, two basic working formulæ are obtained for the modular-ratio method of design of concentrically-loaded columns and other struts, namely:

Compressive stress in concrete due to thrust $F$:

$$f_{ce} = \frac{F}{A + (m - 1)A_{se}}. \quad \ldots \quad (2.2a)$$

Safe thrust on concentrically-loaded member $i$:

$$P_0 = [A + (m - 1)A_{se}]p_{ce}. \quad \ldots \quad (2.2b)$$

By substituting appropriate values for $A$, the stress in, or safe thrust on, a member of any cross-section is obtained. For example, for a member of square cross-section (Fig. 2),

$$P_0 = [d^2 + (m - 1)A_{se}]p_{ce}. \quad \ldots \quad (2.2c)$$

The modular-ratio method of analysis was at one time the common basis for the design of concentrically-loaded reinforced concrete columns, but the load-factor or ultimate-load method is now more generally used. The modular-ratio method is developed in the foregoing since it is the basis of more complex analyses and illustrates the interaction between the concrete and the steel bars embedded therein. A comparison of the two methods is given in Example No. 1 (page 34).

**Load-factor Method.**

The load-factor or ultimate-load method of designing a column or strut subjected to a concentric load is based on determining the smallest thrust or compressive load which causes the member to fail and then assessing the safe thrust or load as a fraction of the load that would cause failure. Just before failure in compression of a short strut the compressive stress in the concrete is almost equal to its crushing strength $u$ (as measured by
crushing 6-in. cubes at twenty-eight days); at the same time the com-
pressive stress in the reinforcement is almost the yield-point or equivalent
yield stress \( \sigma_y \) of the steel. Therefore the safe concentric thrust is given
by an expression of the form \( \frac{A_e \sigma_u}{x} + \frac{A_s \sigma_y}{y} \), in which \( x \) and \( y \) are coefficients
embodying suitable factors of safety, say, about three in \( x \) and two in \( y \).
A working formula can be derived if \( \frac{u}{x} = p_{ce} \) and \( \frac{\sigma_y}{y} = p_{se} \), that is equal
to the safe compressive stresses in the concrete and steel respectively.
Therefore the safe concentric load is \( A_e p_{ce} + A_s p_{se} \), or
\[
P_0 = A_e p_{ce} + A_s (p_{se} - p_{ce}).
\]  
(2.3)
This formula is the basis of the current methods of design of concentrically-
loaded columns and can be simplified for members of regular cross-section
(Fig. 2) as follows.

Rectangular member: \( P_0 = bd p_{ce} + A_s (p_{se} - p_{ce}) \).
Square member: \( P_0 = d^2 p_{ce} + A_s (p_{se} - p_{ce}) \).
(2.3a)

**Comparison of the Alternative Methods.**

Comparison of formulae (2.2c) and (2.3a) shows that the safe load as
calculated by the load-factor method is greater than that calculated by
the modular-ratio method, because in the load-factor method it is assumed
that the resistance of the reinforcement is greater than that assumed when
the modular-ratio method is used. The resistance of the concrete is the
same in both methods.

**Example No. 1.—Safe Load on a Short Column.** Compare the
safe loads on a “short” column 16 in. square of \( 1:2:4 \) ordinary concrete
reinforced with four \( \frac{3}{4} \)-in. bars, the safe load being calculated by the
modular-ratio and load-factor methods. (A “short” column is defined on
page 37.)

Referring to Fig. 2, \( d = 16 \) in. and \( A_e = 2.41 \) sq. in. The compressive
stresses \( p_{ce} \) and \( p_{se} \) are 760 lb. per square inch in the concrete and 18,000 lb.
in the steel, as described on pages 18 and 22. By the modular-ratio
method, with a modular ratio \( m \) of 15, substitution in formula (2.2c) gives the safe load
\[
P_0 = 760[16^2 + (14 \times 2.41)] = 220,000 \text{ lb.}
\]
Similarly by the load-factor method, substitution in formulae (2.3a) gives
\[
P_0 = (16^2 \times 760) + 2.41(18,000 - 760) = 235,000 \text{ lb.},
\]
which is nearly 7 per cent. greater than the safe load calculated by the
modular-ratio method. If more longitudinal reinforcement were provided
than in this example, the difference would be even greater.

**Reinforcement in Columns.**

The reinforcement considered in the foregoing comprises the main
longitudinal bars which are provided as shown in Fig. 2. The limiting
proportions of the amount of main reinforcement in columns are described on page 120. The main bars are in compression and therefore each bar acts as a slender strut which, if not otherwise secured, tends to buckle outwards since the resistance of the thin cover of concrete outside the bars is negligible. The bars must therefore be restrained by being tied together by steel binders extending around them as indicated in Fig. 2. The binders are generally separate links, and suitable sizes and spacings are considered in the design of complete columns on page 194. If the binders are spaced very closely, they have a further strengthening effect in that they act as a sheath around the concrete core. When a column is compressed in the direction of its axis the concrete tends to spread sideways and, if this lateral spreading is restrained, a greater axial compression of the concrete in the core can be resisted.

Columns with Helical Binding.

The fact that the binding adds to the strength of a column is made use of in the design of heavily-loaded columns by providing a continuous helix of binding around the longitudinal bars as shown in the column of octagonal cross-section in Fig. 2. The apparent increase in strength of columns with helical binding has been investigated experimentally and the formulae in general use are logical but more or less empirical. Only the concrete in the core, that is within the binding, is effectively restrained by the binding, and therefore the concrete outside the binding is neglected in assessing the safe load on the column.

The safe load is the sum of three factors: (1) The compressive resistance \( P_c \) of the concrete in the core, which is \( A_k p_{oc} \) where \( A_k \) is the cross-sectional area of concrete in the core of diameter \( D_k \), the mean diameter of the helix. Hence \( P_c = A_k p_{oc} = \left( \frac{\pi}{4} D_k^2 - A_{sc} \right) p_{oc} \); (2) The compressive resistance \( P_s \) of the longitudinal reinforcement, which is \( A_{sc} p_{sc} \) as for ordinary columns designed by the load-factor method; and (3) The additional resistance \( P_B \) due to the restraining effect of the helical binding, which is proportional to the amount of binding and to the tensile stress \( f_{et} \) permissible in the binding.

The load \( P_B \) can be estimated as follows. If \( A_{et} \) is the cross-sectional area of the bar or wire forming the helix, the safe circumferential tension is \( A_{et} f_{et} \) in one turn of the helix, or \( \frac{A_{et} f_{et}}{s} \) per unit length of column if \( s \) is the pitch of the turns of the helix. If \( f_{eb} \) is the equivalent axial compressive stress in the concrete to produce \( P_B \) such that \( P_B = A_k f_{eb} = \frac{1}{2} \pi D_k^2 f_{eb} \), the corresponding radial stress acting horizontally is \( \mu' f_{eb} \), where \( \mu' \) is a factor allowing for Poisson's ratio and the elastic strain in the binding; if \( \mu' \) is, say, \( \frac{1}{3} \), the radial stress is \( \frac{1}{3} f_{eb} \) and produces a circumferential
tension in the helical binding of \((\frac{f_{ce}}{D_k})(\frac{s}{D_k})\) per unit length of column. Thus
\[
\frac{A_{st}p_{st}}{s} = (\frac{f_{ce}}{D_k})(\frac{s}{D_k}),
\]
from which \(f_{ce} = \frac{8A_{st}p_{st}}{5D_k}\); by substitution,
\[
P_B = \frac{\pi D_k}{4} \left(\frac{8A_{st}p_{st}}{5D_k}\right) = 2\pi D_k A_{st} p_{st} \left(\frac{1}{s}\right).
\]
The volume of the helix in length \(s\) of the column is \(\pi D_k A_{st}\) and therefore the equivalent area \(A_B\) of the helical binding, that is the volume of binding in unit length of column, is \(\frac{\pi D_k A_{st}}{s}\). Therefore \(A_{st} = \frac{A_B s}{\pi D_k}\). Substituting for \(A_{st}\) in the last expression for \(P_B\) and reducing gives \(P_B = 2p_{st} A_B\). If the stress \(p_{st}\) permissible in the binding is 13,500 lb. per square inch, \(P_B = 27,000 A_B\) lb., which is the empirical value commonly recommended. The foregoing analysis, which is over-simplified, is an indication of a basis for assessing the empirical value of the binding.

The total safe load \(P_0\) on a helically-bound column is therefore \(P_c + P_s + P_B\), that is
\[
P_0 = \left(\frac{\pi D_k}{4}\right)^2 - A_{sc} \right) p_{cc} + A_{sc} p_{sc} + 2p_{st} A_B. \quad \quad (2.4a)
\]
It is convenient to express the amount of binding by \(A_B\) which is half the cross-sectional area of helical binding in a vertical cross-section of unit length (1 ft.) of the column; that is
\[
A_B = \frac{12A_{st}}{s} = \frac{12}{s} \left(\frac{A_B s}{\pi D_k}\right), \quad \text{and} \quad A_B = \frac{\pi D_k}{12} A_{st}.
\]
Substituting in formula (2.4a) and re-arranging,
\[
P_0 = \frac{\pi D_k}{4} p_{cc} + A_{sc} (p_{sc} - p_{cc}) + 0.523 D_k A_B p_{st}. \quad \quad (2.4b)
\]
or, if \(p_{st} = 13,500\) lb. per square inch, \(A_B = \frac{12A_{st}}{s}\), and all units are inches and pounds,
\[
P_0 = 0.7854 D_k p_{cc} + A_{sc} (p_{sc} - p_{cc}) + 7060 A_B D_k. \quad \quad (2.4c)
\]
The corresponding formula in British Standard Code No. 114 gives identical results and, since \(A_B = \frac{\pi D_k}{12} A_{st}\), is
\[
P_0 = p_{cc} A_E + p_{sc} A_{sc} + 27,000 A_B. \quad \quad (2.4d)
\]
Formula (2.4c) is given as being more suitable for the practical design of helically-bound columns. On page 196 the limitations of the size of the bar forming the helix, the pitch of the helix, the proportion of longitudinal reinforcement, and the load \((P_c + P_B)\) supported by the concrete are considered. The example in the following illustrates the difference in the calculated safe load on a column or strut when the effect of helical binding is taken into account and otherwise.

**Example No. 2.—Safe Load on a Column with Helical Binding.** Compare the safe loads, taking into account the helical binding and other-
wise, on a short column of octagonal cross-section 18 in. across the flats
\((d = 18 \text{ in.})\) constructed in \(1 : 2 : 4\) ordinary concrete reinforced with eight
\(\frac{3}{4}\)-in. mild-steel bars longitudinally \((A_{es} = 3.53 \text{ sq. in.})\) and \(\frac{3}{4}\)-in. helical
binding at \(2\frac{1}{4}\)-in. pitch \((A_B = 0.53 \text{ sq. in. per foot})\). Assume that the
diameter \(D_2\) of the core is 16 in.

The safe load on the column taking into account the helical binding
is calculated by substitution in formula (2.4c), that is
\[
P_0 = (0.7854 \times 16^2 \times 760) + [3.53 \times (18,000 - 760)] + (7060 \times 0.53 \times 16)
= 153,000 + 60,860 + 60,000 = 273,860 \text{ lb}.
\]
To calculate the safe load (neglecting the effect of the binding) the entire
cross-sectional area of the octagon is taken into account, and not only
the area of the circular core; that is the gross area \(A\) is \(0.828 \times 18^2 = 268\)
sq. in. Substitution in formula (2.3) gives
\[
P_0 = (268 \times 760) + [3.53(18,000 - 760)] = 204,000 + 60,860 = 264,860 \text{ lb}.
\]
In this example there is a slight advantage in providing helical binding,
although the extra cost of such binding might not be worth while for so
small a gain in load-carrying capacity.

**Slender Struts and Columns.**

The formulæ derived in the foregoing for the safe load on simple com-
pression members apply to a member which is so short that there is no
risk of failure by buckling. If, however, the member is long and flexible
the risk of buckling may be more serious than the risk of the material
failing as a result of direct crushing. It is generally considered that if
the effective length of a reinforced column does not exceed 15 times the
least lateral dimension buckling need not be considered, but if of greater
length, the common method of allowing for this effect is to reduce the
safe load \(P_0\) calculated for a short member, the amount of the reduction
depending on the slenderness of the member.

**Load-reduction Factor and Slenderness.**—If \(R_L\) is the load-
reduction coefficient for a slender or long column, the safe load on such
a column is \(R_L P_0\). Expressions for the reduction coefficient \(R_L\) incorporate
the slenderness factor which is the ratio of the effective length \(l\) to the
least radius of gyration \(k\), and may take the form \(K_1 - K_2 \frac{l}{k}\) in which
\(K_1\) and \(K_2\) are numerical constants. One formula of this type is
\[
R_L = 1.5 - \frac{l}{100k} \quad . \quad . \quad . \quad \quad (2.5a)
\]

For a square column \(k = 0.29d\) and \(R_L\) is \(1.5 - \frac{l}{29d}\) if the reinforce-
ment is neglected. This is almost the same expression as those on which
the reduction coefficients are based in the British Standard Code when
the slenderness factor, expressed as the ratio of \(l\) to \(d\) does not exceed 33,
namely
\[
R_L = 1.5 - \frac{l}{30d} \quad . \quad . \quad . \quad \quad (2.5b)
\]
If the slenderness factor exceeds 33 but does not exceed 57, the corresponding expression is

$$R_L = 0.95 - \frac{l}{60d}$$  \hspace{1cm} (2.5c)$$

In accordance with these expressions, $R_L$ is unity (that is there is no reduction) if $\frac{l}{d} = 15$ and is zero (that is there is no safe load) if $\frac{l}{d} = 57$. Intermediate values are interpolated linearly.

**Effective Length.**—The effective length of a strut or column depends on the degree of restraint at the ends. A column rigidly fixed at both ends is less likely to buckle than a column of the same length with hinged ends. With the recommendations of the B.S. Code as a guide, the effective length $l$ expressed in terms of the actual length $L$ (Fig. 3) are: (a) If the column is hinged at both ends the effective length is the same as the actual length. (b) If the column is fixed in position and direction at both ends, $l$ is $\frac{1}{4}L$. (c) For the intermediate condition of one end being fixed and the other end being fixed in position but imperfectly restrained in direction, $l$ is between $\frac{1}{4}L$ and $L$ depending on the degree of directional restraint. (d) For the more serious case of one end being fixed and the other end imperfectly restrained in position and direction, $l$ is between $L$ and $2L$ depending on the degree of end-restraint. A fixed end is one at which the column is restrained in position and direction so that it can neither rotate at that end nor can one end move laterally relative to the other end. A hinged end is one where the column is restrained in position but not in direction, that is the column is free to rotate at the end but the end cannot move laterally relative to the other end. In the case of columns in multiple-story buildings $L$ is the height of a story.

**Least Lateral Dimension.**—The least lateral dimension is the length $d$ of the side of a square member or the distance between opposite faces of a member the cross-section of which is a regular polygon, or the length $b$ of the narrower side of a rectangular member. If a column with helical binding is designed so that the safe load depends in part on the resistance
of the binding, the least lateral dimension is the diameter of the core. For the foregoing sections, and other sections that are symmetrical about each of two axes mutually at right-angles, the least lateral dimension is easily identified, and substitution of the slenderness ratio in formulae (2.5b) or (2.5c) gives the reduction coefficient. The least lateral dimension of members of cross-sections other than the foregoing, for example, unsymmetrical sections having re-entrant angles, is not so readily identified. It is generally recommended that the least lateral dimension is the thickness of the flange or web of a member of tee, ell, or cruciform cross-section so long as the flanges (or projecting parts) are twice or more long as they are broad.

Radius of Gyration.—In cases where the least lateral dimension is not identifiable, the slenderness factor can be based on the least radius of gyration, which is calculated from the relation \( \sqrt{\frac{\text{moment of inertia}}{\text{equivalent area}}} \). For the purpose of determining the slenderness factor, it is not worth while considering the small effect which the reinforcement has on the radius of gyration. To find the least radius of gyration it is necessary to decide by inspection (or calculation) about which axis the radius of gyration is the smaller. For regular sections (neglecting the reinforcement) the least radius of gyration can generally be calculated readily from well-known expressions for the moment of inertia and area; many published tables give these data and some also give radii of gyration. The radii of gyration of several sections are given in Chapter XIII.

Example No. 3.—Safe Load on a Slender Column. Calculate the reduced safe load on the octagonal column in Example No. 2 if the column is 40 ft. high and is fixed at both ends. Therefore \( L = 40 \times 12 = 480 \) in.; \( l = \frac{4}{3} \times 480 = 360 \) in.

If the safe load is calculated without taking the binding into account, the least lateral dimension is the overall width of the column, that is \( d = 18 \) in., and the slenderness factor \( \frac{l}{d} \) is \( \frac{360}{18^2} \), that is 20, which exceeds 15. Substituting in formula (2.5b), the reduction coefficient \( R_L \) is

\[
1.5 - \frac{360}{30 \times 18} = 0.83
\]

and the reduced safe load is 0.83 \( \times 264,860 = 220,000 \) lb.

If the safe load is calculated with the helical binding taken into account, the least lateral dimension is the diameter of the bound core, that is \( d = 16 \) in. Therefore \( \frac{l}{d} = \frac{360}{16} = 22.5 \) and \( R_L = 1.5 - \frac{360}{30 \times 16} = 0.75 \)

and the reduced safe load is 0.75 \( \times 273,860 = 205,400 \) lb., which is less than the reduced safe load calculated without taking the binding into account.
Chapter III
Resistance to Simple Bending

A structural member subjected to bending only is common, and resistance to such bending is the basis of the design of beams. Bending action produces deformation that results in compressive strains at one side of the member and tensile strains at the opposite side. Intermediately there is a position where there is no strain, that is where the compressive strain is changing to tensile strain. The axis passing through this position is commonly called the neutral axis. The plane of the member in which the neutral axis lies is the neutral plane and is at right-angles to the plane in which bending takes place. Formulae for resistance to bending are derived from the fundamental principles of equilibrium, namely, that the total resistance of the compressive zone on one side of the neutral plane is equal to the total resistance of the tensile zone on the other side, and that the magnitude of the couple formed by the two resistances is equal to the applied bending moment. In this chapter the resistance to bending of reinforced concrete beams is based on the two common methods of analysis, namely, the modular-ratio method and the load-factor method. The analysis is restricted mainly to beams of rectangular and simple tee cross-sections; beams of more complex cross-sections are dealt with in Chapter XIII, Vol. II.

I. MODULAR-RATIO METHOD.

In the design of reinforced concrete beams there are three cases to consider, namely, (i) the bending moment which a beam can resist without exceeding the permissible stresses, that is the maximum safe moment of resistance; (ii) the greatest stresses produced in a beam of known size when subjected to a specified bending moment; and (iii) the size of a beam and the amount of reinforcement required to resist a specified bending moment with stresses not exceeding the permissible stresses. Before determining the resistance of beams, the distribution of the stresses, the position of the neutral plane, and the magnitude of the arm of the resisting couple (generally called the lever arm) are considered.

Distribution of Stresses.

The assumptions made in the modular-ratio method are that at any cross-section of a beam the strain in a longitudinal plane is proportional to the distance of the plane from the neutral plane (that is rectilinear
variation of strain is assumed), and that the concrete and steel are elastic within the range of the permissible stresses. It is also common to neglect the tensile resistance of the concrete and to assume that the reinforcement in the tensile zone resists all the tensile forces; an exception to this assumption is made principally in the design of containers of liquids. The rectilinear variation of strain is illustrated in Fig. 4 for a beam of any general cross-section for the condition of the concrete not being resistant to tensile forces and the variation of stress when so resistant. For both conditions there is a compression zone above the neutral plane and a tensile zone below.

**Concrete Effective in Tension.**—If the concrete is assumed to resist tension, the entire cross-section of the beam for the full depth, that is the overall depth $d$, is subjected to stress. The concrete in the tensile zone is in a state of tensile strain, and therefore tensile stress, similar to a beam of any homogeneous material such as steel. Reinforcement in both the tensile zone and the compressive zone is subjected to a greater stress than, but the same strain as, the concrete which surrounds the steel; as is explained on page 32, the stress in the reinforcement is $m$ times the stress in the surrounding concrete. As in a beam of homogeneous material, the neutral plane passes through the centroid of the cross-section, but there is an important difference in the case of a reinforced concrete beam in which the centroid is not necessarily the centroid of the geometrical cross-section but is the centroid of the equivalent cross-section, which is the geometrical cross-section modified to take into account that an area of reinforcement is equivalent to $m$ times that area of concrete. The condition when the concrete is effective in tension is considered on page 69.

**Concrete Ineffective in Tension.**—It is more common to assume that the whole of the concrete in the tensile zone may have cracked and therefore can offer no resistance and that all the tension is resisted by the reinforcement provided for this purpose. It is generally assumed that

![Fig. 4.—Stresses due to Bending (Modular-ratio Method).](image)
the reinforcement bars in tension are all grouped together at a distance \( d_t \) above that edge of the beam which is in tension. The distance \( d_1 \) from the compressed edge to the centre of the reinforcement in tension is called the effective depth (\( d_1 = \bar{d} - d_t \)). This dimension is important because it is the basis of the calculation of the resistance of the beam in the same way that the overall depth \( \bar{d} \) is the basic dimension if the concrete is assumed to resist tension. In the latter condition the stress in the concrete varies from a maximum compressive stress \( f_{cb} \) at the compressed edge to a maximum tensile stress \( f_{st} \) at the edge in tension, as shown in Fig. 4. If it is assumed that the concrete does not resist tension the variation is from a maximum compressive stress \( f_{cb} \) in the concrete at the compressed edge to no stress at the neutral plane, and a tensile stress \( f_{st} \) in the reinforcement. The latter case is dealt with in the following.

**Position of the Neutral Plane in Relation to the Stresses.**

**In Terms of Maximum Stresses.**—Irrespective of the shape of the cross-section of a beam the position of the neutral plane can be determined in terms of the maximum stresses, that is the maximum compressive stress in the concrete and the tensile stress in the reinforcement, assuming that the concrete does not resist tension. Considering a beam of the general cross-section in Fig. 4, the strain in the reinforcement in tension is \( \frac{f_{st}}{E_s} \), and the maximum strain in the concrete is \( \frac{f_{cb}}{E_c} \). If \( d_n \) is the distance from the compressed edge of the section to the neutral plane, by similar triangles

\[
\frac{f_{cb}}{E_c} : \bar{d}_n = \frac{f_{st}}{E_s} : \bar{d}_1 - d_n, \quad \text{from which, since} \quad m = \frac{E_s}{E_c},
\]

\[
\bar{d}_n = n_1 \bar{d}_1 = \frac{\bar{d}_1}{1 + \frac{f_{st}}{m f_{cb}}}, \quad \ldots \quad (3.1a)
\]

**Neutral-plane Factor.**—It is convenient to express the ratio of the depth \( d_n \) to the neutral plane to the effective depth \( \bar{d}_1 \), by a term \( n_1 \left( \frac{d_n}{\bar{d}_1} \right) \) which is called the neutral-plane factor. Formula (3.1a) can therefore be re-written as

\[
n_1 = \frac{1}{1 + \frac{f_{st}}{m f_{cb}}}, \quad \ldots \quad (3.1b)
\]

**In Terms of the Permissible Stresses.**—Formulas (3.1a) and (3.1b), expressing the position of the neutral plane in terms of the greatest stresses induced in the beam, can be re-written in terms of the permissible stresses, namely, \( p_{cb} \) in the concrete in compression and \( p_{st} \) in the rein-
force in tension, that is the neutral-plane factor is given by

\[ n_1 = \frac{I}{I + \frac{P_{st}}{m f_{cb}}} \quad (3.1c) \]

which is an important formula in design.

**Example No. 4.—Neutral-plane Factor.** Calculate the neutral-plane factor for permissible stresses of 1000 lb. per square inch in compression in ordinary 1:2:4 concrete and 20,000 lb. per square inch in tension in mild steel bars. Substituting \( P_{st} = 20,000 \text{ lb.}, \) \( f_{cb} = 1000 \text{ lb. per square inch, and } m = 15 \) in formula (3.1c) gives

\[ n_1 = \frac{I}{20,000} = 0.0428. \]

Since these stresses are those commonly permissible in the design of buildings, this numerical value is of importance.

**Example No. 5.—Position of Neutral Plane.** Determine the position of the neutral plane in a beam 20 in. deep if the centre of the reinforcement in tension is \( \frac{1}{8} \) in. from the edge in tension and the permissible stresses are as in Example No. 4. Therefore \( d_1 = 20 - \frac{1}{8} = 18\frac{1}{8} \text{ in.} \)

From formula (3.1a), \( d_n = \frac{18.5}{20,000} = 0.0428 \times 18.5 = 7.93 \text{ in.} \)

**Maximum Stresses and Ratio of Maximum Stresses.**—Formula (3.1b) can be transposed to express the compressive stress if the tensile stress and value of \( n_1 \) are known, and vice versa; that is

\[ f_{cb} = \frac{n_1 f_{st}}{m(I - n_1)}; \quad f_{st} = \frac{(I - n_1)m f_{cb}}{n_1}. \quad (3.1d) \]

Another useful formula is that expressing the ratio of stresses and is derived by re-arranging formulae (3.1d) to give

\[ \frac{f_{st}}{f_{cb}} = \frac{m(I - n_1)}{n_1} = m\left(\frac{I}{n_1} - 1\right). \quad (3.1e) \]

**Position of the Neutral Plane in Relation to the Amount of Reinforcement.**

The position of the neutral plane can also be expressed in terms of the amount of reinforcement.

**Rectangular Beam with Reinforcement in Tension and Compression.**—Consider the rectangular beam (Fig. 5) with reinforcement in tension and compression, having an overall depth \( d \), an effective depth \( d_1 \), and a breadth \( b \). The cross-sectional area of the reinforcement in tension is \( A_{st} \) and of the reinforcement in compression \( A_{sc} \). The rectilinear variation of strain results in the stress diagram as shown in Fig. 5. The
area of concrete in compression is \(d_nb - A_{sc}\), and the average compressive stress in the concrete is \(\frac{1}{2}f_{cb}\). The compressive resistance of the concrete is therefore \(\frac{1}{2}f_{cb}d_nb - (f)A_{sc}\). For conditions of equal strain the compressive stress in the compression reinforcement is \(m\) times the stress in the surrounding concrete, that is \((\frac{d_n - d_2}{d_n})mf_{cb}\), in which \(d_2\) is the distance from the compressed edge to the centre of the reinforcement in compression.

![Diagram](image)

**Fig. 5.—Rectangular Beams (Modular-ratio Method).**

The compressive resistance of the reinforcement in compression is therefore \((\frac{d_n - d_2}{d_n})mf_{cb}A_{sc}\). The total compressive resistance of the concrete and reinforcement must, for equilibrium, be equal to the tensile resistance, that is, \(A_{st}f_{st}\). The equation of equilibrium is

\[
\frac{1}{2}f_{cb}(d_nb) - (\frac{d_n - d_2}{d_n})f_{cb}A_{sc} + (\frac{d_n - d_2}{d_n})mf_{cb}A_{sc} = A_{st}f_{st}.
\]

Denoting the ratio of the reinforcement in tension and compression by \(r_t\) and \(r_c\) respectively (that is \(A_{st} = r_tbd_1\) and \(A_{sc} = r_cbd_1\)), and denoting \(d_2\) by \(f_2d_2\), and substituting for these terms and \(d_n = n_1d\) in the foregoing equation, reducing and transposing gives

\[
\left[\frac{n_1}{2} + \frac{n_1 - f_2(m - 1)r_c}{n_1}\right]f_{cb}bd_1 = r_tbd_1f_{st},
\]

from which \(\frac{n_1}{2} + \frac{n_1 - f_2(m - 1)r_c}{n_1} = r_t(m - 1)r_c\). Substituting for \(\frac{f_{st}}{f_{cb}}\) from formula (3.1e), gives the quadratic equation

\[
\frac{1}{2}n_1^2 + n_1[(m - 1)r_c + r_m] - [r_m + f_2r_c(m - 1)] = 0,
\]

the positive root of which is

\[
n_1 = -[(m - 1)r_c + r_m] + \sqrt{[(m - 1)r_c + r_m]^2 + 2[r_m + f_2r_c(m - 1)]}.
\]

To a reasonable degree of accuracy \(m - 1 = m\), and therefore

\[
n_1 = \sqrt{m^2(r_t + r_c)^2 + 2m(r_t + f_2r_c)} - m(r_t + r_c). \quad (3.2a)
\]

**Example No. 6.—Position of Neutral Plane with Compression Reinforcement.** Determine the position of the neutral plane in the
rectangular beam in Fig. 6 with reinforcement to resist tension comprising two 1-in. bars \(A_{st} = 1.57 \text{ sq. in.}\) and reinforced to resist compression by two \(\frac{3}{4}\)-in. bars \(A_{sc} = 0.88 \text{ sq. in.}\) placed \(1\frac{1}{2}\) in. below the compressed edge \(d_2 = 1.375 \text{ in.}\). Therefore \(r_c = \frac{0.88}{10 \times 18.5} = 0.0048\)

and \(m(r_t + r_c) = 15(0.0085 + 0.0048) = 0.20\) approximately;

\[f_a = \frac{1.375}{18.5} = 0.075.\] Substituting in formula (3.2a),

\[n_1 = \sqrt{(0.20)^2 + (2 \times 15)(0.0085 + 0.0075 \times 0.0048)} - 0.20 = 0.353.
\]

\(d_n = 0.353 \times 18.5 = 6.5\) in. approximately.

**Rectangular Beam with Reinforcement in Tension Only.**—If there is reinforcement in tension only, \(A_{st} = 0\) and \(r_c = 0\). Therefore formula (3.2a) becomes

\[n_1 = \sqrt{(mr_t)^2 + 2mr_t - mr_t}. \quad \ldots \quad \ldots \quad (3.2b)\]

**Example No. 7.—Position of Neutral Plane with Tension Reinforcement Only.** Determine the position of the neutral plane in the rectangular beam in Fig. 6 with reinforcement in tension only. \(d_1 = 18\frac{1}{4} \text{ in.}\) and \(b = 10 \text{ in.}\). The reinforcement in tension comprises two 1-in. mild-steel bars \(A_{st} = 1.57 \text{ sq. in.}\). Since

\[r_t = \frac{1.57}{10 \times 18.5} = 0.0085, \quad mr_t = 15 \times 0.0085 = 0.1275.\]

Substituting in formula (3.2b),

\[n_1 = \sqrt{(0.1275)^2 + (2 \times 0.1275) - 0.1275} = 0.39.\]

Therefore \(d_n = 0.39 \times 18.5 = 7.22\) in. This factor and dimension differ from 0.428 and 7.93 in. as determined for certain permissible stresses in Examples Nos. 4 and 5, thereby indicating that with the specified reinforcement these stresses cannot be realised simultaneously.

**Flanged Beams.**—Formule for the position of the neutral plane can be derived for simple flanged beams such as tee-beams and ell-beams (Fig. 7), which are very common in reinforced concrete construction. The provision of reinforcement in compression in a flanged beam of this type is not common, and therefore the case of reinforcement in tension only
is considered. The width and thickness of the flange are \( b \) and \( d_s \) \( (= d_1 \cdot d) \) respectively; the breadth of the rib is \( b_r \). The overall depth \( d \), the effective depth \( d_1 \), and the cover-ratio \( f_1 \) \( (= \frac{d_1}{d}) \) of the reinforcement in tension \( A_{st} \) \( (= \tau.bp.d) \) are similar to the corresponding dimensions for a rectangular beam. Equating the total compressive resistance of the flange (neglecting

![Diagram](image.png)

**Fig. 7.—Flanged Beams (Modular-ratio Method).**

the small part of the rib in compression) to the total tensile resistance, 
\[
\frac{1}{2}(f_{ob} + f_{cb})bd_s = A_{st}f_{et}.
\]

By proportion from the stress diagram, 
\[
f_{cb} = \frac{a_n - d_s}{d_n}f_{cb} \quad \text{and} \quad f_{et} = \left(\frac{d_1}{d_n} - \frac{a_n}{d_n}\right)m_f_{ob}.
\]
Substituting and transposing gives
\[
d_n = \frac{A_{st}m_{d_1} + \frac{1}{2}bd_s^2}{A_{st}m + bd_s^2}, \quad \ldots \quad (3.2c)
\]

The application of this formula is given in Examples Nos. 11 (page 49), 21 (page 57), and 24 (page 59).

For other sections the formulae for the position of the neutral plane based on the dimensions of the section and the amount of reinforcement are too complex to be of practical value, although if a designer has to deal frequently with a certain section it may be worth while to derive the formula and possibly to prepare a graphical representation for everyday use.

**Lever Arm.**

The magnitude of the internal resistance couple, that is the moment of resistance, is the tensile resistance \( F_{et} \) of the reinforcement in tension (or the total compressive resistance \( F_c \) of the concrete and reinforcement in compression) multiplied by the distance between the lines of action of \( F_{et} \) and \( F_c \). This distance is called the lever arm \( l_a \) \( (= a_1.d_1) \); the term \( a_1 \) is the lever-arm factor. The line of action of the tension \( F_{et} \) passes through the centroid of the reinforcement in tension if all the bars are approximately at the same distance from the neutral plane, and the distance to this centroid from the compressed edge is the effective depth \( d_1 \) as in the rectangular and flanged beams in Figs. 5 and 7. For these
common cases the formula for the lever arm are derived in the following. For beams of more complex cross-sections the corresponding formulae are derived in Chapter XIII.

**Rectangular Beam with Tensile Reinforcement Only.**—In a rectangular beam with reinforcement in tension only (Fig. 5) \( h_c \) is the distance from the compressed edge to the centroid of the triangular stress-diagram, which is the line of action of \( F_{cc} \) (the compression in the concrete) and is equal to \( \frac{1}{3}d_n \); therefore

\[
l_a = l_{ac} = d_1 - \frac{1}{3}d_n; \quad a_1 = i - \frac{1}{3}n_1.
\]

(3.3a)

**Example No. 8.—Lever-arm Factor of Beam with Reinforcement in Tension Only.** Determine the lever-arm factor for a rectangular beam with reinforcement in tension only if the permissible stresses are 1000 lb. per square inch in the concrete and 20,000 lb. per square inch in the reinforcement. From Example No. 4, \( n_1 = 0.428 \); substituting in formula (3.3a), \( a_1 = i - \frac{0.428}{3} = 0.857 \), or 0.86 approximately. This numerical value is used frequently and should be remembered.

**Example No. 9.—Lever Arm of Beam with Reinforcement in Tension Only.** Determine the lever arm for the rectangular beam in Fig. 6 with reinforcement in tension only. From Example No. 7, \( d_n = 7.22 \) in.; substituting in formula (3.3a), \( l_a = 18.5 - \frac{7.22}{3} = 16 \) in. approximately. Alternatively, \( a_1 = i - \frac{0.39}{3} = 0.87 \) and

\[
l_a = 0.87 \times 18.5 = 16 \text{ in. approximately}.
\]

**Rectangular Beam with Compression Reinforcement.**—For a rectangular beam with reinforcement in tension and compression (Fig. 5) the lever arm \( l_{ac} \) to the line of action of \( F_{cc} \) is \( d_1 - \frac{1}{3}d_n \) and \( F_{cc} \) is \( \frac{1}{3}b d_n f_{cb} \), that is the area of the triangular stress diagram multiplied by the width of the section. The lever arm \( l_{as} \) to the centre of the reinforcement in compression is \( d_1 - d_2 \). The additional stress in the reinforcement in compression is, by proportion, \( (m - i)(\frac{d_n - d_2}{d_n})f_{cb} \), and the resistance \( F_{oc} \) of this reinforcement is \( (m - i)(\frac{d_n - d_2}{d_n})A_{o cf_{cb}} \). The resultant lever arm \( l_a \) is obtained from the moment relation \( l_a(F_{cc} + F_{oc}) = F_{cc}l_{ac} + F_{oc}l_{as} \).

Substituting \( F_{cc} = \frac{1}{3}n_1 d_1 f_{cb} \), \( F_{oc} = (\frac{n_1 - f_2}{n_1})(m - i)A_{o cf_{cb}} \),

\[
l_{ac} = d_1(1 - \frac{1}{3}n_1), \quad l_{as} = (i - f_2)d_1, \quad \text{and} \quad A_{oc} = r_b d_1,
\]

and transposing gives

\[
a_1 = \frac{l_a}{d_1} = \frac{n_1(i - \frac{1}{3}n_1) + r_b(m - i)(\frac{n_1 - f_2}{n_1})(i - f_2)}{\frac{1}{3}n_1 + r_b(m - i)(\frac{n_1 - f_2}{n_1})}.
\]

(3.3b)
Example No. 10.—Rectangular Beam with Compression Reinforcement. Determine the lever arm for the rectangular beam in Fig. 6 with reinforcement in tension and compression. From Example No. 6, \( n_1 = 0.353 \). Substituting this value and \( r_c = 0.0048, f_s = 0.075 \), and

\[
r_c(m - 1) \left( \frac{n_1 - f_s}{n_1} \right) = 0.0048 \times 14 \left( \frac{0.278}{0.353} \right) = 0.0548 \text{ in formula (3.3b)},
\]

\[
a_1 = \frac{0.176(I - 0.118) + (0.0548 \times 0.925)}{0.176 + 0.0548} = 0.89
\]

and

\[
l_a = 0.89 \times 18.5 = 16.5 \text{ in.}
\]

It is seen from Examples Nos. 8, 9, and 10 that the numerical value of the lever-arm factor is about \( \frac{1}{3} \) (0.875) and this value is very nearly correct for all reasonable proportions of reinforcement. In approximate calculations it is therefore sufficiently accurate to use the approximate value of \( a_1 = \frac{1}{3} \) or \( l_a = \frac{1}{3} d_1 \). A more accurate value for beams with a large proportion of reinforcement in compression is to assume the lever arm to be the mean of \( l_{ae} \) and \( l_{as} \), that is \( a_1 = \frac{1}{2}[(I - \frac{1}{3}n_1) + (I - f_s)] \), or

\[
a_1 = I - \frac{1}{3}n_1 - \frac{1}{3}f_s \text{ (approximately)}. \quad (3.3c)
\]

Flanged Beams.—For tee-beams and ell-beams (Fig. 7), if the small amount of compressive resistance between the soffit of the flange and the neutral plane is neglected, and only reinforcement in tension is provided, the compressive resistance is that due to the trapezoidal block of stress of maximum value \( f_{cb} \) and minimum value \( f_{c1} \). By proportion,

\[
f_{c1} = \frac{d_n - d_s f_{cb}}{d_n} \quad \text{or} \quad d_s = s_1 d_1, f_{c1} = \left( \frac{n_1 - s_1}{n_1} \right) f_{cb}.
\]

It can be shown geometrically that \( h_c = \frac{f_{cb} + 2f_{c1}(d_s)}{f_{cb} + f_{c1} (\frac{1}{3})} \). Since \( l_a = d_1 - h_c \), substituting and transposing gives

\[
l_a = a_1 d_1 = d_1 - \left( \frac{3d_n - 2d_s}{2d_n - d_s} \right) d_s; \quad a_1 = I - \left( \frac{3n_1 - 2s_1}{2n_1 - s_1} \right) s_1. \quad (3.3d)
\]

It is worth while considering some special cases of flanged beams. It is assumed in the foregoing that the neutral plane is below the flange. If it coincides with the soffit of the flange, \( d_n = d_s \) and \( f_{c1} = 0 \), so that \( h_c = \frac{1}{4}d_s \). If the neutral plane lies within the flange, \( h_c = \frac{1}{4}d_n \), which is the condition for a rectangular beam. Therefore a flanged beam in either of these conditions is designed as a rectangular beam of breadth \( b \), which is the width of the flange. The smaller the thickness \( d_s \) is in relation to \( d_n \), the more nearly the trapezoidal stress-diagram approaches a rectangle until in the limit \( h_c = \frac{1}{4}d_s \); since \( h_c \) is always less than \( \frac{1}{4}d_s \), a safe approximation is to consider \( h_c \) to be equal to \( \frac{1}{4}d_s \), which is a common assumption for a flanged beam forming part of a monolithic reinforced concrete floor or the like; on this assumption, therefore,

\[
l_a = d_1 - \frac{1}{3}d_s; \quad a_1 = I - \frac{1}{3}s_1. \quad \quad (3.3e)
\]
Example No. 11.—Comparison of Approximate and More Exact Lever Arms. Compare the approximate and more exact calculated lever arms of the example of a tee-beam in Fig. 7; \( d_1 = 18\frac{1}{2} \) in.; \( d_2 = 4 \) in.; \( b = 36 \) in.; \( b_f = 6 \) in.; \( A_{st} = 1.57 \) sq. in. (two \( x \)-in. bars). The position of the neutral plane is determined by substituting these values and \( m = 15 \) in formula (3.26), that is

\[
d_n = \frac{(1.57 \times 15 \times 18.5) + (0.5 \times 36 \times 4^2)}{(1.57 \times 15) + (36 \times 4)} = 4.3 \text{ in.}
\]

The neutral plane is therefore below the flange, and formula (3.30) can be used for determining the lever arm. By substitution,

\[
l_a = 18.5 - \left[\left(\frac{3 \times 4.3}{2 \times 4.3} - \frac{2 \times 4}{4}\right)\right] = 17.1 \text{ in.}
\]

The approximate lever arm is calculated by substitution in formula (3.36), that is \( l_a = 18.5 - (\frac{4}{2}) = 16.5 \) in., instead of 17.1 in. by the more accurate calculation. The approximate value is generally sufficiently accurate for ordinary problems and, because it is smaller, it errs on the side of safety.

Moment of Resistance.

The moment of resistance \( M_r \) of a structural member is the measure of its capacity to resist bending and is the value of the couple formed by the total compressive resistance \( F_c \) and the total tensile resistance \( F_t \). The total compressive resistance is equal to the compressive resistance of the concrete \( F_{cc} \) plus the resistance of the reinforcement in compression \( F_{sc} \); that is \( F_c = F_{cc} + F_{sc} \). The total tensile resistance must be equal to the total compressive resistance and is the product of the cross-sectional area of the reinforcement in tension and the tensile stress therein, assuming that all the bars comprising this reinforcement are grouped together at a distance \( d_1 \) (the effective depth) from the compressed edge; that is \( F_t = F_{st} = A_{st} f_{st} \). Therefore, in general terms,

\[
\begin{align*}
\text{Moment of compressive resistance:} & \quad M_{rc} = l_a (F_{cc} + F_{sc}) \\
\text{Moment of tensile resistance:} & \quad M_{rt} = l_a A_{st} f_{st} 
\end{align*}
\]

(3.4)

The lever arm is a function (in the mathematical sense) of the depth \( d_n \) to the neutral plane, which in turn is a function of the maximum compressive stress \( f_{cb} \) and the maximum tensile stress \( f_{st} \). The forces \( F_c \) and \( F_t \) are also functions of \( f_{cb} \) and \( f_{st} \) respectively and of \( d_n \). The safe moment of resistance is obviously the greatest value of \( M_r \) for which neither \( f_{cb} \) or \( f_{st} \) exceeds the permissible stress. There are three cases to consider, namely, (i) When the dimensions of the member and the amount of reinforcement are such that the permissible stresses \( \rho_{cb} \) and \( \rho_{st} \) occur simultaneously; this is the condition normally aimed at in design but which rarely occurs exactly. (ii) When there is more reinforcement in tension than is required to satisfy (i); for this condition the safe permissible compressive stress \( \rho_{cb} \) determines the safe moment of resistance since the
tensile stress will always be less than the permissible tensile stress. (iii) When there is less reinforcement in tension than is required to satisfy (i); for this condition the safe tensile stress \( p_{st} \) determines the safe moment of resistance since the compressive stress will always be less than the permissible compressive stress. In the following, the optimum condition of case (i) is analysed first and subsequently the modifications for cases (ii) and (iii) are considered. The application is limited to rectangular and flanged beams. More complex members are dealt with in Chapter XIII.

**Moment of Resistance with Permissible Stresses occurring Simultaneously.**

**"Balanced " Reinforcement.**—For the condition that the permissible stresses \( p_{cb} \) and \( p_{st} \) occur simultaneously, the neutral-plane factor \( n_1 \) is given directly by formula (3.1c) and therefore the compressive resistance \( F_c \) is established. The tensile resistance \( F_t \) is equal to \( F_c \) and is also equal to \( A_{st}p_{st} \); therefore the area of reinforcement in tension \( A_{st} \) necessary to ensure the permissible stresses occurring simultaneously is established, since \( p_{st} \) is a specified stress. When this amount of reinforcement is provided the design can be said to be "balanced", that is the compressive and tensile resistances just balance when the permissible stresses occur, and this amount can therefore be termed the balanced amount of tensile reinforcement \( A_{sb} \), which can be expressed as a proportion, \( r_b = \frac{A_{sb}}{bd_1} \). It is often expressed as a percentage which is generally called the "economic percentage of reinforcement ", but this is a misnomer since it is not always that this amount of reinforcement results in the cheapest beam, the dimensions of which depend on the relative costs of concrete and reinforcement in place in the structure. The balanced proportion is, therefore, merely that ratio for which the concrete and steel in a given beam are subjected to stresses equal to, or in the same ratio as, certain specified stresses.

**Rectangular Beam with Compression Reinforcement.**—Consider a rectangular beam with reinforcement in tension and compression (Fig. 5). The neutral-plane factor is given by formula (3.1c) for specified permissible stresses \( p_{cb} \) and \( p_{st} \), and the lever arms \( l_{ac} \) and \( l_{as} \) are \((1 - \frac{1}{3}n_1)d_1 \) and \((1 - f_2)d_1 \) respectively. The components \( F_{cc} \) and \( F_{sc} \) of the total compressive resistance are as given in the derivation of formula (3.3b). Therefore the moment of compressive resistance is \( F_{cc}l_{ac} + F_{sc}l_{as} \), which gives

\[
M_{rc} = M_{rt} = \left[ \frac{n_1}{2}(1 - \frac{n_1}{3}) + r_c(m - 1)(\frac{n_1 - f_2}{n_1})(1 - f_2) \right] p_{cb} bd_1^2. \tag{3.4a}
\]

The total compressive resistance \( F_{cc} + F_{sc} \) is

\[
\left[ \frac{n_1}{2} + r_c(m - 1)(\frac{n_1 - f_2}{n_1}) \right] p_{cb} bd_1.
\]
The tensile resistance is $A_{st}P_{st}$ or $r_cP_{st}d_1$, and must be equal to $F_{se} + F_{sc}$. Therefore the "balanced" amount of reinforcement in tension is $\frac{F_{se} + F_{sc}}{P_{st}}$, that is

$$A_{sb} = \left[ \frac{n_1}{2} + r_c(m - I)\left(\frac{n_1 - f_s}{n_1}\right)\frac{P_{cb}}{P_{st}} \right] d_1 = r_b d_1$$

in which $r_b = \left[ \frac{n_1}{2} + r_c(m - I)\left(\frac{n_1 - f_s}{n_1}\right)\frac{P_{cb}}{P_{st}} \right]$. (3.4b)

Example No. 12.—Moment of Resistance of Rectangular Beam with "Balanced" Reinforcement. Determine the moment of resistance of, and the "balanced" amount of reinforcement in tension required in, the rectangular beam in Fig. 8 with two $\frac{1}{4}$-in. bars in compression, if stresses of 1000 lb. and 20,000 lb. per square inch are to be induced simultaneously.

From Example No. 4, $n_1 = 0.428$; as in Example No. 6, $r_c = 0.0048$ and $f_s = 0.075$; assume $d_1$ to be 18.5 in.; $b = 10$ in. By substitution in formula (3.4a),

$$M_{rc} = \left[ \frac{0.428}{2} (I - 0.143) + (0.0048 \times 14) \left(\frac{0.353}{0.428}\right) \times 0.925 \right] 1000$$

$$\times 10 \times 18.5^2 = 803,000 \text{ in.-lb.}$$

By substitution in formula (3.4b),

$$A_{st} = A_{sb} = \left[ \frac{0.428}{2} + (0.0048 \times 14) \left(\frac{0.353}{0.428}\right) \right] \times \frac{1000}{20,000} \times 10 \times 18.5 = 2.49 \text{ sq. in.}$$

say, four $\frac{1}{4}$-in. bars (2.41 sq. in.), for which $d_1$ is slightly greater than the assumed value of 18.5 in.

Moment-of-resistance Factor of Rectangular Beam with Tensile Reinforcement Only.—For a rectangular beam with reinforcement in tension only (Fig. 5), $A_{st} = 0$, $r_c = 0$, and $I_a = (I - \frac{1}{4}n_1)d_1$. Therefore the moment of compressive resistance is $\frac{1}{4}n_1(I - \frac{1}{4}n_1)P_{cb}d_1^2$. The factor $\frac{1}{4}n_1(I - \frac{1}{4}n_1)P_{cb}$ is generally termed the moment-of-resistance factor, and if it is denoted by $Q_c$

$$M_{rc} = Q_cP_{cb}d_1^2; \quad Q_c = \frac{1}{4}n_1(I - \frac{1}{4}n_1)P_{cb}$$

and

$$r_b = \frac{n_1P_{cb}}{2P_{st}}; \quad A_{st} = r_b d_1 = \frac{n_1P_{cb}}{2P_{st}}d_1.$$. (3.4c)
It is seen that \( Q_e \) is dependent only on the permissible stresses \( p_{cb} \) and \( p_{st} \) since \( n_1 \) is derived from formula (3.1c). Values of \( Q_e \) for various combinations of stresses \( p_{cb} \) and \( p_{st} \) are given in Table A on page 123.

**Example No. 13.—Moment-of-resistance Factor and "Balanced" Reinforcement.** Determine the value of the moment-of-resistance factor of, and the "balanced" proportion of reinforcement in, a rectangular beam with reinforcement in tension only for permissible stresses of 1000 lb. and 20,000 lb. per square inch. From Example No. 4, \( n_1 = 0.428 \) and \( a_1 = 1 - \frac{1}{2}n_1 \) is 0.86. Substituting in formula (3.4c),

\[
Q_e = \frac{1}{2} \times 0.428 \times 0.86 \times 1000 = 184 \text{ lb. per square inch}
\]

\[
r_b = \frac{1}{2} \times 0.428 \times \frac{1000}{20,000} = 0.00107.
\]

**Example No. 14.—Moment of Resistance and "Balanced" Reinforcement.** Determine the moment of resistance of a rectangular beam which is 20 in. deep and 10 in. wide (Fig. 8), with reinforcement in tension only and determine the "balanced" amount of reinforcement if the maximum stresses are to be 1000 lb. and 20,000 lb. per square inch simultaneously.

Substituting the values of \( Q_e \) and \( r_b \) from Example No. 13 in formulæ (3.4c), and assuming that \( d_1 = 18.5 \text{ in.} \),

\[
M_{rc} = 184 \times 10 \times 18.5^2 = 630,000 \text{ in.-lb.}
\]

and

\[
A_{sh} = 0.0107 \times 10 \times 18.5 = 1.98 \text{ sq. in.}
\]

that is, say, two \( \frac{1}{4} \)-in. bars, with which \( d_1 \) is 18.19 in. allowing, say, \( \frac{1}{4} \)-in. cover of concrete. Therefore \( M_{re} = 184 \times 10 \times 18.19^2 = 609,000 \text{ in.-lb.} \)

**Flanged Beams.—** For a tee-beam and ell-beam (Fig. 11) with no reinforcement in compression and in which the neutral plane is below the soffit of the flange, the compressive resistance of the concrete \( F_{ce} \) is \( \left( \frac{p_{eb} + f_{e1}}{2} \right) d_{eb} \) in which \( f_{e1} \) is \( \left( \frac{d_n - d_s}{d_n} \right) p_{eb} \). The tensile resistance of the reinforcement \( F_{ts} \) is \( A_{st} p_{st} \) or \( r_b d_1 p_{st} \) . The lever arm \( l_a \) may be calculated by formula (3.3d); if \( l_a \) is calculated approximately from formula (3.3e), the moment of the compressive resistance is

\[
\frac{p_{eb}}{2} \left( 1 + \frac{d_n - d_s}{d_n} \right) \left( d_1 - \frac{d_s}{2} \right) d_{eb}.
\]

Therefore, since \( d_s = s_1 d_{1} \) and \( d_n = n_1 d_{1} \),

\[
M_{rc} = (2d_n - d_s) \left( d_1 - \frac{d_s}{2} \right) \frac{p_{eb} b d_{s}}{2d_n}
\]

\[
= \left( 2n_1 - s_1 \right) \left( 1 - \frac{s_1}{2} \right) \frac{p_{eb} s_1}{2n_1} b d_{1}^2,
\]

\[(3.4d)\]

in which \( d_n \) is from formula (3.2c).

Since the compressive resistance is \( \left( 1 + \frac{d_n - d_s}{d_n} \right) p_{eb} b d_s \), the cross-sectional area of the reinforcement in tension required is

\[
A_{st} = \frac{A_{sb} \left( 2d_n - d_s \right) p_{eb} b d_{s}}{p_{st}} \frac{M_{rc}}{(d_1 - \frac{1}{2} d_s) p_{st}}
\]

or

\[
r_b = \frac{(2n_1 - s_1) p_{eb} s_1}{p_{st} n_1} b d_{1}.
\]

\[(3.4e)\]
Since the stress at the bottom of the flange is generally likely to be small, the approximate moment of compressive resistance, assuming \( f_{c1} = 0 \), and \( l_a = d_1 - \frac{d}{2} \), is given by

\[
M_{rc} = \frac{1}{2}bd_1p_{eb}(d_1 - \frac{d}{2}) \approx \text{approximately. (3.4f)}
\]

**Example No. 15.—Moment of Resistance and “Balanced” Reinforcement of Tee-beam.** Determine the moment of resistance of, and the corresponding “balanced” amount of reinforcement in, the tee-beam in Fig. 8 if stresses of 1000 lb. and 20,000 lb. are to occur simultaneously. From Example No. 4, \( n_1 = 0.428 \). Assuming that \( d_1 = 3 \text{ in.} \), \( d_1 = 17 \text{ in.} \), and \( d_n = 0.428 \times 17 = 728 \text{ in.} \); therefore the neutral plane is below the flange and formulae (3.4a) and (3.4e) apply. Substituting \( d_n = 728 \text{ in.} \), \( d_2 = 4 \text{ in.} \), \( d_1 = 17 \text{ in.} \), \( p_{eb} = 1000 \text{ lb. per square inch} \), and \( b = 36 \text{ in.} \),

\[
M_{rc} = (14.56 - 4)(17 - 2)\left(\frac{1000 \times 36 \times 4}{2 \times 728}\right) = 1570,000 \text{ in.-lb.}
\]

and \( A_{st} = A_{sb} = \frac{1,570,000}{15 \times 20,000} = 523 \text{ sq. in., say, six } 1\text{1}\text{ in. bars arranged as in Fig. 8; for this arrangement } d_1 = 17\frac{1}{2} \text{ in. instead of the assumed value of } 17 \text{ in. and is sufficiently accurate. The breadth of the rib should be } 8 \text{ in. to allow sufficient space between, and sufficient cover of concrete outside, the bars.}

**Moment of Resistance with less than the “Balanced” Amount of Reinforcement in Tension.**

With the “balanced” amount of reinforcement in tension, the total compressive resistance at the permissible compressive stress is equal to the total tensile resistance at the permissible tensile stress. If the amount of reinforcement in tension is less than the “balanced” amount, it follows from the basic condition that the compressive and tensile resistances must be equal, that the maximum compressive stress \( f_{eb} \) must be less than the permissible compressive stress \( p_{eb} \) if the tensile stress is equal to the permissible tensile stress \( p_{st} \). Therefore, as seen from formula (3.1c), \( n_1 \) is less than the value corresponding to the permissible stresses, and consequently the lever arm is slightly greater. Since in this case the tensile resistance determines the safe moment of resistance, it is generally accurate enough if the positions of the neutral plane and the lever arm are calculated for the condition when the permissible stresses are induced simultaneously. A more accurate result is obtained if in the calculation of \( n_1 \), and therefore of \( a_1 \), the actual amount of reinforcement is taken into account. With this value of \( a_1 \) the moment of tensile resistance is \( a_1d_1A_{st}p_{st} \). The maximum compressive stress in the concrete is given by substituting \( p_{st} \) for \( f_{st} \) in the appropriate formula (3.1d).

**Rectangular Beam with Compression Reinforcement.**—For a rectangular beam with reinforcement in tension and compression, the
approximate value of \( n_1 \) is obtained from formula (3.1c). More accurately \( n_1 \) would be obtained from formula (3.2a). The lever-arm factor is obtained from formula (3.3b) or approximately from formula (3.3c). The moment of resistance \( M_{rt} \) is calculated from formula (3.4a).

Example No. 16.—Moment of Resistance of Rectangular Beam with Compression Reinforcement and Less than “Balanced” Amount of Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 13 if the reinforcement in tension and compression both comprise two \( \frac{1}{4} \)-in. bars \( (A_{st} = A_{sc} = 0.88 \text{ sq. in.}) \).

It is shown in Example No. 12 that the “balanced” amount of reinforcement in tension for a beam of this size with the specified amount of reinforcement in compression is 2.49 sq. in.; therefore this is also a case of the reinforcement in tension being less than the “balanced” amount.

The effective depth \( d_1 \) is 18\( \frac{1}{8} \) in.; therefore \( r_t = r_c = \frac{0.88}{10 \times 18.625} = 0.0047 \), and \( m(r_t + r_c) = 0.141; f_2 = \frac{1.375}{18.625} = 0.074 \). Substitution in formula (3.2a) gives the more accurate value of \( n_1 \) as

\[
n_1 = \sqrt{(0.141)^2 + (2 \times 15)(0.0047 + (0.074 \times 0.0047))} - 0.141 = 0.273.
\]

Substitution in formula (3.3b) gives

\[
a_1 = \frac{0.137(1 - 0.091) + \left(0.0047 \times \frac{0.273 - 0.074}{0.273}\right)0.926}{0.137 + \left(0.0047 \times \frac{0.273 - 0.074}{0.273}\right)} = 0.912.
\]

Therefore substitution in formula (3.4) gives

\[
M_{rt} = 0.912 \times 18.625 \times 0.88 \times 20,000 = 298,000 \text{ in.-lb}.
\]

This moment should be compared with the moment if it be calculated with the approximate value of \( n_1 = 0.428 \) from formula (3.1c) and the approximate value of \( a_1 \) from formula (3.3c), namely,

\[
a_1 = 1 - \frac{0.428}{6} - \frac{0.074}{2} = 0.892,
\]

which in this example varies little from 0.912 calculated by the more laborious method. The maximum compressive stress in the concrete is calculated by substituting \( n_1 = 0.273 \) and \( f_{st} = 20,000 \text{ lb. per square inch} \) in formula (3.1d), that is \( f_{cb} = \frac{0.273 \times 20,000}{15 \times 0.727} = 500 \text{ lb. per square inch} \).

Rectangular Beam with Tension Reinforcement Only.—Similarly for a rectangular beam with tensile reinforcement only, \( n_1 \) may be obtained approximately from formula (3.1c) or more accurately from formula (3.2b); \( a_1 \) is obtained from formula (3.3a) in either case, and the moment of resistance is obtained from formula (3.4).

Example No. 17.—Moment of Resistance of Rectangular Beam with Tension Reinforcement only and with Less than “Balanced” Amount of Reinforcement. Determine the moment of resistance of the
rectangular beam in Fig. 9 with reinforcement in tension only if this reinforcement comprises two \( \frac{3}{4} \)-in. bars \( (d_1 = \frac{3}{4} \text{ in.}; d_2 = 18.625 \text{ in.}) \) and the permissible tensile stress is 20,000 lb. per square inch.

It is shown in Example No. 14 that the "balanced" amount of reinforcement is 1.98 sq. in.; since the cross-sectional area of two \( \frac{3}{4} \)-in. bars is 0.88 sq. in., this is a case where the reinforcement provided is less than the "balanced" amount. By the approximate method \( n_1 = 0.428 \) (as in Example No. 4), and therefore from formula (3.2a) \( a_1 = 0.86 \) (as in Example No. 8). From formula (3.4),

\[
M_{rt} = 0.86 \times 18.625 \times 0.88 \times 20,000 = 282,000 \text{ in.-lb.}
\]

More accurately, using formula (3.2b) with \( r_t = \frac{0.88}{10 \times 18.625} = 0.0047 \) and \( m_{rt} = 0.0705 \),

\[
\begin{align*}
\frac{n_1}{n_2} &= \sqrt{0.0705^2 + (2 \times 0.0705)} - 0.0705 = 0.311; \\
a_1 &= 1 - \left( \frac{1}{4} \times 0.311 \right) = 0.896; \\
M_{rt} &= 0.896 \times 18.625 \times 0.88 \times 20,000 = 294,000 \text{ in.-lb.},
\end{align*}
\]

which is slightly greater than the approximate value. The compressive stress in the concrete is less than the permissible stress and is given by substituting \( n_1 = 0.311 \) and \( f_{ct} = f_{st} = 20,000 \text{ lb. per square inch} \) in formula (3.1d); that is \( f_{cb} = \frac{0.311 \times 20,000}{15 \times 0.689} = 603 \text{ lb. per square inch} \).

**Flanged Beams.**—For a tee-beam and ell-beam the lever arm \( a_1d_1 \) is approximately \( d_1 = \frac{1}{4}d_2 \), and this value is substituted in formula (3.4) to give the safe moment of resistance. The lever arm can be calculated more accurately from formula (3.3d) using the value of \( d_1 \) calculated from formula (3.2c).

**Example No. 18.**—Moment of Resistance of Tee-beam with Less than "Balanced" Reinforcement. Determine the moment of resistance of the tee-beam in Fig. 9 reinforced with two 1-in. bars \( (A_{st} = 1.57 \text{ sq. in.}) \) which is less than the "balanced" amount of reinforcement in tension as established in Example No. 15. The approximate lever arm is 18.5 \(- 2 = 16.5 \text{ in.} \). Therefore

\[
M_{rt} = 1.57 \times 20,000 \times 16.5 = 520,000 \text{ in.-lb.}
\]
Moment of Resistance with More than "Balanced" Amount of Reinforcement in Tension.

If the amount of reinforcement in tension exceeds the balanced amount, it is obvious that the tensile stress in the reinforcement will be less than the permissible tensile stress when the greatest compressive stress in the concrete is equal to the permissible compressive stress. Thus the neutral plane is lower (that is, \( d_n \) is greater) and the lever arm is less than if the permissible stresses occurred simultaneously. The safe moment of resistance is the moment of compressive resistance \( M_{rc} \), which is the total compressive resistance multiplied by the lever arm, that is \( F_{o da} \), and can be evaluated approximately by assuming that \( d_n \) has the value corresponding to the permissible stresses and substituting in the appropriate formulæ for \( l_n \) and \( F_{oc} + F_{sc} \).

**Rectangular Beams.**—A more accurate method for rectangular beams (Fig. 5) is to evaluate \( n_1 \) directly from formulæ (3.2a) or (3.2b) and \( M_{rc} \) from formulæ (3.4a) or (3.4c).

**Example No. 19.—Moment of Resistance of Rectangular Beam with Tension Reinforcement Only and More than "Balanced" Amount.** Determine the moment of resistance of the rectangular beam in Fig. 10 with reinforcement in tension only comprising two \( 1\frac{1}{2} \)-in. bars

\[
(A_{st} = 2.45 \text{ sq. in.}) \quad \text{which exceeds the "balanced" amount of 1.98 sq. in.}
\]
determined in Example No. 14.

\[
d_1 = 18\frac{1}{2} \text{ in.}; \quad n_t = \frac{2.45}{10 \times 18.125} = 0.0135; \quad m_r = 0.203.
\]

Substituting in formula (3.2b),

\[
n_1 = \sqrt{(0.203)^2 + (2 \times 0.203)} - 0.203 = 0.467.
\]

If the permissible compressive stress is 1000 lb. per square inch, substituting in formula (3.4c) gives

\[
M_{rc} = \left( \frac{1}{4} \times 0.467 \right) \left[ 1 - \left( \frac{1}{4} \times 0.467 \right) \right] 1000 \times 10 \times 18.125^2 = 648,000 \text{ in.-lb.}
\]

The tensile stress in the reinforcement is less than the permissible stress and is obtained by substituting \( n_1 = 0.467 \) in formula (3.1d):

\[
f_{st} = \frac{0.533 \times 15 \times 1000}{0.467} = 17,100 \text{ lb. per square inch.}
\]
RESISTANCE TO SIMPLE BENDING

Example No. 20.—Moment of Resistance of Rectangular Beam with Compression Reinforcement and More than "Balanced" Amount of Tension Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 10 reinforced with two 14-in. bars in tension and with two 14-in. bars in compression.

\[ A_{sc} = 0.88; \quad r_e = \frac{0.88}{10 \times 18.125} = 0.00485; \]

as in Example No. 19, \( r_t = 0.0135; \)

\[ m(r_t + r_e) = 15(0.0135 + 0.00485) = 0.275; \]

\[ d_2 = 1\frac{1}{4} \text{ in.;} \quad f_2 = \frac{1.375}{18.125} = 0.076. \]

Substituting in formula (3.2a),

\[ n_2 = \sqrt{(0.275)^2 + (2 \times 15)[0.0135 + (0.076 \times 0.00485)]} = 0.275 = 0.427. \]

Substituting in formula (3.4d),

\[ M_{re} = \left[ \frac{0.427}{2} \left( 1 - \frac{0.427}{3} \right) + (0.00485 \times 14) \left( \frac{0.351}{0.427} \times 0.924 \right) \right] \times \frac{1000}{10 \times 18.125^2} = 770,000 \text{ in.-lb.} \]

Flanged Beams.—For a tee-beam and ell-beam (Fig. 7) the moment of compressive resistance is calculated directly from formula (3.4d) with \( d_n \) calculated from formula (3.2c); or more accurately, substitute \( I_a \) calculated from formula (3.3d) for the term \( (d_1 - \frac{1}{4}d_3) \) in formula (3.4d).

Example No. 21.—Moment of Resistance of a Tee-beam with More than "Balanced" Amount of Tension Reinforcement. Determine the moment of resistance of the tee-beam in Fig. 10 if it is reinforced in tension with six 14-in. bars, which exceeds the balanced amount (5.23 sq. in.) of tensile reinforcement established in Example No. 15 for a tee-beam of the same dimensions. \( d_t = 3 \text{ in.}; A_{st} = 7.36 \text{ sq. in.} \)

Substituting in formula (3.2c),

\[ d_n = \frac{(7.36 \times 15 \times 17) + (0.5 \times 36 \times 4^2)}{(7.36 \times 15) + (36 \times 4)} = 8.5 \text{ in.,} \]

which means that the neutral plane is below the flange. Therefore approximately, by substitution in formula (3.4d),

\[ M_{re} = [(2 \times 8.5) - 4](17 - 2) \frac{1000 \times 36 \times 4}{2 \times 8.5} = 1,650,000 \text{ in.-lb.} \]

More accurately, substitution in formula (3.3d) gives

\[ I_a = 17 - \left[ \frac{(3 \times 8.5) - (2 \times 4)}{2 \times 8.5 - 4} \right] \]

\( = 15.2 \text{ in.} \)

(compared with the approximate lever arm of \( 17 - 2 = 15 \text{ in.} \)). Substituting in formula (3.4d) modified,

\[ M_{re} = [(2 \times 8.5) - 4] \frac{15.2 \times 1000 \times 36 \times 4}{2 \times 8.5} = 1,680,000 \text{ in.-lb.}, \]

which is so near to the approximate moment that the extra labour in the more accurate calculation is not justified.

E
Stresses due to an Applied Bending Moment.

In the foregoing the safe moment of resistance only has been considered, and this moment is equal to the greatest bending moment that can be applied without exceeding either the permissible compressive stress, the permissible tensile stress, or both. Any bending moment less than this greatest bending moment produces stresses less than the permissible stresses. Evaluation of the stresses produced by a specified bending moment is often required in practice, especially in checking a design. As before, the total compressive resistance is equal to the total tensile resistance but the couple represented by these forces is now equal to the applied bending moment \( M \), that is

\[ M = Fv \theta_a = Fv \frac{t}{a} \quad \ldots \quad (3.5) \]

A method of determining the stresses produced by a bending moment is to evaluate, by the formulæ in preceding sections, the stresses corresponding to the safe moment of resistance, \( M_{rt} \) or \( M_{rc} \) depending on whether the amount of reinforcement in tension is less than or greater than the "balanced" amount. The stresses produced by the bending moment are then the stresses corresponding to the safe moment of resistance decreased or increased in the proportion \( \frac{M}{M_{rt}} \) or \( \frac{M}{M_{rc}} \). A method more readily applicable to beams of regular cross-section is to determine \( n_1 \) and \( a_1 \). Since \( M_{rt} = M = a_1 d_1 A_{st} f_{st} \), whatever be the stresses or proportion of tensile reinforcement,

\[ f_{st} = \frac{M}{a_1 d_1 A_{st}} \quad \ldots \quad (3.5a) \]

The maximum compressive stress in the concrete is then obtained by substituting \( f_{st} \) and \( n_1 \) in formula (3.1d).

**Rectangular Beam.**—For a rectangular beam (Fig. 5) \( n_1 \) is evaluated from formula (3.2a) or (3.2b) and \( a_1 \) from formula (3.3b) or (3.3a) respectively.

**Example No. 22.**—Stresses in Rectangular Beam with Tension Reinforcement Only. Calculate the stresses produced by a bending moment of 500,000 in.-lb. in the rectangular beam in Fig. 11 with reinforcement in tension only. \( A_{st} = 1.57 \) sq. in.; \( m_{tr} = \frac{15 \times 1.57}{10 \times 18.5} = 0.128 \).
By substitution in formula (3.2b),
\[ n_1 = \sqrt{0.128^2 + (2 \times 0.128)} - 0.128 = 0.394. \]
From formula (3.3a), \( a_1 = 1 - (\frac{2}{3} \times 0.394) = 0.869. \)
By substitution in formula (3.5a),
\[ f_{st} = \frac{500,000}{0.869 \times 18.5 \times 1.57} = 19.750 \text{ lb. per square inch}. \]
By substitution in formula (3.1d),
\[ f_{cb} = \frac{0.394 \times 19.750}{15 \times 0.606} = 857 \text{ lb. per square inch}. \]

**Example No. 23.—Stresses in Rectangular Beam with Compression Reinforcement.** Calculate the stresses produced by a bending moment of 500,000 in.-lb. in the rectangular beam in Fig. 11 with reinforcement in tension and compression. \( A_{st} = A_{sc} = 1.57 \text{ sq. in.}; \)
\[ r_t = r_c = \frac{1.57}{10 \times 18.5} = 0.0085; \ m(r_t + r_c) = 0.255; \]
\[ f_s = \frac{1.5}{18.5} = 0.081. \]
By substitution in formula (3.2a),
\[ n_1 = \sqrt{0.255^2 + (2 \times 1.5)(0.0085 + (0.081 \times 0.0085)) - 0.255 = 0.330. \]
By substitution in formula (3.3b),
\[ a_1 = \frac{0.165(1 - 0.110) + (0.0085 \times 14) \left(\frac{0.249 \times 0.919}{0.330}\right)}{0.165 + (0.0085 \times 14) \left(\frac{0.249}{0.330}\right)} = 0.90. \]

By substitution in formula (3.5a),
\[ f_{st} = \frac{500,000}{0.90 \times 18.5 \times 1.57} = 19,100 \text{ lb. per square inch}. \]
By substitution in formula (3.1d),
\[ f_{cb} = \frac{0.330 \times 19.100}{15 \times 0.670} = 627 \text{ lb. per square inch}. \]

**Flanged Beams.**—For a tee-beam or ell-beam (Fig. 7), \( n_1 = \frac{d_n}{d_1}; \ d_n \) is calculated from formula (3.2c); \( a_1 \) is calculated from formula (3.3d) or can be determined approximately directly from formula (3.3e). Substituting \( a_1 \) in formulas (3.5a) gives the tensile stress \( f_{st} \) produced by the bending moment \( M \). The maximum compressive stress in the concrete is calculated by substituting \( f_{st} \) and \( n_1 \) in formula (3.1d) or this stress can be determined approximately by substituting in formula (3.4f) transposed thus:
\[ f_{cb} = \frac{2M}{bd_s(d_1 - \frac{1}{3}d_s)}. \]  

**Example No. 24.—Stresses in a Tee-beam.** Calculate the stresses produced by a bending moment of 750,000 in.-lb. in the tee-beam in Fig. 12. \( A_{st} = 2.36 \text{ sq. in.} \) The lever arm is approximately
\[ 18.5 - 2 = 16.5 \text{ in.} \]
By substitution in formula (3.5a)

\[ f_{st} = \frac{750,000}{16.5 \times 2.36} = 19,300 \text{ lb. per square inch.} \]

From formula (3.2c),

\[ d_n = \frac{(2.36 \times 15 \times 18.5) + (0.5 \times 36 \times 4^2)}{(2.36 \times 15) + (36 \times 4)} = 5.25 \text{ in.}, \]

which means that the neutral plane is below the flange. Therefore

\[ n_1 = \frac{5.25}{18.5} = 0.283. \]

By substitution in formula (3.1d),

\[ f_{eb} = \frac{0.283 \times 19,300}{15 \times 0.717} = 508 \text{ lb. per square inch,} \]

or more approximately, and more easily because there is no need to calculate \( d_n \), by substitution in formula (3.5b),

\[ f_{bc} \geq \frac{2 \times 750,000}{36 \times 4 \times 16.5} = 630 \text{ lb. per square inch;} \]

this stress differs from that determined by the less approximate calculation but coincidence of the two stresses is approached as \( d_n \) becomes more nearly equal to \( d_s \).

**Design of Beams to Resist a Specified Bending Moment.**

Most structural design is a process of trial and error; that is the size of a member is assumed and the resulting stresses checked in one of a number of ways depending on the type of member. For example, in the simple case of a solid concrete slab which is designed as a beam, say, 1 ft. wide, a thickness is assumed in order that the weight, and therefore the bending moment due to the weight of the slab, can be assumed. The thickness required to resist the bending moment can then be determined and the assumed thickness altered as necessary. For a rectangular beam without reinforcement in compression, a breadth \( b \) and weight are assumed and the depth required to resist the bending moment can then be calculated; the resultant dimensions are altered if the ratio of breadth to depth is unreasonable or if there is insufficient resistance to shearing forces. The design of a rectangular beam that is restricted in size is more direct, since the moment of resistance of the concrete can be evaluated; this moment, subtracted from the applied bending moment, gives the moment to be resisted by the reinforcement in compression. The thickness of the flange of a tee-beam or ell-beam is generally decided by the resistance required for the slab forming the flange, and the size of the rib is generally determined by the resistance required to shearing force; design is primarily concerned with the calculation of the amount of reinforcement required in tension and checking the maximum compressive stress. For most other sections, all dimensions are generally specified or assumed, and design
comprises calculating the amount of reinforcement required and checking the compressive and shearing stresses. In the following are given the basic formulae for the foregoing procedures; these formulae are easily derived from those given previously in this chapter. The bending moment to be resisted in each case is $M$.

**Rectangular Beam with Tension Reinforcement Only.**—For a beam of known or assumed breadth $b$ and with reinforcement in tension only (Fig. 5), the minimum effective depth and the reinforcement required are given by formulae (3.4c) transposed thus:

$$
\begin{align*}
  d_1 &= \sqrt{\frac{M}{Q_c b}}; \\
  Q_c &= \frac{M}{b d_1^2} = \frac{n_1}{2} \left( 1 - \frac{n_1}{3} \right) p_{cb} \\
  A_{st} &= \frac{M}{(1 - \frac{1}{3} n_1) d_1 p_{st}}.
\end{align*}
$$

(3.6a)

In these formulae $n_1$ is the neutral-plane factor corresponding to the permissible stresses and is obtained from formula (3.1c).

**Rectangular Beam with Compression Reinforcement.**—For a rectangular beam with tension and compression reinforcement (Fig. 5) the procedure is to calculate the moment of the compressive resistance of the concrete $M_c$ for the known or assumed values of $b$ and $d_1$ and the permissible stresses $p_{cb}$ and $p_{st}$. The residual moment $M_s$ must be resisted by reinforcement in compression, the additional stress in which is $(m - 1) \left( \frac{d_n - d_2}{d_n} \right) p_{cb}$, and the lever arm $l_{as}$ of which is $d_1 - d_2$. Therefore

$$
\begin{align*}
  M_c &= Q_c b d_1^2; \\
  M_s &= M - M_c; \\
  l_{ac} &= d_1 - \frac{1}{3} d_n; \\
  l_{as} &= d_1 - d_2; \\
  A_{sc} &= \frac{M_s}{(m - 1) \left( \frac{d_n - d_2}{d_n} \right) l_{as} p_{cb}}; \\
  A_{st} &= \frac{M_c}{l_{ac}} + \frac{M_s}{l_{as}} p_{st}.
\end{align*}
$$

(3.6b)

**Flanged Beams.**—For a tee-beam or ell-beam (Fig. 7) the dimensions $d$, $d_1$, $d_2$, $b$, and $b_r$ are known or assumed. The only unknown, $A_{st}$, is obtained from the general expression that the moment is equal to $d_1 d_1 A_{st} p_{st}$, which, transposed, gives

$$
A_{st} = \frac{M}{(d_1 - \frac{1}{3} d_2) p_{st}}.
$$

(3.6c)

The maximum compressive stress $f_{cb}$ in the flange can be checked by calculating $d_n$ from formula (3.2c) and proceeding as in Example No. 24.

Examples of the application of the foregoing formulae are given in the consideration of the design of slabs and beams in Chapters IX and X respectively.

### II. LOAD-FACTOR METHOD.

The basic difference between the modular-ratio and load-factor methods of determining the resistance of reinforced concrete beams is that in the
modular-ratio method conditions when the beam carries the design load are considered, whereas in the load-factor (or ultimate-load) method conditions immediately before failure are considered. A factor in support of the modular-ratio method is that it takes into account the probable stresses in the concrete and reinforcement due to the working load, the margin of safety being included in the assessment of the permissible stresses which, as explained on page 34, are fractions of the strength or other property of the materials considered separately. The safe resistance of a beam calculated in accordance with the load-factor method is the least bending moment which would cause failure divided by a selected factor of safety. It is therefore claimed in favour of the load-factor method that the factor of safety is known whereas in the modular-ratio method the factor of safety is indefinite because the assessment of the stresses does not take into account the true interaction between the concrete and the reinforcement. Analysis by the load-factor method is based on the results of tests and therefore includes the effect of many of the factors excluded from a purely theoretical analysis. Many proposals have been made to establish a satisfactory load-factor method to replace the modular-ratio method, and the method described in the following is based on that given in the British Standard Code of Practice No. 114 (1957).

Basis of Analysis by the Load-factor Method.

In the load-factor method the modular-ratio is not used and direct proportionality of the compressive stress to the strain of the concrete is not assumed. Tests show that the strain varies rectilinearly with the distance from the neutral plane but, since the elastic modulus of concrete increases as the compressive stress increases beyond ordinary working values, the

![Diagram of Load-factor Method Applied to Beams](image)

**Fig. 12.—Load-factor Method Applied to Beams.**

probable distribution of compressive stress when the member is nearing failure is as shown in Fig. 12. This "block" of compressive stress is of indefinite shape and can be replaced by an equivalent rectangular block (Fig. 12). The stress ordinate of the rectangular block is shown by tests to be about two-thirds of the compressive strength \( \sigma \) of works cubes of the
concrete. The depth $d_n$ of the block is such that the total compressive resistance at near-failure is equal to the total resistance of the reinforcement in tension at near-failure, allowance being made for any reinforcement in the compressive zone. The greatest value of $d_n$ is $\frac{1}{4}d_1$. The greatest tensile stress in the reinforcement in tension at near-failure of the beam is the yield-point stress or the equivalent yield stress of the steel. The load factors (or factors of safety) generally adopted are three against failure primarily due to crushing of the concrete and two against failure primarily due to yielding of the reinforcement in tension.

An important requirement is that the stresses under working conditions shall not be so great as to cause excessive cracking. Therefore under the working load the stress in the reinforcement in tension should not exceed the ordinary tensile stress permissible in the steel. Similarly the equivalent compressive stress in the concrete, considered as uniform over the entire compression zone, should not (under working conditions) exceed two-thirds of the ordinary compressive stress permissible in bending. Also, the stress in the reinforcement in compression should not exceed the ordinary compressive stress permissible in steel and should not exceed

$$50,000 \left( \frac{d_n - d_2}{d_n} \right) \text{ lb. per square inch};$$

this limitation is based on tests which show that the stress in reinforcement in compression varies approximately linearly from zero if the bars are at the neutral plane to a maximum of 100,000 lb. per square inch (if the steel has a sufficiently great yield stress) if the bars are at the top face of the compression zone. Design formulae for beams of common regular cross-sections are derived in the following from the foregoing bases.

Rectangular Beams.

When a beam is about to fail, the total resistance of the reinforcement in tension of cross-sectional area $A_{st}$ is $A_{st}p_{yu}$, if $p_{yu}$ is the yield-point stress or equivalent yield stress. Applying a factor of safety of two, the safe tensile resistance is $\frac{1}{2}A_{st}p_{yu}$, which is $A_{st}p_{et}$ (as in the modular-ratio method) since the ordinary permissible tensile stress $p_{et}$ is half the yield point or equivalent yield stress. This expression satisfies the condition that the tensile stress should not exceed the permissible tensile stress. The safe moment of tensile resistance is $A_{st}p_{et}l_a$ ($l_a$ is the equivalent lever arm).

Rectangular Beam with Tension Reinforcement Only.—In the case of a rectangular beam with reinforcement in tension only, the lever arm is $d_1 - \frac{1}{2}d_n$ because the equivalent block of stress is rectangular. The safe moment of tensile resistance $M_{rt}$ is $A_{st}p_{et}(d_1 - \frac{1}{2}d_n)$. Near failure the total compressive resistance is the volume of the equivalent block of compressive stress which, for a rectangular beam of breadth $b$, is $\frac{3}{2}wud_n b$. Dividing by a factor of three, the safe compressive resistance is $\frac{3}{2}wu d_n b$, that
is $\frac{3}{2}p_{cb}d_nb$ since the compressive stress $p_{cb}$ permissible in bending is one-third of the crushing strength. This expression satisfies the condition that the equivalent uniform compressive stress should not exceed $\frac{3}{2}p_{cb}$. Since the total tensile resistance is equal to the total compressive resistance, 

$$A_{st}p_{st} = \frac{3}{2}p_{cb}d_nb,$$

from which

$$d_n = \frac{3A_{st}p_{st}}{2p_{cb}}.$$ 

Therefore the lever arm $(d_1 - \frac{1}{b}d_n)$ is $d_1 - \frac{3A_{st}p_{st}}{4p_{cb}}$. For the limiting value of $d_n$, that is $\frac{1}{b}d_1$, the lever arm is $\frac{3}{2}d_1$, and thus the safe moment of the compressive resistance is $\frac{3}{2}p_{cb}(\frac{3}{2}d_1)(\frac{3}{2}d_1)$, that is $\frac{9}{4}p_{cb}bd_1^2$. Therefore the formulae for determining the moment of resistance are

$$M_{re} = \frac{1}{b}p_{cb}bd_1^2 = Q_{CL}bd_1^2.$$ 

$$M_{st} = A_{st}p_{st}\left(d_1 - \frac{3A_{st}p_{st}}{4b_{cb}}\right). \quad (3.7)$$

Values of $Q_{CL} (= \frac{1}{b}p_{cb})$ are given in Table A (page 123) for comparison with the corresponding factor $Q_C$ in the modular-ratio method.

Transposing formula (3.7) results in the following formulae for determining the effective depth $d_1$ and the cross-sectional area of the reinforcement in tension $A_{st}$ required for a beam to resist a bending moment $M$:

$$d_1 = 2\sqrt{\frac{M}{p_{cb}}}; \quad A_{st} = \frac{2b_{cb}}{3p_{st}}\left[ d_1 - \sqrt{d_1^2 - \frac{3M}{b_{cb}}} \right]. \quad (3.7a)$$

**"Balanced" Design.**—For a "balanced" design, that is when the maximum permissible stresses $p_{st}$ and $p_{cb}$ occur simultaneously, the cross-sectional area $A_{st}$ of the reinforcement in tension is determined by equating the tensile and compressive resistances to give $A_{st} = \frac{2}{3} \frac{p_{cb}}{p_{st}}d_nb$. Substituting $d_n = \frac{1}{b}d_1$ and $r_b = \frac{A_{st}}{bd_1}$, the proportion, and amount, of reinforcement in tension for a "balanced" design are respectively

$$r_b = \frac{1}{3} \frac{p_{cb}}{p_{st}}; \quad \text{and} \quad A_{st} = \frac{1}{3} \frac{p_{cb}}{p_{st}}d_1. \quad (3.7b)$$

If the proportion of reinforcement in tension is less than $r_b$, the tensile resistance determines the safe moment of resistance of the beam, which is $M_{rt}$; if the proportion is greater, the compressive resistance determines the safe moment of resistance, and is $M_{re}$ as given by formula (3.7).

**Example No. 25.**—"Balanced" Reinforcement in Rectangular Beam with Tension Reinforcement Only. Determine the "balanced" proportion of reinforcement for a rectangular beam with reinforcement in tension only if the permissible stresses are 1000 lb. per square inch in the concrete and 20,000 lb. per square inch in the reinforcement.

By substitution in formula (3.7b), $r_b = \frac{1000}{\frac{3 \times 20,000}{1}} = 0.0167$. Compare with $r_b = 0.0107$ for the modular-ratio method as determined in Example No. 13.

**Example No. 26.**—Moment of Resistance and "Balanced" Amount of Reinforcement in Rectangular Beam with Tension
Reinforcement Only. Determine the moment of resistance of, and the "balanced" amount of reinforcement in, the rectangular beam in Fig. 13, 20 in. deep and 10 in. wide, with reinforcement in tension only, if the permissible stresses are 1000 lb. and 20,000 lb. per square inch. Assuming that \( d_t = 1\frac{1}{2} \) in., \( d_1 = 18\frac{1}{2} \) in., and substituting in formula (3.7),

\[
M_{rt} = \frac{1}{3} \times 1000 \times 10 \times 18.5^2 = 857,000 \text{ in.-lb.}
\]

Since \( r_b = 0.0167 \), as determined in Example No. 25,

\[
A_{st} = A_{sb} = 0.0167 \times 10 \times 18.5 = 3.08 \text{ sq. in.,}
\]

or by substitution in formula (3.7b),

\[
A_{st} = A_{sb} = \frac{1000}{3 \times 20,000} \times 10 \times 18.5 = 3.08 \text{ sq. in.,}
\]

say, four 1-in. bars for which \( d_1 = 18\frac{1}{2} \) in. is satisfactory. (Compare with two \( 1\frac{1}{4} \)-in. bars and a moment of resistance of 609,000 in.-lb. as determined by the modular-ratio method in Example No. 14 for a beam of the same size.)

Example No. 27.—Moment of Resistance of Rectangular Beam with Less than "Balanced" Amount of Tension Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 9 with reinforcement in tension only comprising two \( 1\frac{1}{4} \)-in. bars if the permissible tensile stress is 20,000 lb. per square inch.

\[
\frac{0.88}{10 \times 18.625} = 0.0047
\]

which, being less than 0.0167 as calculated in Example No. 25, signifies that the tensile resistance determines the safe moment of resistance. Substituting in formula (3.7) gives

\[
M_{rt} = 0.88 \times 20,000(18.625 - \frac{3 \times 0.88 \times 20,000}{4 \times 10 \times 1000}) = 306,000 \text{ in.-lb.}
\]

Compare with 294,000 in.-lb. as calculated by the modular-ratio method in Example No. 17.

Example No. 28.—Moment of Resistance of Rectangular Beam with More than "Balanced" Amount of Tension Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 10 with reinforcement in tension only comprising three \( 1\frac{1}{4} \)-in. bars if the permissible compressive stress is 1000 lb. per square inch. \( A_{st} = 3.68 \text{ sq. in.} \), which exceeds the "balanced" amount as determined in Example No. 26. Therefore the compressive resistance determines the safe moment of resistance. Substitution in formula (3.7), with \( d_1 = 18\frac{1}{2} \) in., gives

\[
M_{rc} = \frac{1}{3} \times 1000 \times 10 \times 18.125^2 = 824,000 \text{ in.-lb.}
\]

Rectangular Beam with Compression Reinforcement.—For a rectangular beam with reinforcement in tension and compression (Fig. 12) the effect of the reinforcement in compression of cross-sectional area \( A_{se} \) is to increase the safe compressive resistance by an amount \( A_{se} \varphi_{sc} \) (\( \varphi_{sc} \) is the permissible compressive stress); this amount does not take into account the small reduction in area of the concrete in compression due to the area occupied by the reinforcement in compression, but the reduction is negligible. The lever arm of the reinforcement in compression is \( d_1 - d_2 \); therefore the safe moment of compressive resistance is given by

\[
M_{rc} = \frac{1}{3} \varphi_{sc} b d_1^2 + A_{se} \varphi_{sc}(d_1 - d_2).
\]

(3.8)
For formula (3.8) to apply, that is for the moment of resistance to be dependent on the crushing of the concrete and not on the failure of the reinforcement in tension, it is necessary for the tensile resistance to be not less than the compressive resistance, that is for $A_{st}p_{st}$ to be not less than $\frac{1}{4}P_{c}b d_{1} + A_{sc}p_{sc}$. Therefore the limiting amount of reinforcement in tension is given by

$$A_{st} = A_{sb} = \frac{1}{P_{st}}\left(\frac{P_{c}b d_{1}}{4} + A_{sc}p_{sc}\right). \quad \quad (3.8a)$$

For the design of a beam of given size to resist a bending moment $M$, formula (3.8) can be transposed with $M_{re} = M$ to give the cross-sectional area of reinforcement in compression required, namely,

$$A_{sc} = \frac{M - \frac{1}{4}P_{c}b d_{1}}{(d_{1} - d_{2})p_{sc}}. \quad \quad (3.8b)$$

The cross-sectional area of the reinforcement in tension is then determined from formula (3.8a).

Example No. 29.—Moment of Resistance and “Balanced” Amount of Reinforcement in a Rectangular Beam with Compression Reinforcement. Determine the moment of resistance of, and the “balanced” amount of reinforcement in tension required in, the rectangular beam in Fig. 13 with reinforcement in compression comprising two

\[\text{Fig. 13.—Examples of Beams with “Balanced” Amount of Reinforcement (Load-factor Method.)}\]

\[\frac{3}{8}\text{-in. bars, if the permissible stresses are 1000 lb. per square inch in the concrete and 18,000 lb. per square inch in tension and 18,000 lb. per square inch in compression in the reinforcement.} \]

$A_{sc} = 0.88 \text{ sq. in.}$. Assuming that $d_{1} = 18\frac{3}{8} \text{ in.}$, substitution in formula (3.8) gives

$$M_{re} = \left(\frac{3}{4} \times 1000 \times 10 \times 18.5\right) + (0.88 \times 18,000)(18.5 - 1.375) = 1,129,000 \text{ in.-lb.}$$

Substitution in formula (3.8a) gives

$$A_{st} = \frac{1}{18,000}\left[\left(\frac{1000}{4} \times 10 \times 18.5\right) + (0.88 \times 18,000)\right] = 3.44 \text{ sq. in.},$$

say, four $1\frac{3}{4}$-in. bars, for which $d_{1} = 3\frac{3}{8}$ in. if $1\frac{3}{8}$-in. cover of concrete is provided. Therefore

$$M_{re} = \frac{1}{4}(1000 \times 10 \times 18.19) + (0.88 \times 18,000)(18.5 - 1.375) = 1,257,000 \text{ in.-lb.}$$
RESISTANCE TO SIMPLE BENDING

Compare with a moment of resistance of 803,000 in.-lb. and four \( \frac{3}{4} \)-in. bars as determined by the modular-ratio method in Example No. 12.

Example No. 30.—Moment of Resistance of Rectangular Beam with Compression Reinforcement and Less than “Balanced” Amount of Tension Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 9 with reinforcement in compression. Since the amount of reinforcement in tension is less than the amount required for a “balanced” design (Example No. 29), the tensile resistance of the two \( \frac{3}{4} \)-in. bars alone determines the safe moment of resistance. The lever arm of the compression in the concrete is not less than \( \frac{3}{4} \times 18 \frac{1}{2} \) in., say, 14 in., and the lever arm of the reinforcement in compression is \( 18 \frac{1}{2} - 1 \frac{3}{8} = 17 \frac{1}{2} \) in. Assuming that the resultant lever arm is about 15 in.,

\[
M_{rc} = 0.88 \times 20,000 \times 15 = 264,000 \text{ in.-lb.}
\]

Compare with the moment of resistance 298,000 in.-lb. as determined by the modular-ratio method in Example No. 16.

Example No. 31.—Moment of Resistance of Rectangular Beam with Compression Reinforcement and More than “Balanced” Amount of Tension Reinforcement. Determine the moment of resistance of the rectangular beam in Fig. 10 with reinforcement in tension comprising four \( \frac{1}{2} \)-in. bars and reinforcement in compression comprising two \( \frac{3}{4} \)-in. bars. Since \( A_{st} = 4.91 \) sq. in., comparison with Examples Nos. 29 and 30 shows that more than the amount of reinforcement in tension for a “balanced” design is provided, and therefore the safe moment of resistance is determined by the compressive resistance. Substitution in formula (3.8) gives

\[
M_{rc} = \left( \frac{1}{2} \times 1000 \times 10 \times 18 \cdot 125^2 \right) + (0.88 \times 18,000)(18 \cdot 125 - 1 \cdot 375) = 1,090,000 \text{ in.-lb.}
\]

Flanged Beams.

For a tee-beam or ell-beam without reinforcement in compression (Fig. 12), the safe moment of tensile resistance is similar to that of a rectangular beam, that is \( M_{rt} = A_{st}p_{st}l_a \), the lever arm \( l_a \) being \( d_1 - \frac{1}{2}d_s \), approximately; as in the modular-ratio method it is assumed that the depth of the compression zone is equal to the thickness of the flange, that is

\[
M_{rt} = A_{st}p_{st}(d_1 - \frac{1}{2}d_s) = A_{st}p_{st}d_1(1 - \frac{1}{2}s_1). \quad (3.9a)
\]

For the limiting condition that \( d_s = \frac{1}{2}d_1 \), the safe compressive resistance is \( \frac{3}{2}p_{cb}[(b - br)d_s + \frac{1}{2}d_1br] \). The lever arm is

\[
\frac{\frac{3}{2}p_{cb}[(b - br)d_s(d_1 - \frac{1}{2}d_s) + \frac{3}{2}d_1br(d_1 - \frac{1}{2}d_1)]}{\frac{3}{2}p_{cb}[(b - br)d_s + \frac{1}{2}d_1br]},
\]

that is

\[
l_a = \frac{(b - br)(d_1 - \frac{1}{2}d_s) + \frac{3}{2}d_1^2br}{(b - br)d_s + \frac{1}{2}d_1br}.
\]

Multiplication of the compressive resistance by the lever arm gives an expression for the safe moment of compressive resistance, which is

\[
M_{rc} = \left\{ \frac{br}{4b} + \frac{1}{3}(1 - \frac{br}{b}) \left[ \frac{2d_s}{d_1} - \left( \frac{d_s}{d_1} \right)^2 \right] \right\} p_{cb}bd_1^2. \quad (3.9b)
\]
If the small amount of compression below the flange is neglected, the compressive resistance is \( \frac{2}{3} p_{cb} d_s b \) and the lever arm is \( d_1 - \frac{1}{3} d_s \). Therefore the safe moment of compressive resistance and the corresponding cross-sectional area of reinforcement in tension required to produce a "balanced" design are given by

\[
M_{rc} = \frac{1}{3} p_{cb} d_s b (2d_1 - d_s) ; \quad A_{st} = A_{sb} = \frac{2p_{cb} d_s b}{3p_{st}} .
\]  

(3.9c)

If less than this amount of reinforcement in tension is provided, the reinforcement in tension determines the safe moment of resistance, which is given by formula (3.9a).

If the applied bending moment \( M \) is less than \( M_{rc} \) as given by formula (3.9b), the amount of reinforcement in tension can be determined from

\[
A_{st} = \frac{M}{(d_1 - \frac{1}{3} d_s)p_{st}} .
\]

(3.9d)

which is the same expression as formula (3.6c) for the modular-ratio method.

Example No. 32.—Moment of Resistance and "Balanced" Amount of Reinforcement in a Tee-beam. Determine the moment of resistance of, and the "balanced" amount of reinforcement in tension required in, the tee-beam in Fig. 13 if the permissible stresses are 1000 lb. and 20,000 lb. per square inch. Substituting \( d_1 = 17 \frac{1}{8} \) in. and other terms in formula (3.9c),

\[
M_{rc} = \frac{1}{3} \times 1000 \times 4 \times 36(34.25 - 4) = 1,450,000 \text{ in.-lb.}
\]

\[
A_{st} = A_{sb} = \frac{2 \times 1000 \times 4 \times 36}{3 \times 20,000} = 4.8 \text{ sq. in., say, five } 1\frac{1}{4}\text{-in. bars.}
\]

Compare with the moment of resistance of 1,570,000 in.-lb. and six \( 1\frac{1}{4} \)-in. bars determined by the modular-ratio method in Example No. 15.

Example No. 33.—Moment of Resistance of Tee-beam with Less than "Balanced" Amount of Tension Reinforcement. Determine the safe moment of resistance of the tee-beam in Fig. 9 reinforced in tension by two 1-in. bars. Comparison with Example No. 32 shows that the reinforcement is less than the amount required for a "balanced" design. Therefore the tensile resistance determines the safe moment of resistance. If \( p_{st} \) is 20,000 lb. per square inch, substitution in formula (3.9a) gives

\[
M_{st} = 1.57 \times 20,000(18.5 - 2) = 520,000 \text{ in.-lb., which is exactly the same result as is obtained by the modular-ratio method in Example No. 18.}
\]

Example No. 34.—Moment of Resistance of a Tee-beam with More than "Balanced" Amount of Reinforcement. Determine the safe moment of resistance of the tee-beam in Fig. 10 if it is reinforced in tension with six \( 1\frac{1}{4} \)-in. bars, which exceeds the amount required for a "balanced" design. Therefore the compressive resistance determines the safe moment of resistance, and, if \( p_{cb} \) is 1000 lb. per square inch, substitution in formula (3.9c) gives

\[
M_{rc} = \frac{1}{3} \times 1000 \times 4 \times 36(34 - 4) = 1,440,000 \text{ in.-lb.}
\]
Comparison of Modular-ratio and Load-factor Methods.

The difference in the calculated moments of resistance and designs of reinforced concrete beams in accordance with the modular-ratio method and the load-factor method can be seen by examining the corresponding formulae, and in particular by comparing the numerical examples in the foregoing. The principal conclusions are that for beams without reinforcement in compression and with small proportions of reinforcement in tension there is little difference between the two methods, but the load-factor method gives the greater resistance if the proportion is such that "balanced" design is obtained. For example, with safe stresses of 1000 lb. per square inch in the concrete and 20,000 lb. per square inch in the steel, the moment-of-resistance factor $Q_e$ for a rectangular beam without reinforcement in compression is 184 lb. per square inch by the modular-ratio method, whereas by the load-factor method the factor $Q_{CL}$ is $\frac{1}{4} \times 1000$, that is 250 lb. per square inch, which means that the moment of resistance is 36 per cent. greater. (See Table A on page 123 for values of $Q_e$ and $Q_{CL}$.) For beams with reinforcement in compression the load-factor method gives a greater safe moment of compressive resistance, as not only is the resistance of the concrete considered to be greater but greater value is placed upon the resistance of the reinforcement in compression. In the modular-ratio method the stress in the reinforcement in compression is related to the compressive stress in the surrounding concrete, which stress is generally much less than the compressive stress permissible in the reinforcement as reckoned in the load-factor method.

In all rectangular beams the amount of reinforcement required to balance the greatest compressive resistance is, however, much greater when calculated by the load-factor method than by the modular-ratio method, and a true comparison of cost cannot be made without considering the relative costs of concrete, steel, and shuttering.

In the case of tee-beams and ell-beams, there is no practical difference between the results of the two methods if the thickness of the flange is determined by factors other than the resistance of the beam. The amount of reinforcement generally decides the safe moment of resistance and, since the lever arm is the same by both methods, the safe moment of tensile resistance is the same by both methods.

III. CONCRETE RESISTANT TO TENSION.

In the foregoing analyses it is assumed that the concrete in the tension zone of a beam has cracked and therefore offers no resistance to tensile
forces. In some structures, notably those containing liquids, cracking is objectionable and the beams and slabs comprising such structures are designed on the assumption that the concrete does not crack, and the tensile stresses in the concrete are therefore limited as described on page 18. In the following, the resistance of reinforced concrete beams is considered on the assumption that the concrete can resist tension. The distribution of stress (Fig. 4) is as described on page 41. The relationship between the applied bending moment (or moment of resistance) and the stresses is the same as for a homogeneous material, that is

\[
\frac{\text{bending moment}}{\text{moment of inertia}} = \frac{\text{stress}}{\text{distance from the neutral plane}};
\]

or the maximum stresses (at the edge of the beam) are given by

\[
\frac{\text{bending moment}}{\text{section modulus}}.
\]

The safe moment of resistance is therefore expressed by the product of the permissible stress and the section modulus. The difference between reinforced concrete and a homogeneous material is taken into account in evaluating the modulus of the section, and the reinforcement is reckoned to be \( m \) times as effective as an equal area of concrete. The entire cross-section of a beam is subjected to stress; that is the overall depth \( d \) is effective and not only the effective depth as in the previous analyses. The neutral plane passes through the centroid of the equivalent cross-section (see page 41).

**Modulus of Section.**

The definition of the section modulus \( Z \) of a beam of any cross-section is (moment of inertia about the centroid) \( \div \) (distance from centroid to edge). If the centroidal moment of inertia is \( I \) and the distance from the centroid of the entire cross-section to the compressed edge is \( X \) or \( d_n \), and therefore \( d - d_n \) from the tensioned edge, where \( d \) is the overall depth of the beam, the section moduli are

\[
\text{For the compressed edge: } Z_c = \frac{I}{d_n}.
\]

\[
\text{For the tensioned edge: } Z_t = \frac{I}{d - d_n}.
\]

(3.10)

Formulae for \( I \) and \( d_n \) for several reinforced concrete sections are given in Chapter XIII, Vol. II. For common sections the terms in formula (3.10) can be readily calculated.

**Rectangular Beam with Tension and Compression Reinforcement.**—For a rectangular beam with reinforcement in tension and com-
pressure (Fig. 14) the geometrical properties of the cross-section are readily derived and are

Equivalent area: \( A_e = bd + (m-1)(A_{st} + A_{sc}) \).

Position of centroid:

\[
X = \bar{d}_n = \left\{ \frac{1}{2}bd^2 + (m-1)[A_{sc}d_2 + A_{st}(d-d_2)] \right\} \frac{1}{A_e}.
\]

Centroidal moment of inertia:

\[
I = \frac{1}{6}[\bar{d}_n^3 + (d-\bar{d}_n)^3]b + (m-1)[A_{sc}d_n^3 - d_2^3] + A_{st}(d - d_n - d_2)^3.
\]

Section moduli: \( Z_e = \frac{I}{\bar{d}_n} \); \( Z_t = \frac{I}{d - \bar{d}_n} \).

Simplifications can be made for certain common conditions. Tables giving the section moduli of many plane regular figures can be used to obtain the section modulus of the gross cross-section, to which must be added an allowance for the reinforcement. For example, for a rectangular beam

![Diagram showing the centroids and section moduli for different types of beams](image)

**Fig. 14.—Beams with Concrete Effective in Tension.**

with symmetrical reinforcement, \( A_{sc} = A_{st} \), \( d_2 = d_4 \), and \( d_n = \frac{1}{2}d \). Consequently \( Z_e = Z_t = Z \). To the section modulus of a rectangle, \( \frac{1}{6}bd^2 \), must be added the effect of the additional moment of inertia of the reinforcement, which is \( (A_{st} + A_{sc})\left(\frac{1}{2}d - d_2\right)^3 \), and the additional section modulus is

\[
\frac{(m-1)(A_{st} + A_{sc})\left(\frac{1}{2}d - d_2\right)^3}{\frac{1}{2}d}.
\]

Therefore the total section modulus is given by

\[
Z = \frac{bd^2}{6} + \frac{(m-1)(A_{st} + A_{sc})\left(\frac{1}{2}d - d_2\right)^3}{\frac{1}{2}d}.
\]

Since the position of the centroid is not affected greatly by different ratios of \( A_{st} \) to \( A_{sc} \), formula (3.10b) is accurate enough for most cases of beams with reinforcement in tension and compression.

**Rectangular Beam with Tensile Reinforcement Only.**—For a rectangular beam with reinforcement in tension only, formulae (3.10a) apply
with \( A_{sc} = 0 \), that is

\[
\begin{align*}
A_e &= b d + (m - 1) A_{st}; \quad d_n = \left[ \frac{1}{2} b d^3 + (m - 1) A_{st} d_1 \right] \frac{1}{A_e} \\
I &= \frac{1}{2} [d_n^3 + (d - d_n)^3] b + (m - 1) A_{st} (d_1 - d_n)^2 \\
Z_c &= \frac{I}{\bar{d}_n}; \quad Z_t = \frac{I}{d - d_n}
\end{align*}
\]

(3.10c)

Formulas (3.10c) are complex expressions for a beam of simple cross-section and it is in general sufficiently accurate to assume that the centroid of the equivalent area is at the centroid of the rectangular cross-section, that is the position of the neutral plane is such that \( d_n = \frac{1}{2} d \). With this simplification,

\[
Z = Z_c = Z_t = \frac{b d^2}{6} + \frac{A_{st} (m - 1) (\frac{1}{2} d - d_1)^2}{\frac{1}{4} d}
\]

(3.10d)

**Flanged Beams.**—For a tee-beam or ell-beam (Fig. 14) it is likewise sufficiently accurate to assume that the neutral plane passes through the centroid of the gross area, neglecting the reinforcement. With this simplification, the geometrical properties of a flanged beam with reinforcement in tension only are:

\[
\begin{align*}
A_e &= b r d + (b - b_r) d_8 + (m - 1) A_{st}. \\
d_n &= \left[ b r d^2 + (b - b_r) d_8^2 + 2 (m - 1) A_{st} d_1 \right] \frac{1}{2 A_e}. \\
I &= \frac{1}{2} b r [d_n^3 + (d - d_n)^3] + (b - b_r) d_8 \left[ \frac{d_8^2}{12} + (d_n - \frac{1}{2} d_8)^2 \right] \\
&+ A_{st} (m - 1) (d_1 - d_n)^2. \\
Z_c &= \frac{I}{\bar{d}_n}; \quad Z_t = \frac{I}{d - d_n}.
\end{align*}
\]

(3.10e)

It should be noted that flanged beams designed for the concrete to resist tension may have the tensile zone in the flange, in which case \( Z_c \) and \( Z_t \) apply to the tensioned and compressed edges respectively, that is the top of the flange and the bottom of the rib respectively.

**Moment of Resistance.**

The safe moment of resistance of a beam, the entire area of which is effective, is given by the smaller of the two expressions

\[
M_{rc} = Z_c \rho_{cb}; \quad M_{rt} = Z_t \rho_{ct},
\]

(3.11)

in which \( \rho_{cb} \) and \( \rho_{ct} \) are respectively the compressive and tensile stresses permissible in the concrete. Since the permissible tensile stress is in general only a fraction of the permissible compressive stress, the safe moment of resistance is generally \( Z_t \rho_{ct} \). For a beam of regular cross-section the moment of resistance is obtained by multiplying the permissible
stress by the appropriate section modulus as determined by the corresponding formulæ (3.10a) to (3.10c).

The examples in the following relate to beams of the same cross-sections as the beams previously considered in which the concrete does not resist tension; in these examples the permissible stresses in the concrete in bending are assumed to be 1000 lb. per square inch in compression and 300 lb. per square inch in tension. The permissible tensile and compressive stresses in the reinforcement are not critical since in both cases they are \( m \) times the stress in the surrounding concrete, and, if \( m = 15 \), the stress will not exceed 15,000 lb. per square inch.

**Example No. 35.—Moment of Resistance of Rectangular Beam with Tension Reinforcement Only.** Determine the moment of resistance of the rectangular beam in Fig. 6 with reinforcement in tension only: \( d = 20 \) in., \( d_t = 1\frac{1}{8} \) in., \( d_1 = 18\frac{1}{4} \) in., \( A_{st} = 1\cdot57 \) sq. in. Substitution in the approximate formulæ (3.10d) gives

\[
Z_c = Z_t = \frac{10 \times 20^2}{6} + \frac{(14 \times 1\cdot57)(10 - 1\cdot5)^2}{10} = 826 \text{ in.}^3
\]

\[
M_{rt} = 826 \times 300 = 247,800 \text{ in.-lb.}
\]

**Example No. 36.—Moment of Resistance of Rectangular Beam with Compression Reinforcement.** Determine the moment of resistance of the rectangular beam in Fig. 6 with reinforcement in tension and compression: \( d = 20 \) in., \( d_t = 1\frac{1}{8} \) in., \( d_1 = 1\frac{1}{4} \) in., \( A_{st} = 1\cdot57 \) sq. in., \( A_{sc} = 0\cdot88 \) sq. in. Formulae (3.10a) are applicable, but it is sufficiently accurate to use formula (3.10b) with \( d_s = d_t = 1\frac{1}{8} \) in. Substitution gives

\[
Z = \frac{10 \times 20^2}{6} + \frac{14(1\cdot57 + 0\cdot88)(10 - 1\cdot5)^2}{10} = 915 \text{ in.}^3
\]

\[
M_{rt} = 915 \times 300 = 274,500 \text{ in.-lb.}
\]

Comparison with the more accurate value should be made by using formulæ (3.10a).

**Example No. 37.—Moment of Resistance of Tee-beam.** Determine the moment of resistance of the tee-beam in Fig. 7 with reinforcement in tension only: \( d = 20 \) in., \( d_t = 1\frac{1}{8} \) in., \( d_1 = 18\frac{1}{4} \) in., \( b = 36 \) in., \( b_r = 6 \) in., \( d_s = 4 \) in., \( A_{st} = 1\cdot57 \) sq. in.

Substitution in formulæ (3.10e) gives

\[
A_s = (6 \times 20) + (36 - 6)4 + (14 \times 1\cdot57) = 262 \text{ sq. in.}
\]

\[
d_n = \frac{6 \times (20)^2 + 30(4)^2 + (2 \times 14)(1\cdot57 \times 18\cdot5)}{2 \times 262} = 7\cdot05 \text{ in.}
\]

\[
I = \frac{5}{12}[7\cdot05^3 + (20 - 7\cdot05)^3] + (30 \times 4)\left[\frac{4^4}{12} + (7\cdot05 - 2)^2\right]
\]

\[+ (14 \times 1\cdot57)(18\cdot5 - 7\cdot05)^2 = 11,140 \text{ in.}^4;
\]

\[
Z_c = \frac{11,140}{7\cdot05} = 1580 \text{ in.}^3; \quad Z_t = \frac{11,140}{20 - 7\cdot05} = 860 \text{ in.}^3
\]

Substitution in formula (3.11) gives

\[
M_{rc} = 1580 \times 1000 = 1,580,000 \text{ in.-lb.},
\]

and

\[
M_{rt} = 860 \times 300 = 258,000 \text{ in.-lb.}
\]

Therefore \( M_{rt} \) is the smaller and is the safe moment of resistance.
If the bending action is such that the flange is in tension (instead of being in compression) the following apply:

\[ M_{rc} = 860 \times 1000 = 850,000 \text{ in.-lb}, \]

and \[ M_{rt} = 1580 \times 300 = 474,000 \text{ in.-lb}. \]

The safe moment of resistance is again determined by the tensile resistance.

**Stresses due to an Applied Bending Moment.**

The general expressions for the maximum compressive and tensile stresses in the concrete due to an applied bending moment \( M \), assuming that the concrete does not crack are, respectively,

\[ f_{cb} = \frac{M}{Z_c}; \quad f_{ct} = \frac{M}{Z_t}; \quad \ldots \ldots \quad (3.11a) \]

in which \( Z_c \) and \( Z_t \) are obtained from formulae (3.10a) to (3.10c) for beams of known cross-section and reinforcement. If any of the factors in these formulae are not known, as is usual when designing a beam to resist a specified bending moment, such factors are generally assumed since the formulae for direct calculation of any of the unknowns are too cumbersome for practical use.

**Example No. 38.—Stresses in Tee-beam.** Determine the stresses in the tee-beam in Fig. 7 if it is subjected to a bending moment of 200,000 in.-lb. which produces compression in the flange. From Example No. 37, \( Z_c = 1580 \text{ in.}^3 \) and \( Z_t = 860 \text{ in.}^3 \). Therefore, substitution in formulae (3.11a) gives

At top of flange: \( f_{cb} = \frac{200,000}{1580} = 127 \text{ lb. per square inch (compression)}. \)

At bottom of rib: \( f_{ct} = \frac{200,000}{860} = 233 \text{ lb. per square inch (tension)}. \)
CHAPTER IV

RESISTANCE TO AXIAL THRUST COMBINED WITH BENDING

Due to the monolithic nature of a reinforced concrete structure, many members are subjected to bending simultaneously with a thrust parallel to their axes. Common examples are arches, columns in buildings and other framed structures, chimneys, towers, and the piers or other supports of waterside structures. The intensity of stress in such members is not uniform over the entire cross-section, as is the case of the compressive stress due to a concentric axial thrust acting alone, since the stress at any point is a combination of the stresses due to the thrust and the bending. In the case of a homogeneous material, such as steel, the combination is a simple operation, but it is less so in the case of reinforced concrete.

Combined Stresses.

In general, the resultant stress at any part of a member subjected simultaneously to thrust and bending is the algebraical sum of the stresses due to a concentric axial load and a bending moment. The stresses due to the concentric load are entirely compressive as is described on page 31. The stresses due to bending vary from a maximum compressive stress at one edge of the member to a maximum tensile stress at the opposite edge, as is described on page 41. Therefore if the stresses due to the thrust predominate the combined stresses are entirely compressive but, if the bending moment is large compared with the thrust, part of the member will be in compression and part in tension. The evaluation of the resultant stresses, or, conversely, the amount of combined thrust and bending a reinforced concrete member can resist, may be made by either the modular-ratio (equal-strain) method or by the load-factor (ultimate-load) method. Both methods are considered in this chapter. The common assumption that the concrete has no tensile strength is made in the case of members in which the stresses are compressive and tensile. The condition of the concrete being able to resist tension is also considered in relation to the modular-ratio method.

The combined stress at any point in a member of homogeneous material is the algebraical sum of

\[
\frac{\text{concentric thrust}}{\text{cross-sectional area}} \quad \text{and} \quad \frac{(\text{bending moment}) \times (\text{distance of point from centroidal axis})}{\text{moment of inertia about the centroidal axis}}.
\]

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These expressions are applicable to reinforced concrete members only if the stresses are entirely compressive or if the concrete can resist tension and if the equal-strain method of analysis is adopted; for these conditions the equivalent cross-sectional area and the centroidal moment of inertia (as established on pages 71 and 72 for simple sections and in Chapter XIII, Vol. II, for complex sections) are substituted for the corresponding terms in the general expression.

The maximum combined stresses, if they are entirely compressive or if the concrete is assumed to resist tension, are therefore given by

\[
\frac{\text{concentric thrust}}{\text{equivalent area}} \pm \frac{\text{bending moment}}{\text{equivalent section modulus}}.
\]

If the combined stresses are partly compressive and partly tensile and the concrete is not assumed to resist tension, conditions are more complex. The analysis proceeds by replacing the bending moment \( M \) by an equivalent eccentric thrust equal to the concentric thrust \( N \) acting at a distance \( e \) such that the eccentricity \( e \) is \( \frac{M}{N} \) if it is measured from the centroid of the stressed area, which may not be the same as the centroid of the entire cross-section. Before proceeding with the analysis by the alternative methods the term "eccentric thrust" must be examined.

**Eccentric Thrust.**

A thrust is eccentric if its line of action does not pass through the centroid of the cross-section of the member. This condition produces non-uniform distribution of stress similar to a bending moment acting simultaneously with a concentric thrust. Because of this similarity the analysis is simplified if the condition of bending moment and thrust acting simultaneously is transposed to an equivalent condition of eccentric load.

**Thrusts and Moments in the Same Plane.**—The following consideration applies when the member is subjected to thrusts and moments acting in one plane. The external actions on a member subjected to bending and compression are due to one or more of the following causes.

1. A number of thrusts \( P_1 \) to \( P_n \) acting parallel to the axis of the member at distances \( g_1 \) to \( g_n \) from, say, the edge at which the compressive stress is greatest as in Fig. 15a. If the line of action of any thrust is outside the section the dimension \( g \) is positive, and if it is within the section \( g \) is negative. Examples of thrusts outside the section are the loads from a crane-beam or other weight on a corbel or bracket on the side of a column. Examples of thrusts within the section are the loads from beams bearing centrally or eccentrically on a column, or the load from a column above that being considered, or the weight of the column itself. The thrust in an arch or a member of a framed structure may be either outside or within the section.
(2) A bending moment \( M \), which is assumed to be positive if it acts in a clockwise direction as in Fig. 15a. Examples of such moments are those imposed by beams constructed monolithically with a column, or by forces due to wind, or the bending moment on an arch.

(3) If the bending moment \( M \) is partly or entirely due to the restraint of beams constructed monolithically with a column, it is accompanied by a vertical thrust \( P_d \), the position of the line of action of which is indefinite, and it is proper to consider that \( P_d \) acts at the centroid \( O \) (Fig. 15a) of the stressed area of the member, since only in this position is \( P_d \) equivalent to a concentric thrust producing no additional bending moment on the column, the entire bending effect being included in \( M \).

The loads \( P_1 \) to \( P_n \) can be replaced by a load \( \sum_{i=1}^{n} P \) acting at a distance \( X + \frac{\sum_{i=1}^{n} P_i g_i}{\sum_{i=1}^{n} P} \) from \( O \), the centroid of the stressed area, as in Fig. 15b. The bending moment \( M \) can be replaced by an equivalent couple comprising two opposite forces each equal to the total load \( N \) on the member; one force acts at \( O \) and the other at a distance \( \frac{M}{N} \) from \( O \) (Fig. 15b). The total load \( N \) on the member is \( \sum_{i=1}^{n} P + P_d + N - N \), that is \( \sum_{i=1}^{n} P + P_d \). The total moment about \( O \) of these loads is \( N \left( \frac{M}{N} \right) + \sum_{i=1}^{n} P \left( X + \frac{\sum_{i=1}^{n} P_i g_i}{\sum_{i=1}^{n} P} \right) \). The

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**Fig. 15.—Eccentric Thrust and Moments.**
total load \( N \) and moment \( M \) can therefore be replaced by a single load \( N \)
acting at a distance \( e \) from O (Fig. 15c) such that

\[
N = \sum_{i}^{n} P_{i} + P_{a}; \quad e = e_{1}d = \frac{M}{N} \left( \frac{\sum_{i}^{n} P_{i}}{N} + \frac{\sum_{i}^{n} P_{f}g_{i}}{\sum_{i}^{n} P_{i}} \right).
\] (4.1)

If the stresses are compressive over the entire cross-section, \( \sigma \) can be
evaluated because all the terms in formula (4.1) are known, \( \bar{X} \) being the
distance to the centroid of the entire equivalent area. If tensile and
compressive stresses are induced, \( \bar{X} \) is the distance to the centroid of
the stressed area, which is not known until the position of the neutral plane
is known, and this depends on the relative magnitude of the maximum
stresses.

**Concentric Thrust and Moment.**—In the common case of a con-
centric thrust and moment only acting (that is \( P_{a} \) and \( M \) respectively),
the eccentricity about O is given by

\[
\varepsilon = \varepsilon_{1}d_{1} = \frac{M}{P_{a}}. \quad \ldots \quad \ldots \quad \ldots \quad (4.1a)
\]

**Example No. 39.—Eccentricity.** Determine the eccentricity of the
thrust equivalent to a concentric thrust of 100,000 lb. and a moment of
1,000,000 in.-lb.

Substitution in formula (4.1a) gives \( \varepsilon = \frac{1,000,000}{100,000} = 10 \) in.

**Example No. 40.—Equivalent Eccentric Thrust.** Determine the
equivalent eccentric thrust and eccentricity if the overall depth \( d \) of the
member in Example No. 39 (Fig. 15d) is 20 in., the distance \( \bar{X} \) from
the greatest compressed edge is 11 in., and, in addition to the bending moment
and concentric thrusts, the member is subjected to a thrust of 10,000 lb.
at 12 in. inside the compressed edge \( (P_{1} = 10,000 \) lb., \( g_{1} = -12 \) in.), and
50,000 lb. at a point 10 in. outside the compressed edge \( (P_{2} = 50,000 \) lb.,
\( g_{3} = +10 \) in.). Also \( P_{a} = 100,000 \) lb.; \( M = 1,000,000 \) in.-lb.

Substitution in formulae (4.1) gives \( N = (10,000 + 50,000) + 100,000
= 160,000 \) lb.

\[
\varepsilon = \frac{60,000}{160,000} \left[ 11 + \frac{-(10,000 \times 12) + (50,000 \times 10)}{60,000} \right] + \frac{1,000,000}{160,000} = 12.75 \text{ in.}
\]

I. MODULAR-RATIO METHOD.

There are two principal cases to be considered: (i) compressive stresses
only; and (ii) compressive and tensile stresses. The latter case is further
subdivided into (a) concrete having no resistance to tension, and (b) con-
crete resistant to tension.

**Compressive Stresses only.**

In this simple case the entire section of the member is resistant, the
maximum and minimum compressive stresses in the concrete being given
by the general relation

\[
\frac{\text{total load } N}{\text{equivalent area } A_e} = \frac{\text{total moment } N\theta}{\text{section modulus } Z_e}, \text{ that is }
\]

\[
f_{c(\text{max.})} = \left(\frac{1}{A_e} + \frac{\epsilon}{Z_e}\right)N; \quad f_{c(\text{min.})} = \left(\frac{1}{A_e} - \frac{\epsilon}{Z_e}\right)N.
\]

in which \(Z_e\) and \(Z_t\) are the equivalent section moduli (as described on page 70) referred to the edges where \(f_{c(\text{max.})}\) and \(f_{c(\text{min.})}\) respectively occur.

The distribution of the direct and bending stresses is shown in Fig. 16.

\[
\begin{align*}
\text{(COMPRESSIVE STRESS DUE TO THRUST N)} + \text{(COMPRESSIVE \\
\text{& TENSILE STRESSES DUE TO MOMENT})} = \text{(RESULTANT \\
\text{COMPRESSIVE STRESSES DUE TO M & N})}
\end{align*}
\]

Fig. 16.—Stresses due to Bending and Thrust: Compressive Stresses Only.

For the stresses to be entirely compressive, \(f_{c(\text{min.})}\) must be positive, and the limit in this case is \(f_{c(\text{min.})} = 0\), that is when \(\epsilon = \frac{Z_e}{A_e}\); for this condition \(f_{c(\text{max.})} = 2\frac{N}{A_e}\) if \(Z_c = Z_t\). For a rectangular member without reinforcement this limit occurs when \(\epsilon = \frac{d}{b}\), but the effect of the reinforcement (which is generally to increase the section modulus by a greater percentage than the increase in the effective area) is to raise the limit to, say, \(\frac{d}{b}\) for moderate amounts of reinforcement. Considering formulae (4.2), the total load \(N\) and the eccentricity \(\epsilon\) are evaluated from formulae (4.1). If the resultant stresses are all compressive, \(\epsilon\) is a fraction of the distance \(X\) and therefore the term \(\sum_{1}^{n} \frac{P}{A_p}\) is generally negative. For members of common cross-sections the section moduli \(Z_t\) and \(Z_e\) are calculated from formulae series (3.10) on page 70, and the effective area from formulae series (2.1) on page 32; for other cross-sections the corresponding formulae in Chapter XIII are applicable. The application of the formulae to members of rectangular cross-section only are considered in the following.

**Rectangular Member with Unsymmetrical Reinforcement.**—The cross-section of a rectangular member with unsymmetrical reinforcement, such as the column of a framed structure or an arch rib, is shown in
Fig. 17. Formulæ (3.10a) give the equivalent area $A_e$ and the section moduli $Z_e$ and $Z_t$ if $X$ is substituted for $d_n$. Alternatively formulæ (3.10b) may give the section moduli sufficiently accurately. Substitution in formulæ (4.2) gives the greatest and least compressive stresses in the concrete. It is easier in this case to use formulæ (3.10a) and (3.10b) separately rather than combine them into one complex expression.

**Rectangular Member with Symmetrical Reinforcement.**—In the case of a rectangular member with symmetrical reinforcement (Fig. 17)

![Diagram](image)

$A_e$ is given by formulæ (3.10a) and $Z (= Z_e = Z_t)$ by formula (3.10b). Substitution in formula (4.2) and rearrangement gives an expression for the maximum and minimum compressive stresses, namely,

$$
\frac{f_{c(\text{max})}}{f_{c(\text{min})}} = \left( \frac{1}{bd} + \frac{m-1}{(A_{st} + A_{se})} \right) \frac{6\varepsilon}{bd^2 + 12d(m-1)(A_{st} + A_{se})(\frac{1}{2} - \frac{d}{a})^2} \sqrt{N}. \quad (4.2a)
$$

**Rectangular Member with Reinforcement near One Face Only.**—In the case of a rectangular member with reinforcement near one face only as in Fig. 17 formulæ (3.10c) apply for the equivalent area and section moduli, but the approximate formula (3.10d) gives in general the section modulus sufficiently accurately. If the value of $Z$ as calculated from formula (3.10d) is acceptable, the maximum and minimum compressive stresses are obtained from formula (4.2a) with $A_{se} = 0$. The examples which follow show the method of applying the foregoing formulæ to calculate the maximum stresses; for simplification and comparison, the cross-sections of the members considered are those in Fig. 6 on page 45.

**Example No. 41.—Rectangular Member with Unsymmetrical Reinforcement.** Calculate the maximum compressive stresses in the rectangular member in Fig. 6 with unequal amounts of reinforcement near each face (that is unsymmetrical reinforcement) if the member is subjected to a bending moment $M$ of 150,000 in.-lb. simultaneously with a concentric thrust $N$ of 100,000 lb.; \( \varepsilon = \frac{150,000}{100,000} = 1.5 \text{ in.} \) Substitution in formula
(3.10a) gives \( A_e = (10 \times 20) + 14(1.57 + 0.88) = 234 \) sq. in. Applying formula (3.10b), the section modulus is 915 in.\(^3\) as calculated in Example No. 36. Therefore, substitution in formula (4.2) gives

\[
\begin{align*}
&f_{c}^{(\text{max.})} = \left( \frac{I}{234} \pm \frac{1.5}{915} \right) 100,000 = 428 \pm 164 = 592 \text{ lb. per square inch} \\
&f_{c}^{(\text{min.})} (\text{maximum}) 
\end{align*}
\]

(1.5) and 264 lb. per square inch (minimum). The more accurate stresses obtained by using \( Z_e \) and \( Z_t \) as calculated from formula (3.10a) should be determined for comparison.

Example No. 42.—Rectangular Member with Reinforcement near One Face Only. Calculate the maximum and minimum compressive stresses in the rectangular member in Fig. 11 with reinforcement at one face only, if the member is subjected to a bending moment \( M \) of 150,000 in.-lb. and a concentric thrust \( N \) of 100,000 lb.; \( e = 1.5 \) in. Substitution in formula (3.10c) gives

\[
A_e = (10 \times 20) + (14 \times 1.57) = 222 \text{ sq. in.}
\]

From Example No. 35, \( Z = Z_e = Z_t \) (approximately) = 826 in.\(^3\). Substitution in formula (4.2) gives

\[
\begin{align*}
&f_{c}^{(\text{max.})} = \left( \frac{I}{222} \pm \frac{1.5}{826} \right) 100,000 = 450 \pm 182 = 632 \text{ lb. per square inch} \\
&f_{c}^{(\text{min.})} (\text{maximum}) 
\end{align*}
\]

This result is obtained also by substituting \( b = 10 \) in., \( d = 20 \) in., \( (m - r) = 14, A_{st} = 1.57 \) sq. in., \( A_{se} = 0 \), and \( a = d_t = 11.4 \) in. in formula (4.2a). For comparison, the stresses should also be determined by calculating \( Z_e \) and \( Z_t \) more accurately by using formula (3.10c).

**Compressive and Tensile Stresses:**

**Concrete Resists Tension.**

When the stress \( f_{c}^{(\text{min.})} \) calculated from formula (4.2) is negative, tensile stresses are produced in part of the member. If it is assumed that the concrete will resist tensile forces, the neutral plane passes through the centroid of the entire cross-section and the distribution of stresses is as in Fig. 18, that is the bending stresses exceed the stresses due to the direct thrust. Formula (4.2) is still applicable and the method of calculating the stresses in members of rectangular cross-section is as given in the preceding paragraphs.

Example No. 43.—Rectangular Member with Unsymmetrical Reinforcement. Calculate the maximum stresses in the member in Example No. 41 if the bending moment is 450,000 in.-lb.,

\[
A_e = 234 \text{ sq. in. and } Z = 915 \text{ in.}^3
\]

as before. Substitution in formula (4.2) gives the maximum compressive and tensile stresses as

\[
\begin{align*}
&f_{cc}^{(\text{max.})} = \left( \frac{I}{234} \pm \frac{4.5}{915} \right) 100,000 = 428 \pm 492 = 920 \text{ lb. per square inch (compression)} \\
&f_{ct}^{(\text{max.})} = 64 \text{ lb. per square inch (tension)}
\end{align*}
\]
Fig. 18.—Bending and Thrust: Compressive and Tensile Stresses (Modular-ratio Method). Concrete Effective in Tension.

Alternatively, substitution can be made directly into formula (4.2a) to give the same result.

Compressive and Tensile Stresses: Concrete Does not Resist Tension.

The assumption that the concrete cannot resist tensile forces is the basis of the design of most reinforced concrete members subjected to bending. If the concrete is assumed to have cracked because it has no resistance to tension, the distribution of the stresses is as shown in Fig. 19, the tensile forces being resisted entirely by the reinforcement. Although Fig. 19 illustrates a rectangular member, the distribution of the stresses applies to members of any cross-section, the general case of
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which is analysed in Chapter, XIII Vol. II. The factors required in addition to those when the stresses are entirely compressive are the effective depth \( d_1 \), the position \( X' \) of the centroid of the stressed section, and the position of the neutral plane \( d_n = n_1 d_1 \). All reinforcement below the neutral plane is in tension; reinforcement and concrete above the neutral plane are in compression. Calculation of the stresses proceeds by first determining the positions of the centres of action of the total tensile and compressive forces \( F_{st} \) and \( F_c \), which are at distances \( d_1 \) and \( h \) respectively below the compressed edge. The position of the centroid of the stressed area, which is at a distance \( X' \) below the compressed edge, can be obtained in terms of \( d_n \), as are also the dimensions \( d_1 \) and \( h \). The first statical condition is that there must be equilibrium when the section is subjected to the action of the three forces, that is the total tensile resistance \( F_{st} \), the total compressive resistance \( F_c \), and the total applied eccentric thrust \( N \) which acts at a distance \( \varepsilon \) from the centroid of the stressed area. Therefore

\[
F_{st} - F_c + N = 0.
\]

The terms \( N \) and \( \varepsilon \) are obtained from formula (4.1) or (4.1a). The second statical condition is that the moments, about any plane, of the forces acting at the section must balance. Taking moments about the line of action of \( F_{st} \), this condition is expressed by

\[
N(\varepsilon - X' + d_1) - F_c(d_1 - h) = 0.
\]

From these two conditions are obtained the basic formulae

\[
F_c = \frac{N(\varepsilon - X' + d_1)}{d_1 - h}; \quad \text{and} \quad F_{st} = F_c - N. \quad (4.3)
\]

Since \( F_c \) is a function of the maximum compressive stress \( f_{cb} \) and \( F_{st} \) is a mathematical function of the tensile stress \( f_{st} \), formulae for the maximum stresses can be derived from formulae (4.3).

In the case of a member of known dimensions subjected to a known thrust and moment (or eccentricity), there are three unknown factors, namely, the stresses \( f_{cb} \) and \( f_{st} \) and the position \( d_n \) of the neutral plane. Three formulae are required to determine these unknown factors; two formulae are those for the stresses derived from formula (4.3), and the third is formula (3.1a) on page 42 which establishes the value of \( d_n \) relative to the calculated stresses \( f_{cb} \) and \( f_{st} \). Thus \( d_n \) cannot be determined until \( f_{cb} \) and \( f_{st} \) are known, and vice versa.

**Trial-and-error Method.**—It is evident that the relation between \( d_n, f_{cb} \), and \( f_{st} \) is complex and is in fact a third-degree relation which in general is not worth while deriving. It is convenient in practice to adopt a trial-and-error method in which a value for \( d_n \) is assumed and the stresses with this value are calculated. The trial value is then checked by substituting \( f_{cb} \) and \( f_{st} \) in formula (3.1a); if there is any discrepancy between the assumed and calculated values of \( d_n \), a second trial value is assumed and the problem reworked until sufficient agreement is obtained. The
analysis of the general case, which is given in the following, is not often required but is the basis of the analysis of some special cases which are given in Chapter XIII, Vol. II. The analysis of members of simple cross-section are given in this chapter.

A rectangular member with reinforcement in tension and compression is shown in Fig. 19. If a position $d_n$ for the neutral plane is assumed, numerical values for the terms in formula (4.3) are readily obtained. Since the centre of tension is at the centre of the group of bars $A_{st}$, $d_1$ is established. The total tension $F_{st}$ is $A_{st}f_{st}$, the stress $f_{st}$ being unknown at this stage. It is convenient to divide $F_s$ into two components: the compressive resistance $F_{sc}$ of the concrete and the additional compressive resistance $F_{se}$ of the reinforcement in compression. The value of $F_{cc}$ is the area of the triangular diagram of stress multiplied by the breadth $b$ of the member, that is $\frac{1}{2}bd_nf_{cb}$, say $J_{cfb}$; the stress $f_{cb}$ is unknown at this stage. The resistance $F_{se}$ is the additional stress $(m - 1)\left(\frac{d_n - d_2}{d_n}\right)f_{cb}$ in the reinforcement in compression multiplied by $A_{sc}$, say, $J_{sfb}$. The distances of the line of action of the compressive forces $F_{cc}$ and $F_{sc}$ from the reinforcement in tension are $l_{ac} = d_1 - \frac{1}{2}d_n$ and $l_{as} = d_1 - d_2$ respectively. The position of the centroid of the stressed area can be determined by taking moments of the component stressed areas about the centre of the reinforcement in tension and dividing by the total equivalent stressed area, that is

$$x' = d_1 - \frac{bd_n(d_1 - \frac{1}{2}d_n)}{bd_n + (m - 1)A_{scl}}.$$  \hspace{1cm} (4.3a)

The sum of the moments of the forces about the line of action of $F_{st}$ is $F_{cc}l_{ac} + F_{sc}l_{as} - N(e + d_1 - X')$. Substituting $J_{cfb}$ for $F_{cc}$ and $J_{sfb}$ for $F_{sc}$, formula (4.3) can be developed to give

$$f_{cb} = \frac{N(e + d_1 - X')}{J_{eac} + J_{jas}}; \quad f_{st} = \frac{(J_e + J_s)f_{cb} - N}{A_{st}}.$$ \hspace{1cm} (4.3b)

in which $J_e = \frac{1}{2}bd_n$ and $J_s = (m - 1)\left(\frac{d_n - d_2}{d_n}\right)A_{sc}$.

Substitution in formula (3.1b) of the stresses $f_{cb}$ and $f_{st}$ calculated from formula (4.3b) provides a check of the assumed value of $d_n$.

The procedure for using this method for determining the stresses in a member of known dimensions is as follows. Assume $d_n$. Evaluate $J_e$ and $J_s$, $l_{ac}$ and $l_{as}$. Determine $X'$ from formula (4.3a). Substitute these values in formula (4.3b). With the stresses $f_{cb}$ and $f_{st}$ so determined, check the trial value of $d_n$ by substitution in $\frac{d_1}{1 + \frac{f_{st}}{m_{fcb}}}$; if the two values of $d_n$ differ, select another trial value, evaluate again those factors affected by $d_n$, recalculate the stresses, and check again; the second result should be satisfactory.
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Example No. 44.—Calculation of Stresses by Trial-and-error Method. Calculate the maximum compressive and tensile stresses in the rectangular member with unequal amounts of reinforcement in tension and compression in Fig. 6 on page 45 if the member is subjected to a bending moment of 150,000 in.-lb. and a concentric thrust of 15,000 lb. (Compare with Example No. 41.) The data given are $d = 20$ in., $b = 10$ in., $d_1 = 1\frac{1}{8}$ in., $d_2 = 1\frac{3}{8}$ in., $A_{st} = 1.57$ sq. in., $A_{sc} = 0.88$ sq. in., and $e = \frac{150,000}{15,000} = 10$ in. For clarity, the calculation can be set out as in the following. The stresses are determined for the first trial value of $d_n$ and then, if necessary as in this example, a second trial is made.

\[
\begin{align*}
1st \ trial & & 2nd \ trial \\
\bar{d}_n &= 9 \text{ in.} & \bar{d}_n &= 10 \text{ in.} \\
I_{ac} &= d_1 - \frac{1}{3}d_n & 18.5 - 3 = 15.5 \text{ in.} & 18.5 - 3.33 = 15.17 \text{ in.} \\
I_{as} &= d_2 - d_1 & 18.5 - 1.38 = 17.12 \text{ in.} & 17.12 \text{ in.} \\
J_c &= \frac{1}{3}bd_n & \frac{1}{3} \times 10 \times 9 = 45 \\
J_s &= (m-1) \left( \frac{d_2 - d_1}{\bar{d}_n} \right) A_{sc} & 1 \left( \frac{9 - 1.38}{9} \right) 0.88 = 10.4 \\
F_c &= F_{sc} + F_{ce} = (J_c + J_s)f_{ce} & 55.4f_{ce} \\
J_{c,ac} &= 45 \times 15.5 = 698 & 50 \times 15.17 = 759 \\
J_{s,as} &= 10.4 \times 17.12 = 178 & 10.6 \times 17.12 = 183 \\
J_{c,as} + J_{s,as} &= 876 & 942 \\
\bar{d}_1 - X' &= (\text{formula 4.3a}) & (90 \times 14) + (12.3 \times 17.12) \\
& & (10 \times 9) + (14 \times 0.88) + (15 \times 1.57) \\
& & = 147.1 = 11.7 \text{ in.} \\
& & \frac{1471}{125.9} = 11.7 \text{ in.} \\
f_{ce} &= (\text{formula 4.3b}) & 15,000 \left( 10 + \frac{11.7}{11.7} \right) \\
& & \frac{15,000}{876} = 372 \text{ lb. per sq. in.} \\
& & \frac{15,000}{942} = 164 \text{ lb. per sq. in.} \\
f_{st} &= (\text{formula 4.3b}) & (55.4 \times 372) - 15,000 \\
& & 1.57 \\
& & = 3600 \text{ lb. per sq. in.} \\
& & = 3700 \text{ lb. per sq. in.} \\
\text{Check } \bar{d}_n &= (\text{formula 3.1c}) & 18.5 \left( \frac{3600}{15 \times 372} \right) = 11.2 \text{ in.} \\
& & \frac{18.5}{1 + \frac{3600}{15 \times 372}} = 10.7 \text{ in.} \\
\end{align*}
\]

The value $\bar{d}_n = 10.7$ in. is sufficiently near to the assumed value of 10 in., especially as the stresses are small.

Rectangular Member with Reinforcement in Tension Only.—The trial-and-error method in the foregoing can be simplified to apply to a rectangular member with reinforcement in tension only, since in this case $J_5$, $I_{as}$, $A_{sc}$, and $d_2$ are omitted and therefore

\[
f_{ce} = \frac{N(e - X' + d_1)}{\frac{1}{3}bd_n(1 - \frac{1}{3}d_n)}; \quad \text{and} \quad f_{st} = \frac{\frac{1}{3}bd_nf_{ce} - N}{A_{st}}. \quad (4.3c)
\]

Rectangular Member with Symmetrical Reinforcement.—The common case of a column in a building is a rectangular member with
symmetrical reinforcement \((A_{se} = A_{st})\) and can be dealt with by the trial
and-error-method described in the foregoing, but, because of its frequent
occurrence, a direct solution with the aid of tables or charts is more con-
venient. There are so many variable factors that a series of charts applicable
to all rectangular members would be unwieldy. The basis of most
charts is that equations relating the dimensions of the sections, the position
of the neutral plane, the stresses, and the eccentricity of the load, to the
expressions of equilibrium can be derived if formula \((3.1c)\), relating the
neutral-plane factor to the stresses, is taken into account. The formulæ
that follow are examples of such basic equations in which
\[
d_n = n_1 d_1, \quad r_s = \frac{A_{st} + A_{se}}{b \bar{d}}, \quad A_{st} = A_{se}, \quad \text{and} \quad \bar{d}_s = \bar{d}_t.
\]
\[
e = \frac{n_1^2 (\frac{1}{2} d - \frac{1}{2} n_1 d_1) + \frac{1}{2} r_s (d - 2 \bar{d}_t) \left[(d - 2 \bar{d}_t)(m - \bar{x}) - n_1 \bar{d}_1 + \bar{d}_s\right]}{n_1^2 \bar{d}_1 + r_s (m - \bar{x})(2n_1 \bar{d}_1 - d) + n_1 \bar{d}_1 + \bar{d}_s}
\]
\[
f_{ce} = \frac{M}{K_2 d \bar{d}_1^3};
\]
\[
K_2 = \left(\frac{\bar{d} - n_1}{4d_1 - n_1}\right) n_1 + \frac{r_s (d - 2 \bar{d}_t)}{4n_1 d_1^3} [m - \bar{x} (d - 2 \bar{d}_t) - n_1 \bar{d}_1 + \bar{d}_s].
\]
\[(4.3d)\]

In these formulæ it is assumed that \(X\) is \(\frac{1}{2} d\), an assumption which is
not always accurate since \(X\) is likely to vary from about \(0.4d\) to \(0.65d\).
If \(e\) exceeds \(0.5d\) the error is small, but it may be considerable for smaller
values of \(e\), although as \(e\) approaches \(\frac{1}{2} d\) the error is again reduced until
the condition of compression over all the section is reached, and then \(X\)
does actually equal \(\frac{1}{2} d\). Many published charts are based on formulæ
\((4.3d)\) but all have limitations. Those in the "Report on Formulæ for
Computation of Stresses" (Institution of Structural Engineers) are based
on formulæ \((4.3d)\) with \(m = 15\), \(A_{st} = A_{se}\) and specific values of \(\bar{d}_s / \bar{d}_1\).

**Large Eccentricity.**—The complex calculations described in the fore-
going are not justified in a case of members subjected to a large moment
compared with the thrust, that is if the eccentricity is large, say, if \(e\) exceeds
\(d\), since in such a case the stresses are largely determined by the bending
moment and the thrust causes only slight modification of these stresses.
In this case, therefore, the stresses due to the bending moment may be
calculated first as explained on page 58 for a beam subjected to bending
only. The maximum compressive stress should then be increased by a stress
\(f_{ce}\) and the tensile stress in the reinforcement decreased by a stress
\(m f_{ce}\) where, for a rectangular member,
\[
f_{ce} = \frac{N}{bd_n + m A_{st} + A_{se}(m - x)}
\]
\[(4.3e)\]
in which \(d_n\) is the position of the neutral plane with the stresses due to
bending only. Formula \((4.3e)\) gives simply the compressive stress which is
the thrust \(N\) divided by the equivalent stressed area.
Example No. 45.—Stresses in Member with Large Eccentricity.

Calculate the maximum compressive and tensile stresses in the rectangular member (Fig. 6) in Example No. 44 by the approximate method. The position of the neutral plane, if the member is subjected to bending only, is determined in Example No. 6 (page 45) to be \( d_n = 6.5\) in., and the lever arm \( l_n \) is determined in Example No. 10 (page 48) to be \( 16.5\) in.

Substituting in formula (3.5a):

\[ f_{st} \text{ (due to bending)} = \frac{150,000}{16.5 \times 1.57} = 5800 \text{ lb. per square inch.} \]

Substituting in formula (3.1d):

\[ f_{cb} \text{ (due to bending)} = \frac{6.5 \times 5800}{15(18.5 - 6.5)} = 209 \text{ lb. per square inch.} \]

Substituting in formula (4.3e):

\[ f_{cc} = \frac{15,000}{(10 \times 6.5) + (1.57 \times 15) + (0.88 \times 14)} = 147 \text{ lb. per square inch.} \]

The maximum compressive stress is \( 209 + 147 = 356 \text{ lb. per square inch} \)
and the tensile stress is \( 5800 - (15 \times 147) = 3590 \text{ lb. per square inch.} \)

These stresses should be compared with those obtained by the more accurate but lengthy calculation in Example No. 47, namely, 344 lb. and 3700 lb. per square inch; the differences are not significant in this case in which \( e = 10 \text{ in.} = \frac{1}{3}d \), but would be more marked if \( e \) were considerably less than \( d \).

**Design of Rectangular Members.**—The foregoing methods determine the stresses produced by a specified thrust and moment acting on a member of known cross-section, but problems of design can be dealt with on a similar basis but with the value of \( d_n \) as determined from formula (3.1c) for the known permissible stresses.

To design a rectangular member with tensile reinforcement only, an effective depth \( d_1 \) is assumed; the breadth \( b \) and the amount of reinforcement \( A_{st} \) required are then given by formula (4.3c) transposed and with \( X = \frac{1}{3}d \) approximately, that is

\[ b = \frac{N(e - \frac{1}{3}d + d_1)}{\frac{1}{3}d_n(d_1 - \frac{1}{3}d_n)p_{cb}}; \quad \text{and} \quad A_{st} = \frac{\frac{1}{3}bd_{n}p_{cb} - N}{p_{st}}. \quad (4.4a) \]

If the value of \( b \) obtained thus is unsuitable another trial value of \( d_1 \) may give suitable proportions. If convenient dimensions are not obtained in this way a suitable cross-section may be derived by reducing \( p_{st} \), thereby increasing \( d_n \) and the area of concrete in compression, and thereby reducing \( b \) but increasing \( A_{st} \).

For the design of a rectangular member with reinforcement in tension and compression it is necessary to know, or to assume, the dimensions \( b \) and \( d_1 \). By transposing and expanding formula (4.3b), but with \( X = \frac{1}{3}d \), the areas of reinforcement required are obtained from

\[ A_{sc} = \frac{N(e - \frac{1}{3}d + d_1) - \frac{1}{3}bd_{n}(d_1 - \frac{1}{3}d_n)p_{cb}}{(m - 1)(\frac{d_{n} - d_2}{d_n})(d_1 - d_2)p_{cb}}; \quad (4.4b) \]

\[ A_{st} = \frac{(J_0 + J_4)p_{cb} - N}{p_{st}}. \]
If these values of $A_{se}$ and $A_{st}$ are unsuitable, for example if the total reinforcement exceeds, say, 8 per cent. or is less than 1 per cent. of the gross area of the concrete, or if $A_{se}$ greatly exceeds $A_{st}$, the tensile stress should be reduced and the problem reworked; or other values of $b$ and $d$ should be considered, if possible. It is obvious that large variations in the amounts of reinforcement can result from modifying $\rho_{st}$; a reduction in this stress increases $A_{st}$, but reduces $A_{se}$ since more concrete is included in the compression zone by the consequent increase of $\bar{d}_n$. For any given cross-section there is a value of $\rho_{st}$ which gives a minimum value of $A_{st} + A_{se}$, and this produces the most economical design since the amount of concrete is constant with given values of $b$ and $d$.

In formulae (4.4a) and (4.4b), $\bar{X}$ is assumed to be $\frac{1}{2}d$; when the design of an important member is complete the stresses should be calculated by substituting the value of $\bar{X}$ as determined from formula (4.3a).

Example No. 46.—Determination of Amount of Reinforcement.

Determine the amount of reinforcement in tension and compression required in a rectangular member subjected to a bending moment of 400,000 in.-lb. and a concentric thrust of 50,000 lb., if $d = 20$ in. and $b = 10$ in. The permissible stresses are $\rho_{st} = 20,000$ lb. per square inch and $\rho_{eb} = 1000$ lb. per square inch; $n_1 = 0.426$ from formula (3.1c).

Assume that $d_t = d = 1\frac{1}{2}$ in. Therefore

$$d_1 = 18.5 \text{ in.}, \quad d_n = 0.426 \times 18.5 = 7.87 \text{ in.}$$

$$\epsilon = \frac{400,000}{50,000} = 8 \text{ in.}$$

Substitution in formula (4.4b) gives

$$A_{se} = \frac{50,000(8-10+18.5)-(\frac{1}{2} \times 10 \times 7.87)(18.5 - 2.62)1000}{14 \times 7.87 \times 17 \times 1000} = 1.04 \text{ sq. in.;}$$

provide, say, two $\frac{3}{8}$-in. bars in compression. Substitution in formula (4.4b) for $A_{st}$ with $J_c = \frac{1}{2} \times 10 \times 7.87 = 39.4$ and

$$J_s = 14 \times \frac{6.37}{7.87} \times \frac{1.04}{1} = 11.7,$$

gives $A_{st} = \frac{(39.4 + 11.7)1000 - 50,000}{20,000} = 0.056 \text{ sq. in.}$; provide the minimum amount of reinforcement in tension of about 0.5 per cent., that is $0.005 \times 10 \times 18.5 = 0.925$ sq. in., say, two $\frac{3}{8}$-in. bars (0.88 sq. in.). Since the reinforcement in tension is not being used fully effectively, it would probably be more economical to design this member with a smaller permissible tensile stress. Assume a tensile stress of 15,000 lb. per square inch for which $d_n = \frac{18.5}{1 + \frac{15,000}{15 \times 1000}} = 9.25$ in., and

$$A_{se} = \frac{50,000(8-10+18.5)-(\frac{1}{2} \times 10 \times 9.25)(18.5 - 3.08)1000}{14 \times \frac{7.75}{9.25} \times 17 \times 1000} = 0.56 \text{ sq. in.}$$

Therefore provide two $\frac{3}{8}$-in. bars in compression (0.6 sq. in.).

$$J_c = \frac{1}{2} \times 10 \times 9.25 = 46.25; \quad J_s = 14 \times \frac{7.75}{9.25} \times 0.5 = 6.59.$$
Therefore \( A_{st} = \frac{(46.25 + 6.59)1000 - 50,000}{15,000} = 0.19 \text{ sq. in.} \); provide the minimum amount of reinforcement, say two \( \frac{3}{4} \)-in. bars as before. The second calculation results in the more economical member because less reinforcement in compression is shown to be justified.

II. LOAD-FACTOR METHOD.

The basis of the load-factor method of calculating the resistance of members subjected to thrust and bending simultaneously is similar to that for the resistance to bending described on page 61. There are two principal cases, namely, when the tensile resistance and when the compressive resistance of the member determines the safe thrust and moment. There is also an intermediate case, which is the borderline or limiting condition between the two principal cases. The analyses in the following apply to rectangular members and are based on the method recommended in British Standard Code No. 114.

Resistance Determined by Tensile Strength.

When the bending moment predominates, that is when the eccentricity is large, failure is due to yielding of the reinforcement in tension. The conditions as failure approaches are shown in Fig. 20 and are such that the stress in the reinforcement in tension and compression is the yield stress of the steel. The forces in these reinforcements are \( A_{st}p_{sy} \) and \( A_{sc}p_{sy} \) respectively. Expressed in terms of the permissible stresses \( p_{st} \) and \( p_{sc} \), which should not exceed half the yield-point or equivalent yield stress, the forces are \( 2A_{st}p_{st} \) and \( 2A_{sc}p_{sc} \) respectively. Applying a load factor of two, the safe resistances are \( A_{st}p_{st} \) and \( A_{sc}p_{sc} \) respectively.

The compressive resistance of the concrete near failure is equivalent to a rectangular block of stress of depth \( 0.85d_n \) and, with the stress ordinate equal to two-thirds of the crushing strength \( \nu \) of works cubes of concrete,
the compressive resistance of the concrete is \( \frac{3}{4}u(0.85d_n b) \). Since the stress \( \rho_{cc} \) permissible in direct compression in concrete is \( 0.76 \times \frac{1}{4} \times u \) (see pages 17 and 18), \( \rho_{cc} = 0.253u \); therefore \( \frac{3}{4}u = 2.63\rho_{cc} \). The total compression is therefore \( 2.63\rho_{sc}(0.85d_nb) \). If a load-factor of 2.63 is applied, the safe compressive resistance of the concrete is \( 0.85d_nb \).

Referring to Fig. 20, in which \( P \) is the safe thrust acting at an eccentricity \( e \), to produce equilibrium the algebraical summation of the tensile and compressive forces must balance, that is

\[
0.85d_nb\rho_{cc} + A_{sc}\rho_{sc} - A_{st}\rho_{st} - P = 0.
\]

Similarly, algebraical summation of the moments of these forces about the line of action of the tensile resistance gives

\[
0.85d_nb\rho_{cc}[d_1 - \frac{1}{2}(0.85d_n)] + A_{sc}\rho_{sc}(d_1 - d_2) - P(e - \frac{1}{2}d + d_1) = 0.
\]

**Rectangular Member with Symmetrical Reinforcement.**—If these two summations are simplified to apply to a rectangular member with symmetrical reinforcement, that is \( A_{sc} = A_{st} = \frac{1}{2}A_s \) in which \( A_s \) is the cross-sectional area of the total reinforcement, they become

\[
0.85d_nb\rho_{cc} + \frac{1}{2}A_s(\rho_{sc} - \rho_{st}) - P = 0
\]

and

\[
0.85d_nb\rho_{cc}(d_1 - 0.425d_n) + \frac{1}{2}A_s\rho_{sc}(d_1 - d_2) - P(e - \frac{1}{2}d + d_1) = 0.
\]

Combining these summations to eliminate \( d_n \) and transposing gives an expression for the safe thrust \( P \) acting at an eccentricity \( e \), namely,

\[
P = \rho_{cc}\left\{ U + \sqrt{U^2 + A_s(d_1 - d_2)\rho_{sc} + Y'(2d_1b - Y')} \right\}
\]

in which

\[
U = b(\frac{1}{4}d - e) - Y' \quad \text{and} \quad Y' = \frac{A_s(\rho_{st} - \rho_{sc})}{2\rho_{cc}}.
\]

Formula (4.5a) is applicable when the safe thrust is determined by the tensile resistance, that is when the applied thrust is less than the limiting eccentric thrust \( P_0 \) as calculated in the following.

**Limiting Eccentric Load.**

The limiting condition, between the conditions that the tensile or compressive resistance determines the safe thrust, is obtained when the stress in the reinforcement is equal to the yield stress \( \rho_{sy} \) of the steel at the same time as the stress in the concrete reaches the stress equivalent to the limiting strain, which is generally assumed to be 0.0033. The strain diagram in Fig. 20 shows that the relation between the strain in the reinforcement in tension and the maximum strain in the concrete is

\[
\frac{\rho_{sy}}{E_s} : d_1 - d_n = \frac{1}{300} : d_n,
\]

in which \( E_s \) is the ordinary elastic modulus of mild steel (that is \( 30 \times 10^6 \) lb. per square inch) or the secant modulus of high-yield-stress steel at a stress
of \( 2\rho_{st} \). Substituting \( 2\rho_{st} \) for \( \rho_{sy} \) and transposing, the depth \( d_n \) to the neutral plane is \( \frac{d_1}{1 + \frac{600\rho_{st}}{E_s}} \). The factor \( 0.85d_n \) is therefore \( \frac{0.85d_1}{1 + \frac{600\rho_{st}}{E_s}} \) and, if this expression is denoted by \(Xd_1\), and substituted in the equations of equilibrium of forces and moments, the resulting expressions for a rectangular member are

\[
Xd_1 d_2 + A_{sc}\rho_{sc} - A_{st}\rho_{st} - P_b = 0,
\]

and

\[
Xd_1 d_2 (1 + \frac{1}{2}X) + A_{sc}\rho_{sc}(d_1 - d_2) - P_b(\sigma_b - \frac{1}{2}d + d_1) = 0,
\]

in which \( P_b \) is the safe limiting thrust acting at the limiting eccentricity \( e_b \) about the centroid of the cross-section of the member.

**Rectangular Member with Symmetrical Reinforcement.**—In the case of a rectangular member with symmetrical reinforcement, substituting \( A_{sc} = A_{st} = \frac{1}{2}A_s \) and \( d_1 = d_2 \), and transposing, the equilibrium equations become

\[
P_b = Xd_1 d_2 - \frac{1}{2}A_s(\rho_{st} - \rho_{sc})
\]

\[
e_b = \frac{X(1 - \frac{1}{2}X)\rho_{sc}d_1^2 + \frac{1}{2}A_s\rho_{sc}(d_1 - d_2)}{P_b} - \frac{d_1 - d_2}{2}
\]

in which \( X = \frac{0.85}{1 + \frac{600\rho_{st}}{E_s}} \).

\[ (4.5b) \]

If the applied thrust exceeds \( P_b \), the resistance is determined by the compressive strength and the analysis is as in the following.

**Resistance Determined by Compressive Strength.**

The safe concentric thrust \( P_0 \) on a reinforced concrete member is determined on page 33 by the load-factor method of analysis. The safe thrust \( P \) acting at an eccentricity \( e \) is generally less than \( P_0 \). Tests show that if the compressive strength of the member determines the safe resistance, the decrease in safe load, namely \( P_0 - P \), is proportional to the moment \( Pe \). At the limiting eccentric thrust \( P_b \) acting at the limiting eccentricity \( e_b \), the decrease is \( P_0 - P_b \), and therefore for any thrust \( P \) greater than \( P_b \) and applied at any eccentricity \( e \), the relation is

\[
\frac{P_0 - P}{P_0 - P_b} = \frac{P_0 - P_b}{P_b e_b}.
\]

Transposition gives the safe load

\[
P = \frac{P_0}{1 + \left( \frac{P_0}{P_b} - 1 \right) \frac{e}{e_b}}. \quad \quad \quad (4.5c)
\]

**Examples.**

**Example No. 47.—Limiting Thrust and Eccentricity.** Calculate the limiting thrust and eccentricity for the rectangular member in Fig. 20;
\( d = 20 \text{ in.}, \quad d_1 = 18 \frac{1}{2} \text{ in.}, \quad d_2 = 1 \frac{3}{4} \text{ in.}, \quad b = 10 \text{ in.} \) Mild steel reinforcement: 
\( E_s = 30 \times 10^6 \text{ lb. per square inch}, \quad p_{st} = 20,000 \text{ lb.}, \quad p_{sc} = 18,000 \text{ lb. per square inch}; \) and \( A_{st} = A_{se} = 1 \cdot 57 \text{ sq. in.}; \) therefore \( A_s = 3 \cdot 14 \text{ sq. in.} \) Ordinary quality concrete \( 1 : 2 : 4; \quad p_{cc} = 760 \text{ lb. per square inch.} \)

Substitution in formula (4.5b) gives 
\[
X = \frac{0.85}{1 + \frac{600 \times 20,000}{30 \times 10^6}} = 0.607.
\]

\[
P_b = (0.607 \times 18 \cdot 5 \times 10 \times 760) - (\frac{1}{2} \times 3 \cdot 14)(20,000 - 18,000) = 82,060 \text{ lb.}
\]

\[
\varepsilon_b = \frac{(0.607 \times 0 \cdot 70 \times 760 \times 10 \times 18 \cdot 5^2) + (1 \cdot 57 \times 18,000 \times 17)}{82,060} = \frac{12}{12} = 10 \cdot 8 \text{ in.}
\]

**Example No. 48.—Safe Thrust at Large Eccentricity.** Calculate the safe thrust on the member in Example No. 47 (Fig. 20) if the thrust acts at an eccentricity of 15 in.

Since \( \varepsilon \) is greater than \( \varepsilon_b \) (= 10 \cdot 8 \text{ in.}) as determined in Example No. 47), the safe thrust must be less than \( P_b \) (= 82,060 lb.) and therefore formulae (4.5a) are applicable. By substitution,

\[
Y' = \frac{3 \cdot 14(20,000 - 18,000)}{2 \times 760} = 4 \cdot 13 \text{ sq. in.}
\]

And

\[
U = 10(10 - 15) - 4 \cdot 13 = -54 \cdot 13 \text{ sq. in.}
\]

Therefore

\[
P = 760 \left( -54 \cdot 13 \right)
\]

\[
+ \sqrt{\frac{-54 \cdot 13^2 + (3 \cdot 14 \times 10 \times 17 \times 18,000)}{760} + 4 \cdot 13 \left[ (2 \times 18 \cdot 5 \times 10) - 4 \cdot 13 \right]}
\]

\[
= 58,300 \text{ lb.}
\]

**Example No. 49.—Safe Thrust at Small Eccentricity.** Calculate the safe thrust on the member in Example No. 47 if the thrust acts at an eccentricity of 6 in.

Since \( \varepsilon \) is less than \( \varepsilon_b \) (=10 \cdot 8 \text{ in.}) the safe thrust exceeds \( P_b \) (=82,060 lb.) and therefore formula (4.5c) is applicable.

Substitution in formula (2.3a) on page 34 gives

\[
P_s = (760 \times 10 \times 20) + 3 \cdot 14(18,000 - 760) = 206,300 \text{ lb.}
\]

Substitution in formula (4.5c) gives

\[
P = \frac{206,300}{1 + \left( \frac{206,300}{82,060} - 1 \right) \cdot \frac{6}{10 \cdot 8}} = 112,000 \text{ lb.}
\]

Suitable cross-sections for members required to resist a specified thrust and bending moment can be determined directly by the load-factor method only by means of unwieldy formulae. It is therefore necessary to assume dimensions and reinforcement and, by calculations similar to those in the foregoing examples, to determine the safe thrust and compare this thrust with the applied thrust. If there is much difference the dimensions or reinforcement or both should be decreased or increased as necessary.
CHAPTER V
RESISTANCE TO AXIAL PULL WITHOUT BENDING AND WITH BENDING

There are two dissimilar conditions to be analysed in considering a reinforced concrete member subjected to concentric pull or to axial pull combined with bending, namely, if the concrete in the tensile zone is considered to have cracked and if the concrete is considered to resist tension as in the case of the design of containers of liquids. Both conditions are analysed in the following in accordance with the modular-ratio method.

I. CONCRETE RESISTANT TO TENSION.

An example of a member subjected to concentric pull without bending and in which the concrete is considered to resist tension is the wall of a cylindrical tank. This condition might also apply to ties, such as those in industrial structures, in which cracking might be objectionable because of the risk of corrosion of the reinforcement. Examples of parts of structures subjected to axial pull and bending simultaneously, where the concrete is considered to resist tension, are the walls of rectangular containers of liquids.

Resistance to Concentric Pull.

If the concrete is considered to resist tension, a concentric pull is similar to a concentric thrust in so far that the line of action of the force passes through the centroid of the equivalent area of the entire cross-section of the member. The position of the centroid is determined as
explained on page 32. The effect of a concentric pull is to extend the member uniformly (Fig. 21). The extension, and consequently the strain, of the reinforcement and the concrete are identical and therefore the tensile stresses are proportional to the elastic moduli of the two materials. By comparison with the corresponding condition of concentric thrust, if a reinforced concrete member of equivalent area \( A_e \) is subjected to a concentric pull \( F_T \), the tensile stress \( f_{ct} \) in the concrete is \( \frac{F_T}{A_e} \) assuming that the concrete does not crack. The tensile stress \( f_{ct} \) in the reinforcement is \( m f_{ct} \) and is always less than the tensile stress permissible in the steel since \( f_{ct} \) must be small to avoid cracking. If \( A \) is the gross area of the cross-section of the concrete and \( A_s \) is the total cross-sectional area of the reinforcement,

\[
f_{ct} = \frac{F_T}{A + A_s (m - 1)}.
\]

If the tensile stress permissible in the concrete is \( \phi_{ct} \), the safe pull on a concentrically-loaded member is given by

\[
F_{T(max)} = [A + A_s (m - 1)] \phi_{ct}.
\]

Rectangular Member with Unsymmetrical Reinforcement.—In the case of a rectangular member with unsymmetrical reinforcement (Fig. 22a) subjected to a concentric pull \( F_T \), the formulae which follow are readily derived.

Position of centroid: \[
X = \frac{\frac{1}{2}bd^2 + (m - 1)(A_{e1}d_1 + A_{e2}d_2)}{bd + (m - 1)(A_{s1} + A_{s2})}.
\]

Tensile stress: \[
f_{ct} = \frac{F_T}{bd + (m - 1)(A_{s1} + A_{s2})}.
\]

Safe pull: \[
F_{T(max)} = [bd + (m - 1)(A_{s1} + A_{s2})] \phi_{ct}.
\]

Rectangular Member with Symmetrical Reinforcement.—In the case of a rectangular member with symmetrical reinforcement the centroid is at the centre of the section if \( d_1 = d_2 \). Therefore \( X = \frac{1}{2}d \), \( A_{e1} = A_{e2} = \frac{1}{2}A_e \), and \( f_{ct} = \frac{F_T}{bd + (m - 1)A_e} \); \( F_{T(max)} = [bd + (m - 1)A_e] \phi_{ct} \).

Example No. 50.—Wall with Symmetrical Reinforcement and Subjected to a Concentric Pull. Calculate the safe concentric pull which can be resisted by the 6-in. wall in Fig. 22b if the tensile stress permissible in the concrete is 200 lb. per square inch. Consider 1 ft. length of wall; \( b = 12 \text{ in.} \), and \( d = 6 \text{ in.} \). With \( \frac{1}{4} \text{-in.} \) bars at 6-in. centres near both faces, \( A_s \) \( = A_{s1} + A_{s2} \) \( = 2 \times 0.392 = 0.784 \text{ sq. in. per foot}. \) By substitution in formula (5.1c),

\[
F_{T(max)} = [(12 \times 6) + (14 \times 0.784)]200 = 16,600 \text{ lb. per foot}.
\]
Eccentric Pull and Bending.

Consideration of the external actions on a member subjected to bending and axial pulls simultaneously is similar to the corresponding cases of eccentric thrusts, regard being paid to the opposite direction of the action of the concentric or eccentric pulls. The most common condition, which is the only case considered in the following, is the simple case of a bending moment and a concentric pull acting simultaneously. In Fig. 23 it is assumed that a clockwise bending moment $M$ and a concentric pull $F_T$ are acting. As in the case of thrusts, the force and moment may be replaced by a single pull $F_T$ acting at a distance $a$ from the centroid such that the eccentricity is given by

$$
e = e_1d = \frac{M}{F_T}. \quad \quad \quad (5.2)$$

If the concrete is not cracked the position of the centroid can be established from the appropriate formulae in Chapter XIII, Vol. II, for members of irregular cross-section and from formula in (5.1b) for rectangular members.

It is convenient in some of the analyses in the following to determine the eccentricity $e_s$ of the force $F_T$ about the centre-line of the group of reinforcement bars nearer the line of action of $F_T$, that is

$$e_s = e + X - d_1. \quad \quad \quad (5.2a)$$

There are two principal conditions due to the simultaneous action of axial pull and bending, namely, when the resultant stresses are entirely tensile, and when compressive and tensile stresses are produced. The two cases are dealt with in the following.
BASIC REINFORCED CONCRETE DESIGN

Axial Pull and Bending: Tensile Stresses Only.

When the concrete is considered to resist tension and tensile stresses only are produced, the condition is analogous to the case of compressive stress all over the cross-section, but observing that the eccentric force $F_T$ is a pull. Therefore the maximum and minimum tensile stresses (Fig. 24)

$$
\begin{align*}
\left[ \text{TENSILE STRESSES DUE TO PULL } f_T \right] + \left[ \text{TENSILE AND COMPRESSIVE STRESSES DUE TO MOMENT } \frac{M}{f_{ct}} \right]
= f_{ct}^{(\text{resultant})} - f_{ct}^{(\max)} = f_{ct}^{(\max)} - f_{ct}^{(\min)}
\end{align*}
$$

Fig. 24.—Stresses due to Concentric Pull and Bending: Concrete Effective in Tension. Tensile Stresses Only.

are the summation of the tensile stress due to the concentric pull and the tensile and compressive stresses due to the moment, and

$$
\begin{align*}
f_{ct}^{(\text{min.})} &= F_T \left( \frac{1}{A_e} - \frac{e}{Z_e} \right) ; \quad f_{ct}^{(\text{max.})} = F_T \left( \frac{1}{A_e} + \frac{e}{Z_e} \right).
\end{align*}
$$

The section moduli $Z_t$ and $Z_e$ refer to the edges where the maximum and minimum tensile stresses respectively occur; the effective area and section moduli are obtained from the same formulæ as are used for bending and thrust (with compressive stresses only); the eccentricity is derived from formula (5.2). The limit of application of this case is obviously when $f_{ct}^{(\text{min.})}$ is zero, that is when $e = \frac{Z_t}{A_e}$. Simplifications of the terms in formula (5.3) for members of regular cross-section are self-evident.

Fig. 25.—Concentric Pull and Bending: Concrete Effective in Tension. Tensile Stresses Only.
Rectangular Member with Unsymmetrical Reinforcement.—An important case of combined bending and tension which commonly occurs in the design of walls of watertight containers (if they are assumed not to be cracked) is a rectangular member with unsymmetrical reinforcement as in Fig. 25a. The equivalent area \( A_e \), the position \( \bar{X} \) of the centroid, and the centroidal moment of inertia are given by formulae (3.10a) in Chapter III. Therefore with \( A_{e1} = A_{et} \), \( A_{e2} = A_{ec} \), and \( \bar{X} = d_n \), formulae (5.3) become

\[
\begin{align*}
\sigma_{et(max.)} &= F \left\{ \frac{1}{bd + (m - 1)(A_{e1} + A_{e2})} + \frac{e(d - \bar{X})}{I} \right\}, \\
\sigma_{et(min.)} &= F \left\{ \frac{1}{bd + (m - 1)(A_{e1} + A_{e2})} - \frac{e\bar{X}}{I} \right\},
\end{align*}
\]

and if \( \bar{X} = \frac{1}{2}d \) approximately and \( d_1 = d_2 \),

\[
I = \frac{bd^3}{12} + (m - 1)[(A_{e1} + A_{e2})(\frac{1}{2}d - d_2)^2].
\]

Formulae (5.3a) are readily simplified for members with symmetrical reinforcement or with tension reinforcement only.

Example No. 51.—Wall with Unsymmetrical Reinforcement and Subjected to Bending and a Pull. Calculate the stresses in the 6-in. wall in Fig. 25b if it is subjected to an axial pull of 5000 lb. per foot and a bending moment of 5000 in.-lb. per foot. \( e = \frac{5000}{5000} = 1 \) in., which, being small in relation to \( d = 6 \) in., indicates that tensile stresses only are likely. Consider 1 ft. length of wall; \( b = 12 \) in., \( A_{e1} = 0.392 \) sq. in. per foot; \( A_{e2} = 0.196 \) sq. in. per foot; \( d_1 = d_2 = 1 \) in.; \( d_1 = 5 \) in. For this type of member it is sufficiently accurate to assume that \( \bar{X} = \frac{1}{2}d = 3 \) in., and, by substitution in formulae (5.3a),

\[
I = \frac{12 \times 6^3}{12} + 14(0.392 + 0.196)(3 - 1)^3 = 249 \text{ in.}^4
\]

\[
\sigma_{et(max.)} = 5000 \left\{ \frac{1}{(12 \times 6) + (14 \times 0.588)} + \frac{1 \times 3}{249} \right\}
\]

\[
= 5000(0.01245 + 0.01205) = 123 \text{ lb. per square inch.}
\]

\[
\sigma_{et(min.)} = 5000(0.01245 - 0.01205) = 2 \text{ lb. per square inch.}
\]

If this example be re-worked with the more accurate values of \( I \) and \( \bar{X} \) obtained from formulae (3.10a), it will be found that \( \bar{X} = 3.12 \) in. (instead of 3 in.) and the maximum stress \( \sigma_{et(max.)} \) is 124 lb. per square inch (tension) and the minimum stress is 4 lb. per square inch (compression, but practically zero); by comparison, it is seen that the additional arithmetic required for the more accurate method is not worth while.

Axial Pull and Bending: Tensile and Compressive Stresses.

If the eccentricity \( e \) exceeds \( \frac{Z_e}{A_e} \left( = \frac{I}{X_A} \right) \), compressive and tensile stresses are produced. The analysis proceeds similarly to that for bending.
and compression but modified to allow for a pull instead of a thrust. For members subjected to a bending moment $M$ simultaneously with a pull $F_T \left( e = \frac{M}{F_T} \right)$, the distribution of stresses is as in Fig. 26 assuming that

\[ f_{ct} = F_T \left( \frac{1}{A_e} + \frac{e}{Z_e} \right), \quad f_{cb} = F_T \left( \frac{1}{A_e} - \frac{e}{Z_e} \right). \]  

**Rectangular Member with Unsymmetrical Reinforcement.**

In the case of a rectangular member with unsymmetrical reinforcement the formulae corresponding to (5.3a) are

\[ f_{ct} = F_T \left[ \frac{I}{bd + (m - 1)(A_{et} + A_{sc})} + \frac{e(d - X)}{I} \right], \quad f_{cb} = F_T \left[ \frac{I}{bd + (m - 1)(A_{et} + A_{sc})} - \frac{eX}{I} \right], \]

in which $X$ and $I$ are obtained from formulae (3.10a) in Chapter III, or $I$ can be calculated approximately from the appropriate formula in series (5.3a).

**Example No. 52.—Wall with Unsymmetrical Reinforcement and Subjected to Bending and a Pull.** Calculate the stresses in the wall in Example No. 51 (Fig. 25b) if the direct pull is 5000 lb. per foot and the bending moment is 15,000 in.-lb. per foot. $e = \frac{15,000}{5000} = 3$ in., which in relation to $d = 6$ in. indicates that tensile and compressive stresses are
likely. Substituting the approximate value, as before, for \( \bar{X} (= \frac{1}{3}d = 3 \text{ in.}) \) and \( I (= 249 \text{ in.}^4) \) in formulae (5.4a),

\[
\begin{align*}
 f_{ct} &= 5000\left[0.01245 + \frac{3 \times 3}{249}\right] = +242 \text{ lb. per square inch (tension)}, \\
 f_{cb} &= 5000[0.01245 - 0.036] = -118 \text{ lb. per square inch (compression)}.
\end{align*}
\]

The direct design of members for this condition is not straightforward and it is best to assume the dimensions and reinforcement, calculate the stresses, and make suitable alterations to the assumptions if the stresses are too great or too small compared with the permissible stresses.

II. CONCRETE INEFFECTIVE IN TENSION.

Examples of reinforced concrete members subjected to concentric thrust without bending and in which the concrete is assumed not to resist tension, include vertical ties in which cracking is not objectionable and the walls of cylindrical containers of dry materials. Horizontal ties, such as those between pile-caps in foundations or between the opposite walls of rectangular containers, are not in tension only since the weight of the tie and any load that the tie may carry accidentally or purposely causes bending. Vertical ties may be subjected to tension only such as in a bowstring girder and for a floor or the like which is suspended from a structure above instead of being supported from below. Other examples of members subjected to axial pull and bending simultaneously are more common, and, in addition to horizontal ties, occur in the case of the walls of rectangular containers of dry materials.

Resistance to Concentric Pull.

The condition of concentric pull, in the case of the concrete not resisting tensile forces, is dissimilar to that of a concentric thrust in so far that its line of action passes through the centroid of the reinforcement only and not the centroid of the equivalent area of the entire cross-section of the member. The position of the centroid of two unequal groups of reinforcement \( A_{s1} \) and \( A_{s2} \) (Fig. 27a) is given by

\[
\bar{X} = \frac{d_2 A_{s1} + A_{s2} d_2}{A_{s1} + A_{s2}}.
\]  

(5.5)

For a member with symmetrical reinforcement, that is \( A_{s1} = A_{s2} \), \( \bar{X} = \frac{1}{2}d \) if \( d_2 = d_1 \). If the concrete cracks the entire pull \( F_X \) is resisted by the total reinforcement \( A_s (= A_{s1} + A_{s2}) \). The formulae for the tensile stress in the reinforcement and the safe concentric pull are therefore

\[
\begin{align*}
 f_{st} &= \frac{F_X}{A_s}; \\
 F_{X(max.)} &= A_s p_{st},
\end{align*}
\]

(5.6)

in which \( p_{st} \) is the tensile stress permissible in the reinforcement. These
formulae apply not only to the design of tie-beams, cracking of which may not be detrimental to the strength, but also to ties and walls of cylindrical containers for the condition that, should the concrete crack, the entire tensile force can be resisted safely by the reinforcement.

Example No. 53.—Wall with Unsymmetrical Reinforcement and Subjected to Concentric Pull. Calculate the position of the centroid of, and the stresses in, the reinforcement in the 6-in. wall in Fig. 25b, if the wall is subjected to a concentric pull of 5000 lb. per foot assuming that the concrete does not resist tension. As in Example No. 51, \( A_{s1} = 0.392 \text{ sq. in.}, \ A_{s2} = 0.196 \text{ sq. in.}, \) and \( A_s = 0.588 \text{ sq. in. per foot}; \ d_1 = d_2 = 1 \text{ in.}; \ d_1 = 5 \text{ in.} \). Substituting in formula (5.5) and (5.6),

\[
X = \frac{(0.392 \times 5) + (0.196 \times 1)}{0.588} = 3.66 \text{ in.},
\]

and

\[
f_{st} = \frac{5000}{0.588} = 8500 \text{ lb. per square inch.}
\]

Axial Pull and Bending: Tensile Stresses Only.

If the tensile resistance of the concrete is neglected, the axial pull and bending moment are resisted entirely by the tensile forces in the reinforcement. The first step is to determine the position of the centroid of the reinforcement. With symmetrical reinforcement \( X = \frac{1}{2}d \), but for the common case of a member with unsymmetrical reinforcement this position is given by formula (5.5). The eccentricity of the equivalent pull about this centroid is, as before, \( \frac{M}{F_T} \). If tensile forces in the groups of reinforcement \( A_{s1} \) and \( A_{s2} \) (Fig. 27b) be denoted by \( F_{s1} \) and \( F_{s2} \) respectively, the two summations for equilibrium are \( F_{s1} + F_{s2} = F_T = 0 \) and

\[
F_{s1}(d_1 - d_2) - F_T(e + X - d_2) = 0.
\]

The greater tensile stress \( f_{st(max)} \) occurs in the group of reinforcement \( A_{s1} \) nearer the line of action of \( F_T \); the smaller tensile stress \( f_{st(min)} \) occurs in
AXIAL PULL AND BENDING

the other group $A_{s2}$. Since $F_{s1} = A_{s1}f_{st(max)}$ and $F_{s2} = A_{s2}f_{st(min)}$, substitution in the foregoing summations and transposition give

$$f_{st(max)} = \frac{F_T(e + X - d_2)}{A_{s1}(d_1 - d_2)}; \quad f_{st(min)} = \frac{F_T - A_{s1}f_{st(max)}}{A_{s2}} \quad (5.7)$$

The limit for this case is when $f_{st(min)} = 0$, that is if $e$ exceeds $x' (= d_1 - X)$ compressive stresses are produced. In the limiting case of $e = d_1 - X$ the line of action of the equivalent eccentric pull coincides with the centre of the group of reinforcement $A_{s1}$, and the tensile stress in this group is simply $\frac{F_T}{A_{s1}}$; there is no stress in the group $A_{s2}$.

Example No. 54.—Wall with Unsymmetrical Reinforcement Subjected to Bending and a Pull. Calculate the tensile stress in the reinforcement in the 6-in. wall in Fig. 25b if subjected to a concentric pull of 5000 lb. per foot and a bending moment of 5000 in.-lb. per foot as in Example No. 51, but neglecting the tensile resistance of the concrete.

$$e = \frac{5000}{5000} = 1 \text{ in.}$$

Substituting $A_{s1} = 0.392 \text{ sq. in.}, A_{s2} = 0.196 \text{ sq. in.},$ $A_{s} = 0.588 \text{ sq. in. per foot}, d_1 = 5 \text{ in. and } d_2 = 1 \text{ in. in formulae (5.5) and (5.7) gives } X = 3.66 \text{ in. (as in Example No. 53)}$ and

$$f_{st(max)} = \frac{5000(1 + 3.66 - 1)}{0.588 \times 4} = 7770 \text{ lb. per square inch.}$$

$$f_{st(min)} = \frac{5000 - (0.392 \times 7770)}{0.196} = 1020 \text{ lb. per square inch.}$$

Axial Pull and Bending: Tensile and Compressive Stresses.

If the bending moment is large compared with the axial pull, compressive and tensile stresses are likely to be produced. The compressive forces are resisted by the concrete and the reinforcement in compression, if any, and the tensile forces are resisted entirely by the reinforcement in tension if the concrete is considered not to resist tension. The distribution of the stresses is as shown in Fig. 28, but the position of the centroid of the stressed area is not known until the position of the neutral plane and the maximum stresses are established. Assuming a value of $X$ and with $e = \frac{M}{F_T}$, the equilibrium summations are

$$F_{st} + F_{ce} + F_{se} - F_T = 0, \quad \text{and} \quad F_T(e + X - d_2) - F_{ce}I_{ac} - F_{se}I_{as} = 0.$$

Rectangular Member with Unsymmetrical Reinforcement.—In the case of a rectangular member with unsymmetrical reinforcement as in Fig. 28, the dimension $X$ is evaluated from formula (4.3a), as for axial thrust and bending on page 84, with a trial value of $d_2$. From the summations of equilibrium, formulae for the maximum stresses are derived
in a manner similar to the case of bending and axial thrust, and are

\[ f_{cb} = \frac{F_T(e + X - d_1)}{J_{oc} + J_{oa} f_{cb}}; \quad f_{st} = \frac{F_T + (J_c + J_s) f_{cb}}{A_{st}} \]

in which \( J_c = \frac{1}{3}bd_n; \quad J_s = (m - 1) \left( \frac{d_n - d_2}{d_n} \right) A_{sc}; \)
\( l_{ac} = d_1 - \frac{1}{3}d_n; \quad l_{as} = d_1 - d_2. \)

\( (5.8) \)

Rectangular Member with Reinforcement in Tension Only.—
A rectangular member with reinforcement in tension only is a common case for walls of rectangular containers. The analysis for this case is applicable even if reinforcement is placed near both faces of the wall,

Fig. 28.—Concentric Pull and Bending: Concrete Ineffective in Tension. Tensile and Compressive Stresses. Rectangular Member with Reinforcement in Tension and Compression.

since the reinforcement in the compression zone may not always be effective in resisting compression because it is not secured against buckling. If the eccentricity is small the stresses can be evaluated by formulæ (5.8) modified to

\[ f_{cb} = \frac{F_T(e + X - d_1)}{J_{oc}}; \quad f_{st} = \frac{F_T + J_c f_{cb}}{A_{st}}. \]

\( (5.8a) \)

Approximate Analysis.—If \( e \) exceeds, say, \( d_1 \), the approximate analysis in the following is sufficiently accurate. Measure \( e \left( \frac{M}{F_T} \right) \) from the centre of the section (Fig. 29). The distance \( e_s \) from the centre of the reinforcement in tension to the line of action of \( F_T \) is \( e + \frac{1}{3}d - d_1 \). Taking moments about \( F_{cc} A_{st} f_{st} l_{ac} = F_T(e_s + l_{ac}) \), in which \( l_{ac} = d - \frac{1}{3}d_n \) (or assume that \( l_{ac} = \frac{3}{2}d_1 \), which is sufficiently accurate in cases where this method is admissible). Therefore

\[ f_{st} = \frac{F_T}{A_{st}} (1 + \frac{e_s}{l_{ac}}); \quad \text{or} \quad A_{st} = \frac{F_T}{f_{st}} (1 + \frac{e_s}{l_{ac}}). \]

\( (5.8b) \)
The compressive stress is not generally critical, especially if the thickness $d$ is determined by the resistance to bending only.

**Example No. 55.—Wall with Unsymmetrical Reinforcement and Subjected to Bending and a Pull.** Calculate the tensile stress and maximum compressive stress in the 6-in. wall in Fig. 25b if the wall is subjected to a concentric pull of 5000 lb. per foot and a bending moment of 25,000 in.-lb. per foot, assuming that the concrete does not resist tension and that the reinforcement in compression $A_{ec}$ is tied in to prevent buckling; $e = \frac{25,000}{5000} = 5$ in., which indicates that tensile and compressive stresses are likely to be produced. Therefore formulae (5.8) are applicable with $b = 12$ in., $d_1 = 5$ in., $A_{st} = 0.392$ sq. in., $A_{ec} = 0.196$ sq. in. per foot; $l_{as} = 4$ in. Assume that $d_a = 1.5$ in.; substitution in formula (4.3a) gives

$$X = 5 - \frac{(12 \times 1.5)(5 - 0.75) + (14 \times 0.196 \times 4)}{(12 \times 1.5) + (14 \times 0.196) + (15 \times 0.392)} = 1.72 \text{ in.}$$

Substitution in formulae (5.8) gives

$$J_c = \frac{1}{2} \times 12 \times 1.5 = 9; \quad J_s = 14 \times \left(\frac{1.5 - 1}{1.5}\right) \times 0.196 = 0.92;$$

$$l_{ae} = 5 - 0.5 = 4.5 \text{ in.}$$

$$f_{cb} = \frac{5000(5 + 1.72 - 5)}{9 \times 4.5 + (0.92 \times 4)} = 195 \text{ lb. per square inch.}$$

$$f_{st} = \frac{5000 + (9 + 0.92)195}{0.392} = 17,300 \text{ lb. per square inch.}$$

The corresponding value of $d_n$ is $\frac{5}{1 + \frac{17,300}{15 \times 195}} = 0.72 \text{ in.}$, which differs sufficiently from the value of 1.5 in. assumed to warrant re-working with another trial value of $d_n$, say 1 in., for which $X = 1.85 \text{ in.}, J_c = 6$, and $l_{ae} = 4.67 \text{ in.}$. Substitution in formulae (5.8) gives, as before, $f_{cb} = 340 \text{ lb.}$ and $f_{st} = 18,000 \text{ lb. per square inch, for which stresses } d_n = 1.08 \text{ in., which is practically equal to the second value assumed.}$

The method in the foregoing is admissible only if the reinforcement in compression is well tied in; if this is not so, formulae (5.8a) should be used neglecting $A_{ec}$, or the approximate method represented by formula (5.8b) should be applied as in the example which follows.

**Example No. 56.—Stresses in a Wall Calculated by the Approximate Method.** Calculate the tensile stress in the reinforcement in the 6-in. wall in Example No. 55, neglecting the reinforcement in compression. As before, $e = 5$ in., and $e_s = 5 + \left(\frac{1}{6}\times 6\right) - 5 = 3 \text{ in.}; l_{ae} = \frac{4}{7} \times 5 = 4\frac{3}{7} \text{ in. approximately.}$ Substituting $e_n, l_{ae}, A_{st} = 0.392 \text{ sq. in.}, \text{ and } F_T = 5000 \text{ lb.}$ in formula (5.8b), $f_{st} = \frac{5000}{0.392} \left(1 + \frac{3}{4\frac{3}{7}}\right) = 21,500 \text{ lb. per square inch, which is excessive for mild-steel bars, in which the permissible stress is generally } 20,000 \text{ lb. per square inch.}$ The stress is, however, calculated by an approximate method which gives a stress exceeding slightly that calculated by a more exact method.
CHAPTER VI
RESISTANCE TO SHEARING FORCES

Consideration of the resistance to shearing of a reinforced concrete member is largely empirical. A shearing force acting at right-angles to the axis produces shearing stresses at right-angles and parallel to the axis, and results in tensile stresses acting at an angle to the axis as in members of other materials. Since concrete can resist only a small amount of tension, shearing forces must generally be resisted by reinforcement provided for this sole purpose.

Shearing Stresses.

Homogeneous Material.—Consider a beam, or other member, of uniform rectangular cross-section (Fig. 30) and of homogeneous material such as uncracked concrete. The neutral plane is at the mid-depth of the member \( \bar{d} = \frac{1}{2} d \) since it passes through the centroid of the section. Consider two parallel planes AB and CD at right-angles to the axis of the member and a very small distance \( \delta l \) apart, and consider the equilibrium of the part of the member ACFE of width \( b \) lying between these planes.

![Fig. 30.—Shearing Stresses in Homogeneous Rectangular Beams.](image)

The horizontal forces acting upon this element are \( C_0 \) the total the compressive force (due to bending) on AE, \( C_1 \) the total compressive force (due to bending) on CF, and the total shearing force on the plane EF. If the bending moment at AB is \( M_0 \), the compressive stress at A is \( \frac{M_0}{I} \frac{d}{2} \); at E, it is \( \frac{M_0}{I} (d - x) \). Therefore the average stress is \( \frac{M_0}{2I} (d - x) \), and \( C_0 \) is \( \frac{M_0}{2I} (d - x) \). Similarly if \( M_1 \) is the bending moment at CD, \( C_1 \) is \( \frac{M_1}{2I} (d - x) \). If the shearing stress on plane EF is \( q_s \), the total shearing force on plane EF is \( 8l b q_s \), which, for equilibrium, must be equal to the difference between the opposite forces \( C_0 \) and \( C_1 \), that is

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must be equal to $8l \cdot bq_x$. Substituting $I = \frac{bd^3}{12}$ and transposing, $q_x$ is

$$\frac{xb}{2I}(d - x)(M_0 - M_1)$$

Since $\frac{M_0 - M_1}{8l}$ is the rate of change of bending moment which is equal to the shearing force $Q$, the intensity of horizontal shearing stress $q_x$ on a plane at a distance $x$ below the edge in compression of the member is

$$\frac{6x(d - x)Q}{bd^3}.$$ Since the intensities of the horizontal and vertical shearing stresses at any point are equal, it follows that there is no vertical shearing stress at the edge in compression ($x = 0$), and that the vertical shearing stress is a maximum at the neutral plane ($x = \frac{1}{2}d$), where the greatest stress is $1.5 \frac{Q}{bd}$. The stress $\frac{Q}{bd}$ is the average vertical shearing stress; therefore the maximum shearing stress is $1.5$ times the average shearing stress. Since the expression for $q_x$ is a second-degree equation, it follows that the distribution of shearing stresses on a member of homogeneous material is parabolic as in Fig. 30.

**Fig. 31.**—Shearing Stresses in Reinforced Rectangular Beam.

**Reinforced Concrete Member of Uniform Depth.**—Consider a reinforced concrete member of rectangular section and of uniform depth (Fig. 31a). The concrete below the neutral plane is assumed to be unable to resist tensile stresses due to bending because of possible cracking. An analysis similar to the foregoing establishes that the distribution of shearing stress above the neutral plane is parabolic, varying from zero at the edge in compression ($x = 0$) to a maximum stress $q$ at the neutral plane ($x = d_n$). Below the neutral plane the horizontal shearing stresses are constant down to the reinforcement in tension, because, considering two planes as before, the horizontal shearing force on any plane below the neutral plane is equal to the change in the tensile force in the reinforcement in the length $8l$; that is $8l \cdot bq$ is equal to $T_0 - T_1$, in which $T_0 = \frac{M_0}{I_a}$ and $T_1 = \frac{M_1}{I_a}$. Therefore $q$ is

$$\left(\frac{M_0 - M_1}{8l}\right) \frac{1}{I_a b}$$

since $\frac{M_0 - M_1}{8l}$ is $Q$.

$$q = \frac{Q}{I_a b}.$$ (6.1)
Formula (6.1) is the primary expression for the intensity of shearing stress in a rectangular reinforced concrete member of uniform depth.

**Rectangular Reinforced Concrete Member of Non-uniform Depth.**—The expression corresponding to formula (6.1) but for a rectangular reinforced concrete member of non-uniform depth (Fig. 31b) is derived by considering that the lever arms \( l_0 \) of \( T_0 \) and \( T_1 \) are \( a_1 d_0 \) and \( a_1 d_1 \) respectively. If the lower edge slopes at an angle \( \alpha \) with respect to the top edge, \( d_0 = d_1 - 8l \tan \alpha \). If \( M_0 \) is greater than \( M_1 \) the statement of equilibrium is

\[
q b \cdot 8l = T_0 - T_1 = \frac{M_0}{a_1 d_0} - \frac{M_1}{a_1 d_1} = \frac{1}{a_1} \left( \frac{M_0}{d_0} - \frac{M_1}{d_1} \right).
\]

Therefore \( q \) is

\[
\frac{M_0 - M_1}{8l} + \frac{M_1}{d_1} \tan \alpha = \frac{Q}{ba_1 d_1 \left( \frac{8l}{d_1} \tan \alpha \right)}.
\]

Since \( \frac{M_0 - M_1}{8l} = Q \), and \( \frac{8l}{d_1} \tan \alpha \)

is negligible, the shearing stress is given by

\[
q = \frac{Q + \frac{M_1}{d_1} \tan \alpha}{l_0 b}.
\]  

(6.1a)

Formula (6.1a) is applicable when the bending moment decreases as the depth increases, that is the condition in Fig. 31b where \( M_0 \) is greater than \( M_1 \) but \( d_0 \) is less than \( d_1 \); this condition occurs commonly at haunches adjacent to the supports of a freely-supported beam. If the converse obtains, that is when the bending moment increases as the depth increases \((M_1 > M_0)\), a condition which occurs at the haunch adjacent to the support of a continuous beam, the corresponding formula, which is derived similarly to formula (6.1a), is

\[
q = \frac{Q - \frac{M_1}{d_1} \tan \alpha}{l_0 b}.
\]  

(6.1b)

**Example No. 57.—Rectangular Beam of Uniform Depth.** Calculate the shearing stress in a rectangular reinforced concrete beam 10 in. wide and of a uniform depth of 20 in. if the shearing force \( Q \) is 20,000 lb.

Assume that \( d_1 = 20 - 1\frac{1}{2} = 18\frac{1}{2} \) in., and \( l_0 = \frac{7}{3} \times 18\frac{1}{2} = 16\frac{1}{2} \) in.

By substitution in formula (6.1),

\[
q = \frac{20,000}{16.25 \times 10} = 123 \text{ lb. per square inch.}
\]

**Example No. 58.—Rectangular Beam of Non-uniform Depth.** Calculate the shearing stress in a rectangular beam, the soffit of which is inclined at a slope of 1 in 3, if the bending moment \( M_1 \) is 100,000 in-lb. and the shearing force \( Q \) is 20,000 lb. at a vertical plane where the beam is 20 in. deep. The width is 10 in. and the bending moment increases as the depth increases.

Assume that \( d_1 = 18\frac{1}{2} \) in. and \( l_0 = 16\frac{1}{2} \) in. Substituting \( \tan \alpha = \frac{1}{3} \) in formula (6.1b),

\[
q = \frac{20,000 - \frac{100,000}{18.5} \times \frac{1}{3}}{16.25 \times 10} = 112 \text{ lb. per square inch.}
\]
Reinforcement to Resist Shearing Forces.

When the effects of shearing forces cause excessive tensile stresses in the concrete, the practice in Great Britain is to neglect the tensile resistance of the concrete and to resist the entire shearing force by reinforcement so placed that it is in tension. The boundary between the case of the concrete being able to resist the shearing force and the contrary case is the safe shearing stress to which the concrete can be subjected; this stress is related to the tensile strength of the concrete, which in turn is related to its compressive strength. In general the safe shearing stress is considered to be about one-tenth of the safe compressive stress in bending as described on page 19. Therefore if the shearing stress, as calculated by formula (6.1), (6.1a), or (6.1b), exceeds the safe stress, reinforcement must be provided. This reinforcement is generally in the form of bars inclined to the axis of the member as in Fig. 32, or binders (also called links or stirrups) as in Fig. 33. There is reason to believe that the effectiveness of a given amount of reinforcement to resist shear increases as the shearing stress calculated by formula (6.1) decreases, but in current British practice this fact is not taken into account.

**Fig. 32.**—Reinforcement to Resist Shearing: Inclined Bars.

The empirical British method of considering the resistance of inclined bars is to consider them as the inclined tension members of a lattice girder,
the compression members, or imaginary struts, of which are formed by the resultant inclined compressive forces in the concrete, as in Fig. 32a. The resistance to shearing force at any vertical plane is the sum of the vertical components of the tensile forces $T_w$ in the inclined bars and the compressive forces $C_w$ in the imaginary struts. For equilibrium at point A, the vertical component of $C_w$ must be equal to the vertical component of $T_w$, that is $T_w \sin \theta$. If the tensile stress in the inclined bar of cross-sectional area $\delta A_{st}$ is $f_{sw}$, $T_w = f_{sw} \cdot \delta A_{st}$ and the vertical component is $f_{sw} \cdot \delta A_{st} \cdot \sin \theta$. For equilibrium at A, the horizontal components of $T_w$ and $C_w$ and the tensile force $F_T (= f_{st} \cdot \delta A_{st})$ in the horizontal part of the bar must be in equilibrium, that is $T_w \cos \theta = C_w \cos \theta = F_T$. Substituting

$$T_w = f_{sw} \cdot \delta A_{st}, \quad C_w = T_w \frac{\sin \theta}{\sin \beta} \quad \text{(because } C_w \sin \beta = T_w \sin \theta),$$

and $F_T = f_{st} \cdot \delta A_{st}$, gives $f_{sw} \cdot \delta A_{st} \left( \cos \theta + \frac{\sin \theta}{\tan \beta} \right) = f_{st} \cdot \delta A_{st}$. Therefore

$$f_{sw} = \frac{f_{st} \cdot \delta A_{st} \cdot \sin \theta}{\cos \theta + \frac{\sin \theta}{\tan \beta}} \quad \text{and the shearing resistance } T_w \sin \theta \text{ is } \frac{f_{st} \cdot \delta A_{st} \cdot \sin \theta}{\cos \theta + \frac{\sin \theta}{\tan \beta}}$$

that is $\frac{f_{st} \cdot \delta A_{st}}{\cot \theta + \cot \beta}$. If the spacing of the inclined bars is $x_1 d_w$, in which $d_w$ is the height as shown on Fig. 32a, $\cot \beta = x_1 - \cot \theta$, whence

$$Q = \frac{f_{st} \cdot \delta A_{st}}{x_1}. \quad \ldots \quad \ldots \quad (6.2a)$$

It is obvious that for any given angle of inclination $\theta$ there is a spacing $x_1 d_w$ which gives the maximum value of $Q (= f_{sw} \cdot \delta A_{st} \cdot \sin \theta)$, which occurs when $f_{sw}$ is equal to the maximum safe tensile stress $p_{st}$, that is when $p_{st} \cdot \delta A_{st} \cdot \sin \theta = \frac{p_{st} \cdot \delta A_{st}}{x_1}$, that is when $x_1 = \frac{1}{\sin \theta}$. Commonly $\theta$ is 45 deg. (Fig. 32b) and $x_1$ is then $\sqrt{2}$; or if $\theta = 30$ deg., $x_1 = 2$; or if $\theta = 60$ deg., $x_1 = \sqrt{3}$.

If $\theta = 45$ deg. and the common spacing $x_1 d_w$ is $1.41 d_w$ (Fig. 32b),

$$Q = 0.71 p_{st} \cdot \delta A_{st}. \quad \ldots \quad \ldots \quad (6.2b)$$
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For the spacing in Fig. 32c, which is also sometimes adopted

\[ x_1 d_w = 2d_w, \]
\[ Q = \frac{1}{2} \rho_{st} \delta A_{st}. \]  \hspace{1cm} (6.2c)

The inclined bars in Figs. 32a, b, and c are arranged in a single system since the bars and the corresponding struts form a single lattice. If two or more systems are superimposed, as is common when the shearing forces acting on a member are great, the resistance is double or treble, as the case may be, that of a single system. For example, for the double system in Fig. 32d, with \( \theta = 45 \) deg.,
\[ Q = 1.41 \rho_{st} \delta A_{st}. \]  \hspace{1cm} (6.2d)

For the arrangement in Fig. 32e, with \( \theta = 45 \) deg.,
\[ Q = \rho_{st} \delta A_{st}. \]  \hspace{1cm} (6.2e)

For the general case of inclined bars not conforming to any of the cases illustrated, the shearing resistance can be calculated from formula (6.2a) if the inclined bar is formed by bending up (or down) the reinforcement in tension, in which the tensile stress is \( f_{st} \).

Example No. 59.—Effect of Spacing of Bars on Resistance to Shearing. Compare the resistance to shearing force of a bar of 1 in. diameter inclined at 45 deg., if the bar and an identical adjacent bar are arranged as in Fig. 32b, c, d, or e, and if the permissible tensile stress \( \rho_{st} \) in the horizontal or inclined part of each bar is 20,000 lb. per square inch.
\( \delta A_{st} = 0.785 \) sq. in.

(i) Single system; spacing 1.41\( d_w \). By substitution in formula (6.2b),
\[ Q = 0.71 \times 20,000 \times 0.785 = 11,100 \text{ lb.} \]

(ii) Single system; spacing 2\( d_w \). By substitution in formula (6.2c),
\[ Q = 0.5 \times 20,000 \times 0.785 = 7850 \text{ lb.} \]

(iii) Double system; spacing 0.7\( d_w \). By substitution in formula (6.2d),
\[ Q = 1.41 \times 20,000 \times 0.785 = 22,200 \text{ lb.} \]

(iv) Double system; spacing \( d_w \). By substitution in formula (6.2e),
\[ Q = 20,000 \times 0.785 = 15,700 \text{ lb.} \]

As is explained in the preceding text, this example shows that bars spaced at 1.41\( d_w \) in a single system, or 0.7\( d_w \) in a double system, are more effective than if arranged at wider spacings.

Binders.

The resistance of binders (Fig. 33) to shearing force is considered in the same manner as for inclined bars, except that in this case the members in tension of the assumed lattice girder are vertical but the imaginary "struts" in compression are inclined. The effective depth of the "girder" is the lever arm \( l_a \) of the reinforced concrete member. Consider binders spaced at a distance apart equal to the lever arm; for equilibrium at points A and B, the vertical component of \( C_w \) must equal the tensile force in the vertical member, which for a single binder is \( f_{sw} A_w \) in which \( f_{sw} \) is the tensile stress in the binder and \( A_w \) is the cross-sectional area of both arms.
of the binder. With such an arrangement the vertical plane cuts only one inclined "strut" and the resistance to shearing force is therefore the vertical component of the compressive force in the imaginary strut, that is \( f_{su} A_w \). If, as is common, the binders are spaced closer than \( l_a \), say \( s \), then the effect is that of a series of superimposed lattice girders, the number of such "girders" being \( \frac{l_a}{s} \) and the shearing resistance, if the permissible tensile stress is \( p_{st} \), is therefore given by

\[
Q = \frac{p_{st} A_w l_a}{s}.
\]

This is the basic formula for evaluating the shearing resistance of single binders, but in practice it is more convenient to substitute \( V \) for the basic shearing value of the binders, such that

\[
Q = V l_a; \quad V = \frac{p_{st} A_w}{s}.
\]

(6.3a)

If a double system (Fig. 33) or other multiple system of binders is provided the shearing resistance is double (or other multiple) of that given by formula (6.3a).

**Example No. 60.—Resistance of Binders to Shearing.** Calculate the resistance to shearing force of \( \frac{1}{2} \)-in. double binders at 6-in. centres in a beam 20 in. deep and 10 in. wide if the permissible tensile stress is 20,000 lb. per square inch; and compare this resistance with that of the concrete alone if the permissible shearing stress in the concrete, neglecting the reinforcement, is 100 lb. per square inch.

Substituting in formula (6.3a) \( s = 6 \) in., \( A_w = 2 \times 0.11 = 0.22 \) sq. in., \( l_a = \frac{1}{2}(20 - 1\frac{1}{2}) = \text{say, } 16\frac{1}{2} \text{ in.} \) and \( p_{st} = 20,000 \text{ lb. per square inch} \) gives the resistance of the reinforcement, namely,

For single binders,

\[
V = \frac{0.22 \times 20,000}{6} = 733; \quad Q = 733 \times 16\frac{1}{2} = 11,900 \text{ lb.}
\]

For double binders, \( Q = 2 \times 11,900 = 23,800 \text{ lb.} \)

Substituting \( q = 100 \text{ lb. per square inch}, b = 10 \text{ in.}, l_a = 16\frac{1}{2} \text{ in.} \) in formula (6.1) transposed gives the resistance of the concrete, namely,

\[
Q = 100 \times 16\frac{1}{2} \times 10 = 16,250 \text{ lb.}
\]
CHAPTER VII
BOND BETWEEN CONCRETE AND REINFORCEMENT

Adhesion of the concrete to the reinforcement bars is essential for the proper action of a reinforced concrete member. A bar subjected to a tensile or compressive stress will slip, that is move relative to the surrounding concrete, if there is insufficient adhesion between the two materials. It is therefore necessary to provide sufficient length of bar to ensure that the total adhesion or bond on the area of contact between the steel and the concrete provides sufficient resistance to slipping. To prevent failure by slipping the total bond must be greater than the force in the bar.

Average Bond Stress.—The amount by which the bond is greater than the force in the bar must be enough to provide an adequate margin of safety, which is generally obtained by adopting a safe average bond stress. This stress is a fraction of the average bond stress just before failure; permissible stresses are given on page 22. Emphasis is placed on the term "average" in this connection because the interaction between the concrete and the bar is complex.

The Action of Bond.—The phenomenon of bond is not completely understood, but the complexity can be readily perceived if some of the factors involved are considered. The concrete surrounding a bar subjected to a tensile force is generally cracked, although the cracks may be minute and only visible upon very close inspection; across the width of a crack, however small, the bar is not in contact with the concrete. Also, immediately on either side of the crack there can be no adhesion between the bar and the concrete, otherwise it would not be possible for the crack to form without fracturing the steel. Therefore in the vicinity of a crack there is a short length of bar to which the concrete does not adhere (Fig. 34a). Between cracks the concrete adheres to the bar, and the degree of adhesion probably waxes and wanes from crack to crack. These minute
cracks are a characteristic of most reinforced concrete members (except concentrically-loaded columns and similar members), and if they remain minute "hair" cracks they are not a structural defect. Their presence is due to the fact that the common safe working stress in the reinforcement produces potential strain of the concrete so great that the corresponding tensile stress in the concrete would far exceed the tensile strength of the concrete and consequently the concrete would fail in tension. If the reinforcement is so distributed that this potential strain is well distributed, the resulting cracks are never likely to be of consequence. The fact that a bright steel bar embedded in concrete remains bright when it is removed even after many years seems to prove that bond is a physical and not a chemical phenomenon. The fact that concrete shrinks when setting and hardening may account to some degree for adhesion, since the contracting material tends to grip the bars. It might be thought that a bar in tension would act contrarily and cease to have contact with the surrounding concrete due to the reduction in diameter which accompanies axial tensile strain, but apparently this effect is inconsiderable. The adhesion to a bar with loosely adhering mill scale, oil, or other foreign material is likely to be much less than that to a bright bar, which in turn is less than that to a bar with a slightly rusty surface. The resistance to slipping is increased considerably if the bar is deformed in some way, as in the case of a twisted square bar or a twisted ribbed bar, where the occurrence of some degree of mechanical anchorage is self-evident. Failure of bond may have disastrous results; therefore precautions are sometimes taken to secure a bar from slipping by providing a positive anchor at its end in addition to a length sufficient to provide adequate bond. Since bond cannot be readily analysed theoretically, practice is based almost entirely on the results of tests and experience. Failures of bond, either in actual structures or in test members, point the way to sound design in this respect, and lead to the recommendations given in codes and regulations. The rules and formulae given in the following are in general based on the recommendations of British Standard codes.

Resistance to Tensile Force.

**Bond-length.**—The basic expression for the bond resistance of a reinforcement bar subjected to tension is

\[
\left[ \left( \frac{\text{surface area of}}{\text{bond length of bar}} \right) \times \left( \frac{\text{safe average}}{\text{bond stress}} \right) + \left( \frac{\text{resistance of}}{\text{end anchor}} \right) \right].
\]

This resistance must be not less than the tensile force \( F_{et} \) in the bar, which is the product of the tensile stress \( f_{et} \) in the bar and the cross-sectional area of a round bar of diameter \( D \). Therefore \( F_{et} \) is \( \frac{1}{4} \pi D^2 f_{et} \). The surface area of a length \( l \) of the bar is \( \pi D l \). Therefore, for equality of resistance for a
straight round bar without an end anchor, \( \frac{1}{4} \pi D^2 f_{st} = \pi D s_b \), in which \( s_b \) is the safe average bond stress. The least length \( l_0 \) of bar required to obtain this equality is \( \frac{f_{st} D}{4s_b} \) or, if \( l_0 \), which is termed the bond-length, is expressed in terms of the diameter of the bar,

\[ l_0 = ND, \text{ in which } N = \frac{f_{st}}{4s_b}. \]  

(7.1)

For the common stresses of \( f_{st} = 20,000 \text{ lb. per square inch in mild steel} \) and \( s_b = 120 \text{ lb. per square inch in plain round bars}, \) \( N \) is 41\( \frac{2}{3} \), that is the minimum safe bond-length is equal to, say, "42 diameters", which is the usual way of referring to the bond-length of a bar. For bars which are not circular, the diameter is generally assumed to be the diameter of a circle having the same cross-sectional area as the bar.

It is seen that the bond-length depends solely on the ratio of the tensile stress in the bar and the safe average bond stress; to avoid bond-lengths being too short when the tensile stress is small, the minimum length should be twelve diameters and this length should be provided beyond the section at which theoretically there is no stress. It is sometimes recommended that in a beam the bars in tension should be continued for a distance equal to six diameters beyond the section where there is no stress and then bent into the compression zone at an angle of about 45 deg., and this is desirable where practicable.

**Deformed Bars.**—The numerical values in the foregoing apply to plain round bars of mild steel. The bond resistance of a deformed bar, such as a twisted square bar, twisted ribbed bar, or other bar with projections or indentations, is greater than that of a plain round bar of the same size. British Standard Code No. 114 defines a deformed bar as one for which tests demonstrate that the average adhesion is at least 10 per cent. greater than that of a plain round bar, in which case it is recommended that the safe bond stress be increased by 25 per cent., that is the minimum bond-length is three-quarters of the bond-length of a plain round bar of the same nominal diameter.

**Anchorage of Reinforcement Bars.**

The British Standard Code recommends that the bond-length calculated in accordance with formula (7.1) may be regarded as the greatest length required (subject to the requirement of a minimum length of twelve diameters). If an anchor is provided at the end of a bar, this length may be reduced by the length equivalent to the resistance of the anchor. It is thought by the writer, however, that this reduction is inadvisable in view of the importance of bond resistance, and it is suggested that complete anchorage should be provided in addition to the length calculated by
formula (7.1). Common forms of anchors are shown in Fig. 35, and the resistances of these are discussed in the following.

**Hooks and Bends.**—The most common anchor is a semi-circular hook, the dimensions of which should be as shown in Fig. 35d in order that the compressive stresses in the concrete inside the hook are not excessive. The anchorage value of such a hook is generally considered to be equivalent to a straight length of sixteen diameters (Fig. 35a), that is four diameters for each 45 deg. of bend. Less effective, but also a common form of anchorage, is a 45-deg. bend (Fig. 35c) which has a value of twelve diameters, and therefore requires in addition a straight length of four diameters to give the same value as the semi-circular hook. Considered from the same point of view, a 90-deg. bend (Fig. 35b) has a value of eight diameters and requires an additional straight length of eight diameters to provide an anchorage equivalent to a straight length of sixteen diameters. It is important to realise that the resistance to bond of a bend such as that in Fig. 35b is not increased if the length of the straight part of the bend exceeds four diameters because there is no tensile force in this part unless the concrete within the curved part of the bend has crushed; therefore failure may have occurred before the additional anchor begins to work fully.

**Mechanical Anchors.**—It is impracticable sometimes to provide the minimum lengths shown in Fig. 35, in which case a mechanical anchor should be provided. The simplest form of such an anchor is a 45-deg. bend or semi-circular hook with an anchor-pin inside the bend (Fig. 36a and b) which has proved to be effective; arbitrary but reasonable values of the bond resistance are certainly in excess of 16 diameters. These forms or equivalent anchorages are most useful at the end supports of beams and at the junction of walls, and in other cases where a short length of straight bar

![Fig. 35.—Dimensions and Equivalent Bond-lengths of Bends and Hooks.](image)
and a restricted anchorage only can be provided. Where space is extremely restricted, a plate anchor (Fig. 36c) may provide the combined effect of the bond-length and anchor, but there must be a sufficient mass of concrete in front of the plate to withstand the thrust exerted by the plate. The area of the plate (after deducting the area of the hole through which the bar passes) multiplied by the safe pressure on the concrete must be not less than the tensile force in the bar. Since the end of the bar is threaded the cross-sectional area at the bottom of the threads determines the safe force in the bar, and the end of the bar should be swaged up to compensate for the loss of area due to the threads. The safe local pressure at such an anchor can generally be considered to be, say, 50 per cent. greater than the normal safe stress in direct compression in the concrete.

Tensile Bond-lengths and Anchors.

Minimum Bond-lengths.—The application of the foregoing rules regarding bond-lengths and anchors are illustrated comparatively in Fig. 37 in which are given the minimum bond-lengths for a tensile stress of 20,000 lb. per square inch \( [F_{st} = 20,000(\frac{1}{4})D^2] \) and a bond stress of 120 lb. per square inch, which stresses, as described on pages 21 and 22, apply to mild steel and ordinary 1:2:4 structural concrete and are in
accordance with the recommendations of British Standard Code No. 114 for buildings. The slightly longer lengths, for these stresses, considered necessary in general practice as explained in the foregoing, are given in brackets.

Examples of Provision for Bond.—In the designs given in Part II and Part III the foregoing rules are applied to several cases of bars in tension. Other examples of suitable and ineffective methods of anchoring bars are illustrated in Fig. 38.

The detail at the support of a freely-supported beam is shown in

![Fig. 38.—Provision of Bond and Anchorage.](image)

*Fig. 38a.* If the beam is supported by another beam the length of bar within the supporting beam is generally too short to provide sufficient bond-length; therefore the provision of a semi-circular hook is seen to be almost valueless. Similarly the provision of a right-angle bend, although providing slightly more length of straight bar within the beam, is likewise insufficient and, as explained previously, the value of the vertical part of this type of anchorage beyond a length of four diameters is doubtful and should not be relied upon. A satisfactory anchor in this case is a 45-deg. bend with an anchor-bar as illustrated.

The bond of the bars in a base for a reinforced concrete column (*Fig. 38b*) is an important factor in the design of the reinforcement. The stress in the reinforcement is greatest at the plane X—X as explained on page 220. In a small base, the distance from this plane to the beginning of the hook
on the bar is short and therefore the bond-length is short, and the size of the bar is therefore likely to be determined by the bond-length available.

**Example No. 61a.—Column Base.** In a base 5 ft. square and similar to that in Fig. 38b, the amount of reinforcement required at a stress of 20,000 lb. per square inch is 0-6 sq. in. per foot width of the base. Determine suitable bars.

The amount required can be provided by bars of \( \frac{3}{4} \)-in. diameter spaced at 12-in. centres, \( \frac{3}{4} \)-in. bars at 9-in. centres, \( \frac{4}{4} \)-in. bars at 6-in. centres, or \( \frac{1}{2} \)-in. bars at 4-in. centres. Generally the most economical design is that incorporating large bars at the widest practicable spacing, in this case, say, \( \frac{3}{4} \)-in. at 9-in. centres. If the column supported by the base is 12 in. wide, the maximum bond-length available is about 20 in., which is about twenty-seven diameters if \( \frac{3}{4} \)-in. bars are provided. This length is insufficient because a length of not less than 42 diameters and a semi-circular hook are required for a stress of 20,000 lb. per square inch, as in Fig. 37. The diameter of the largest suitable bar is therefore 20 in. \( \div 42 \), that is about \( \frac{1}{2} \) in.; hence \( \frac{1}{2} \)-in. bars at 4-in. centres should be provided.

The junction of the outer wall and a partition wall of a rectangular container is shown in Fig. 38c. The internal pressure tends to force the outer wall away from the partition wall and the horizontal bars in the partition wall are therefore in tension. The greatest tensile stress occurs at the plane at the end of the corner splays. The size of these splays and the thickness of the outer wall are generally such that the bond-length of a horizontal bar is insufficient to develop the necessary maximum stress even though small bars are used. Therefore the bars projecting from the partition wall must be anchored around a vertical bar provided for the purpose.

**Example No. 61b.—Junction of Walls of a Container.** At the junction of 6-in. walls similar to that in Fig. 38c, there are 6-in. by 6-in. splays, \( \frac{1}{2} \)-in. horizontal bars, and \( \frac{2}{4} \)-in. cover of concrete over the horizontal bars; the bond-length available from the plane of greatest stress is about 10 in., that is twenty diameters. If the maximum stress is 20,000 lb. per square inch and the anchorage is equivalent to a right-angle bend, it is seen from Fig. 37 that a length of fifty diameters is required; therefore the bond-length available is quite inadequate and a mechanical anchor as shown must be provided.

**Anchorage of Reinforcement to Resist Shearing.**

The effective bond and anchorage of reinforcement provided to resist shearing force are important. It is shown on page 105 that the greatest shearing stress in a reinforced concrete beam occurs in the zone from the neutral plane to the reinforcement in tension. From the neutral plane to the edge in compression the shearing stresses decrease.

**Inclined Bars.**—It is reasonable to suppose that the greatest tensile stress in an inclined bar provided to resist shearing occurs at the neutral
plane. Therefore the bond-length required might be measured from the neutral plane as shown by the dimensions \( l \) in Fig. 39 for beams subjected to positive and negative bending moments. British Standard Code No. 114 recommends that the straight horizontal part of the bar only should be considered as the bond-length, that is the dimensions \( l \) in Fig. 39. Satisfactory methods of design would be to adopt the minimum bond-lengths in Fig. 37 for the dimension \( l \), or the greater minimum lengths in Fig. 37 for the dimension \( l_s \).

**Binders.**—It is not always possible to provide sufficient bond-length above the neutral plane for a binder provided to resist shearing forces except in the case of a very small bar or wire forming the binder in a very deep beam. Therefore some form of mechanical anchorage is essential and generally the detail shown in Fig. 39 is convenient and satisfactory, since there are generally bars in the top of the beam around which the binder can be anchored. The recommendation for the minimum anchorage of binders in British Standard Code No. 114 is also illustrated in Fig. 39.

**Radii of Bends.**

The radius of the bend in a reinforcement bar is determined by the crushing strength of the concrete within the bend. The recommendation of British Standard Code No. 114 is that the local stress in the concrete shall not exceed three times the stress permitted in the concrete in direct compression, that is \( 3f_{ct} \). Consider a right-angle bend of internal radius \( R \) in a bar of diameter \( D \). The resistance of the concrete is \( 3f_{ct}RD \). If the tensile stress in the bar at the beginning of the bend is \( f_{st} \), the tensile
BOND BETWEEN CONCRETE AND REINFORCEMENT

force in the bar is \( \frac{1}{2} \pi D^2 f_{st} \). Since the resistance must be not less than this force, equating the two expressions results in the minimum radius being \( \frac{\pi f_{st}}{12 \rho_{ce}} D \). In 1:2:4 concrete \( \rho_{ce} \) is 760 lb. per square inch, and the greatest stress \( \rho_{st} \) in a mild-steel bar is 20,000 lb. per square inch. Therefore the minimum radius of a bend for these conditions is \( \frac{\pi}{12} \times \frac{20,000}{760} D \), that is 7D, or seven times the diameter of the bar. At an anchor at the end of a bar the tensile stress is generally very much less than the maximum permissible stress, and in this case the radius may be very much less than 7D, or other radius to suit the actual stresses \( \rho_{st} \) and \( \rho_{ce} \). It is generally recommended that the radius should in no case be less than 2D, as is shown in Fig. 35, which corresponds to a stress of about 5000 lb. per square inch. If the stress at the beginning of the hook or bend forming an anchor is likely to exceed this amount, it is advisable to increase the radius of the bend accordingly.

Local Bond Stress of Beam Bars in Tension.

The foregoing considerations deal with the prevention of slipping of the ends of bars in tension due to failure of adhesion between the concrete and steel. It is equally important to ensure that the bars providing the reinforcement in tension in a beam should be able to act with the concrete. Consider two planes \( X-X \) and \( Y-Y \) close together, as in Fig. 34b, and cutting a group of bars in tension. If the bending moment at \( X \) is \( M_0 \) and at \( Y \) is \( M_1 \), the total tensile force in the reinforcement is \( \frac{M_0}{l_a} \) at \( X \) and \( \frac{M_1}{l_a} \) at \( Y \). The change in force between \( X \) and \( Y \) is therefore \( \frac{M_0 - M_1}{l_a} \), and it is balanced by the shearing stresses in the surrounding concrete. To transmit this force to the concrete, the total safe local adhesion must be not less than the change in force. If \( s_b_1 \) is the local bond stress, the total adhesion for the length \( 8l \) is \( s_b_1 \). 8l. \( \Sigma \pi D \), in which \( \Sigma \pi D \) is the sum of the perimeters of the bars in the group. If \( \Sigma \pi D \) is denoted by \( o \), equating the total adhesion to the change in force gives \( s_b_1 = \frac{M_0 - M_1}{8l}.o \); but \( \frac{M_0 - M_1}{8l} \) is the rate of change of bending moment which is equal to the shearing force \( Q \). Therefore

\[
s_{b_1} = \frac{Q}{l_a o}. \quad \ldots \quad \ldots \quad \ldots \quad (7.2)
\]

The local bond stress in the reinforcement in tension near the ends of beams, where the shearing force is generally greatest and the number of bars is least, is calculated by formula (7.2) and compared with the permissible local bond stress given on page 22. If the permissible stress is
exceeded, the sum of the perimeters of must be increased by increasing the number of bars at the end of the beam.

Example No. 62a.—Local Bond Stress for Bars in a Beam. Calculate the local bond stress for the bars in a beam 20 in. deep \((l_b = 16\frac{1}{2} \text{ in.})\) if the shearing force is 20,000 lb. adjacent to the support where there are two \(\frac{3}{4}\)-in. mild steel bars in tension. Substituting \(o = 2 \times \pi \times \frac{3}{4} = 4.73\) in. in formula (7.2),

\[
s_{b_1} = \frac{20,000}{16 \cdot 25 \times 4.73} = 260 \text{ lb. per square inch.}
\]

The permissible local bond stress for 1:2:4 concrete of ordinary quality is 180 lb. per square inch (see page 22); therefore it is necessary to arrange the reinforcement in the beam in this example so that at least three \(\frac{3}{4}\)-in. bars are carried to the supports, in which case \(o = 7.1\) in., and

\[
s_{b_1} = \frac{20,000}{16 \cdot 25 \times 7.1} = 174 \text{ lb. per square inch,}
\]

which is satisfactory.

Resistance to Compressive Force.

For bars in compression the bond due to the adhesion of the concrete to the surface of the bar is augmented by the indeterminate resistance to thrust of the concrete against the end of the bar. Therefore a length slightly less than the minimum bond-length given by formula (7.1) for bars in tension is satisfactory, and the recommendation in this respect in British Standard Code No. 114 is that the bond-length should be not less than

\[
l = ND \quad \text{where} \quad N = \frac{f_{se}}{5s_b}
\]

in which \(D\) is the diameter of the bar, \(s_b\) is the permissible bond stress as described on page 22, and \(f_{se}\) is the compressive stress in the bar. No anchor is required at the ends of a bar in compression, but it must be ensured that bars ending near the face of a member are not likely to push through the concrete covering the end of the bar. The least bond-length in compression is generally recommended to be twelve diameters, and this length should be provided beyond the point at which it is estimated that there is no compressive stress in the bar.

Example No. 62b.—Bond-length of a Bar in Compression. Determine the bond-length required for a \(\frac{3}{4}\)-in. mild steel bar if the compressive stress is 18,000 lb. per square inch and the permissible bond stress is 120 lb. per square inch. By substitution in formula (7.3),

\[
l = \frac{18,000}{5 \times 120} \times \frac{7}{8} = 30 \times \frac{7}{8} = \text{say, 27 in.}
\]
Continuity of Reinforcement.

Should a single reinforcement bar be either awkward to fix in position or too long to be supplied or transported, two or more bars may be used instead. The most common method of transferring the force in one bar to another adjacent bar is to overlap the ends of the two bars. The length of the overlap should be not less than the bond-length as calculated from formula (7.1) for bars in tension or from formula (7.3) for bars in compression. Reasonable minimum lengths, as recommended in British Standard Code No. 114 are thirty diameters for a bar in tension and twenty-four diameters for a bar in compression. If the two bars are of different diameter, the length of the overlap may be based on the diameter of the smaller bar.

Overlapping Bars in Tension.—If a number of bars in tension lie parallel to one another it is preferable for the laps to be staggered so that adjacent bars do not overlap at the same section, thereby avoiding congestion and the possibility of developing a plane of weakness in the member. Where an overlap occurs in the tensile zone of the member, as in the wall of a cylindrical tank, it is preferable to provide a semi-circular hook as an anchor. The method of providing continuity by butting the ends of two bars in tension and laying a separate length of bar of the same size alongside is not generally recommended; if this type of splice is provided, the splice-bar should overlap both the abutting bars for a length in excess of the lengths calculated by formula (7.1), and an anchor should be provided at the end of each abutting bar and at the ends of the splice-bar. A modification of this method of producing continuity sometimes adopted in large beams to avoid congestion of bars is to arrange the ends of the main bars to occur at different positions along the beam and to provide one or more additional bars throughout the beam to act as the splice-bar at a number of breaks. To avoid excessive length, it may be necessary to have a break also in the splice-bar, but at this break one of the main bars would act as the splice-bar.

Overlapping Bars in Compression.—Overlaps in bars in compression occur most commonly in columns and at the supports of continuous beams as shown in the typical details on page 3. In both these cases one bar is cranked to lie alongside the other. The slope of this crank must be "easy", say, 1 in 12, otherwise there is a danger of a defect arising due to the rapid change of direction of the axial force in the bar. The provision of separate splice-bars in place of overlapping bars in compression is not generally desirable except in the case of a small column surmounting a much larger column.

Turnbuckles and Welding.—Where space is limited, continuity of bars in tension or compression can be effected by providing a physical connection of the two parts of the abutting bars either by providing a
turnbuckle or by welding. Both methods are much more expensive than overlapping the bars or providing splice-bars and are only justified in important members such as large bridge girders or arches where a congestion of large bars would occur if overlaps were provided.

If a screwed turnbuckle as in Fig. 40a is provided the bars should be swaged up at the ends so that cutting the screw-threads does not decrease the effective cross-sectional area of the bar.

A form of welded butt-joint is also illustrated in Fig. 40b. If the joint is of this or other accepted form and the weld is made in accordance with established practice and by competent operators, there is not likely to be a reduction of the strength of the bar at the joint. There are regulations, such as those of the London County Council, applying to welded joints in reinforcement.

Cover of Concrete over Reinforcement.

For a reinforcement bar to act properly in conjunction with the concrete it is essential that there be sufficient concrete around the bar. Therefore the space between two parallel bars and the distance from the face of a bar to the outer face of the member must be not less than certain minima which have been established by experience.

**Cover.**—The distance from the bar to the nearest face of a member is called the cover of concrete and, in addition to ensuring that there is sufficient concrete to enable the necessary bond-resistance to develop, the cover protects the bar from atmospheric and other corrosive influences. In general the cover should be equal to a distance not less than the diameter of the bar, but in the case of small bars this rule results in insufficient cover and the minima in the following are generally recommended for the main bars in interior members of an ordinary building or similar structure: Slabs, walls, and the like, \( \frac{3}{4} \) in.; Beams and the like, 1 in.; Columns and the like, \( 1\frac{1}{2} \) in. (or 1 in. for small columns). The cover over binders outside the main bars in beams and columns should be not less than \( \frac{1}{2} \) in. The cover over the ends of bars should be not less than twice the diameter of the bar. For exposed members and for interior members subjected to
fumes or other corrosive influences, greater cover than in the foregoing should be provided and acceptable values are: Slabs, $\frac{1}{2}$ in.; Main bars in beams, columns, and the like, $1\frac{1}{2}$ in.; Binders, $\frac{3}{4}$ in. For reinforced concrete in the ground, such as foundations, retaining walls, and the like, the cover over all bars including binders should be not less than $1\frac{1}{2}$ in. but a greater cover is desirable. For marine structures at least 2 in. of concrete should be provided over all reinforcement. For structures containing liquids, the cover should be not less than $1$ in. or the diameter of the bar, whichever is the greater.

**Space between Bars.**—The horizontal space between parallel bars should be not less than the diameter of the bar, and in no case less than $1$ in., in order that there shall be no difficulty in placing the concrete between the bars. If the size of the largest pieces of coarse aggregate exceeds $\frac{7}{8}$ in., the space should be not less than the largest size plus about $\frac{1}{4}$ in.; that is if the aggregate is graded from $1\frac{1}{2}$ in. downwards the space should be not less than $1\frac{3}{4}$ in. If concrete is consolidated by mechanical vibration the distance between adjacent bars should be such that the vibrating tool can be inserted.

The vertical space between parallel bars in members such as beams should be not less than $\frac{1}{4}$ in., but a greater space is desirable with bars larger than $1$ in.

---

**Table A.—Moment-of-resistance Factors ($Q_o$)**

This table applies to beams without reinforcement in compression.

<table>
<thead>
<tr>
<th>Modular-ratio Method ($m = 1.5$)</th>
<th>Load-factor Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile Stress in Reinforcement ($f_o$)</td>
<td></td>
</tr>
<tr>
<td>Compressive Stress in Concrete ($f_c$)</td>
<td>12,000</td>
</tr>
<tr>
<td>250</td>
<td>84</td>
</tr>
<tr>
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<tr>
<td>250</td>
<td>351</td>
</tr>
<tr>
<td>250</td>
<td>431</td>
</tr>
</tbody>
</table>

$Q_o = \frac{M_0}{M_r}$

$Q_o$ = Moment of resistance

$M_r$ = Moment of resistance at section

$\alpha_i = \frac{m}{1 + \frac{m}{m_o}}$

$\alpha_o = \frac{m}{1 + \frac{m}{m_o}}$

$\alpha_i = \frac{1}{1 + \frac{m}{m_o}}$
### Table B.—Cross-sectional Areas of Plain Round Bars.

<table>
<thead>
<tr>
<th>DIAMETER</th>
<th>3/8&quot;</th>
<th>1/2&quot;</th>
<th>5/8&quot;</th>
<th>3/4&quot;</th>
<th>7/8&quot;</th>
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<td>0.192</td>
<td>0.250</td>
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<td>0.400</td>
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<td>0.744</td>
<td>0.936</td>
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<tr>
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<td>1.884</td>
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</tbody>
</table>

### CROSS SECTIONAL SPACINGS

- **A_s** = CROSS-SECTIONAL AREA (SQ. IN.)
- **D** = DIAMETER OF BAR (IN.)
- **N** = NUMBER OF BARS
- **S** = SPACING (OR PITCH) OF BARS (IN.)

**Specify the desired values for each parameter.**
PART II

SIMPLE STRUCTURAL MEMBERS

The principles of design and the permissible stresses described in Part I and the formulae derived therein are applied in this part to the design of simple basic structural elements, namely, solid slabs, rectangular beams, tee-beams, columns, walls, and foundations. It is first necessary to consider, as in Chapter VIII which follows, the loads which such members are required to support, and the bending moments and shearing forces to which they will be subjected as a result of the loads. Although each member is considered separately, it must be remembered that each is a constituent of a monolithic structure in which each member works in conjunction with the other members.

ADDITIONAL NOTATION FOR PART II

The symbols given in the following are those in Chapters VIII to XII and are additional to those in Part I (page 9).

DIMENSIONS AND PROPERTIES OF SECTIONS.

$A_{sc'}$: Cross-sectional area of vertical reinforcement in unit length of wall.
$B$: Width of square or rectangular base; diameter of circular base.
$B_1, B_2$: Widths of a double rectangular base. $b$: Width at top of pyramidal base.
$b'$: Width of primary beam.
$D$: Diameter of loaded area of circular or polygonal base; diameter of shaft or plinth of chimney; thickness of wall. $d_c$: Width of column at base.
$d'$: Thickness of base (ft.). $d_0'$: Effective depth of base at critical plane for shearing (ft.). $d''_f$: Effective depth of base (ft.).
$e$: Eccentricity of load on base; width of slab carrying a concentrated load. $e_p$: Eccentricity of load $P$. $e_x, e_y$: Eccentricities of load on a raft.
$F_H$: Horizontal load or force. $f$: Distance from concentrated load to edge of slab. $g$: Width of area under a concentrated load.
$H$: Height (general). $h$: Length of area under a concentrated load; distance from a pile to the centroid of a group of piles; height to centre of action of horizontal force. $h_2$: Distance between braces of a trestle.
$I$: Moment of inertia. $I_{AB}, I_{BC}$, etc.: Moments of inertia of members AB, BC, etc. $I_g$: Moment of inertia of any column in a group.
$L$: Length of rectangular base. $L_1, L_2$: Widths of parts of a double rectangular base; widths at ends of a trapezoidal base. $L_{11}, L_{22}$: Widths of trapezoidal base under loads $P_1$ and $P_2$ respectively. $l$: Distance between loads or piles; length of a member; span of a beam. $l_{AB}, l_{BC}$, etc.: Lengths of members AB, BC, etc. $l_c$: Length of end span of slab. $l_n$: Length of a brace. $l_p, l_q$: Spans of primary and secondary beams respectively. $l_k$: Span of a slab; spacing of secondary beams. $l_1, l_2, l_n$, etc.: Distance between loads in a group.

125
\( N \): Number of columns or piles in a group. \( N_x \): Number of columns in a row. \( n \): Number of loads on a strip base.

\( X \): Position of centre of gravity of block of ground pressure. \( X, Y \): Co-ordinates of centre of gravity of loads. \( x \): Distance of column from centre-line of group; distance from nearer support to centre of area under a concentrated load. \( x, y \): Co-ordinates of centroid of a raft; distances from a load to the edges of a base or from faces of a column to the edges.

\( \Delta \) (delta): Deflection of head of column.

**Forces and Moments.**

\( K \): Stiffness factor. \( K_{AB}, K_{BC}, \text{etc.:} \) Stiffness factors of members AB, BC, etc. \( k, k_1, k_2, \text{etc.:} \) Bending moment coefficients (denominator).

\( M \): Bending moment in general or on a beam or slab; moment acting on a base or on a group of piles. \( M_{AB}, M_{BA}, \text{etc.:} \) Bending moment at ends A and B respectively of member AB. \( M_8 \): Bending moment at end of beam assuming that both ends are fixed. \( M_4 \): Difference of values of \( M_8 \) of two beams at a common support. \( M_{neg} \): Negative bending moment. \( M_x \): Bending moment on any column at distance \( x \) from centre of a group.

\( P \): Net load imposed on a foundation (excluding weight of foundation). \( P_A \): Weight of base A. \( P_B \): Weight of base B, or of pile-cap, or of any foundation. \( P_\ell \): Total weight of tie-beam or similar. \( P_{max}, P_{min} \): Loads on columns of trestle. \( P_s \): Total pressure on area of base outside chimney shaft, or plinth. \( P_T \): Total load on the ground under a foundation (including the weight of the foundation). \( P_W \): Safe load on load-bearing wall. \( P_x \): Additional force on column at distance \( x \). \( P_1, P_2 \): Loads imposed on bases A and B respectively. \( P_{1(max)}, P_{1(min)} \), etc.: Maximum and minimum loads imposed on bases A and B respectively, etc. \( \dot{p} \): Intensity of wind pressure or of pressure imposed on the ground. \( \dot{p}_B \): Weight of foundation of unit area. \( \dot{p}_b \): Maximum net pressure imposed on the ground at edge of base of chimney. \( \dot{p}_D \): Net pressure on ground at edge of chimney shaft or plinth. \( \dot{p}_{max}, \dot{p}_{min} \): Maximum and minimum pressures on ground due to eccentric load. \( \dot{p}_n \): Net reaction of ground. \( \dot{p}_s \): Net pressure on ground at critical plane for shearing. \( \dot{p}_x, \dot{p}_y \): Intensity of net reaction of ground at \( x \) and \( y \) respectively. \( \dot{p}_0 \): Net imposed pressure on ground (excluding weight of base). \( \dot{p}_1, \dot{p}_2 \): Maximum and minimum net reaction of ground. \( q_a \): Safe load-bearing capacity of ground.

\( R \): Upward force on a balanced foundation. \( R_A, R_B \): Loads on ground at A and B respectively (balanced and tied foundations). \( R_{max}, R_{min} \): Maximum and minimum values respectively of \( R \). \( R_W \): Stress-factor for a wall depending on its dimensions.

\( V_H \): Velocity of wind at height \( H \) ft.; \( V_1 \): Velocity of wind at 33 ft.

\( W \): Safe load on a pile; total load on a beam or slab or trestle; total concentrated load. \( W_D \): Total dead load. \( W_s \): Total load (secondary edge-beam). \( W_4 \): Total load (interior secondary beam). \( W_1 \): Concentrated load on unit length. \( w \): Intensity of uniformly-distributed load on a primary beam or on unit length of a wall footing. \( w_B, w_b \): Weight of unit length of primary and secondary beam respectively. \( w_D, w_L \): Intensity of dead and live load respectively. \( w_s \): Intensity of total load on a slab.

\( \beta \) (beta): Coefficient relating to \( M_{neg} \).
CHAPTER VIII
LOADS, BENDING MOMENTS, AND FORCES

Before commencing the design of a structural member, or checking a design, it is necessary to determine the actions, such as bending, shearing, twisting, thrust, or pull, to which the member is likely to be subjected. These actions depend on the loads to be supported by, or the pressures to be resisted by, the structure of which the member forms a part, that is the loads imposed on floors and roofs, pressures due to wind, retained earth, contained materials and the like, and the weight of the structure itself. Each member plays a part in transferring the loads to the foundation.

Loads and Pressures.

Loads may be conveniently considered as permanent loads (generally called dead loads) and transient or temporary loads (generally called live or imposed loads). A structural member must be designed to carry the total load, which is the sum of the dead and live loads. The minimum loads to which ordinary structures may be subjected are specified in codes, regulations, and similar documents. The magnitude of some loads are given in subsequent chapters in which the design of specific structures is considered. It is shown in the following that an exact assessment of the loads and pressures to which a structure is likely to be subjected is not always easy, and part of a designer's work is to select reasonable loads so that the structure is neither unsafe because the load assumed is too small nor too costly because it is too great. The degree of accuracy necessary when assessing the dead load depends on the type of structure or member. For example, in the case of a reinforced concrete roof or a bridge of large span the dead load may exceed the greatest live load; consequently the effects of the total load may be due mainly to the dead load, the estimation of which must therefore be fairly exact. In the case of a warehouse floor, a bridge of small or medium span, or a tank, the transient loads may be many times greater than the dead load and a small inaccuracy in the dead load will have little effect on the total design load. If the ratio of live load to dead load is, say, three, an error of 10 per cent. in the dead load of a structure results in an error of only 2½ per cent. in the total load. If the ratio is, say, a half, the resultant error is about 7 per cent.

If the stability of a structure depends on the weight of the structure or part of the structure, the assessment of the dead load should be as accurate as possible. For some structures of this type, such as chimneys, tall towers, and retaining walls, the live load is the pressure of the wind
or retained earth, and, since such pressures cannot be assessed closely, extreme accuracy in assessing the dead load may be illogical. It is, however, a sound axiom that, when assessing loads or estimating the numerical magnitudes of other factors, as many items as practicable should be included at their exact value, thereby reducing the overall percentage of error due to the uncertain magnitudes of other items.

Permanent or Dead Loads.

The primary permanent or dead load is the weight of the member itself plus the weight of finishes or coverings applied for decorative, protective, insulation, acoustic, or other purposes. The dead load also includes the weights of permanent walls and other parts of a structure supported directly or indirectly by the member being considered. Common weights of structural and building materials, such as concrete, mortar, flooring, tiling, asphalt, timber, steel, masonry, bricks, and the like are given in a British Standard(8.1). *

Weight of Concrete.—The weight of concrete varies considerably and depends on the density of the aggregate and, for reinforced concrete, on the proportion of steel reinforcement. A convenient unit weight to assume for reinforced concrete made with well-graded gravel or crushed stone is 144 lb. per cubic foot since with this density the weight (in pounds) of one foot length of the member is numerically equal to the cross-sectional area (in square inches) of the member; that is a member 20 in. deep by 10 in. wide weighs 200 lb. per foot, or a slab 6 in. thick weighs $6 \times 12 = 72$ lb. per square foot. This unit weight should not, however, be used indiscriminately since most codes recommend a minimum unit weight of 150 lb. per cubic foot, which is more accurate for ordinary vibrated structural reinforced concrete. If the volume of the reinforcement is more than, say, 2 per cent. of the volume of the concrete, a greater weight should be assumed, such as 160 lb. to 170 lb. per cubic foot for concrete with 5 per cent. of reinforcement.

If broken brick, pumice, clinker, vermiculite, foamed slag, expanded clay, or other lightweight aggregate is used, a concrete of appreciably lower density is produced. Likewise cellular concrete produced by aeration, foaming or other chemical processes, or by the omission of sand (as in no-fines concrete) is much lighter. These lightweight concretes may have a density as small as 20 lb. per cubic foot. Since in general the lower the density of concrete the less its strength, most lightweight concretes have small crushing strengths and may not therefore be suitable for structural reinforced concrete. There are exceptions, and lightweight concrete is produced having crushing strengths comparable with concrete with gravel aggregate and is used mainly for reinforced or prestressed roof slabs.

* References thus (8.1) relate to the Bibliography on page 151.
Heavy concrete is produced by using aggregates of great density, such as barytes, magnetite, haematite and other iron ores, and the like. Such concrete, which is used for kentledge, shielding from atomic radiation, and other purposes where heaviness is required, may have a density up to 250 lb. per cubic foot, or up to 350 lb. per cubic foot if steel shot, steel filings and turnings, or similar materials, are used.

**Transient or Live Loads.**

The estimation of the magnitude of transient, temporary or live loads is generally less exact than that of dead loads. Live loads can be classified according to their apparent degree of accuracy, namely, specified loads, calculated loads, indefinite loads and influences analogous to loads.

**Specified Loads.**—Specified loads include the loads imposed on floors and roofs of buildings which are given in codes, building by-laws and regulations\(^{(8.2)}\), and are expressed as the intensity of a uniformly-distributed load. The magnitude of the load depends on the purpose for which the roof or floor is to be used, and varies from 30 lb. (or less) per square foot for flat roofs to upwards of 200 lb. per square foot for warehouse floors. A floor is rarely subjected to a uniform load as in general the weights of goods stored in warehouses, furniture and fittings in offices, machines in factories, vehicles in garages, and people in dance halls are irregularly disposed, there being no load on some parts of the floor and a high concentration of load elsewhere. Heavy weights, such as safes, may be moved across floors. The magnitudes of specified loads, however, are based on experience and in general represent the uniform load which has the same effect as that of the probable actual load. The loads recommended in British Standard codes are not mandatory, but those in building by-laws and regulations of local authorities are compulsory minima. In all cases, therefore, it is necessary to exercise discretion to determine whether a floor or roof should be designed for the minimum load recommended or specified or whether a greater load should be considered. This is particularly the case for the floors of warehouses the loading on which due to different goods may vary extensively.

Live or rolling loads are generally specified for bridges, and in Britain these loads for railway and road bridges are given in a British Standard\(^{(8.3)}\), which also gives guidance for the estimation of the important secondary effects of such loads, such as impact, centrifugal action, lurching, and longitudinal forces due to tractive effort and braking. Not all bridges will have to support loads as great as those imposed by main roads and main lines of railways, and in such cases it is necessary to determine the weights of vehicles likely to cross the bridge. It may be that not more than one of the heaviest vehicles is likely to be on the bridge at one time; it is not therefore necessary to design such structures for the most adverse
condition of the bridge being fully covered by the heaviest vehicles. The Ministry of Transport specify a normal load for road bridges and an abnormal load, comprising one heavy vehicle only, for some main-road bridges\(^{(8,4)}\).

**Calculated Loads and Pressures.**—The pressures and weights imposed by the contents of tanks, reservoirs, and other structures containing liquids are known most exactly since in these cases the horizontal and vertical pressures are based on simple precise hydrostatic principles. The pressures are known less exactly when containers are for the storage of coal, grain, cement, crushed stone, dry sand, or other dry granular materials. The well-established formulæ in Chapter XVIII, Vol. II, for such materials give working rules for the calculation of the probable horizontal pressures. The pressures due to granular materials submerged or floating in liquids can also be determined by a combination of formulæ for dry materials and liquids. The accuracy of calculated pressures due to contained dry materials depends on the angle of friction and the unit weight of the material in bulk; both these factors may not be known accurately, and therefore a wide variation of estimated pressures may be obtained by assuming different probable angles and weights. This inaccuracy is additional to any inaccuracy in the formulæ used to calculate the pressure. It is therefore very necessary to know not only the probable minimum angle of friction and maximum weight but also the derivation of the formula used so that the effect of variations from the conditions in a particular case can be considered; these effects can best be determined by examination of the records of the numerous tests made to investigate such pressures. Formulae for pressures should not be accepted without question, especially in the case of retaining walls, the primary load on which is the pressure of the earth on the back of the wall. Wide variations of this pressure are likely because of the nature of the ground (most importantly whether the material is granular like sand or cohesive like clay), the moisture content and possible later change in moisture content, the slope of the bank behind the wall, and other physical conditions.

**Loads due to Wind, Waves, Ships and Machinery.**—More indefinite are the loads and pressures due to wind, waves, ships, machinery, and the like, and in this respect experience is often the only guide.

The intensity of wind pressure on a plane surface (\(p\) lb. per square foot) depends on the velocity of the wind (\(V_H\) miles per hour). A common relationship which has been determined by experiment is

\[
p = 0.003 V_H^2. \\
\text{(8.1a)}
\]

The average pressure is less when the wind blows against a circular or octagonal surface (it is reduced by about one-third on a circular shape), but the ratio of height to breadth of the structure has an influence, the higher the ratio the smaller being the reduction. The probable maximum
velocities of winds are obtainable from meteorological records for the
district or may be specified, either as velocities or as equivalent pressures,
by local authorities. Since the velocity of wind generally increases with
height above the ground, it is necessary to adjust the local records for
height, and a formula derived from an expression given by the Meteorolo-
gical Office is

\[ V_H = V_1 \left( \frac{H}{33} \right)^{0.13} \]

in which \( V_H \) is the velocity in miles per hour at height \( H \) ft. if the specified
velocity at the standard height of 33 ft. (10 metres) is \( V_1 \) miles per hour.
In Great Britain it is rarely that velocities exceeding 100 miles per hour
need be considered, and common pressures for design purposes vary from
7 lb. per square foot to 30 lb. per square foot. In the absence of more
specific values of wind pressure, the recommendations and data in British
Standards can be adopted, with discretion, to determine the pressures
on buildings\(^{8,2}\), chimneys\(^{8,2}\), towers\(^{8,2}\), transmission poles\(^{8,4}\), and
bridges\(^{8,3}\).

The pressure of waves on marine structures is almost impossible to
deduce theoretically and, because of the importance of this subject, exten-
sive practical investigations have been made. The relevant data can-
not be expressed summarily but it is unlikely that a designer who has not
had experience of marine structures would be called upon to design such
a structure or one required to resist the blows from vessels when berthing
and the pulls from mooring ropes of vessels when berthed.

Other structures cannot be designed without experience because of the
indefiniteness of the loading or because secondary effects such as impact,
vibration, acceleration, gyration, high or low temperature, and corrosive
or abrasive action may be more severe than the direct effects of the static
loads. Many industrial structures are in this category, especially those
supporting machinery. The suppliers of the machines may state the static
load, but the effect on the structure must be assessed by the structural
designer, to which end it is almost imperative that he should see a machine
working under similar conditions. The designer should not always expect
to be given particulars which the suppliers may think not to be important,
and therefore should determine whether the machine is vibratory, gyratory,
or can subject the structure to shock and whether the process is accom-
panied by high or low temperatures or corrosive or abrasive actions. It is
not unknown for structures to be defective because the designer has been
unaware of such secondary effects and therefore has not taken means to
resist them.

**Other Effects**.—Other influences analogous to loads in so far that they
set up stresses in the structure must be considered and allowed for if
necessary. One of the most common and important is the shrinkage that
occurs when concrete hardens, complementary to which are the expansion
and contraction of structural members due to changes of ambient temperatures and wetness. Some means of dealing with these effects are described in Chapter XIV, Vol. II. The effects of corrosive atmosphere are offset in general by providing greater cover of concrete over the reinforcement than is usual for structural members in ordinary atmosphere, and in extreme cases by using special cement, such as sulphate-resistant cement. Another common cause of stress, which is as difficult to assess as to avoid with certainty, is unequal settlement of the foundations of a structure due to compression of the ground or to subsidence caused by mining. It is fortunate that the monolithic nature of a reinforced concrete structure may offset wholly or in part the ill effects of such settlement by redistributing the load on the foundation, but the stresses induced in so doing are additional to the ordinary stresses due to the loads, and, even if no defect arises due to the resultant higher stresses, the factor of safety of parts of the structure may be seriously reduced.

**Probability of Loading.**—An important aspect of live loading on structures is that the greatest load may not produce the greatest bending moments and forces. The most common example (considered on page 142) is the load imposed on a continuous beam, the bending moments at some parts of which are greatest when some spans only are loaded. This aspect must also be taken into account when designing underground structures, because the horizontal pressures due to the earth are indefinite. Normally in designing retaining walls, culverts, tunnels, underground tanks, and the like, a reasonable value is assumed for horizontal pressure due to the earth, but sometimes this pressure may be greatly increased or it may not be present at all. For example, the drainage behind a wall may be ineffective and allow water to accumulate, under which condition the wall will be subjected to hydrostatic pressures which may be double the assumed earth pressure; it is reasonable therefore for the wall to be designed to withstand the probable earth pressures at the ordinary permissible stresses, but to ensure that under the most severe possible condition, which is likely to be transient or not to occur at all, higher stresses, that is a smaller factor of safety, would be permissible. This matter of the probability of loading is serious in the case of underground tanks, since the external ground pressures act contrary to the internal pressures due to the contents of the tank. The internal pressures due to liquids can be calculated very accurately, but earth pressures can be assessed only indefinitely. If the assessment of the pressure of the earth is too high, the net outward pressure on the wall may exceed the pressure assumed in the design, and transitory or permanent conditions may arise in which there may be no ground pressure; in this case the tank may be liable to burst unless it is designed to withstand the whole internal pressure without relief from the external pressure.
Distribution of Loads.

Having determined the nature and magnitude of the loads or pressures to which a structure is subjected, the next step is to consider how these loads or pressures affect each member comprising the structure. In general, the loads are applied externally to a structure, and the resultant forces are transferred from one member to another until they reach the foundation. In each stage of this transference the loads or forces are probably transmitted to two or more members, and an important part of the design of each member is to determine how much of the load each is required to support. Examples of the distribution of load on structures of various types are considered in Volume II, but in the following the comparatively simple case of a building is considered. The principal load, apart from the weight of the building itself, comprises the live loads imposed on the floors and roof. In ordinary construction comprising slabs, beams and columns, as illustrated on pages 2 and 3 and shown diagrammatically in Fig. 41, the live load is imposed directly on the slab, and is transferred to a series of parallel secondary beams which in turn transfer the load to the primary or main beams, by which it is transferred to columns and so down to the foundation.

**Loads on Secondary Beams.**—If the weight of the slab and the live load assumed to be uniformly imposed thereon is $w_s$ lb. per square foot, and the spacing of the secondary beams is $l_x$ ft., as shown in Fig. 41, the load imposed on each interior secondary beam is $w_s l_x$ lb. per linear foot of beam. The weight of the beam, say, $w_b$ lb. per foot, must be added to this load to give the design load. If the total load on each interior secondary beam of span $l_s$ is denoted by $W_t$ and the total load on each secondary beam at the edge of the floor (which beam supports a width of only $\frac{1}{2}l_x$ ft. of slab), by $W_e$, it follows that

$$W_t = (w_s l_x + w_b)l_s; \quad W_e = (\frac{1}{2}w_s l_x + w_b)l_s.$$  \hspace{1cm} (8.2a)

These loads are uniformly distributed as shown in Fig. 42a.

**Loads on Primary Beams.**—In the example in Fig. 41 each primary beam, or main beam, supports two lines of interior secondary beams, although other arrangements may comprise one or any other number of lines of beams. The secondary edge-beams and the remaining interior secondary beams are supported directly by the columns. The load on each primary beam, as shown in Fig. 42b, comprises therefore two concentrated loads $W$ lb. and its own weight $w_B$ lb. per foot, which is distributed uniformly. Each concentrated load on a primary edge-beam is equal to half the load on an interior secondary beam. Each concentrated load on an interior primary beam is equal to twice this amount since the beam assists in supporting two secondary beams at each junction. Since the weight of the primary beam is only a small proportion of the total load,
it is generally sufficiently accurate to add a weight \( w_{blx} \) to each concentrated load. Therefore, each concentrated load is as follows.

On interior primary beam: \( W = W_t + w_{blx} \).

On primary edge-beam: \( W = \frac{1}{4}W_t + w_{blx} \). \hspace{1cm} (8.2b)

**More Exact Distribution.**—The foregoing simple consideration is commonly adopted, but a more exact distribution of the loads between the secondary and primary beams may be worth while sometimes for the sake of economy. The load on the strip of floor immediately over any beam is carried directly by the beam, and the load on the slab (including the weight

---

**Fig. 41.—**Part-plan of Typical Floor construction: Slab Spanning in One Direction.
of the slab) immediately against the side of a primary beam is carried entirely by that beam, and only at some distance from the beam is the entire load carried by the secondary beams. The distribution of the load to the secondary and main beams is therefore probably as shown by the trapeziums and 45-deg. triangles in Fig. 41. To avoid the complexity of several triangular loads, the triangular areas may be reduced to an equivalent rectangular area, that is if the width of the primary beam is \( b' \) ft. the uniformly-distributed load on an interior primary beam is that on a strip of floor of width \( l_x + b' \) ft. The total uniformly-distributed load is increased to \( w_s(l_x + b') + w_B \) lb. per foot. The concentrated loads are reduced accordingly since the total load on each secondary beam is now only \( (w_s l_x + w_b) \) lb. per foot multiplied by the average length \( l_s - (l_x + b') \) ft.

An additional refinement can also be included in the case of a series of continuous secondary beams the supports at the end of which are primary

![Fig. 42.— Loads on Beams.](image)

edge-beams. It is well known that if a beam is freely supported on one support and fixed at the other the load on the free support is only \( \frac{2}{3} \) of the total uniformly-distributed load on the beam, and the load on the fixed support is the remaining \( \frac{1}{3} \) of the total load, instead of half on each support as is the case where conditions are the same at each support. The end span of a series of continuous reinforced concrete beams is something between these two extremes, that is it is not quite freely supported at the end nor quite rigidly fixed at the penultimate support. Therefore the loads on the end and penultimate supports are probably 0.4 and 0.6 respectively of the total load instead of \( \frac{2}{3} \) or \( \frac{1}{5} \) and \( \frac{1}{3} \) or \( \frac{1}{5} \). Summarising these loads gives the following.

Secondary beam.—Span \( l_s \). Total load uniformly distributed on length \( l_s - (l_x + b') \) ft.

\[
\begin{align*}
\text{Interior beam:} & \quad W_i = (w_s l_x + w_b)[l_s - (l_x + b')] \text{ lb. } \\
\text{Edge-beam:} & \quad W_e = (\frac{1}{2}w_s l_x + w_b)[l_s - (l_x + b')] \text{ lb. }.
\end{align*}
\]  (8.2c)
Primary Beam.—Span \( l_p \).

Interior beam: Each concentrated load = \( W_i \) (or \( 1.1 W_i \) on a penultimate beam).

\[
\text{Uniformly-distributed load throughout span} = w = w_e (l_e + b') + w_B \text{ lb. per foot.}
\]

\[
\text{(8.2d)}
\]

Edge-beam: Each concentrated load = \( 0.4 W_i \).

\[
\text{Uniformly-distributed load throughout span} = w = w_e (\frac{3}{4} l_e + b') + w_B \text{ lb. per foot.}
\]

The total load on each secondary or primary edge-beam must include the weight of the wall, windows, or any external cladding which the beam supports.

The more complex distribution of loading represented by formulæ (8.2c) and (8.2d) is optional for a floor comprising slabs designed to span in one direction only, but a somewhat similar distribution is obligatory in the case of a floor comprising panels spanning in two directions as described in Volume II. The total load on each primary beam of the type of floor in Fig. 41 is practically the same whether formulæ (8.2a) and (8.2b) or formulæ (8.2c) and (8.2d) are applied, but the decrease in bending moment due to the reduced loads in formulæ (8.2c) and (8.2d) is greater than the increase due to the larger uniformly-distributed load; therefore less reinforcement is required than if formulæ (8.2a) and (8.2b) were used. In the case of a single floor it may not be worth while using the more complex formulæ (8.2c), but in a multiple-story building a small saving on each floor may result in a substantial saving. In any case, avoidance of a little more arithmetic is rarely a valid reason for providing more reinforcement or other material than is necessary.

**Loads on Columns.**—The load on a column at the level of any floor is the sum of the loads transmitted thereto by the beams supported by the column and the load from the column above the column being considered. In the example in Fig. 41, each interior column supports the ends of two interior secondary beams and two interior primary beams, that is the load on the column from one floor is equivalent to the sum of the total loads on one interior secondary beam and one interior primary beam. Each exterior column along one side of the building supports the ends of one interior secondary beam and two primary beams, that is the load from one floor is equivalent to the sum of the total load on one primary edge-beam and half the total load on an interior secondary beam. Similarly the load on each exterior column along the adjacent side of the building is the sum of the total load on one secondary edge-beam and half the total load on an interior primary beam. Each corner column supports the ends of one primary edge-beam and one secondary edge-beam, that is the load from one floor is the sum of half the total load on each of these beams. Instead of considering the load from each beam separately, it is generally
sufficiently accurate to consider the approximate area of floor supported by each column, that is $l_i l_p$ for each interior column, $\frac{1}{2} l_i l_p$ for each exterior column, and $\frac{1}{4} l_i l_p$ for each corner column. The load from each floor is then approximately the appropriate area multiplied by the intensity of loading $w$, increased to allow for the weights of the beams, as in Example No. 74 on page 181.

**Bending Moments.**

The loads and pressures to which a structure is subjected produce bending moments on most members, exceptions being members such as concentrically-loaded columns which transmit the load directly to the foundation. Common members subjected to bending include the slabs and beams of floors and roofs, the beams and columns of ordinary buildings and other framed structures, arches, walls of containers and retaining structures, and most foundations.

**Statically-determinate Members.**—Bending moments are readily calculated if the member is statically-determinate, that is if there is no interaction between the member and its supports or adjacent members. Such members are not common in monolithic reinforced concrete construction, but occur more frequently in precast construction if the joints between adjacent members are not capable of transmitting bending moments. Large monolithic structures that are generally statically determinate include bridge girders of one span, simple retaining walls, some types of cantilevered bridges, three-hinge arches, and chimneys (vertical cantilevers). The bending moments on all such structures are calculated from the simple rules that the reactions of the supports are in statical equilibrium with the applied loads, and that the bending moment at any plane is the algebraical sum of the moments of the loads and reactions on one side of the plane.

**Statically-indeterminate Members.**—Most members in monolithic construction are statically indeterminate because the bending moments depend on the elastic deformation of the member and adjacent members. Particular cases of statically-indeterminate members and structures, such as slabs spanning in two directions, framed structures, two-hinged and fixed arches, walls of tanks, special forms of roofs, and some types of retaining walls are considered in Volume II. The frames of buildings are considered in Chapter XI. In this chapter, consideration is given to statically-indeterminate reinforced concrete members in general, and continuous beams and slabs in particular.
General Consideration of a Statically-indeterminate Member.

The general case of a statically-indeterminate member partially or wholly restrained at both ends is illustrated in Fig. 43 where at (a) is shown a member AB of span \( l \) carrying a load \( W \).

**Bending Moments.**—The restraint moments \( M_{AB} \) at support A and \( M_{BA} \) at support B are determined by one of the theoretical or approximate methods of analysis described in Volume II, and depend on the interaction of contiguous members. The bending-moment diagram shown at (c) in Fig. 43 is the result of the algebraical addition of the diagram assuming statical determinancy (that is, in this case, free support) and the diagram of the restraint moment acting alone. The deformation corresponding to these bending moments is given at (d) in Fig. 43, which also shows the positions of the planes of contrafлексure. The maximum positive bending moment occurs on a plane at a distance \( x \) from support A and is represented by \( \frac{Wl}{k_2} \). At this plane, the "free-support" moment is \( \frac{Wl}{k_1} \), which is not
necessarily the maximum "free-support" moment, and the restraint moment is \( \frac{WL}{k_3} \). Therefore \( \frac{WL}{k_3} = \frac{WL}{k_1} - \frac{WL}{k_2} \). The "free-support" moment \( \frac{WL}{k_1} \) can be determined fairly accurately. The restraint moment \( \frac{WL}{k_3} \), which is equal to \( (M_{AB} - M_{BA}) \frac{l - x}{l} \), is only as accurate as \( M_{AB} \) and \( M_{BA} \), and, as already explained, it is improbable that these restraint moments can be determined very accurately; in fact the exact value is never likely to be known, so that some approximation must be accepted. Fortunately, in monolithic reinforced concrete construction this does not matter to some extent because the bending moments appear to adjust themselves in accordance with the resistances provided. Thus if \( M_{AB} \) and \( M_{BA} \) are underestimated, \( \frac{WL}{k_3} \) will be underestimated; consequently \( \frac{WL}{k_2} \) will be overestimated. Within limits, the member will be quite safe if the moment of resistance provided at the supports conforms to the underestimated values of \( M_{AB} \) and \( M_{BA} \) so long as the moment of resistance in the span is not less than the overestimated value of \( \frac{WL}{k_2} \).

If due to \( M_{AB} \) or \( M_{BA} \) being underestimated there is a tendency for the member to be overstressed near the supports, there will tend to be greater deformation in these parts, but this deformation can only occur if the part of the member in the span becomes more deformed. This additional deformation in the span, however, will be resisted if there is excessive strength in this part. It therefore follows that a deficiency of resistance to negative bending moment at the supports can be compensated for by an excess of resistance to positive bending moment in the span (and vice versa). In an extreme case the resistance at midspan could be made equal to the entire "free-support" positive bending moment \( \frac{WL}{k_1} \), in which case it would be theoretically possible to ignore the negative bending moment \( M_{AB} \) and \( M_{BA} \) at the supports, but this neglect would be likely to cause severe cracking of the concrete over the supports which might be objectionable although not necessarily a sign of structural unsoundness. To avoid misuse of this principle of compensatory resistances, in practical design the amount by which the bending moment can be adjusted is limited to 15 per cent. of the calculated bending moment as described later.

**Shearing Forces.**—The shearing forces on a beam or other member are related to the bending moment through the well-known relation that shearing force is the rate of change of bending moment. The shearing force at any plane of a statically-determinate member is the algebraical sum of the loads and reactions on one side of the plane and is sometimes called the static shearing force. The shearing force at any plane of a statically-indeterminate member is the algebraical sum of the varying shearing force considering the member as statically determinate and the uniform shearing
force due to the difference in the restraint bending moment at each end of the member. The total shearing force calculated on this basis is called the elastic shearing force. The component and combined diagrams of these shearing forces are shown for the general case of a restrained statically-indeterminate member at (b) in Fig. 43, from which it will be seen that the reaction of the support at which the restraint is greatest is greater than the static reaction, and the reaction of the other support is correspondingly less. In the extreme case of one end of the member being rigidly fixed and the other entirely freely supported (that is unrestrained), the amount by which the resultant reaction (and therefore the shearing force) exceeds the static shearing force is 25 per cent. for a member carrying a uniformly-distributed load; that is the static shearing reaction at each end is \( \frac{1}{3} W \), but, due to the elastic shearing force, the reaction at the fixed end is increased to \( \frac{4}{3} W \) and reduced at the free end to \( \frac{2}{3} W \). This effect is taken into account in the derivation of formulae (8.2c) and (8.2d) for the distribution of loading on to beams supporting other beams.

It is obvious that at the inner support of an end span of a continuous beam the additive effect of the elastic shearing force is too great to be ignored (as it is sometimes). At the same time, however, the deductive effect towards the outer support should not be taken fully into account unless it is absolutely certain that conditions are such that the beam will act as assumed. For example, if a continuous beam is truly freely supported at the outer support, reduction of the shearing force by the amount of the elastic shearing force may be justified. If a beam is supported on and is monolithic with a column or a very substantial beam, a free support is not provided although the bending moment may be, and in common practice is, calculated on the basis of a free support at the end. It is advisable therefore to err on the side of safety and to provide at all sections shearing resistance not less than the static shearing force or the elastic shearing force, whichever is the greater. Although resistance to bending is able, to a certain extent, to redistribute the bending moments favourably, it is not the case that excess of shearing resistance at one plane can compensate for a deficiency elsewhere.

Continuous Beams.

Methods of Calculation.—The bending moments due to loads on beams of one span, if freely supported at both ends, are determined from the principles of simple statics. For beams continuous over two or more spans, which are common in reinforced concrete structures, the bending moments caused by the applied load are calculated by one of the classical methods such as slope-deflection, the theorem of three-moments, or the principle of least work. These theoretical methods are generally tedious, and often the accuracy of the results is not commensurate with the labour
involved because of the inaccuracy of the assumptions it is necessary to make. These assumptions are that

(i) The supports are at the same relative level before and after loading; if some of the supports of the beam are columns and other supports are beams, as is usual in floors, it is unlikely that the deflection of all the supports will be the same.

(ii) The supports are knife-edge supports, that is they have no width and do not offer resistance to the rotation of beams deforming due to irregularity of the applied loads. In all structures the support has some appreciable width, and in monolithic construction the support offers resistance to the rotation of the supported beam either by the resistance to bending of a supporting column or the resistance to twisting of a supporting beam.

(iii) The moment of inertia is uniform throughout. If haunches are provided at the supports of a beam the moment of inertia is not uniform. Even if the beam is of uniform depth throughout the amount of reinforcement varies throughout, and flexural conditions in the span are by no means similar to those near supports over which the beam is continuous; therefore the assumption of uniform moment of inertia is rarely valid.

These assumptions can be modified by taking into account actual conditions, but this procedure introduces further complexities into a calculation that is already too complex for everyday use. There is therefore justification for the use of approximate methods of calculation, such as the method of moment distribution which comprises successively closer approximations and is popular since it requires little mathematical or theoretical knowledge and the result, which is obtained in a direct manner without the solution of simultaneous equations as in other methods, is sufficiently accurate for most structures. The application of this method is given on page 148.

Bending-moment Coefficients.—Since the bending moment $M$ at any section of any beam is proportional to the total load $W$ and the span $l$, the moment can be expressed as $\frac{Wl}{k}$ as is indicated in the foregoing section. The purpose of all analyses is to determine the value of the bending-moment coefficient $\frac{1}{k}$. Owing to the many indeterminate factors in the calculation of bending moments on continuous beams approximate values of $k$ are often sufficient, especially in the case of floor slabs spanning over two or more spans, and secondary beams and some primary beams of floors. In establishing the values of this coefficient, the possible variation of the incidence of the live or imposed load must be taken into account since, when the ratio of imposed load to dead load is high, there is considerable difference between the bending moments produced when all spans are loaded and the greater bending moments when only some spans are loaded.
For beams supporting a uniformly-distributed load the value of \( k \) for the maximum bending moment varies from 8 for a freely-supported (statically-determinate) beam to 12 or 24 for a beam fixed at both ends (statically-indeterminate); for a beam carrying a load concentrated at mid-span the corresponding values are 4 and 8. Thus \( k \) generally lies between the wide limits of 4 and 24, but the conditions of a particular problem generally indicate a much narrower range, so that upon inspection it is often possible to estimate a probable value of \( k \) which is near enough for a preliminary calculation and may serve as a rough check on the results of a more complex final calculation.

**Effect of Live Load.**—The incidence of live load to produce the greatest bending moments is illustrated in Fig. 44. Whereas the dead load operates on all spans simultaneously, the most adverse case for the live load is when certain spans only are loaded. It is not worth while to consider more than the two spans adjacent to a support as loaded to calculate the greatest negative bending moment at the support, or more than three alternate spans loaded to calculate the greatest positive bending moment at the middle of the central span of the three; the effect of the loads on the remaining alternate spans is to increase the bending moments at the critical sections, but the increase is negligible (not more than 5 per cent.) in relation to the effect of other assumptions. It should be noted that all the spans of a series of continuous beams supporting the bottom of a tank are loaded simultaneously by the liquid contents; therefore the live load in this case is analogous to a dead load.

**Reduction of Negative Bending Moments.**—Negative bending moments calculated by any method, other than by the assumption of approximate coefficients, may be reduced by not more than 15 per cent. if the positive bending moments (due to the same loading that produces the negative bending moments) in the two adjacent spans are increased by the same numerical amount. This adjustment is justified because factors such as the beam being monolithic with its supports, the width of the supports, and the deflection of intermediate supports exceeding that of end supports, tend
to reduce the negative bending moments but are generally neglected in the calculation. On the other hand, if the moment of inertia of the beam is greater near the supports than at midspan, as occurs in beams with haunches at the supports, the bending moments at the supports are greater than those obtained by calculation based on the assumption of uniform moment of inertia; unless variation of moment of inertia is taken into account it is not advisable to reduce the negative bending moments in this case. The reduction of negative bending moments and the corresponding increase of the positive bending moments is also justified on the ground that the ultimate load-carrying capacity of a continuous beam may depend more on the resistance to positive than to negative bending moments, as is explained on page 139. The practical advantage gained from the foregoing reduction in the case of floor beams is that the congestion of reinforcement at the support is reduced because less reinforcement in tension and compression is required. The increase of positive bending moments at midspan increases the amount of reinforcement in tension necessary at this position, but this does not generally increase the complexity. There is generally sufficient compressive resistance in the slab forming the compression flange of the tee-beam at midspan to resist the increased bending moments, whereas at the support the narrow rib of the beam does not provide much resistance to compression and reinforcement in compression may be necessary.

**Approximate Bending Moments for Equal Spans.**—In the simplest case of beams continuous over equal spans, such as secondary beams of three or more equal spans, approximate bending moments based on approximate values of \( k \) are generally sufficiently accurate. Some examples of such coefficients for dead and live loads imposed as uniformly-distributed loads or loads concentrated at midspan or at the third-points of the span are given in Fig. 46. End spans and interior spans are shown separately. The coefficients are derived from a simplified elastic theory of continuous beams and are adjusted to decrease the theoretical negative bending moments by not more than 15 per cent. and to increase the positive bending moments accordingly. These approximate bending moments are applicable to beams of unequal spans so long as the inequality is not greater than 15 per cent. of the longest span.

More approximate coefficients can be used in the case of continuous solid slabs supported by and constructed monolithic with beams. In such cases the assumption of "knife-edge" support is far from actuality and it is sufficiently accurate to assume values of \( \frac{L}{k} \) equal to one-twelfth for the positive bending moment at midspan and the negative bending moment at the supports of an interior span, and one-tenth for the positive bending moment at midspan and the negative bending moment at the interior support of an end span also, as shown in Fig. 46.
Fig. 45.—Moment-distribution applied to Continuous Beams (See Example No. 65, page 149.)
Negative Bending Moments in the Span.—If the ratio of live load to dead load exceeds, say, \( \frac{1}{2} \), the occurrence of negative bending moment throughout the span is possible due to live load being imposed on the two spans adjacent to an unloaded span. The resistance to twisting of the supports precludes the possibility of the greatest theoretical negative bending moment being produced, and the probable negative bending moment is given by

\[
M_{\text{neg.}} = (\beta w_L - w_D) \frac{l^2}{24},
\]

(8.3)

in which \( w_L \) and \( w_D \) are the intensities of the live and dead loads respectively, and \( \beta \) is a coefficient which is equal to \( \frac{2}{3} \) for beams and \( \frac{1}{3} \) for solid slabs. The smaller coefficient results in a smaller bending moment on
solid slabs, which is likely because the torsional restraint of the supporting beams is relatively greater than in the case of a beam supported on a beam.

**Unequal Spans.**—Approximate methods of calculating bending moments should not be used for beams or slabs continuous over unequal spans or for beams carrying irregular loads. Such cases must be calculated either from the elastic theory or, more conveniently, by the moment-distribution method described on page 148.

**Beams Monolithic with Columns.**—The effect of a beam being monolithic with supporting columns is more important as regards the columns than the beam and is considered in detail on page 198. It is sufficient to state here that if a continuous beam is monolithic with a column at the end supports, resistance to negative bending moments should be provided at these supports; it is not generally worth while adjusting the positive and other negative bending moments on the beam to allow for this effect. The bending moments at an interior support where a beam is monolithic with the supporting column is generally slightly less than the bending moment calculated on the assumption of "knife-edge" supports; therefore this effect is generally neglected as regards the beam.

**Example No. 63.—Bending Moments on a Slab (Approximate Method).** Determine the approximate bending moments on a 6-in. slab spanning over a number of beams spaced at intervals of 10 ft. The live load imposed on the slab is 200 lb. per square foot.

The weight of a 6-in. slab is 75 lb. per square foot if reinforced concrete is assumed to weigh 150 lb. per cubic foot. The total load is therefore $200 + 75 = 275$ lb. per square foot. Applying the approximate coefficients of $\frac{1}{12}$ and $\frac{1}{10}$ the bending moments are as follows.

The negative bending moment over each supporting beam (except the end and penultimate beams) and the positive bending moment at the middle of each interior span is

$$\frac{275 \times 10^2}{2} \times 12 = 27,500 \text{ in.-lb. per foot width.}$$

The negative bending moment over the penultimate supporting beam and the positive bending moment at about the middle of the end span is

$$\frac{275 \times 10^2}{10} \times 12 = 33,000 \text{ in.-lb. per foot width.}$$

The negative bending moment at the middle of each span is calculated by substitution in formula (8.3) to give

$$\left(\frac{200}{2} - 75\right) \frac{10^2}{24} \times 12 = 1250 \text{ in.-lb per foot width.}$$

The numeral 12 is introduced into each of the foregoing calculations to convert the units of the bending moments from foot-pounds to inch-pounds.

**Example No. 64.—Bending Moments on Beams (Approximate Method).** Determine the approximate bending moments on the beams supporting the slab in Example No. 63 if the beams are 18 in. deep below
the slab and 12 in. wide and are continuous over a number of equal spans each 30 ft. long.

The weight of the beam is \(1\frac{1}{2} \times 1 \times 150 \text{ lb.} = 225 \text{ lb. per foot.}\) From formula (8.2a) the uniformly-distributed dead load on the beam is \((75 \times 10) + 225 = 975 \text{ lb. per foot,}\) and the live load is \(200 \times 10 = 2000 \text{ lb. per foot.}\)

Adopting the coefficients in Fig. 46, the bending moments are as in the following.

**Interior Span.**

Maximum positive bending moment at midspan

\[
M = +\left(\frac{975}{24} + \frac{2000}{12}\right)30^2 \times 12
\]

\[
= +2,250,000 \text{ in.-lb.}
\]

Negative bending moment at supports

\[
M = -\left(\frac{975}{12} + \frac{2000}{9}\right)30^2 \times 12
\]

\[
= -3,270,000 \text{ in.-lb.}
\]

**End Span.**

Maximum positive bending moment at midspan

\[
M = +\left(\frac{975}{12} + \frac{2000}{10}\right)30^2 \times 12
\]

\[
= +3,040,000 \text{ in.-lb.}
\]

Negative bending moment at inner support

\[
M = -\left(\frac{975}{10} + \frac{2000}{9}\right)30^2 \times 12
\]

\[
= -3,450,000 \text{ in.-lb.}
\]

Negative bending moment throughout span.—By substitution in formula (8.3),

\[
M_{neg.} = -[(\frac{1}{2} \times 2000) - 975\frac{30^2}{24}] \times 12 = -161,100 \text{ in.-lb.}
\]

**Bending Moments on Continuous Beams Calculated by the Moment-distribution Method.**

The calculation of the bending moments at the supports of a beam continuous over two or more spans may be algebraical or may be entirely arithmetical as described in its simplest form in the following. The theoretical bases of the operations of distribution and carrying over will be readily understood by those familiar with moment-distribution methods, but others should study the theory and more complex analyses given in publications dealing with this subject. In the following example the convention of signs conforms to that commonly used for continuous beams, namely, bending moments producing tension in the bottom of the beams (as at mid-span) are considered to be positive, and those producing tension in the top (as over supports) are negative.
Example No. 65.—Bending Moments on a Beam (Moment-distribution Method). Calculate the bending moments on the beam in Fig. 45 (pages 144 and 145) where four spans of a continuous beam, including the left-hand free support, are shown.

The first step is to draw diagrams of the beam for the dead load and live load separately. On this diagram are entered the fixed-end bending moments for each loaded span; these bending moments are the negative bending moments at the end of each span assuming the beam to be fixed rigidly at each support. The numerical value of the fixed-end moment on a beam supporting a symmetrical load is the total area of the free-bending-moment diagram divided by the span. For example, if the total load $wl$ is uniformly distributed throughout the span, the fixed-end moment is $\frac{1}{2} \left( \frac{wl^2}{8} \right) \frac{1}{l}$, that is $\frac{wl^2}{12}$ at each support; similarly if the total load $W$ is concentrated at midspan, the fixed-end moment is $\frac{1}{2}wl$; other cases including non-symmetrical loading should be calculated from first principles or obtained from tables.

In Fig. 45 the fixed-end bending moments and subsequent bending moments are given in units of 100 ft.-lb. The beam, however, is not fixed rigidly at the supports but is assumed to be free to rotate about each support, although continuous over interior supports; therefore the bending moments to the left and right of a support are identical. The fixed-end moments to the left and right of a support differ because of the inequality of the spans and loads. The difference, or unbalanced moment, is dealt with by distributing it on to the adjacent spans; that is the smaller fixed-end moment must be increased, and the greater must be decreased. The amount depends upon the relative stiffness of the two spans meeting at the support. This stiffness is expressed by the ratio $\frac{\text{moment of inertia} I}{\text{length of span} l}$.

Since only relative stiffnesses are required, only relative moments of inertia need be used; this may save much arithmetic because if, for example, the beam has the same width throughout all the spans, the relative moments of inertia are proportional to the cube of the depth. In Fig. 45 it is assumed that the depth and width are the same in each span; therefore the stiffness is inversely proportional to the span. The distribution factor for any span as regards one support is the ratio of the stiffness of the span to the sum of the stiffnesses of the two spans meeting at the support. The distribution factors are shown in Fig. 45, as is the next step of distributing the unbalanced moment to each span. The distribution factor, for example for span $l_2$ as regards support C, is

$$\frac{I_2}{I_2 + I_3} = \frac{24}{18 + 24} = \frac{4}{7};$$

therefore the distribution factor for span $l_3$ as regards support C is $1 - \frac{4}{7} = \frac{3}{7}$. Similarly for other supports.

If a bending moment is applied to one end of a restrained member, a bending moment equal to half the applied bending moment, but causing opposite curvature, is produced at the other end of the member. These bending moments are called carry-over bending moments. The distributed moment is such an applied bending moment, and therefore each distributed
moment causes a carry-over bending moment at the opposite support to which it is applied. Consider the dead-load bending moments at support C. The fixed-end bending moments (negative) in 100 ft.-lb. units are 324 to the left of the support and 576 to the right. The unbalanced bending moment is 576 - 324 that is 252 (negative). Since the right-hand bending moment is the greater this is reduced by \( \frac{2}{3} \times 252 = 108 \), that is a positive bending moment of 10,800 ft.-lb. is applied. The left-hand fixed-end bending moment is increased by \( \frac{2}{3} \times 252 = 252 - 108 = 144 \), that is a negative bending moment of 14,400 ft.-lb. is applied to the left-hand side. An interim summation at this stage shows that the bending moments to the left and right of each support are equal, that is, they balance. But the effect of this first distributive operation is to cause carry-over bending moments. The distributed bending moment at the right-hand side of support B is a positive bending moment of 72 units of 100 ft.-lb.; therefore there is at the left-hand side of support C a carry-over negative bending moment of \( \frac{1}{3} \times 72 = 36 \) units. Similarly the distributed bending moment at the left-hand side of support D is a positive bending moment of 128 units, which produces a carry-over negative bending moment of 64 units at the right-hand side of support C. Thus the bending moments at support C are unbalanced again by the amount of 64 - 36 = 28 units, which are again distributed to produce a balanced negative bending moment of 52,000 ft.-lb. Successive operations of carry-over and distribution can be continued, the result after each operation resulting in smaller differences. When the omission of a carry-over operation after distribution results in little difference in the bending moments at the supports, it is not necessary to proceed farther. Actually, in practical problems of continuous beams, sufficient accuracy is generally attained after three distributions and two intermediate carry-over operations, and this is the extent to which the example in Fig. 45 is carried.

In the consideration of dead load all the spans are loaded simultaneously, but for live load each span should be considered separately as shown for span BC in Fig. 45. The same number of operations is carried out, but fixed-end bending moments occur only at the ends of the loaded span. The maximum bending moment at any support is then obtained by adding the bending moments of like sign for the dead and live loads. For example, for the maximum negative bending moment at support C, the summation includes the bending moments due to dead load on all spans and the bending moment due to live load on span BC, and on span CD if there were a live load on CD. For the maximum positive bending moment on span BC, the summation for the corresponding negative bending moments at supports B and C include the bending moments due to dead load on all spans, and those due to the live load on span BC (the span being considered), and on span DE if there were a live load on this span; if there were another span to the left of support A, the effect of the live load on that span also would have been included. The positive bending moment is obtained by plotting to scale the diagram of the free bending moment for the total load on the span and superimposing the diagram of the negative bending moment.

**Direct Forces.**

Direct forces may be either concentric forces as described in Chapters II and V or eccentric forces as dealt with in Chapters IV and V, and may
be either thrusts or pulls. Truly concentric forces are unlikely to occur in practice, although in many members it is common to assume such to be the case. For example, concentric thrust is assumed to occur on interior columns of buildings in which the beams are symmetrically disposed. Tie-beams and symmetrically-loaded dividing walls in containers are likewise usually considered to be concentrically loaded although the pressures are in practice likely to be different in each compartment. Discrepancies between the actual and assumed conditions in these and similar cases are considered to be taken into account in assessing permissible stresses and other factors, in the method of calculating the resistance to direct load (Chapters II and V), and in the factor of safety of the structure.

BIBLIOGRAPHY FOR CHAPTER VIII

(8.1).—"Schedule of Unit Weights of Building Materials." B.S. No. 648. (Dead loads; reference should be made to the Appendixes to this Standard in which some important deviations from average weights are given.)

(8.2).—"Code of Functional Requirements of Buildings. Chapter V: Loading." B.S. Code No. 3. (Imposed loads on floors and roofs; wind pressures. Explanatory notes in Appendix should not be overlooked.)

(8.3).—"Girder Bridges: Part 3A. Loads." B.S. No. 153. (Loads and wind pressures on road and railway bridges.)

(8.4).—"Reinforced Concrete Poles for Electrical Transmission and Traction Systems." B.S. No. 607. (Wind pressures.)
CHAPTER IX
SIMPLE SLABS

There are several types of reinforced concrete slabs used for floors, roofs, walls, and the like, namely, ordinary solid slabs of uniform thickness; slabs with hollow blocks inserted to reduce the amount of concrete and the weight; ribbed slabs in the form of contiguous tee-beams, which also result in a reduced amount of concrete and are lighter in weight than corresponding solid slabs; and precast slabs, which may be solid slabs but are generally inverted channels or hollow members, and the use of which economises in shuttering. There are four principal structural forms of solid slabs which are generally cast in place, namely, cantilevered slabs, slabs spanning in one direction only, slabs supported along all edges in such a way that they can span in two directions mutually at right-angles, and slabs without supporting beams but carried directly on columns and called flat slabs, beamless floors, or mushroom floors. Slabs may be of one span or continuous over two or more spans. The simple cases of solid slabs spanning in one direction, cantilevered solid slabs, and hollow-block slabs are dealt with in this chapter. More complex slabs, such as slabs spanning in two directions and flat slabs, are considered in Chapter XV, Vol. II.

Precast slabs are of many types and, although they can be designed to suit a special structure, they are generally proprietary designs. The design of a simple solid slab necessitates consideration, as in the succeeding paragraphs, of the following items in the order given: calculation of the maximum bending moments; calculation of the thickness and the principal reinforcement; detail design including arranging the principal and secondary reinforcement.

Solid Slabs Supported Freely along Two Opposite Edges.

Panels of solid slabs supported freely on two opposite edges (Fig. 47) are less common than continuous slabs but are simple to design. The supports may be brick walls, steel beams, or separate reinforced concrete beams, in which cases the support is practically "free", that is unrestrained, and the slab is statically determinate and is subjected to positive bending moments only. In this consideration of slabs, positive bending moments are those producing tension at the bottom of the slab and negative bending moments are those producing tension at the top. The bending-moment diagram for this condition when the imposed load is uniformly distributed, and the arrangement of the principal reinforcement, are given in Fig. 47a. The maximum positive bending moment is \( \frac{1}{8}w_{s}l_{s}^{2} \), in which \( w_{s} \) is the
intensity of the total load, that is the sum of the imposed live load $w_L$ and the dead load $w_D$ which includes the weight of the slab and any finishes on the top or to the soffit.

If a solid slab is constructed monolithically with reinforced concrete supporting beams or with concrete encasing steel beams, or if the ends of the slab are built into brick walls, as in Fig. 47b, the condition of free support does not obtain strictly since the supports restrain the freedom of action of the ends of the slab. Nevertheless it is usual to calculate the

![Diagram of Slab Analysis](image)

**Fig. 47.—Freely-supported Slabs Spanning in One Direction.**

positive bending moment on the assumption that the slab is supported freely, but reinforcement is provided to resist any small negative bending moment that may result from the end restraint. The maximum positive bending moment is, as before, $\frac{1}{8}w_Ll_x^2$, but provision should be made at the ends to resist a negative bending moment of, say, $\frac{w_Dl_x^2}{24}$. The thickness of the slab and the amount of principal reinforcement is then determined as in the example which follows. To ensure that the slab is sufficiently stiff to prevent excessive deflection, the ratio of span to thickness
of a freely-supported solid slab spanning in one direction should not exceed 30.

Example No. 66.—Freely-supported Slab. Design a solid slab to span 10 ft. and to support a live load of 100 lb. per square foot. The slab is supported on brick walls. The permissible stresses are 1000 lb. per square inch in 1:2:4 concrete of ordinary quality and 20,000 lb. per square inch in mild-steel reinforcement.

If the slab is assumed to be 6 in. thick the weight is 75 lb. per square foot, and the total load is 100 + 75 = 175 lb. per square foot. By substitution in \( \frac{1}{8}wL^2a^3 \), the maximum positive bending moment is

\[ \frac{1}{8} \times 175 \times 10^2 \times 12 = 26,250 \text{ in.-lb. per foot width.} \]

For the stresses specified, it is shown in Example No. 12 (page 51) that the moment-of-resistance factor \( Q_e \) is 184 lb. per square inch and the lever-arm factor is 0.86. By substitution in formula (3.6a) in Chapter III, the effective depth \( d_1 \) of the slab should be not less than

\[ \sqrt{\frac{26,250}{184 \times 12}} = 3.44 \text{ in.} \]

The thickness of the slab should therefore be not less than 3.44 in. + (1-in. cover) + (half the diameter of the bar, say, \( \frac{1}{2} \) in.) = 4.19 in., say, 4.5 in., for which \( d_1 \) is actually 3.75 in. The ratio of span to thickness is \( \frac{10 \times 12}{4.5} = 27 \), which is less than the limiting ratio of 30.

The bending moment of 26,250 in.-lb. per foot allows for the weight of the slab to be 75 lb. per square foot, but the weight of a 4.5-in. slab is only 56 lb. per square foot; therefore the total load is actually 156 lb. per square foot and the bending moment is only 23,400 in.-lb. per foot. By substitution in formula (3.6a) in Chapter III,

\[ A_{st} = \frac{23,400}{20,000 \times 0.86 \times 3.75} = 0.363 \text{ sq. in. per foot,} \]

which is provided by \( \frac{1}{2} \)-in. bars at 6-in. centres. These bars comprise the main transverse reinforcement, which is arranged as in Fig. 47a, if the slab is laid on brick walls or as in Fig. 47b if the ends of the slab are supported in recesses in the walls. For reasons explained on page 161, longitudinal bars are required and the cross-sectional area of this secondary reinforcement should be not less than 0.15 per cent. of the area of the concrete. In this example, the area should be \( \frac{0.15}{100} \times 4.5 \times 12 = 0.081 \) sq. in. per foot width of slab, which is provided by \( \frac{1}{4} \)-in. bars at 7.5-in. centres.

Solid Slabs Continuous over Supports along Two Opposite Edges.

If a slab is continuous over two or more spans it is not statically determinate. The bending moments may be calculated as described for continuous beams on page 140, but the apparent precision in such calculation is not justified in the case of solid monolithic slabs since the assumptions
are by no means realistic. Because each support may settle elastically by a different amount depending on the amount of live load that will produce the maximum bending moment at a critical section, the common assumption that all supports remain level is unlikely to be true. The nearest approach to the common assumption that the supports are "knife-edge" supports, that is that they offer no resistance to rotation of the slab at the support, is a slab supported on un-encased steel beams, but in monolithic reinforced concrete construction the supporting beams offer considerable resistance to the twisting to which they are subjected if the slab tends to rotate at a support as illustrated in Fig. 48. Because of this

![Fig. 48.—Rotation of Beams Supporting Slabs.](image)

restriction of rotation, deformation of one span is not freely transmitted to adjacent spans; therefore the load on one span does not affect adjacent spans so much as the simplified analysis indicates. For the foregoing reasons, slabs continuous over equal spans can be designed safely for approximate bending moments, the values of which depend on the magnitude of the product of the total load and the span divided by a non-dimensional coefficient $k$, that is

$$M = \frac{Wl_x}{k} = \frac{w_s l_x^2}{k}, \quad \ldots \quad (9.1)$$

in which $M$ is the bending moment on unit width of slab of span $l_x$ supporting a total load of $W$ per unit width or a uniformly-distributed total load of intensity $w_s$ per unit area. If the units are pounds per foot or pounds per square foot for the load and feet for the span, the bending moment is in foot-pounds.

**Uniformly-distributed Load, Equal Spans.**—As is stated on page 143, approximate values of $k$ for slabs carrying a uniformly-distributed load and continuous over two or more equal spans are 10 for the midspan of an end span and a penultimate support, and 12 for the midspan of an interior span and at other interior supports. For more accurate computation, which is advisable if the live load is two or more times the dead load, the separate coefficients for live and dead load given in Fig. 45 in Chapter VIII for beams continuous over three or more spans are applicable. The coefficients are repeated in graphical form in Fig. 49a. There is no difficulty in relating these coefficients to those obtained by an "exact" analysis of continuous beams subjected to live and dead loads, due regard
being paid to the incidence of the live load to produce the greatest bending moments.

Since the maximum bending moment on the end span and at the penultimate support of a slab continuous over a number of supports is greater than that in the adjacent interior spans, and since in practice it is preferable to have a slab of the same thickness throughout all spans, there is an advantage in making the end spans \( l_e \) shorter where possible than the interior spans \( l_s \). For a slab carrying a uniformly-distributed load \( w_s \), the ratio of \( l_e \) to \( l_s \) should be such that \( \frac{w_s l_e^2}{10} \) is equal to \( \frac{w_s l_s^2}{12} \), which is so if \( l_e \) is about 0·9\( l_s \). This slight inequality is less than 15 per cent., which is the limit within which spans can be considered to be equal, and enables the same thickness to be provided throughout and the same amount of

**Fig. 49.—Bending Moments on Continuous Slabs.**
reinforcement to be provided at each critical section, which is a great practical advantage. If end and interior spans are equal in length, and a slab of the same thickness is provided throughout, the greater bending moment over the penultimate support should determine the thickness of the slab and the maximum amount of reinforcement, and the reinforcement should be varied to suit the smaller bending moments at other critical sections; this results in an economical slab because the cost of the extra concrete is more than offset by the saving in reinforcement and the simplicity of the shuttering to the soffit.

If the ratio of live load to dead load exceeds two, negative bending moments may occur throughout an unloaded interior span, and the approximate formula (8.3) on page 146 applies with $\beta = \frac{1}{3}$.

**Uniformly-distributed Load on Unequal Spans.**—For unequal spans values of $k$ can be calculated from the theory of continuous beams (see page 147) and then adjusted to allow for variations from the theoretical assumptions. For floor and roof slabs this procedure is not worth the time and labour required and, except in cases of a heavy live load or great inequality of spans, the following rules are sufficiently accurate for a slab supporting a uniformly-distributed load.

If the spans do not differ by more than 15 per cent. of the longest span, the bending moment may be calculated as though all spans are equal to the longest span.

If the difference exceeds 15 per cent., but if the longest span is not more than, say, double the shortest, the bending moments given in Fig. 49b are approximately correct.

If a short span occurs between two long spans, the negative bending moments at the supports should be calculated as if all the spans were equal to the long spans, and resistance should be provided throughout the short span for at least the mean of the two negative bending moments and for the positive bending moment as shown in Fig. 49c.

**Ratio of Span to Thickness.**—In general the bending moment to be resisted determines the thickness of the slab, but to prevent excessive deflection the ratio of span to thickness of a continuous solid slab spanning in one direction should not exceed 35. A continuous slab is stiffer than a freely-supported slab and therefore a ratio larger than 30, which is that for a freely-supported slab, is acceptable. If the tensile stress in the reinforcement is much less than that ordinarily permissible these ratios may be increased, but competent advice should be taken before doing so.

**Solid Slabs Supported along Two Opposite Edges and Supporting Concentrated Loads.**

The coefficients and rules in the foregoing relate to well-established cases of slabs supporting uniformly-distributed loads. The analysis is less
definite if the load is concentrated on a small area or on a transverse or longitudinal strip of slab, and for these cases empirical rules are generally applied.

**Freely-supported Slabs.**—For freely-supported slabs (Fig. 50a) the British Standard Code recommends the width \( e \) of slab to be assumed to assist in carrying the load as

\[
g + 2.4x \left( 1 - \frac{x}{l_x} \right)
\]

in which \( g \) is the width of the loaded area and \( x \) is distance from the nearer support to the centre of the loaded area. For the general condition of a total load \( W \) concentrated on an area of width \( g \) and of length \( b \) in the direction of the span \( l_x \), the total maximum bending moment is

\[
\frac{(l_x - x) W x}{l_x} - \frac{W h}{2.4}
\]

and the bending moment on unit width of slab is the maximum bending moment divided by \( e \), that is if the units are feet and pounds,

\[
M = \frac{W}{e} \left[ \left( 1 - \frac{x}{l_x} \right) x - \frac{h}{8} \right] \text{ ft.-lb. per foot width}
\]

\[ (9.2) \]

If the load is near an unsupported edge such that the dimension \( f \) in Fig. 50a is less than \( 1.2x \left( 1 - \frac{x}{l_x} \right) \),

\[
e = f + g + 1.2x \left( 1 - \frac{x}{l_x} \right)
\]

\[ (9.2a) \]

If the load extends as a strip across the width of the slab as in Fig. 50b, for example, if the slab supports a brick wall, \( h = l_x \) and \( x = \frac{1}{2} l_x \). The total bending moment is \( \frac{W l_x^2}{8} \), and therefore

\[
M = \frac{W l_x^2}{8e} \text{ ft.-lb. per foot; } e = g + 0.6l_x
\]

or if near an edge, \( e = f + g + 0.3l_x \).
If the loaded area is a strip at right-angles to the span as in Fig. 50c, and the load is $W_1$ lb. per foot, the bending moment is given by

$$M = W_1\left[\left(1 - \frac{x}{l_x}\right)x - \frac{h^2}{8}\right] \text{ ft.-lb. per foot.} \quad (9.2c)$$

**Continuous Slabs.**—If the slab is continuous over the supports the bending moments are not easy to assess, but the bending moments calculated from the foregoing formulae can be used as a guide and may be reduced by not more than the following percentages to allow for continuity over three or more spans.

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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Load concentrated</td>
<td>15 per cent.</td>
<td>20 per cent.</td>
<td>30 per cent.</td>
<td>35 per cent.</td>
</tr>
<tr>
<td>at midspan</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Load extending as a</td>
<td></td>
<td></td>
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<tr>
<td>strip over the full</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>length of the panel</td>
<td>20 per cent.</td>
<td>35 per cent.</td>
<td>7(\frac{1}{2}) per cent.</td>
<td>15 per cent.</td>
</tr>
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Reductions for intermediate conditions can be assessed from the foregoing, which are based on comparisons of the effect of a load at midspan and a uniformly-distributed load on continuous beams, and in which it is assumed that one or more spans are loaded in the sequence that produces the most adverse bending moments. The foregoing formulae and rules are applicable if the panel is unsupported along the two shorter edges. If, however, there are supports along these edges, although their effect may be ignored in computing the transverse bending moment due to uniformly-distributed load, the transverse and longitudinal bending moments produced by concentrated loads should be calculated as described in Volume II for slabs spanning in two directions.

**Longitudinal Bending of Slabs Designed to Span in One Direction.**

If a rectangular panel of slab is supported on four sides and the length of the panel exceeds twice the breadth, it is generally assumed that it spans entirely in one direction, that is that the load is transferred entirely to the supports along the longer sides of the panel. The restraining effect of the supports along the shorter sides results in the deformation shown in Fig. 51, and the consequent longitudinal bending relieves the transverse bending by an indefinite amount; this relief is generally ignored, but the longitudinal bending must not be ignored. The tensile forces produced by the longitudinal bending moments are greatest over the supports along the shorter sides and on the underside of the slab where the curvature is sharpest, and are resisted by reinforcement provided near the top of the slab over the supports, and by other reinforcement, called distribution
reinforcement, in the bottom of the slab at right-angles to the main transverse reinforcement.

**Distribution Reinforcement.**—For long narrow panels it is not possible to calculate the amount of distribution reinforcement required, and arbitrary amounts are recommended or specified in various codes and regulations. In the floor and roof slabs of buildings the cross-sectional area recommended indirectly in the British Standard Code is 0.15 per cent. of the cross-sectional area of the concrete. This amount is reasonable for ordinary slabs subjected to uniformly-distributed loads, but much more should be provided if the load is concentrated on small areas, such as is the case of wheels on bridge decks or garage floors, or machines on floors of industrial buildings, or if the slab is subjected to temperatures higher than normal. In slabs forming the deck of a bridge the Ministry of

![Fig. 51.—Deformation of Slab Designed to Span in One Direction.](image)

Transport requires the distribution reinforcement to be the following percentages of the main transverse reinforcement: 40 per cent. for spans of 4 ft., 50 per cent. for 6 ft., 55 per cent. for 8 ft., and 60 per cent. for 10 ft., with a maximum amount of 1/2 sq. in. per foot width of slab. The British Standard Code for structures containing liquids recommends 0.3 per cent. of the gross cross-sectional area of the concrete.

**Reinforcement over Supports along Shorter Edges.**—Current codes give little guidance to the amount of reinforcement over the supports along the shorter edges of a rectangular panel, but the D.S.I.R. Code of 1934 recommended a minimum of 0.3 per cent. of the gross cross-sectional area of the concrete slab and it is reasonable to provide this amount. If the ratio of the longer side to the shorter side of the panel of slab is between 2 and 3, the amount of longitudinal reinforcement can be calculated by considering the slab to span in two directions as described in Volume II, but the amounts of longitudinal reinforcement should be not less than the minima given in the foregoing.

**Shearing Stresses in Solid Slabs.**

The shearing stresses in a solid slab carrying a uniformly-distributed load are generally negligible, but this may not be the case if the span of the slab is short, or if the slab is subjected to a great intensity of load
or to a heavy concentrated load. Consider a solid slab to resist a bending moment of \( \frac{w_{s}l}{2} \) with stresses of 1000 lb. per square inch in the concrete and 20,000 lb. per square inch in the reinforcement. The effective depth \( d_1 \) required is \( \sqrt{\frac{w_{s}l}{2} \times \frac{12}{12 \times 184 \times 12}} \), that is 0.0212l/\sqrt{w_{s}}. \) If the permissible shearing stress without reinforcement to resist the shearing force is 100 lb. per square inch, the limiting shearing force is 0.86d_1 \times 12 \times 100 \text{ lb.}, that is 21.96d_1 \sqrt{w_{s}} \text{ lb. per foot width of slab.} \) The shearing force is \( \frac{1}{2}w_{s}l \) lb. per foot. Equating these two forces, the greatest intensity of load for which the shearing stress does not exceed 100 lb. per square inch is 1925 lb. per square foot. Such an intensity of load is exceptional in buildings, and even in warehouses; since this load represents a head of water of about 37 ft., it is not unlikely in containers of liquids and other storage structures.

**Arrangement of Reinforcement.**

The most common arrangement of transverse and longitudinal bars in a solid slab designed to span in one direction and continuous over a number of supports is as shown in the example in Fig. 52. Alternate transverse bars near the bottom of the slab in the span are bent up to near the top over the supports and are augmented as necessary by short lengths of straight bars to provide resistance to the greater bending moment over the supports. At an end support, where the slab is nominally freely supported, alternate bars are bent up to provide resistance to a nominal negative bending moment; alternatively, one arm of the binders in the edge-beam can be arranged to project into the slab as shown in Fig. 53. The other arrangements of transverse bars shown in Fig. 53 require more steel but are less costly in labour since all the bars are straight. The arrangement at (a) is an optional alternative to that in Fig. 52, but that at (b) is essential if the live load exceeds twice the dead load in order to

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**Fig. 52.—Solid Slab (Example No. 68).**
provide resistance to possible negative bending moments throughout the slab.

**Minimum Reinforcement.**—The British Standard codes recommend that the minimum reinforcement transversely or longitudinally should be not less than 0.15 per cent. of the cross-sectional area of the concrete of slabs in buildings and 0.3 per cent. of slabs in structures containing liquids.

![Diagram](image)

*Fig. 53.—Slab Reinforcement.*

The codes also recommend that the spacing of the main transverse bars should not exceed three times the effective depth of the slab. The spacing of the longitudinal distribution bars should not exceed five times the effective depth, but generally a smaller spacing is more practicable. No recommendation is given for the spacing of the bars near the top of the slab at right-angles to the supports along the shorter sides, but a maximum spacing of three times the effective depth is reasonable.

**Designs of Continuous Slabs Spanning in One Direction.**

The examples which follow show the procedure for the design of a slab spanning in one direction, continuous over a number of supports, and supporting a uniformly-distributed load, and give effect to the recommendations in the foregoing. For comparative purposes the designs are based on the modular-ratio and load-factor methods and incorporate mild-steel and high-yield-stress reinforcement. In practice the calculations would be much shorter than given in the examples since the explanatory matter would be omitted.

**Example No. 67.—Modular-ratio Method; Mild-steel Bars.** Design a slab forming part of a warehouse floor, the imposed load on which is 150 lb. per square foot. The slabs are in panels 10 ft. wide and 30 ft. long similar to those in Fig. 41 in Chapter VIII. The permissible stresses are 1000 lb. per square inch in 1 : 2 : 4 concrete of ordinary quality, 20,000 lb. per square inch in mild-steel bars, and 30,000 lb. per square inch in high-yield-stress reinforcement in the form of welded meshes of cold-drawn wire.
The least thickness for a continuous slab spanning in one direction over spans of 10 ft. each is \( \frac{10 \text{ ft.} \times 12}{35} = 3.43 \text{ in.} \); therefore any slab not thinner than 3\( \frac{1}{2} \) in. will be satisfactory from the point of view of deflection but, as is seen in the following, a greater thickness is required in all cases to resist the bending moments. The dimensions are as in Fig. 52 for a design in accordance with the modular-ratio method and reinforced with mild-steel bars.

**Load.** Dead: Slab (assumed 4\( \frac{1}{2} \) in.) = 57 lb. per square foot

Ceiling and floor finishes, say, = 18 " " " 

Total dead load = 75 " " " " 

Live: Imposed load = 150 " " " " 

Total load = 225 " " " " 

**Bending moments.**—Apply the coefficients in Fig. 49a. Span \( l_x = 10 \text{ ft} \)

Interior spans (midspan): + \( \frac{150}{12} + \frac{75}{24} \times 12 = +18,750 \) in.-lb. per foot.

Interior supports: - \( \frac{150}{9} + \frac{75}{12} \times 12 = -27,500 \) " " " "

Penultimate supports: - \( \frac{150}{10} + \frac{75}{12} \times 12 = -29,000 \) " " " "

End spans (midspan): + \( \frac{150}{10} + \frac{75}{12} \times 12 = +25,500 \) " " " "

End supports: - \( \frac{225}{24} \times 10^2 \times 12 = -11,250 \) " " " "

**Thickness.**—Substituting in formula (3.6a) with \( Q_e = 184 \) (from Example No. 12 in Chapter III) and the greatest bending moment, that is the bending moment at a penultimate support, \( d_1 = \sqrt{\frac{29,000}{184 \times 12}} = 3.63 \text{ in.} \); say, 4\( \frac{1}{2} \)-in. slab, as assumed in the calculation of the load. The actual effective depth, with \( \frac{1}{2} \)-in. bars and \( \frac{1}{2} \)-in. cover of concrete, is 3.75 in., and the lever arm is therefore 0.86 \( \times 3.75 = 3.23 \) in.

**Reinforcement.**—Substitution in formula (3.6b) in Chapter III gives the areas in the following.

Interior spans (midspan): \( \frac{18,750}{20,000 \times 3.23} = 0.290 \text{ sq. in. per foot} \)

Interior supports: \( \frac{27,500}{20,000 \times 3.23} = 0.426 \) " " " "

Penultimate supports: \( \frac{29,000}{20,000 \times 3.23} = 0.450 \) " " " "

End spans (midspan): \( \frac{25,500}{20,000 \times 3.23} = 0.395 \) " " " "

End supports: \( \frac{11,250}{20,000 \times 3.23} = 0.174 \) " " " "

Since there are more interior spans than end spans, the requirements of the midspan of the interior spans should control the arrangement of
the reinforcement. A large variety of bars of different sizes and spacing can be chosen to provide 0.29 sq. in. per foot width of slab, such as (i) \( \frac{1}{8} \)-in. bars at 3-in. centres, (ii) \( \frac{3}{8} \)-in. bars at 4-in. centres, and (iii) \( \frac{5}{8} \)-in. bars at 8-in. centres, beyond which the recommended maximum spacing of \( 3 \times 3.75 = 11\frac{1}{2} \) in. is exceeded. The objection to (i) and (ii) is the numerous small bars which require more labour for bending and fixing as well as being more costly per ton than larger bars. Therefore the choice is \( \frac{1}{4} \)-in. bars at 8-in. centres near the interior spans; if alternate bars are bent up at each end of each span as shown in Fig. 52, the cross-sectional area provided over each interior support is 0.295 sq. in.; therefore there is a deficiency of \( 0.426 - 0.295 = 0.131 \) sq. in., which can be made up by providing additional short lengths of \( \frac{1}{4} \)-in. bars at 16-in. centres in the top. At the middle of the end spans the cross-sectional area of 0.395 sq. in. can be provided by \( \frac{1}{4} \)-in. bars at 6-in. centres. Therefore over the penultimate supports there are \( \frac{1}{4} \)-in. bars at 12-in. centres bent up from the end spans and at 16-in. centres bent up from the first interior span, providing a cross-sectional area of \( 0.196 + 0.148 = 0.344 \) sq. in.; there is therefore a deficiency of \( 0.450 - 0.344 = 0.106 \) sq. in., which for convenience can also be provided by short lengths of \( \frac{1}{4} \)-in. bars at 16-in. centres near the top. Bending up alternate bars at the end support provides \( \frac{1}{4} \)-in. bars at 12-in. centres which are ample for the nominal requirements at this position.

The longitudinal reinforcement should be not less than 0.15 per cent. of \( (4\frac{1}{4} \times 12) = 0.0015 \times 54 \) sq. in., that is 0.081 sq. in. per foot, which is provided by \( \frac{1}{4} \)-in. bars at 11-in. centres. The reinforcement near the top of the slab over the main beams should be about 0.3 per cent. of 54 sq. in., that is 0.162 sq. in. per foot, which is provided by \( \frac{3}{4} \)-in. bars at 53-in. centres. The transverse and longitudinal bars are arranged as in Fig. 52. There is no need to provide reinforcement to resist negative bending moments in the span because, in this example, the live load does not exceed twice the dead load.

**Example No. 68.—Load-factor Method; Mild-steel Bars.** Design the slab in Example No. 67 in accordance with the load-factor method incorporating mild-steel reinforcement.

Assuming a 4-in. slab the total dead load is 68 lb. per square foot and the maximum bending moment, that is the negative bending moment at the penultimate supports, is \( \frac{150}{9} + \frac{68}{10} \) \( 10^2 \times 12 = 28,200 \) in.-lb. per foot. From formula (3.7a) in Chapter III,

\[
d_1 = 2 \sqrt{\frac{28,200}{1000 \times 12}} = 3.06 \text{ in.};
\]

therefore a 4-in. slab is satisfactory, and the actual effective depth is 3.25 in. The area of reinforcement in tension required is obtained from formula (3.7a), that is

\[
A_{st} = \frac{2 \times 12 \times 1000}{3 \times 20,000} \left[ 3.25 - \sqrt{(3.25)^2 - \frac{3 \times 28,200}{12 \times 1000}} \right]
\]

\[= 0.552 \text{ sq. in. per foot}.\]

The bending moments at other critical sections are calculated as in Example No. 67 (but with \( w_D = 68 \) lb. per square foot) and the correspond-
ing cross-sectional areas of reinforcement required and provided are as follows.

<table>
<thead>
<tr>
<th>Bending moment (in.-lb. per foot)</th>
<th>$A_{st}$ required (sq. in. per foot)</th>
<th>Reinforcement provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior spans (midspan): $+18,400$</td>
<td>$0.332$ bottom</td>
<td>$\frac{1}{4}$-in. bars at 7-in. centres.</td>
</tr>
<tr>
<td>Interior supports: $-26,800$</td>
<td>$0.516$ top</td>
<td>Ditto plus $\frac{1}{4}$-in. bars at 12-in. centres.</td>
</tr>
<tr>
<td>Penultimate supports: $-28,200$</td>
<td>$0.552$ top</td>
<td>$\frac{1}{4}$-in. bars at 14-in. and 10-in. centres plus $\frac{1}{4}$-in. bars at 16-in. centres.</td>
</tr>
<tr>
<td>End spans (midspan): $+24,800$</td>
<td>$0.468$ bottom</td>
<td>$\frac{1}{4}$-in. bars at 5-in. centres.</td>
</tr>
<tr>
<td>End supports: $-10,900$</td>
<td>$0.180$ top</td>
<td>$\frac{1}{4}$-in. bars at 10-in. centres.</td>
</tr>
</tbody>
</table>

Longitudinal reinforcement: $0.15$ per cent. of $(4 \times 12)$

$$= 0.0015 \times 48 \text{ sq. in.} = 0.072 \text{ sq. in. per foot}.$$ Provide $\frac{1}{8}$-in. bars at 12-in. centres.

Reinforcement over main beams: $0.3$ per cent. of $48 \text{ sq. in.} = 0.144 \text{ sq. in. per foot.}$ Provide $\frac{1}{8}$-in. bars at 6-in. centres.

The reinforcement should be arranged as in Fig. 52, due notice being taken of the different spacing of the bars. Comparison with Example No. 67 shows that a $4$-in. slab is satisfactory (in place of a $4\frac{1}{4}$-in. slab and that, although slightly less longitudinal reinforcement is required, more transverse reinforcement must be provided.

**Example No. 69.—Modular-ratio Method; High-yield-stress Reinforcement.** Design the slab in Example No. 67 in accordance with the modular-ratio method incorporating high-yield-stress reinforcement in the form of a welded mesh of cold-drawn wire. With working stresses of 1000 lb. per square inch in the concrete and 30,000 lb. per square inch in the reinforcement, the resistance-moment factor is $148 \text{ lb. per square inch}$ (if calculated similarly to Example No. 12 in Chapter III). If a $5$-in. slab is provided the total dead load is $81 \text{ lb. per square foot and the greatest bending moment is}$

$$-\left(\frac{150}{9} + \frac{81}{10}\right) 10^2 \times 12 = 29,700 \text{ in.-lb. per foot.}$$

The effective depth required is $d_1 = \sqrt{\frac{29,700}{12 \times 148}} = 4.09 \text{ in.}$ Therefore a $5$-in. slab is satisfactory; the actual effective depth is $5 - \frac{1}{4}$, say, $0.15 = 4.35 \text{ in.}$; the lever arm is $0.89 \times 4.35 = 3.88 \text{ in.}$ The critical bending moments and areas of the reinforcement, which are calculated as in Example No. 67, are shown at top of page 166.

The meshes would be supplied in flat sheets and would be arranged as in Fig. 54. If the cross-sectional area required did not exceed about $0.2 \text{ sq. in. per foot at any position, the material might be supplied in rolls and laid so as to wave up over the supports, with a consequent economy in steel.}$

Longitudinal reinforcement: Cross-sectional area required $= 0.15$ per cent. of $(12 \times 5) = 0.0015 \times 60 \text{ sq. in.} = 0.090 \text{ sq. in. per foot.}$ The cross wires, at right-angles to the main wires, in B.S. mesh No. 108 have
### Table: Bending Moment and Reinforcement Requirements

<table>
<thead>
<tr>
<th>Interior spans (midspan):</th>
<th>Bending moment (in.-lb. per foot)</th>
<th>$A_{st}$ required (sq. in. per foot)</th>
<th>Reinforcement provided (sq. in. per foot)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+19,050</td>
<td>0.164</td>
<td>B.S. mesh No. 108 (0.169)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>bottom</td>
<td></td>
</tr>
<tr>
<td>Interior supports:</td>
<td>-28,200</td>
<td>0.242</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>top</td>
<td></td>
</tr>
<tr>
<td>Penultimate supports:</td>
<td>-29,700</td>
<td>0.256</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>top</td>
<td></td>
</tr>
<tr>
<td>End spans (midspan):</td>
<td>+26,100</td>
<td>0.224</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>bottom</td>
<td></td>
</tr>
<tr>
<td>End supports:</td>
<td>-11,550</td>
<td>0.149 (at 20,000 lb. per square inch)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>top</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{4}^-\text{in. binders at 6-in centres projecting from edge-beam.}
\]

A cross-sectional area of 0.016 sq. in. per foot; the remaining 0.074 sq. in. per foot required in interior spans is provided by $\frac{1}{4}$-in. mild-steel bars at 12-in. centres near the bottom at midspan, and similar bars should be provided throughout. The cross-sectional area of the reinforcement over the main beams should be 0.3 per cent. of 60 sq. in., that is, 0.180 sq. in., which can be provided by a sheet of B.S. mesh No. 107 or, more economically, by $\frac{1}{4}$-in. mild-steel bars at 5-in. centres.

**Example No. 70.—Load-factor Method; High-yield-stress Reinforcement.** Design the slab in Example No. 67 in accordance with the load-factor method incorporating high-yield-stress reinforcement in the form of a welded mesh of cold-drawn wire. The thickness of the slab is 4 in. as in Example No. 68 and the bending moments are also the same. The cross-sectional areas of the reinforcement are different because the permissible stress is 30,000 lb. per square inch, and the effective depth is greater, say, 3.35 in. because the size of the wires in the mesh is smaller than that of the $\frac{1}{4}$-in. bars in Example No. 68. The data are as follows.

<table>
<thead>
<tr>
<th>Bending moment (in.-lb. per foot)</th>
<th>$A_{st}$ required (sq. in. per foot)</th>
<th>Reinforcement provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior spans (midspan):</td>
<td>+18,400</td>
<td>0.207</td>
</tr>
<tr>
<td></td>
<td>bottom</td>
<td>B.S. mesh No. 106</td>
</tr>
<tr>
<td>Interior supports:</td>
<td>-26,800</td>
<td>0.327</td>
</tr>
<tr>
<td></td>
<td>top</td>
<td></td>
</tr>
<tr>
<td>Penultimate supports:</td>
<td>-28,200</td>
<td>0.350</td>
</tr>
<tr>
<td></td>
<td>top</td>
<td></td>
</tr>
<tr>
<td>End spans (midspan):</td>
<td>+24,800</td>
<td>0.296</td>
</tr>
<tr>
<td></td>
<td>bottom</td>
<td></td>
</tr>
<tr>
<td>End supports:</td>
<td>-10,900</td>
<td>0.174 (at 20,000 lb. per square inch)</td>
</tr>
<tr>
<td></td>
<td>top</td>
<td>$\frac{1}{4}$-in. binders at 7\frac{1}{2}-in. centres projecting from edge-beams.</td>
</tr>
</tbody>
</table>

Longitudinal reinforcement: Cross-sectional area = 0.15 per cent. of 48 sq. in. = 0.072 sq. in.; area of cross wires in B.S. mesh No. 106 is 0.018 sq. in. The remaining 0.054 sq. in. required in the interior spans is provided by $\frac{1}{8}$-in. bars at 18-in. centres. Reinforcement over the main beams is $\frac{1}{8}$-in. bars at 6-in. centres as in Example No. 68.

The reinforcement is arranged similarly to that in Fig. 54, allowance being made for the different meshes and longitudinal bars. Comparison
with Example No. 69 shows that a 4-in. slab is sufficient in place of a 5-in. slab, but the transverse reinforcement is much heavier.

**Example No. 71.—Load-factor Method; High-yield-stress Reinforcement; Thickness of Slab Specified.** Design the interior spans of the slab in the preceding examples in accordance with the load-factor method incorporating high-yield-stress reinforcement in the form of a welded mesh of cold-drawn wire but with the thickness of the slab

![Diagram of Slab Reinforced with Cold-drawn Wire Mesh](image)

Fig. 54.—Slab Reinforced with Cold-drawn Wire Mesh (Example No. 69).

4\(\frac{1}{2}\) in. as required in Example No. 67 for the modular-ratio method and mild-steel bars. The bending moments are therefore the same as in Example No. 67 but the areas of the reinforcement differ because the permissible stress is 30,000 lb. per square inch and the effective depth is about 3.85 in. The data are as follow.

<table>
<thead>
<tr>
<th>Bending moment (in.-lb. per foot)</th>
<th>(A_{st}) required (sq. in. per foot)</th>
<th>Reinforcement provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior spans (midspan):</td>
<td>+18,750</td>
<td>0.183</td>
</tr>
<tr>
<td>Interior supports:</td>
<td>-27,500</td>
<td>0.268</td>
</tr>
<tr>
<td></td>
<td></td>
<td>&quot;          &quot;          No. 105</td>
</tr>
</tbody>
</table>

**Reinforcement**

Longitudinal reinforcement: Cross-sectional area = 0.15 per cent. of 54 sq. in. = 0.081 sq. in. per foot. Cross-sectional area of cross wires in B.S. mesh No. 107 is 0.015 sq. in.; the remaining 0.066 sq. in. is provided by \(\frac{1}{8}\)-in. bars at 15-in. centres. The reinforcement over the main beams is the same as in Example No. 67, namely, \(\frac{1}{8}\)-in. bars at \(\frac{1}{6}\)-in. centres. The reinforcement would be arranged similarly to that in Fig. 54 but allowing for the lighter meshes and different longitudinal reinforcement.

From the foregoing examples it is seen that several structurally sound designs are available for a solid slab required to support a specified load on a given span. Some designs are likely to result in less costly slabs than others. The thicker the slab the greater the lever arm, and consequently, to resist a given bending moment, less reinforcement is required. In some cases the saving in cost of reinforcement more than offsets the increase in cost due to the greater amount of concrete in the thicker slab. If the cost of the slabs in Examples Nos. 67 to 71 are determined at current (1962) prices of concrete and reinforcement, it will be found that the most economical slab is that for which the thickness is determined by the modular-ratio method (with either mild-steel or mesh reinforcement) and the reinforcement is determined by either the modular-ratio or load-factor method.
Cantilevered Solid Slabs.

In the foregoing consideration is given to slabs supported on two or more supports. In modern construction there are many cases of slabs cantilevering horizontally beyond or from a beam or other support, examples being canopies, balconies, and floors where the beams are set in away from the walls of the building. Parapets of bridges, roofs, balconies, terraces, and the like are examples of vertical cantilevered slabs. The design of such slabs is simple and differs from an ordinary slab only in the calculation of the bending moments and the arrangement of the reinforcement. Example No. 72, which follows, gives the essential factors.

**Ratio of Projection to Thickness.**—A cantilevered slab is much more flexible than a similar slab supported on two opposite sides; therefore the ratio of projection to thickness should not exceed twelve, which is much smaller than the greatest ratio of span to thickness recommended for a slab supported on two opposite sides. This thickness, which may be more or less than the thickness required to resist the bending moment, is the least thickness required at the root of the cantilevered slab, but the slab may be tapered to the outer edge so long as the thickness at any point is not less than one-twelfth of the distance from the point to the outer edge.

**Example No. 72.—Canopy.** Design a canopy (roof) projecting 10 ft. to support an imposed load of 30 lb. per square foot with permissible stresses of 1000 lb. and 20,000 lb. per square inch.

The least thickness is \( \frac{10 \text{ ft.} \times 12}{12} = 10 \text{ in.} \) at the root; taper the slab to 3 in. at the outer edge as in Fig. 55, that is the average thickness is \( 6\frac{1}{2} \text{ in.} \).

Load: 6\(\frac{1}{2}\)-in. (average) slab = 82 lb. per square foot.
Asphalt = 10 " " " "
Imposed load = 30 " " " "
Total load = 122 " " " "

Bending moment = \( \frac{1}{12} \times 122 \times 10^2 \times 12 = 73,200 \text{ in.-lb.} \)

Minimum \( d_1 = \sqrt{\frac{73,200}{184 \times 12}} = 5.75 \text{ in.} \), which is less than the effective depth of 10 - \( \frac{1}{2} \) = 9\(\frac{1}{2} \) in. corresponding to the minimum thickness of 10 in. at the root of the cantilever. The cover of concrete provided over the main reinforcement is \( \frac{1}{2} \) in. because the slab is protected by asphalt; otherwise the cover should be at least \( \frac{3}{4} \) in.

\( A_{st} = \frac{73,200}{20,000 \times \frac{3}{8} \times 9\frac{1}{4}} = 0.452 \text{ sq. in. per foot}, \)

which is provided by \( \frac{3}{8} \)-in. bars at 5-in. centres near the top of the slab at the root. Some bars may be curtailed as shown in Fig. 55, since the bending moment decreases rapidly as the distance from the support increases. The distribution bars should vary from 0.15 per cent. of
$10 \times 12 = 0.18$ sq. in. at the root to not less than $0.15$ per cent. of $3 \times 12 = 0.054$ sq. in. at the outer edge; $\frac{1}{8}$-in. bars are provided at 5-in. centres at one end, increasing uniformly to 10 in. (not greater than $5 \times$ say, $2\frac{1}{4}$ in. = 11$\frac{1}{4}$ in.) at the outer edge.

Fig. 55.—Cantilevered Slab (Example No. 72).

It is necessary for the beam or other support of a cantilevered slab to be capable of resisting the twisting induced by the cantilever action, or a counter-moment must be provided at the root of the cantilever in some other way, for example, by continuity with a slab on the other side of the beam.

Hollow-block Slabs.

In a solid slab designed for the condition that the tensile resistance of the concrete is neglected the concrete below the neutral plane is theoretically ineffective, although it serves the purposes of resisting shearing force, surrounding the reinforcement bars in tension, and connecting the bars to the compression zone. It is therefore advantageous from the point of view of reducing the amount of concrete, and thereby saving weight, to omit the ineffective concrete while retaining enough to serve the purposes described. This principle is the basis of ribbed-slab construction shown in Fig. 56. However, the shuttering of the cavities between the ribs of a slab cast in place (Fig. 56b) is costly. Therefore the structural shape required can be obtained by incorporating hollow blocks as permanent shuttering (Fig. 56c). If the blocks* are of burnt clay or lightweight concrete they are light in weight and reasonably cheap. This type of floor is common in office buildings and the like since, in addition to low weight, it has a relatively high degree of thermal and acoustic insulation, and, compared with a ribbed slab with open cavities, the soffit of the blocks provides a plane ceiling. The concrete ribs between the blocks contain the reinforcement in tension. A clay slip-tile is provided at the bottom of the rib to provide continuity of surface similar to that of the soffit of the block. The concrete slab over the blocks forms the compression flange of the slab. If the thickness of the flange is not less than the depth from the compressed edge to the neutral plane, the basis of the design of this type of slab is the same as for a solid slab. Except in the case of very thin slabs, the thickness of the flange is generally less than this depth and the hollow-block slab is designed as a series of tee-beams;

* See B.S. No. 1190: "Hollow Clay Building Blocks."
less complex calculations are sufficiently accurate if the proportions of
the ribs and flange are in accordance with established practice.

**Proportions.**—The width of the rib should be $2\frac{1}{4}$ in. or as much more
as is necessary to encase the reinforcement bars. The depth of the rib
below the flange should not exceed four times the width. The distance
between the centres of adjacent ribs should not exceed 3 ft. The thickness
of the flange should be not less than one-twelfth of the clear distance
between adjacent ribs and in general not less than 1$\frac{1}{8}$ in. If part of the
block is taken into account in assessing the resistance of the slab the least
thickness of the flange is 1$\frac{1}{4}$ in., but if the blocks are jointed together
with mortar the least thickness is 1 in. If the blocks are taken into

![Diagram of slab types and bending moments](image)

Fig. 56.—Ribbed Slabs.

account in calculating the resistance to bending of the composite slab,
their crushing strength should be not less than 2500 lb. per square inch
and it is general to assume that the elastic modulus of the material forming
the block is the same as that of the adjacent concrete. The thickness of
one vertical wall of each block may be taken into account in assessing
the resistance of a rib to shearing force.

**Bending Moments.**—The bending moments on a hollow-block slab
continuous over a number of supports may be calculated as for a solid
slab, but it is not always convenient to provide sufficient compressive
resistance to resist the negative bending moment at the supports. A
common method of design is to calculate the bending moment assuming
the slab to be freely supported and, to reduce the risk of cracking at the
supports, to provide near the top of the slab over the supports nominal
reinforcement equal in area to about one-quarter of the area of the reinforcement in tension at midspan. The bars comprising this reinforcement should extend at least one-tenth of the span on both sides of each support. The foregoing recommendations are taken into account in the example which follows.

**Example No. 73.—Hollow-block Slab.** Design a hollow-block slab (Fig. 56d) to support an imposed load of 150 lb. per square foot over a series of 10-ft. spans (similar to the solid slab in Example No. 67). The permissible stresses are 1000 lb. per square inch in the concrete and 20,000 lb. per square inch in the reinforcement.

Consider the slab to comprise 12-in. by 6-in. hollow clay blocks with a 1\(\frac{1}{2}\)-in. concrete slab over the blocks. If the width of each concrete rib is 2\(\frac{1}{2}\) in., the distance between the centres of adjacent ribs is 14\(\frac{1}{4}\) in.

Load on width of 14\(\frac{1}{4}\) in.: \[\text{lb. per foot}\]

- **1\(\frac{1}{4}\)-in. top slab:** \(\frac{1\cdot25}{12} \times \frac{14\cdot5}{12} \times 150 = 19\)
- **2\(\frac{1}{2}\)-in. rib:** \(\frac{2\cdot5}{12} \times \frac{6}{12} \times 150 = 16\)
- **12-in. by 6-in. block, say:** \(= 26\)
- **Ceiling and floor finish:** \(18 \times \frac{14\cdot5}{12} = 22\)

**Imposed load:** \(150 \times \frac{14\cdot5}{12} = 182\)

**Total:** \(= 265\)

The bending moment at midspan is \(\frac{265 \times 10^2 \times 12}{8} = 39,750 \text{ in.-lb. per } 14\frac{1}{4} \text{ in. width. If the top of the hollow block is assumed to be not less than } \frac{1}{4} \text{ in. thick and is taken into account in assessing the compressive resistance, the effective thickness of the flange is } 1\frac{1}{4} + \frac{1}{4} = 1\frac{3}{4} \text{ in. The thickness of the slab is } 7\frac{1}{4} \text{ in. If the slip-tile is } \frac{1}{4} \text{ in. thick and } \frac{1}{4} \text{ in. cover of concrete is provided below the reinforcement in tension, and if the bars are assumed to be } \frac{1}{4} \text{ in. diameter, the effective depth is } 7\frac{1}{4} - (\frac{1}{4} + \frac{1}{4} + \frac{1}{4}) = 6 \text{ in.}

The lever arm is about 6 - (1\(\frac{3}{4}\) × \(\frac{1}{4}\)) = 5\(\frac{1}{2}\) in.

\(A_{et} = \frac{39,750}{20,000 \times 5\frac{1}{4}} = 0.39 \text{ sq. in.,}

say, two \(\frac{1}{4}\)-in. bars side by side with \(\frac{1}{4}\)-in. cover at the side and \(\frac{1}{4}\)-in. space between the bars; this space should be not less than the diameter of the bar. At least half the reinforcement near the bottom of the rib should be carried through to the support. In this example, one bar extends in the bottom between the supports. The other bar is curtailed at one end and bent up at the other end as shown in Fig. 56d, thus providing reinforcement over the supports; the cross-sectional area provided near the top is 0.196 sq. in., which is more than the recommended minimum of \(\frac{1}{4} \times 0.39 = 0.1 \text{ sq. in.}

The maximum compressive stress, from formula (3.5b) on page 59, does not exceed \(\frac{2 \times 39.750}{14\cdot5 \times 1\frac{3}{4} \times 5\cdot42} = 518 \text{ lb. per square inch; therefore a}
thinner flange could be provided, but this would decrease the lever arm and would require more reinforcement and larger bars near the bottom, thereby increasing the cost.

The shearing force on one rib does not exceed \( 265 \times 10 \times \frac{1}{3} = 1325 \text{ lb.} \) The thickness of a rib plus one wall of a block is, say, \( 2\frac{1}{2} + \frac{1}{3} = 3 \text{ in.} \);

\[
g = \frac{1325}{5\frac{1}{2} \times 3} = 86 \text{ lb. per square inch};
\]

since this is less than 100 lb. per square inch, reinforcement to resist the shearing force is not required.
CHAPTER X

RECTANGULAR AND FLANGED BEAMS

The steps in the design of a reinforced concrete beam are (i) Establish the data, which include the loads, bending moments, shearing forces, permissible stresses, and resistance factors; (ii) Determine by calculation the dimensions and amounts of reinforcement required to resist the bending moments and shearing forces; (iii) Arrange the reinforcement and other details of the beam; and (iv) Check the completed design. It is important that a logical sequence of calculation be adopted to ensure that no factor of importance is overlooked and to avoid unnecessary revisions as the process of trial-and-error proceeds. The stages in accordance with the modular-ratio and load-factor methods of design are set out in general in the following and in detail in the accompanying examples. The emphasis on checking and the economy of the design should be noted, as the former reduces the risk of structural and arithmetical errors, and the latter ensures the production of a practical design.

Design Data.

Loads.—The purpose of a beam is generally to transfer the load on a structure to the main supporting members, such as the columns. The manner in which the load is imposed on the beams depends on the type of structure. In many industrial structures the loads may be concentrated directly on the beams, but in buildings and containers the loads and pressures are supported by or imposed on slabs or walls from which they are transferred to the beams. In some structures the load is supported by the beams both directly and indirectly, as in the case of the deck of a road bridge where the wheels may be on the slab at one time and directly over the beams at another. Consideration of dead and live loads in general is given in Chapter VIII. The dead load includes the weight of the beam. Often the size is not known until the design is complete; therefore a breadth and an overall depth are assumed at the outset.

Ratio of Span to Depth.—To avoid excessive deflection there is a limit to the ratio of the span to the overall depth which should not be exceeded; ratios recommended are twenty for a freely-supported beam, twenty-five for a continuous beam, and ten for a cantilevered beam.

Bending Moments and Shearing Forces.—The most adverse combination of dead load and live load must be considered in calculating the bending moments and shearing forces, and some general methods of doing
so are given in Chapter VIII. Suitable coefficients may be used for calculating the bending moments on secondary beams, but more elaborate methods, such as moment-distribution, should be used for main beams and other important members.

**Effective Span.**—The effective span on which the calculations of the bending moments and shearing forces are based is, in general, the distance between the centres of the supports, but in the case of wide supports it is generally recommended that the span may be assumed to be the clear distance between the faces of the support plus the effective depth of the beam.

**Permissible Stresses.**—The basic permissible stresses in the concrete and reinforcement under ordinary conditions are established as described in Chapter I, but these stresses must be decreased or may be increased if any of the special conditions described in that chapter are applicable.

**Narrow Beams.**—The compressive stress permissible in the concrete and in the reinforcement subjected to compression must also be reduced in narrow beams, that is in beams the length of which between lateral restraints exceeds thirty times the breadth of the compression flange. In such beams the compressive stress in the concrete should not exceed that given by

\[ \sigma_{cb} = \left( \frac{1}{3} - \frac{l}{40b} \right) \sigma_{cb}, \quad \text{... \text{(10.1)}} \]

in which \( \sigma_{cb} \) is the compressive stress otherwise permissible in a beam for which the ratio of length to breadth does not exceed 30. It is not good practice for the ratio of length to breadth to exceed 60. In all narrow beams to which formula (10.1) is applicable it is generally recommended that, whatever the depth, the greatest depth considered in calculating the resistance of the beam should be 8\( b \), and that the entire shearing force should be resisted by reinforcement irrespective of the calculated shearing stress. In general, rectangular beams only have to be considered as narrow beams, because in a flanged beam the width of the flange is either greater than one-thirtieth of its length or should be made this dimension so that the greatest useful compressive resistance is obtained from the concrete. The more complex condition of a narrow beam subjected to an axial thrust simultaneously with a bending moment, as may occur in a framed structure, is dealt with in B.S. Code No. 114.

**Width of Flange.**—The width of the flange of a tee-beam or an ell-beam which is assumed to be effective in contributing to the compressive resistance of the beam should not exceed the least of the dimensions in the following: the actual width of the flange; the distance between the centres of adjacent tee-beams if the slab forming the flange is common to two or more beams; the breadth of the rib plus twelve times the thickness of the flange of a tee-beam, or plus four times that of an ell-beam;
the breadth of the rib of an ell-beam plus half the clear distance between adjacent ribs.

**Resistance Factors.**—Having established the permissible compressive and tensile stresses, the factors for resistance to bending for these stresses should be calculated as described in Chapter III. If the modular-ratio method is used the neutral-plane factor $n_1$, the lever-arm factor $a_1$, and the moment-of-resistance factor $Q_e$ should be evaluated. The factors $n_1$ and $a_1$ are not required in designs in accordance with the load-factor method.

Having determined the design data set out in the foregoing, the design of a beam to resist a known bending moment and shearing force proceeds as described in the following, the formulae being as in Chapter III for resistance to bending, in Chapter VI for resistance to shearing force, and Chapter VII for bond. The symbols in these formulae are summarised on page 9.

**Procedure of Design by the Modular-ratio Method.**

The stages in the design of a continuous beam are given in the following. Simplifications are made in the design of a freely-supported beam as shown in Example No. 74.

Stage No. 1. **Dimensions and Reinforcement at Midspan.**—(a). Determine the effective depth $d_1$ and overall depth $d$ as follows: For a rectangular beam assume a breadth $b$ and calculate the minimum effective depth required from formula $(3.6a)$; $d$ is about $d_1 + 2$ in. If $d$ is restricted and is less than the dimension thus calculated, reinforcement in compression is required and $d_1$ is about $d - 2$ in. For a flanged beam, assume a value for $d$ (a reasonable value is not more than one inch of depth per foot of span); $d_1$ is about $d - 2$ in.; assume the breadth of the rib $b_r$; a reasonable proportion of $b_r$ to the depth of the rib $d - d_s$ is unity for office buildings to one-third for industrial structures.

(b). Determine the approximate lever arm $l_a$. For a rectangular beam without reinforcement in compression, $l_a$ is $a_1d_1$ in which $a_1$ is the factor corresponding to the permissible stresses. For a rectangular beam with reinforcement in compression, $l_a$ is between $l_{as} (= a_1d_1)$ and $l_{as}$ which is about $d_1 - 2$ in. For a flanged beam, $l_a$ is about $d_1 - \frac{1}{3}d_s$.

(c). Check the assumed breadth for resistance to shearing. The shearing stress $q$ is $\frac{Q}{l_0b}$ (or $l_0b_r$) and must not exceed the permissible shearing stress either with or without reinforcement; if the stress $q$ is excessive, increase $b$ for a rectangular beam or $b_r$ for a flanged beam; $b_r$ may be reduced if the shearing stress is small.

(d). Calculate the approximate cross-sectional area $A_{st}$ of the reinforcement in tension at midspan from $\frac{M}{l_0P_{st}}$, and select bars of suitable size
Fig. 57.—Continuous Tee-beam Example No. 75. (See page 182.)
and number. This reinforcement may have to be adjusted for a rectangular beam with reinforcement in compression.

(e). Check that these bars can be accommodated in one or more layers in a beam of the breadth adopted; if not, increase $b$ or $b_r$ or decrease the number of bars by using larger bars. In case of great congestion it may be preferable to decrease $A_{st}$ by increasing $d$ and therefore increasing $l_a$.

(f). With the main bars as now determined, check the effective depth and the lever arm. It may be necessary to increase the number of bars if $d_4$ has been reduced for any reason, for example, due to the bars being in two or more layers.

(g). With the actual effective depth, compare the bending moment at midspan with the moment of the compressive resistance of the concrete $M_{rc}$, which for a rectangular beam is $Q_{ob}d_4$, and for a flanged beam is calculated from formula (3.4d) or more approximately from formula (3.4f). If $M_{rc}$ for a rectangular beam is less than $M$, determine from formulae (3.6b) the amount of reinforcement in compression required at midspan. The maximum amount of reinforcement in compression should be 4 per cent. of the total cross-sectional area of a rectangular beam; any reinforcement in excess of this amount should be neglected in the calculation of the resistance. If $M_{rc}$ is insufficient for a tee-beam, increase $d - d_4$ rather than provide reinforcement in compression.

(h). The dimensions and the reinforcement in tension (and reinforcement in compression if required) at midspan are now established. The weight of the beam is therefore now known and should be compared with the weight assumed. If the actual total load differs by more than, say, 5 per cent. it is advisable to revise the calculations made so far. For important beams it is advisable to revise the calculations with a more exact value of $l_a$ as well as $w$, $d_4$, and $b$ (or $b_r$).

Stage No. 2. Shearing Resistance.—(a). Calculate the shearing resistance of the concrete, that is $ql_4b$ (or $ql_4b_r$). In those parts of the beam where the shearing force exceeds this resistance, reinforcement must be provided to resist the entire shearing force.

(b). Determine the minimum resistance of the binders. The maximum spacing $s$ is $l_a$, but near supports, where reinforcement in compression is generally required, the spacing must not exceed twelve times the diameter of the bars forming this reinforcement. Assume a suitable size for the binders; that is $\frac{1}{4}$ in., $\frac{3}{8}$ in., $\frac{1}{2}$ in., or $\frac{5}{8}$ in., depending on the size of the beam. Calculate the shearing resistance of the minimum amount of binding from formula (6.3), taking into account whether the binders are a single or multiple series, and compare this resistance with the shearing force. If the resistance is insufficient, reduce the spacing or increase the diameter, or, if the surplus shearing force is great, provide inclined bars to resist the shearing force in conjunction with the binders.

(c). Determine which bars from the reinforcement subjected to tension
at midspan are likely to be available to be bent up to act as inclined bars to resist shearing force near the supports and, if they are inclined at 45 deg., calculate the resistance of these bars from formula (6.2a) or (6.2c); if the bars are inclined at any other angle, use formula (6.2). Compare with the shearing force this resistance plus the resistance of the minimum amount of binders. If it is insufficient, reduce the spacing of the binders or provide additional inclined bars, generally by inserting bars near the top of the beam over the supports and bent down in appropriate places to the bottom of the beam.

(d). For important members superimpose the diagram of shearing resistance on the diagram of maximum shearing force. Adjust the reinforcement if necessary to ensure that the diagram of maximum shearing force is covered by the diagram of shearing resistance.

Stage No. 3. Resistance to Negative Bending Moment.—(a). Calculate from (3.6b) the amount of reinforcement in tension and compression required at the supports, and check that the amount of reinforcement in compression required does not exceed 4 per cent.

(b). Compare the amount of reinforcement in tension required with the amount provided by the bars bent up from the bottom, including any extra bars provided to resist shearing. Add bars near the top over the supports to make up any deficiency.

(c). Compare the amount of reinforcement in compression required at the support with the amount provided by the extension from midspan of the available reinforcement in tension. Make up any deficiency by additional bars, or preferably by extending more of the bars in tension; make good any reduction of shearing resistance by additional binders or by additional inclined bars.

(d). If necessary provide resistance to the negative bending moment at midspan [formula (8.3), Chapter VIII] by bars near the top of the beam. If bars are not required near the top to resist bending, provide nominal reinforcement near the top, say, two \( \frac{1}{2} \)-in. bars in small beams, two \( \frac{3}{4} \)-in. bars in beams of medium size, and two 1-in. bars in large beams. It is often advantageous to extend these bars over the supports as by so doing the additional tensile resistance provided may be sufficient without inserting the extra bars described in (b).

(e). In important beams superimpose the diagram of moment of resistance provided (negative and positive) on the diagram of the maximum positive and negative bending moments. Adjust the reinforcement if necessary to ensure that the former diagram covers the latter.

Stage No. 4. Final Checks.—Draw the beam and its reinforcement and check the following.

(a) Check all bond lengths from points of greatest stress and at positions where bars terminate, and ensure that overlaps are sufficient as described in Chapter VII. Check local bond stresses from formula (7.2).
(b). For economy, examine all principal bars to ensure that where practicable each bar provides resistance to two forces at least, such as
(i) Some bars resisting tension near the bottom at midspan of the beam are carried through to the supports to provide resistance to compression;
(ii) Other bars resisting tension near the bottom at midspan are bent up to provide shearing resistance and resistance to tension in the top over the supports; (iii) Extra bars provided to resist tension in the top over the supports are bent down to provide shearing resistance; (iv) Bars near the top at midspan are extended over the supports to resist tension.

(c). Determine the length of the bars and adjust their arrangement if any are too long, say, over 30 ft. in beams of normal size or 40 ft. in large beams.

(d). Examine the arrangement of the bars to ensure that (i) No bar interferes with another; (ii) Bars from intersecting beams, columns, or other members can pass those in the beam being designed; (iii) Spacer bars are provided where necessary to separate layers of bars; (iv) Bars can be readily placed in position; and (v) Sufficient cover of concrete and spaces between bars are provided.

(e). Calculate more or less accurately, as seems warranted by the care with which the operations in the preceding stages have been carried out, (i) The tensile stress in the reinforcement near the bottom at midspan and in the reinforcement over the supports; (ii) The compressive stress in the concrete at midspan; (iii) The amount of reinforcement in compression required at midspan and at the supports, and compare this with the amount provided; and (iv) The shearing resistance provided near the supports (and elsewhere if this is not the only critical plane) and compare with the shearing force. Compare the stresses calculated in (i) and (ii) with the permissible stresses.

Examples of Design by the Modular-ratio Method.

Two common examples of the many types of reinforced concrete beams are given in the following, namely, a freely-supported rectangular beam and a simple continuous tee-beam. The procedure described in the foregoing is followed in the examples with such modifications as are necessary to suit each example. A brief indication is given of the method of calculating the bending moments and shearing force. The basic permissible stresses are assumed to be 1000 lb. per square inch in compression in the concrete, 100 lb. per square inch in shearing in the concrete without reinforcement, 120 lb. and 180 lb. per square inch for the average and local bond stresses respectively, and 20,000 lb. per square inch in tension in mild-steel reinforcement. The design factors for these stresses are, as calculated in Chapter III, \( n_1 = 0.428; a_1 = 0.86; Q_e = 184 \) lb. per square
inch. In practice, the calculations would be abbreviated by the omission of the explanatory matter.

Example No. 74.—Freely-supported Rectangular Beam. Design the freely-supported rectangular beam in Fig. 58 to support a uniformly-distributed load of 1000 lb. per linear foot over a clear span of 15 ft.
Such a beam might be required to support a floor comprising precast slabs. Assume that the beam is 18 in. deep and 10 in. wide.

Load.—Imposed load
Beam weight:
18 in. × 10 in. × 150 lb. per cubic foot = 188 ""  ""
Total load:
w = 1188 ""  ""
Effective span = 15 ft. +, say, 9 in. = 15·75 ft.
Bending moment at midspan: M = \( \frac{1188 \times 15 \times 75^2 \times 12}{8} \) = 442,000 in.-lb.
Shearing force at supports: Q = 1188 × 15 × \( \frac{1}{3} \) = 8910 lb.
Effective depth: \( d_1 = \sqrt{\frac{442,000}{10 \times 184}} = 15·5 \) in.

The overall depth of 18 in. provided is therefore satisfactory, since \( d_1 = 18 \) in. — 1 in. (cover) — \( \frac{1}{3} \) in. (about half a bar-diameter) = 16\( \frac{2}{3} \) in., which is also more than one-twentieth of the span. Note also that the assumed breadth of 10 in. is not less than one-thirtieth of the span and therefore the beam does not have to be considered as a narrow member.

Lever arm: \( l_a = 0·86 \times 15·5 = 14·2 \) in.

Shearing stress: \( q = \frac{8910}{10 \times 14·2} = 63 \) lb. per square inch; therefore no reinforcement is required to resist shearing and nominal binders say, \( \frac{1}{4} \) in. at 12-in. spacing, are suitable.

Reinforcement in tension: \( A_{st} = \frac{442,000}{14·2 \times 20,000} = 1·55 \) sq. in., which is provided by three \( \frac{3}{8} \)-in. bars, two of which are provided throughout in the bottom and the remaining bar is bent up at the supports thereby providing, at little extra cost, considerable extra resistance to shearing which is of value should the beam be subjected to an accidental overload.

The compressive resistance is sufficient since \( d_1 \) is greater than the minimum effective depth required; therefore provide nominal reinforcement near the top, say, two \( \frac{1}{4} \)-in. bars.

Local bond stress. Perimeter of two \( \frac{3}{4} \)-in. bars = \( 2\pi \times 0·875 = 5·5 \) in.
\( s_{t1} = \frac{8910}{14·2 \times 5·5} = 114 \) lb. per square inch, which is less than the permissible stress.

The final checks described in stage (4) should be applied; it will be found that all the requirements are satisfied by the design in Fig. 58.

Example No. 75.—Continuous Tee-beam. Design the tee-beam in Fig. 57 (pages 176 and 177) on the assumption that it is the interior span of a series of beams continuous over a number of equal spans, such as the secondary beams of the floor in Fig. 41. The span of the beam is 30 ft. and adjacent beams are 10 ft. apart. The thickness of the slab forming the flange is 4\( \frac{1}{4} \) in. The slab carries an imposed load of 150 lb. per square foot. Assume that the beam is 12 in. wide and 24 in. deep below the slab; these dimensions are confirmed as the calculations proceed or, if not, they would be adjusted. The overall depth is \( 24 + 4\frac{1}{4} = 28\frac{1}{4} \) in.; the ratio of span to depth is \( \frac{30 \times 12}{28\frac{1}{4}} = 12·6 \), which is less than 25.
**RECTANGULAR AND FLANGED BEAMS**

**Design Data.**

Load.—Apply the simple loading formulae (8.2a) in Chapter VIII. The uniformly-distributed loads are

\[
\text{lb. per foot}
\]

Dead load: \(4\frac{3}{4}\)-in. slab (57) + finishes (say \(20\frac{1}{2}\)) = 77½ lb. \times \text{loft. } = 775
\]

Beam rib (as assumed) = \(\frac{24 \times 12}{144} \times 150 \text{ lb. } = 300
\]

Total dead load:

\(w_D = 1075\)

Live load: Imposed load, \(w_L = 150 \text{ lb. } \times \text{10 ft. } = 1500\)

Total load:

\(w = 2575\)

Bending moments.—Apply the coefficients for equal spans given on page 146.

At midspan: \(M = +\left(\frac{1075}{24} + \frac{1500}{12}\right) 30^2 \times 12 = +1,830,000 \text{ in.-lb.}\)

At supports: \(M = +\left(\frac{1075}{12} + \frac{1500}{9}\right) 30^2 \times 12 = -2,760,000\)

Shearing force: \(Q = 2575 \times 30 \times \frac{1}{2} = 38,625 \text{ lb.}\)

**Stage No. 1.**

Dimensions.—Midspan: \(b_r = 12 \text{ in. } ; d_t = 4\frac{3}{8} \text{ in. } ; d = 4\frac{3}{8} + 24 = 28\frac{3}{4} \text{ in. } ;\)

\(b = (12 \times 4\frac{3}{8}) + 12 = 66 \text{ in. (which does not exceed the spacing of 10 ft. nor one-third of 30 ft.) } ; d_1 = 28\frac{3}{4} \text{ say, } 2\frac{1}{6} \text{ in. } = 27 \text{ in. } ;\)

\(l_a = 27 - (\frac{1}{6} \times 4\frac{3}{8}) = 24\frac{1}{2} \text{ in.}\)

Supports: \(b = 12 \text{ in. } ; d_t = 28\frac{3}{4} \text{ say, } 2\frac{1}{2} \text{ in. } = 25\frac{3}{4} \text{ in.}\)

\(l_{ac} = 0.86 \times 25\frac{3}{4} = 22.2 \text{ in. } ; l_{as} = 25\frac{3}{4} \text{ say, } 2\frac{1}{2} = 23\frac{1}{2} \text{ in. } ;\)

\(l_a (\text{between } l_{ac} \text{ and } l_{as}) = \text{say, } 23\frac{3}{4} \text{ in.}\)

Maximum shearing stress: \(Q = \frac{38,625}{23\frac{3}{4} \times 12} = 136 \text{ lb. per square inch,}\)

which is less than \(4 \times 100 \text{ lb. } = 400 \text{ lb. per square inch, but is greater than } 100 \text{ lb. per square inch; therefore reinforcement to resist}\)

the entire shearing force is required adjacent to the supports: \(b = 12 \text{ in. is satisfactory.}\)

Reinforcement at midspan: \(A_{st} = \frac{1,830,000}{24\frac{1}{2} \times 20,000} = 3.69 \text{ sq. in. } ; \text{ say,}\)

five \(1\)-in. bars. If these bars are provided in one layer as in Fig. 57 with \(1\)-in. cover of concrete, the space between each of the layers of bars is \([12 \text{ in. } - 2 \text{ in. (side covers) } - 5 \text{ in. (bars)}] \div 4 \text{ (spaces) } = 1\frac{1}{4} \text{ in.}, which is not less than the diameter of the bars (1 in.) or the greatest size (\(1\frac{1}{4}\) in.) of the aggregate plus \(\frac{1}{4}\) in., which is also 1 in. Therefore \(b = 12\) in. is satisfactory. The actual effective depth and lever-arm at midspan are also 27 in. and \(24\frac{1}{2}\) in. respectively as assumed.

Compressive resistance at midspan: Approximately from formula (3.4f), \(M_{rc} = \frac{1}{4} \times 66 \times 4\frac{3}{4} \times 1000 (27 - 1\frac{1}{4}) = 3,780,000 \text{ in.-lb., which is more}\)

than sufficient. Therefore the dimensions 24 in. (net depth) and 12 in. (width), and the corresponding weight of the rib as assumed are satisfactory.

**Stage No. 2.**

Shearing resistance of concrete

\(= 100 \times 23\frac{3}{4} \times 12 = 28,400 \text{ lb.}\)

The distance from supports beyond which the concrete resists the entire shearing force is \(\frac{30}{2} - \frac{28,400}{2575} = 4 \text{ ft.}\)
Shearing force at the supports = 38,625 lb.
Resistance of one 1-in. bar (probably available from bottom) and inclined at 45 deg. in a single system is
\[ 0.71 \times 20,000 \times 0.785 = 11,100 \text{ lb}; \]
therefore in double system, the resistance is \[ 2 \times 11,100 \text{ lb.} = 22,200 \text{ lb.} \]
Shearing force to be resisted by binders
\[ V = \frac{16,425}{23.4} = 692; \] spacing of \( \frac{1}{8} \)-in. single binders is
\[ \frac{20,000 \times 0.153}{692} = 4.42 \text{ in.}, \]
say 4-in. centres.
Shearing force at 3 ft. 6 in. from support
\[ = 38,625 - (3.5 \times 2575) = 29,612 \text{ lb.} \]
Resistance provided:
One 1-in. bar inclined at 45 deg. (single system) = 11,100 lb.
\( \frac{1}{4} \)-in. single binders at 4-in. centres
\[ \frac{20,000 \times 0.153 \times 23.4}{4} = 18,200 \]
Total = 29,300
which is sufficiently equal to the resistance required.
Reinforcement to resist the shearing forces is provided as shown in Fig. 58 and, although they are not strictly necessary for a simple beam as in this example, the diagrams of shearing resistance and shearing forces are given.

Stage No. 3.—Bending resistance at supports: \( d_n = 0.428 \times 25.4 = 11 \text{ in.} \);
\( d_2 = 1 \text{ in. (cover)} + 1 \text{ in. (bar)} + \frac{1}{2} \text{ in. (half space)} = 2 \frac{1}{2} \text{ in.}; \)
\( l_{ae} = 25.4 - 2 \frac{1}{2} = 23 \frac{1}{2} \text{ (compared with } 23 \frac{1}{4} \text{ in. assumed).} \)
\( M_{ce} \) (concrete only) = 18.4 \times 12 \times (25.75)^2 = 1,460,000 in.-lb.
\( M_s = 2,760,000 - 1,460,000 = 1,300,000 \text{ in.-lb.} \)
\( A_{se} = \frac{1,300,000}{14 \times \frac{8.75}{11} \times 23.5 \times 1000} = 4.98 \text{ sq. in.}; \)
six 1-in. bars (4.71 sq. in.) are provided by overlapping near the bottom three bars (a) from each of the adjacent spans. All six bars are effective; therefore sufficient bond-length must be provided on each side of the centre-line of the supports. The compressive stress is
\[ 14 \times \frac{8.75}{11} \times 1000 = 11,000 \text{ lb. per square inch,} \]
for which the bond-length, according to formula (7.3), should be \( 18 \frac{1}{4} \times 1 \text{ in.} = 18 \frac{1}{4} \text{ in.}; \) the total overlap must therefore be at least twice this length. Actually an overlap of about 6 ft. is required to ensure that sufficient compressive resistance is provided beyond a distance of 18 \( \frac{1}{4} \) in. from the support. The proportion of reinforcement in compression is
\[ \frac{4.71}{12 \times 28.5} \times 100, \text{ that is about } 1 \frac{1}{10} \text{ per cent., which is less than } 4 \text{ per cent.,} \]
the maximum amount generally recommended. The two bars (b) and (c) in the bottom are available for bending up to provide reinforcement to resist the shearing force.
The amount of reinforcement in tension required at the supports is

$$A_{st} = \left( \frac{1,460,000}{22.2} + \frac{1,300,000}{23.5} \right) \frac{1}{20,000} = 6.06 \text{ sq. in.}$$

Therefore eight 3-in. bars (6.28 sq. in.) are required near the top of the beam over the supports and are provided by one bar (b) [the other bar (b) at each support is bent up too close to the support to be effective], two bars (c) [one bent-up from each span], and five additional 3-in. bars (d) and (e). Bars (d) and (e) are in one layer near the top of the beam. Bars (b) and (c) must therefore be in a second layer, with a space of 1 in. between the two layers to enable the bars in the intersecting main beam to pass between them. The hooks on bar (e) must be in a horizontal plane so as to be contained within the slab.

There is no need to provide resistance to negative bending moment at midspan; therefore the reinforcement at the top between the supports can be nominal, say, two 3-in. bars.

The bending-moment diagram and the diagram of the moment of resistance provided are plotted on Fig. 57 to ensure that the latter diagram covers the former. For a simple beam as in this example, this diagrammatic comparison is not essential but is given as a guide for more complex designs.

Stage No. 4.—The checks described in this stage should be applied to the design in Fig. 57. As the span is 30 ft., it is not practicable to avoid bars exceeding 30 ft. in length; therefore the longest bars, that is bars (c), are about 40 ft. long.

Procedure of Design by Load-factor Method.

The procedure in accordance with the load-factor method is similar in many respects to design in accordance with the modular-ratio method except that the load-factor formulae given in Chapter III are applied. The design data are determined as previously explained and the procedure is as given in the following.

Stage (1).—Dimensions and Reinforcement at Midspan. (a) Assume a breadth $b$ and calculate the minimum effective depth required from formula (3.7a) for a rectangular beam. Otherwise the remarks in Stage 1 (a) for the modular-ratio method are applicable.

(b). Determine the lever arm of the compressive resistance of the concrete that is $l_{ae}$, which is not less than $\frac{2}{3}d_1$ for a rectangular beam and is about $d_1 - \frac{1}{2}d_2$ for a flanged beam.

(c). Check the assumed breadth, by calculating the shear stress as in Stage 1(c) for the modular-ratio method.

(d), (e), and (f). Proceed as in the corresponding calculations for the modular-ratio method, but calculate the cross-sectional area of the reinforcement in tension required at midspan from formula (3.7a) for a rectangular beam with reinforcement in tension only, and from formula (3.9d) for a flanged beam. If reinforcement in compression is required proceed as in (g) to determine $A_{sc}$ and $A_{st}$. 

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(g). With the actual effective depth, compare the moment of the compressive resistance of the concrete with the bending moment at midspan; $M_{re}$ of a rectangular beam is calculated from formula (3.7) and for a flanged beam from formula (3.9b), or more approximately from formula (3.9c). If $M_{re}$ for a rectangular beam is less than $M$, determine the amount of reinforcement in compression from formula (3.8b) and the reinforcement in tension from formula (3.8a). Note the comments in Stage 1 (g) for the modular-ratio method.

Stage (2). Determine the reinforcement to resist the shearing forces as described in Stage 2 (a) to (d) for the modular-ratio method.

Stage (3). Calculate from formulae (3.8b) and (3.8a) respectively the amount of reinforcement in compression and tension required at the supports, and proceed as Stage 2 (a) to (e) for the modular-ratio method.

Stage (4). Carry out the checks described in Stage 4 (a) to (d) for the modular-ratio method. Compare the calculated moments of resistance at midspan and at the supports with the applied bending moments, using formulae (3.7), (3.8), (3.9a), and (3.9b), whichever is appropriate.

Examples of Design by the Load-factor Method.

In the examples which follow the foregoing procedure is applied to the same beams as are designed in Examples Nos. 74 and 75 by the modular-ratio method. Any variation of dimensions and reinforcement required by the load-factor method are given in brackets in Figs. 58 and 57 respectively.

Example No. 76.—Freely-supported Rectangular Beam. Design the freely-supported rectangular beam of 15 ft. span in Example No. 74 (Fig. 58) by the load-factor method. The design data are the same as in Example No. 74.

Effective depth: From formula (3.7a), $d_1 = 2 \sqrt[15]{\frac{442,000}{1000 \times 10}} = 13.3$ in.; an overall depth of 15 in. is satisfactory. The ratio of span to depth is $\frac{15 \times 12}{15} = 12$, which is also satisfactory.

$Actual \quad d = 15 \text{ in.} - 1 \text{ in. (cover)} - \frac{1}{2} \text{ in. (about half bar diameter)} = 13\frac{1}{2} \text{ in.}$

Minimum $l_s = \frac{2}{10} \times 13\frac{1}{2} = 10.1$ in.; $b = 10$ in.

$g = \frac{8010}{10 \times 10} = 88$ lb. per square inch; therefore no reinforcement is required to resist the shearing forces and the breadth of 10 in. as assumed is satisfactory. From formula (3.7a),

$A_{st} = \frac{2 \times 10 \times 1000}{3 \times 20,000} \left[ 13\frac{1}{2} - \sqrt{(13\frac{1}{2})^2 - \frac{3 \times 442,000}{10 \times 1000}} \right] = 2.16$ sq. in.,

which is provided by three 1-in. bars, two of which should extend in the bottom for the full length of the beam and the remaining bar should be
bent-up to resist the shearing force that may be produced by accidental overloading. This arrangement of reinforcement is shown in Fig. 58.

Local bond stress: \( \delta = 2 \times \pi \times 1 \text{ in.} = 6.28 \text{ in.} \);

\[
\delta_{b1} = \frac{8q_{10}}{10 \times 1 \times 6.28} = 140 \text{ lb. per square inch.}
\]

The comments in Example No. 74 apply to this example.

**Example No. 77.—Continuous Tee-beam.** Design the tee-beam, continuous over a number of 30-ft. spans, in Example No. 75 (Fig. 57) by the load-factor method. The design data are the same as in Example No. 75 and the same dimensions of the beam are also assumed.

**Stage No. 1.**—The dimensions at midspan and at the supports are as before except that the lever arm \( l_{ne} \) at the supports is \( \frac{3}{4} \times 25\frac{3}{4} = 19\frac{3}{4} \text{ in.} \); therefore \( l_{a} \) is about \( 21 \text{ in.} \); \( q_{\text{max.}} = \frac{38,725}{21 \times 12} = 154 \text{ lb. per square inch.} \)

Since the approximate lever arm for a tee-beam is the same whether the modular-ratio or load-factor method be used, reinforcement in tension required at midspan is five 1-in. bars arranged as before.

Compressive resistance at mid-span: From formula (3.9c),

\[
M_{rc(\text{approx.})} = \frac{1000 \times 4\frac{1}{2} \times 66}{3} [(2 \times 27) - 4\frac{1}{2}] = 4,900,000 \text{ in.-lb.}
\]

or, more accurately from formula (3.9b),

\[
M_{rc} = \left( \frac{12}{4 \times 66} + \frac{1}{3} \left( 1 - \frac{12}{66} \right) \left[ \frac{9}{27} - \left( \frac{4\frac{1}{2}}{27} \right)^2 \right] \right) 1000 \times 66 \times 27^2
\]

\[= 6,140,000 \text{ in.-lb.}, \text{ which is more than sufficient.} \]

**Stage No. 2.**—The method of determining the reinforcement required to resist shearing forces is the same as in Example No. 75 except that the lever arm near the supports is slightly shorter, but this makes little difference to the calculation.

**Stage No. 3.**—Resistance at supports:

\[
M_{rc} = \frac{1000}{4} \times 12 \times (25.75)^2 = 1,990,000 \text{ in.-lb.},
\]

From formula (3.8b), \( A_{se} = \frac{2,760,000 - 1,990,000}{18,000 \times 23\frac{1}{2}} = 1.82 \text{ sq. in.} \)

which is provided by three 1-in. bars; if three bars are carried through from each span as in Fig. 57, the overlap need be only the minimum of thirty diameters, that is 30 in., in accordance with formula (7.3) in Chapter VII, for a compressive stress of 18,000 lb. per square inch; this length compares with 37 in. (6 ft. provided) in Example No. 75. As in Example No. 75, bars (b) and (c) are available for bending up to resist the shearing force.

From formula (3.8a),

\[
A_{st} = \frac{1}{20,000} \left[ \left( \frac{1000}{3} \times 12 \times 25.75 \right) + (1.82 \times 18,000) \right] = 6.75 \text{ sq. in.}
\]

Note that in this expression the value of \( A_{se} \) is the calculated area (1.82 sq. in.) required and not the actual area (2.36 sq. in.) provided. If the reinforcement in tension (minimum effective area 6.28 sq. in.) over the supports is provided as shown in Fig. 57, there will be a small deficiency, but as the maximum bending moment as calculated occurs at the centre
of the support where the bar (b), which is neglected in Example No. 75, is partially effective, the deficiency is covered. Except for a slightly less overlap of the bars in compression at the supports, the arrangement of reinforcement in Fig. 57 is satisfactory in accordance with the load-factor method.
CHAPTER XI

COLUMNS AND LOAD-BEARING WALLS

Columns and load-bearing walls are similar structurally in that they transmit loads from an upper level vertically to supports or to a foundation at a lower level. The loads may be applied concentrically or eccentrically, and the column or wall may be subjected to bending due to interaction with beams or other members it supports. The calculation of the bending moments and forces produced by the loads on a framework of columns and beams is generally complex. Simple analyses of columns supporting vertical loads in buildings and columns supporting massive structures, towers, and the like and subjected to vertical and horizontal loads are dealt with in this chapter. Some general cases of framed structures and building frames subjected to horizontal loads are given in Chapter XVI. The formulae for, and methods of calculating, the stresses and resistances are given in Chapter II for concentric thrust and in Chapter IV for axial thrust combined with bending and are applied in the following to the design of typical columns and load-bearing walls. Several different designs are suitable for a column to resist a given load, with or without bending, but practical and economical considerations determine the most advantageous design. For convenience, the columns selected to illustrate methods of designing columns in a building frame are those in the typical building, a floor plan of which is given in Fig. 41 in Chapter VIII. A partial cross-section of such a building is shown in Fig. 59.

Fig. 59.—Part Cross-section of Typical Building Frame. (See also Fig. 41, page 134.)

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Concentrically-loaded Columns with Separate Binders.

Consider a typical interior column supporting a floor similar to that of Fig. 41 (in Chapter VIII) and Fig. 59. Since the spans of, and loads on, the beams on both sides of such a column are identical it is usual to ignore the secondary bending to which it may be subjected and to design it as a concentrically-loaded member. A column supporting a symmetrical area of floor is, however, subjected to bending when the live load is imposed on the beam on one side only of the column, but in this case the direct load on the column is less, and the permissible compressive stress for bending combined with direct thrust is greater, than the corresponding load and stress produced by concentric load only. The critical condition is generally when the column is subjected to the maximum concentric load. The difference in the two conditions is demonstrated in the examples which follow.

**Applied Load.**—The load on a column comprises the load transmitted from beams supported directly, the load from the column (if any) above the one being considered, and the weight of the column itself. Instead of considering the load from each beam separately, it is sometimes simpler to consider the load on the area of the part of the floor supported by the column, as shown by shading in Fig. 41 and described on page 136.

**Safe Load.**—The design of a column subjected to concentric load is commonly based on the load-factor method, as described on page 33, in accordance with which the safe load on a "short" column with independent binders is given by formula (2.3). By transposing this formula and including the reduction factor $R_L$ as in formula (2.5) in Chapter II, the gross cross-sectional area $A$ of a column required to carry a concentric load $P$ is

$$A = \frac{P}{\frac{P}{R_L} - A_{sc}(p_{sc} - p_{oo})}.$$  \hspace{1cm} (II.1)

**Main Reinforcement.**—The least amount of main reinforcement generally recommended is 0.8 per cent. of the gross cross-sectional area of the column, and the largest amount is 8 per cent. With the least amount ($A_{sc} = 0.008A$), formula (II.1) becomes

$$A = \frac{P}{(0.992p_{sc} + 0.008p_{oo})R_L}.$$  \hspace{1cm} (II.1a)

If a column of smaller cross-sectional area is provided, more than the minimum amount of reinforcement must be provided. If a column of larger cross-sectional area is provided, the amount of reinforcement need only be 0.8 per cent. of the calculated area $A$ as determined by formula (II.1a). The most costly component in a column is the reinforcement; therefore it is generally more economical to provide the smallest amount of steel although this may result in the use of more concrete.
Permissible Stresses.—The permissible compressive stresses $p_{sc}$ in the reinforcement and $p_{ce}$ in the concrete are as stated on pages 22 and 18, that is for a column of $1:2:4$ concrete of ordinary quality reinforced with mild-steel bars the permissible stresses are 760 lb. per square inch in direct compression in the concrete and 18,000 lb. per square inch in the reinforcement. Substitution in formula (II.1a) gives $A = \frac{P}{898RL}$ for these stresses. Similar numerical values for the denominator in this expression should be calculated for concrete of other qualities and for high-yield-stress reinforcement, as they are of practical value since by their use the size of a column to support a specified load can be most readily determined.

Vertical Bars.—If the size of a column is known the amount of reinforcement required may be calculated directly from formula (II.1) transposed thus:

$$A_{se} = \frac{P}{R_L} - \frac{p_{ce}A}{p_{sc} - p_{ce}}$$  

The main vertical bars should be not less than $\frac{1}{4}$ in. diameter and there should be not fewer than four in a square or rectangular column.

Fig. 60.—Details of Columns.

In square columns with sides greater than 18 in., six or eight bars should be provided. The bars should be disposed symmetrically at the corners or faces of the column as shown by the arrangement of four, six, and eight bars in Fig. 60. If it is likely that the column will be subjected to bending in one particular plane, it is advisable to arrange the bars at the two opposite faces at right-angles to this plane, so that all the
reinforcement is effective in resisting the bending. If there is a column above that being considered, the main bars are extended into the upper column in order to provide splice bars for the reinforcement in the column above, the length of the overlap being not less than 24 times the diameter of the bar in the upper column or a greater length if such is necessary to provide sufficient bond. For example, if the compressive stress in the bar is 18,000 lb. per square inch, as is common in a mild-steel bar, the minimum length according to formula (7.3) in Chapter VII is 30 times the diameter if the column is of 1:2:4 concrete of ordinary quality. Similarly the main bars in the column below extend into the column being considered to act as splice bars or, if there is no lower column, splice bars are embedded in and extend out of the foundation or other substructure. As shown in the example in Fig. 61, bars extending from one column into another of the same size are cranked inwardly so that they do not foul the bars in the upper column. If the upper column is smaller than the lower, the amount by which the bar is cranked must also be sufficient to allow for the reduction in size. Examples of bars cranked when passing from one column to another of the same size or smaller are shown in Fig. 61, in which also are shown suitable proportions for the crank. Bars projecting from a foundation into a column must also be set in from the corners of the column in order to clear the bars in the column. If the difference in size of the upper and lower columns is more than a few inches, the crank may extend so far down into the lower column that the main bars are ineffective; in such a case separate splice bars, as shown in the alternative arrangement at the head of the column in Fig. 61, should be provided. If a column is likely to be subjected to bending, the splice bars, whether they be an extension of the bars in the column below or separate bars, must be in the same plane as the main bars in the column so that they are as fully effective as the main bars in resisting bending.

**Binders.**—The arrangement of the binders around the main bars is governed by several considerations. Primarily the binders prevent the bars from buckling and therefore every bar must be secured to prevent movement outwards. Suitable arrangements of binders in columns with four, six, and eight bars are shown in Fig. 60. If the concrete in the column is to be consolidated by means of a mechanical vibrator inserted in the concrete, it is necessary to leave space in the core of the column for this tool, and therefore some of the alternative arrangements are obviously preferable to others in this respect. The spacing of the binders must also be considered from the point of view of resisting the tendency of the main bars to buckle, and should therefore not exceed twelve times the diameter of the bar secured. The spacing should also not exceed the width of the column and in no case should be greater than 12 in. The size of the bar from which the binder is made should also, for the same reason as in the foregoing, bear some relation to the size of the
Fig. 61.—Column with Separate Binders.
bar secured, and generally should be not less than one-quarter of the
diameter of the main bar, and in no case should be less than \( \frac{3}{8} \) in., except
in very small columns or struts where \( \frac{1}{4} \)-in. wire may be more convenient
than bent bars.

In the example which follows, the matters in the foregoing are con-
considered in relation to the design of a column with independent binders
and subjected to a concentric load.

Example No. 78.—Short Interior Column with Independent
Binders. Design interior column TS in Fig. 59 with independent binders
to carry a floor as in Fig. 41 (in Chapter VIII) if the imposed load on
the floor is 150 lb. per square foot. Assume that the load on the column
is concentric and that the column is in the third story from the top of
a building, each story being 12 ft. high.
The area of floor supported by an interior column is 30 \( \times \) 30 = 900 sq. ft.
The intensity of loading from the floor slab (as in Example No. 67 in
Chapter IX) is 225 lb. per square foot; allowing for the weight of the
beams, the total load is about 250 lb. per square foot. Therefore the
total load on the column is

Load from two floors = 2 \( \times \) 900 sq. ft. \( \times \) 250 lb. = 450,000 lb.
Weight of columns in three stories, say, = 10,000 lb.
Load from roof = 900 sq. ft. \( \times \) say, 100 lb. per sq. ft. = 90,000 lb.

Total concentric load = 550,000 lb.

Assuming that the column is a "short" column \( (R_L = 1) \) and the
permissible stresses are 760 lb. in \( 1:2:4 \) concrete and 18,000 lb. per square
inch in mild steel, \( A = \frac{550,000}{898} = 612 \) sq. in., which area is provided by
a column about 24\( \frac{1}{4} \) in. square if the minimum amount of reinforcement
is provided. A practical size is therefore 24 in. square with slightly more
than the minimum amount of reinforcement. The height of the column
is 12 ft. and the width 24 in.; \( \frac{l}{d} = \frac{12 \times 12}{24} = 6 \); therefore the column
is a "short" column as assumed. From formula \( (11.1b) \),

\[
A_{sc} = \frac{550,000 - (760 \times 24^2)}{18,000 - 760} = 6.51 \text{ sq. in.}
\]

which is provided by eight 1\( \frac{1}{4} \)-in. bars or four 1\( \frac{1}{2} \)-in. bars; as the column
exceeds 18 in. in width, eight bars are preferable, and, since the bending
moment is not likely to be large, the regular arrangement shown in Fig. 61a
and b is suitable. It is assumed that the column in the story below
is also 24 in. square (but of richer concrete than \( 1:2:4 \)) and is reinforced
with eight 1\( \frac{1}{2} \)-in. bars. The minimum length of bar to develop by bond
a compressive stress of 18,000 lb. per square inch is, according to formula
\( (7.3) \) in Chapter VII, \( \frac{18,000}{5 \times 120}D = 30D \). The minimum overlap of the
1\( \frac{1}{2} \)-in. bars in the column below with the 1\( \frac{1}{4} \)-in. bars in the column being
designed is therefore 30 \( \times \) 1\( \frac{1}{4} \) = say, 34 in. To develop a stress of
18,000 lb. per square inch in the 1\( \frac{1}{2} \)-in. bars, the least distance from the
soffit of the main beams to the upper end of the bar must be not less
than 30 \( \times \) 1\( \frac{1}{4} \) = 37\( \frac{1}{2} \) in. If the overall depth of the main beams is assumed
to be 3 ft. and the overlap above the floor is 2 ft. 10 in., the length of
5 ft. 10 in. provided is more than sufficient. It is assumed that the upper column is 27 in. square and is reinforced with eight 
\( \frac{3}{4} \) in. bars. If the \( \frac{1}{4} \) in. bars are cranked and extended into the upper column, the length of the overlap should be not less than \( 30 \times \frac{7}{4} = 27 \) in. The minimum bond-length for the \( \frac{1}{4} \) in. bars at this intersection is 34 in., but the length provided is 3 ft. + 2 ft. 3 in. = 5 ft. 3 in., which also is more than sufficient. It is seen that with deep beams the overlap with the bars in the upper column, in general, is the criterion of the bond resistance required. The cranks in the bars are arranged so that the part forming the splice lies alongside the main bars since in this position they are most effective in resisting any bending moment induced in the column by the main beams. The relative positions of the splice bars and main bars are shown in the cross-sections in Fig. 61b. The cranks in the \( \frac{1}{4} \) in. bars are sideways only and the offset should be not less than \( \frac{1}{8} \) in. + \( \frac{1}{4} \) in. overall, say, 2\( \frac{1}{2} \) in. The cranks in the \( \frac{1}{4} \) in. bars would be placed diagonally to allow for the reduced size of the upper column, and the offset is therefore \( \frac{1}{8} \) in. + \( \sqrt{(\frac{1}{2})^2 + (\frac{1}{4})^2} = \frac{4}{8} \) in. as shown in Fig. 61d. The length of this crank should be not less than \( 12 \times 4\frac{1}{2} \) in. = 4 ft. 1\( \frac{1}{4} \) in., that is the crank would commence below the soffit of the primary beams, which is undesirable; therefore the alternative of separate splice bars, \( \frac{1}{4} \) in. diameter and 2 x 2 ft. 3 in. = 4 ft. 6 in. long, should be adopted as shown in Fig. 61a.

The binders should be spaced at not more than the maximum distance of 12 in. since twelve times the diameter of the vertical bars is \( 13\frac{1}{8} \) in. The diameter of the binder should be not less than \( \frac{1}{8} \) x \( \frac{1}{4} \) in. = \( \frac{1}{4} \) in. Double binders are required to secure each of the eight bars. The position of the lowest and topmost binder as shown in Fig. 61a should be noted.

The vertical edges of columns are chamfered to prevent accidental damage to a sharp corner. If the column is in a building in which vehicles or goods are moved and are likely to hit the column, the vertical edges should be protected by steel guards fixed into the column as in Fig. 60.

The preceding example gives the method of designing a "short" column to carry a fairly heavy load. In the example which follows is given the design of a slender column subjected to a small load.

**Example No. 79.—Slender Interior Column with Independent Binders.** Design interior column QR in Fig. 59 in the top 15-ft. story of a building, to support a concentric load of 90,000 lb. from the roof.

The total load, including the weight of the column, is about 91,000 lb. With \( 1:2:4 \) concrete of ordinary quality and mild-steel bars, \( A \) need not exceed \( \frac{91,000}{898} = 101 \) sq. in. Therefore a column 10 in. square should be satisfactory. For such a column \( \frac{l}{d} = \frac{15 \times 12}{10} = 18 \), which exceeds 15; from formula (2.5b) in Chapter II, the reduction factor \( R_L \) for a slender column is 0.9. Substitution in formula (11.1b) gives

\[
A_{se} = \frac{\frac{91,000}{0.9} - (760 \times 10^3)}{18,000 - 760} = 1.46 \text{ sq. in.},
\]

which is provided by four \( \frac{3}{4} \)-in. bars. If the overall depth of the roof-beams supported by the column is 2 ft., there is just sufficient length of bar above the soffit of these beams to provide a bond length of \( 30 \times \frac{3}{4} = 22\frac{1}{2} \) in. at the upper end. The
diameter of the bar forming the binders must be not less than a quarter of \( \frac{1}{4} = \frac{1}{4} \) in., and the spacing of the binders must not exceed the smaller of 10 in. (the width of the column) or \( 12 \times \frac{3}{2} = 9 \) in.; therefore the binders should be \( \frac{3}{8} \) in. diameter at 9-in. centres.

Concentrically-loaded Columns with Helical Binding.

Formule and rules for the design of columns with helical binding and carrying a concentric load are given on page 35. In the example which follows, these data are applied to the design of an interior column as an alternative to the design in Example No. 78. The limiting amounts of vertical main bars are as given in the foregoing for a column with independent binders. The helical binding should comply with the requirements that the pitch of the helix should not exceed 3 in., nor be less than 1 in.; between these limits the pitch should not exceed one-sixth of the diameter of the core of the column and should be not less than three times the diameter of the main bars.

An important safeguard against excessive stress on the overall cross-section of the column is obtained by conforming to the recommendation that the load supported by the concrete in the core plus the load supported by the helical binding should not exceed the gross cross-sectional area of the column multiplied by half the minimum works-cube crushing strength of the concrete.

Example No. 80.—Octagonal Column with Helical Binding.

Design a column, less than \( 24 \) in. square and with helical binding, to support a total concentric load of \( 550,000 \) lb. as in Example No. 78.

Try a column of octagonal cross-section \( 21 \) in. across the flats, as in Fig. 62, with \( 1:1\frac{1}{4}:3 \) concrete of ordinary quality and mild-steel reinforcement. The permissible stresses are therefore \( f_{ce} = 950 \) lb. per square inch and \( f_{se} = 18,000 \) lb. per square inch. The diameter of the core \( D_k \) is \( 21 - (2 \times 1\text{-in. cover}) - (\text{diameter of bar forming helical binding}) = \text{say}, 18\frac{1}{2} \text{ in.} \). The slenderness ratio is \( \frac{12 \times 12}{18\frac{1}{2}} = 7.8; \) because this is less than 15 the column can be designed as a short column \( (R_L = 1) \). The cross-sectional area of the core is \( \frac{1}{4}\pi(18.5)^2 = 270 \) sq. in. The minimum amount of main reinforcement is 0.8 per cent. of this area, that is 2.16 sq. in.; provide eight \( \frac{3}{4} \)-in. bars \( (A_{sc} = 3.53 \text{ sq. in.}) \). Substitution in the first two terms of formula (2.4c) in Chapter II gives

\[
\text{Load carried by concrete} = 950 \times 270 = 256,500 \text{ lb.}
\]
\[
" \text{ " reinforcement} = 3.53(18,000 - 950) = 60,200 "
\]
\[
\frac{316,700 "}{270}
\]

Therefore the helical binding must increase the load-bearing capacity of the column by \( 550,000 - 316,700 = 233,300 \) lb. The amount of binding required can be determined from the third term in formula (2.4c) transposed to give

\[
A_B = \frac{P_B}{7060D_k} = \frac{233,300}{7060 \times 18.5} = 1.78 \text{ sq. in. per foot.}
\]
The pitch of the helix should be not greater than \( \frac{18.5}{6} = 3.1 \) in., say, 3 in., and should be not less than 1 in. Therefore \( \frac{1}{2} \)-in. helical binding at \( \frac{1}{4} \)-in. pitch \((A_{st} = 1.88 \text{ sq. in.)}\) is suitable. The arrangement of the reinforcement is shown in Fig. 62.

The small contribution by the main reinforcement to the safe load should be noted; it is therefore uneconomical to have much more than

![Diagram showing column with helical binding](image)

Fig. 62.—Column with Helical Binding.

the minimum amount of such reinforcement. The contribution by the binding is considerable and is comparable with that of the concrete. The last step in the calculation is to check that the loads supported by the concrete and binding, namely 256,500 + 233,300 = 489,800 lb., does not exceed the load based on half the crushing strength of the concrete (in this example, \( \frac{950}{0.76} \times 3 = 3750 \text{ lb. per square inch} \)) multiplied by the
gross cross-sectional area of the column, that is,
\[
\left( \frac{1}{4} \times 3750 \right) \times (0.828 \times 27^2) = 687,000 \text{ lb.}
\]

Comparison should be made of the costs of columns of the designs in Example No. 78 (with independent binders) and in this example.

**Bending of Columns due to Vertical Loads in Buildings.**

A beam which is constructed monolithically with a column is prevented from deforming as a freely-supported member, and therefore a restraining bending moment is applied to the beam and acts also on the column.

**Stiffness.**—The bending moments on a column depend on the relative stiffnesses of the beam and column and on the magnitude and disposition of the load on the beam. The stiffness of a structural member is defined by the bending moment which if applied at one end of the member produces unit rotation at that end, assuming that end to be supported and the other end to be fixed. For a column or beam of uniform moment of inertia the numerical value of the stiffness is obtained by dividing the moment of inertia \( I \) by the effective length \( l \). The effective length of a beam is its effective span. The effective length of a column depends on the degree of freedom or continuity at its ends and is discussed on page 3. If the stiffness factor is denoted by \( K \), the factor for a member \( AB \) is

\[
K_{AB} = \frac{I_{AB}}{I_{AB}}. \tag{II.4}
\]

**Exterior Columns.**—The bending moments on columns due to vertical loads on the floors and roof of multiple-story buildings can be calculated from formulæ such as those in the British Standard Code No. 114 which, although approximate, are sufficiently accurate for the purpose. The comprehensive formulæ (II.2c) in Fig. 63 and (II.3a) and (II.3b) in Fig. 64 for the bending moments on multiple-story frames are based on the formulæ in the B.S. Code. Formulæ (II.2a) and (II.2b) for a single-bay single-story frame with hinged or fixed bases are derived from the common elastic theory. Exterior columns, to which all the formulæ in Fig. 63 and formulæ (II.3a) in Fig. 64 apply, are affected more by the interaction of beams and columns than are interior columns, to which formulæ II.3b in Fig. 63 apply. The formulæ in Figs. 63 and 64 are expressed in terms of \( K_{AB}, K_{AC}, \) etc., for the various members.

The effect of a vertical load on a beam is expressed by the fixed-end bending moment \( M_e \), which is the bending moment produced by the load at the support of the beam on the assumption that the beam is rigidly fixed at both supports; that is the fixed-end bending moment is the area of the free-bending-moment diagram for a symmetrical load divided by the span of the beam. The condition of fixity is not common, since the ratio of the stiffness factors at the junction of a beam and column, for
FORMULÆ (11.2a)

\[ M_{AC} = M_{BD} = \frac{K_{AC}}{K_{AC} + \frac{1}{2}K_{AB}} M_e^{(AB)} \]

\[ H = \frac{M_{AC}}{V_{AC}} \]

\[ M_{CA} = M_{DB} = 0 \]

\[ M_e^{(AB)} = \text{FIXED-END B.M. FOR BEAM AB DUE TO W.} \]

FORMULÆ (11.2b)

\[ M_{AC} = M_{BD} = \frac{K_{AC}}{K_{AC} + \frac{1}{2}K_{AB}} M_e^{(AB)} \]

\[ H = \frac{M_{AC} + M_{CA}}{V_{AC}} \]

\[ M_{CA} = M_{DB} = \frac{K_{AC}}{K_{AC} + 2K_{AB}} M_e^{(AB)} \]

FORMULÆ (11.2c)

\[ M_{AC} = \frac{K_{AC}}{K_{AC} + \frac{1}{2}K_{AB}} M_e^{(AB)} \]

\[ M_{CA} = \frac{K_{AC}}{K_{AC} + K_{CE} + \frac{1}{2}K_{CD}} M_e^{(CD)} \]

\[ M_{CE} = \frac{K_{CE}}{K_{AC}} M_{CA} \]

\[ M_{CD} = M_{DC} = M_{CA} + M_{CE} \]

\[ M_e^{(AB)} \text{ or } M_e^{(CD)} = \text{FIXED-END B.M. DUE TO SYMMETRICAL LOAD ON BEAM AB OR CD.} \]

Fig. 63.—Bending Moments on Frames of One Bay.

For example, \( \frac{K_{AC}}{K_{AC} + K_{AB}} \) as in one of the formulæ (11.3a) in Fig. 64, is less than unity. Therefore the end-restraining bending moment is less than \( M_e \). In the uncommon case of the stiffness of the column (say, \( K_{AC} \)) being so very great compared with the stiffness of the beam (say, \( K_{AB} \)) that
$K_{AC}$ is practically equal to $K_{AC} + K_{AB}$, the restraining moment is practically equal to $M_{o}$. In the case of a beam in an intermediate story, the restraining moment is divided between the upper and lower columns in proportion to their stiffnesses.

**Interior Columns.**—The effect of frame action on interior columns is less than on exterior columns. In the case of a symmetrical building frame, bending of interior columns occurs only when the vertical load on

![Diagram showing bending moments on building frames](image)

**Fig. 64.—Bending Moments on Building Frames of Two or More Bays.**

the beam on one side of the column is different from that on the other side. The difference $M_{es}$ of the fixed-end bending moments is then the criterion of the bending of the columns, as is shown in formulae (11.3b) in Fig. 64. If the building frame is unsymmetrical the interior columns are subjected to bending under most conditions of vertical loading, and then $M_{es}$ is the greatest difference of the fixed-end bending moments on the beams meeting at the column.

**Resistance.**—The resistance of a column subjected to bending simul-
COLUMNS AND LOAD-BEARING WALLS

Simultaneously with vertical load may be calculated by the modular-ratio or load-factor methods as described in Chapter IV. Since the load-factor method is commonly used for the determination of the safe concentric load on columns in buildings, the same method is adopted for combined action in the examples which follow, the appropriate formulae being those on pages 89 to 91. As stated in Chapter I, the compressive stresses permissible in concrete under combined action are the same as those permissible in simple bending and are greater than the stresses permissible in direct compression. It is therefore necessary to ensure that a column subjected to bending is not overstressed if the load is considered to act alone as a concentric load as well as ensuring that the greater permissible stress is not exceeded under combined action.

Example No. 81.—Exterior Column in Intermediate Story.
Design exterior column GE in Fig. 59 to support a floor as in Fig. 41 in Chapter VIII if the imposed load on the floor is 150 lb. per square foot. Assume that the column is in the third story from the top of the building (Fig. 64), each intermediate story being 12 ft. high. (Compare with Example No. 78.)

The net area of floor supported by an exterior column is

\[ 30 \times \frac{30}{2} = 450 \text{ sq. ft.} \]

**Total load on column:**

- Load from two floors: \(2 \times 450 \times 250 \text{ lb.} = 225,000 \text{ lb.} \)
- Weight of columns and walls (glass) in three stories: say, \(12,000 \) "
- Load from roof: \(450 \times \text{say, 100 lb.} = 45,000 "

**Total load immediately above level of floor**

\[ = 282,000 " \]

**Permissible stresses:**

- Mild-steel bars: 20,000 lb. per square inch in tension.
  - 18,000 " " " compression.
- 1:2:4 concrete: 760 " " " direct compression.
  - 1000 " " " combined action.

Consider the column to be concentrically loaded (that is neglect bending at this stage). A must be not less than \(282,000 \div 898 = 314 \text{ sq. in.} \); therefore a column 18 in. square would be sufficient to resist the concentric load only. A larger column is therefore required to resist bending and axial load, or an excessive amount of reinforcement is required in an 18-in. column; select a column 24 in. square and ensure that it provides sufficient resistance. To resist axial load only, such as occurs at the plane of contra-flexure in the shaft of the column, the main reinforcement should be not less than 0.8 per cent. of 314 sq. in., that is 2.51 sq. in., which is provided by four 1-in. bars, which should extend throughout the column. Since the bending moment is greatest at the top and bottom of the column, additional bars should be provided near the outer face at the top of the column and, as shown in Fig. 65, should extend into the top of the beam. Similarly additional reinforcement should be provided near the inner face at the bottom of the column as shown by bars (g) at the bottom of column EG; or this extra reinforcement may be provided most conveniently by
extending the bars from the column below if the cross-sectional area of the latter bars is greater than that of the main bars in the column being designed; an example is bars (c) at the bottom of column GE.

**Stiffness factors.**—Neglect the reinforcement in calculating the moments of inertia.
Fig. 66.—Top Lift of Exterior Columns (Example No. 82).
Upper column GE and lower column JG: \( l_{GE} = l_{JG} = 12 \text{ ft.} \);
\( b = d = 24 \text{ in.}; \ K_{GE} = K_{JG} = \frac{24^4}{12 \times 12 \text{ ft.}} = 2304 \text{ in.}^4/\text{ft. units.} \)

Primary beam GH: Cross-section (tee-beam) as in Fig. 65.—The moment of inertia \( I_{GH} \) calculated from first principles or from tables or from formula (3.10e) in Chapter III, is about 90,000 in.\(^4\); \( l_{GH} = 30 \text{ ft.} \);
\[
K_{GH} = \frac{90,000}{30} = 3000 \text{ in.}^4/\text{ft. units.}
\]

**Fixed-end bending moment.**—The total load on the primary beam comprises two concentrated loads from the secondary beams, each load being about 30 ft. \( \times \) 10 ft. \( \times \) 250 lb. = 75,000 lb. plus the weights of the beams, say, 90,000 lb. at each third point of the span; some of this load is distributed, but for this calculation it is accurate enough to assume that the entire load is concentrated at the third-points.

Free bending moment (maximum)
\[
= 90,000 \times \frac{30}{3} \times 12 = 10,800,000 \text{ in.-lb.}
\]

Area of trapezoidal free-bending-moment diagram
\[
= \frac{30 + 10}{2} \times 10,800,000 = 216 \times 10^6 \text{ in.-lb.-ft.}
\]

Fixed-end bending moment
\[
= M_{(GH)} = \frac{216 \times 10^6}{30 \text{ ft.}} = 7,200,000 \text{ in.-lb.}
\]

**Bending moment.**—At the base of upper column GE a formula similar to formula (11.3a) for \( M_{GA} \) in Fig. 64 is applicable, that is
\[
M_{GE} = \frac{K_{GE}}{K_{GE} + K_{JG} + K_{GH}} M_{(GH)} = \frac{2304}{2304 + 2304 + 3000(7,200,000)}
\]
\[
= 2,200,000 \text{ in.-lb.}
\]

The critical sections of the columns are at the soffit of the primary beam (for the lower column) and the level of the top of the floor (for the upper column). Since the maximum calculated bending moments are at the intersection of the axes of the linear frame, that is along the geometrical centre-lines of the beams and columns, the bending moments at the critical sections are less than the maximum bending moment. In this example the primary beams are 3 ft. deep and, assuming that the point of contraflexure is at the mid-height of each column, the bending moment at the level of the top of the floor, which is the bending moment occurring where the load on the upper GE is greatest, is
\[
\frac{6 - 1.5}{6} \times 2,200,000 = 1,650,000 \text{ in.-lb.}
\]

Eccentricity: \( e = \frac{1,650,000}{282,000} = 5.85 \text{ in.} \)

Limiting eccentric load and eccentricity: \( d_1 = 24 - 2 = 22 \text{ in.} \);
\( b = 24 \text{ in.}; d_2 = 2 \text{ in.}; d_1 - d_2 = 20 \text{ in.}; A_4 \) (four 1-in. bars if the additional reinforcement near one face is neglected) = 3.14 sq. in. By substitution in formula (4.5b) in Chapter IV:
\[
X = \frac{0.85}{\frac{600 \times 20,000}{1 + \frac{30 \times 10^6}{30 \times 10^6}}} = 0.607 \text{ (as in Example No. 47)}
\]
\[ P_b = (0.607 \times 22 \times 24 \times 760) - (0.5 \times 3.14)(20,000 - 18,000) = 240,860 \text{ lb.} \]
\[ \varepsilon_b = [(0.607 \times 0.697 \times 760 \times 24 \times 22^2) + (\frac{1}{2} \times 3.14 \times 18,000 \times 20)] \left( \frac{1}{240,860} - \frac{20}{2} \right) = 7.85 \text{ in.} \]

Since the axial load of 282,000 lb. exceeds \( P_b \), the compressive strength determines the resistance and formula (4.5c) is applicable. From formula (2.3a), \( P_\theta = (760 \times 24^2) + 3.14(18,000 - 760) = 492,100 \text{ lb.} \) The safe load at an eccentricity of 5.85 in. is, from formula (4.5c),
\[ P = \frac{492,100}{1 + \left( \frac{492,100}{240,860} - 1 \right) \times \frac{5.85}{7.85}} = 277,000 \text{ lb., which is near enough to} \]

the applied load of 282,000 lb. to be satisfactory since the additional reinforcement near one face in the zone of the maximum bending moment is neglected.

Example No. 82.—Exterior Column in Top Story. Design exterior column CA in the top 15-ft. story of the building in Fig. 59. The total load on the column is 46,000 lb. (compare with the corresponding interior column in Example No. 79). Try a column 12 in. square with four \( \frac{2}{3} \)-in. bars as in Fig. 66. Assume that the moment of inertia of the primary beam of the roof is 40,000 in.\(^4\) and, that \( M_e \) for this beam is, by proportion, \( \frac{125 \text{ lb. per square foot}}{250} \times \frac{1}{\text{ditto}} \times 7,200,000 = 3,600,000 \text{ in.-lb.} \)

Stiffness factors.

Column CA: \( l_{CA} = 15 \text{ ft.}; \ b = d = 12 \text{ in.}; \)
\[ K_{CA} = \frac{12^4}{15 \times 12} = 115 \text{ in.}^4/\text{ft. units.} \]

Primary beam AB: \( I_{AB} = 40,000 \text{ in.}^4; \ l_{AB} = 30 \text{ ft.}; \)
\[ K_{AB} = \frac{40,000}{30} = 1333 \text{ in.}^4/\text{ft. units.} \]

Bending moment at top of column (formula 11.3a):
\[ M_{AC} = \frac{115}{115 + 1333}(3,600,000) = 286,000 \text{ in.-lb. or, say, 260,000 in.-lb.} \]
at the soffit of the beam.

Eccentricity: \( e = \frac{260,000}{46,000} = 5.65 \text{ in.} \)

Limiting eccentric load and eccentricity.—\( d_1 = 12 - 1.5 = 10.5 \text{ in.;} \)
\( b = 12 \text{ in.}; \) \( d_2 = 1.5 \text{ in.}; \)
\( d_1 - d_2 = 8 \frac{1}{2} \text{ in.}; \)
\( A_s = 1.77 \text{ sq. in.} \) By substitution in formula (4.5g), \( X = 0.607 \) as in Example No. 81, and
\[ P_b = (0.607 \times 10 \frac{1}{2} \times 12 \times 760) - \left( \frac{1}{2} \times 1.77 \right)(20,000 - 18,000) = 54,230 \text{ lb.} \]

Since the applied load of 46,000 lb. is less than \( P_b \), the tensile resistance determines the safe eccentric load, and substitution in formulæ (4.5a) in Chapter IV gives
\[ Y' = \frac{1.77(20,000 - 18,000)}{2 \times 760} = 2.33 \]

and
\[ U = 12[\left( \frac{1}{2} \times 12 \right) - 5.65] - 2.33 = 1.87. \]
The safe load $P$ at an eccentricity of 5.65 in. is

$$760 \left(1.87 + \sqrt{(1.87)^2 + (1.77 \times 12 \times 8\frac{1}{8})} \right) \frac{18,000}{760} + 2.33[(2 \times 10 \frac{1}{2} \times 12) - 2.33]$$

which is about 50,000 lb. and exceeds the applied load.

Therefore the section assumed is satisfactory. The arrangement of the reinforcement is shown in Fig. 66.

Example No. 83.—Interior Column subjected to Bending. Consider the resistance of the interior column in Example No. 78 (column ST in Fig. 59, the details of which are given in Fig. 60), if bending is taken into account.

The greatest amount of bending occurs if the live load acts on the part of the floor on one side only of the column. Assume that beam OT is fully loaded and that there is no live load on beam TY (Fig. 59). If the roof and the two floors $N$ and $N - 1$ above the column are fully loaded, the load on the column at the level of floor $N - 2$ is

$$P = 550,000 - [(900 \text{ sq. ft. } \times \frac{1}{2}) \times 150 \text{ lb.}] = 482,500 \text{ lb.}$$

The unbalanced fixed-end bending moment is calculated similarly to the total fixed-end bending moment in Example No. 81 and is due to the difference between the total load of 250 lb. per square foot on beam OT and the dead load of $250 - 150 = 100$ lb. per square foot on beam TY; that is $250 - 100 = 150$ lb. per square foot; by proportion,

$$M_{et(T)} = 7,200,000 \times \frac{150}{250} = 4,320,000 \text{ in.-lb.}$$

Since the sizes and lengths of the two columns and two beams meeting at $T$ are the same as those meeting at $G$, the stiffness factor of each member is the same as in Example No. 81, that is

$$K_{TS} = K_{UT} = 2304 \text{ in.}^4/\text{ft. units}$$
and

$$K_{OT} = K_{TY} = 3000 \text{ in.}^4/\text{ft. units}.$$  

The bending moment at base of the upper column TS (on the centre-line of the beam) is calculated from a formula similar to formula (11.3b) for $M_{DB}$ in Fig. 64, namely,

$$M_{TS} = \frac{K_{TS}}{K_{TS} + K_{UT} + K_{OT} + K_{TY}} M_{et(T)}$$

$$= \frac{2304}{(2 \times 2304) + (2 \times 3000)}(4,320,000) = \text{approx. 950,000 in.-lb.}$$

The bending moment at the critical section is $\frac{3}{8} \times 950,000 = 712,000 \text{ in.-lb.}$

The eccentricity is $\frac{712,000}{482,500} = 1.48$ in. If the limiting eccentric load $P_b$ and the limiting eccentricity $e_b$ are calculated as in Example No. 81, but with $A_s = 5.96$ sq. in. (six 1\frac{1}{2}-in. bars effective in resisting bending), $P_b = 238,000$ lb., and $e_b = 10.1$ in. Since $P$ exceeds $P_b$, the strength of the column in bending is determined by the compressive resistance, and the calculation proceeds as in Example No. 81 to give $P_b = 575,000$ lb. (with eight 1\frac{1}{2}-in. bars effective in resisting concentric compression), and the safe load is 476,000 lb., which is near enough to the applied load of 482,500 lb. to be satisfactory.
Columns Supporting Massive Superstructures.

The bending moments and forces due to horizontal loads, such as wind pressure, on massive structures such as silos supported on a group of columns (Fig. 67a) may be calculated by the simple analysis in the following which, although approximate, is at least as accurate as the assessment of the wind pressure. It is assumed that the superstructure is so massive that it is not deformed by the horizontal pressure of the wind but moves laterally a small distance \( \Delta \). The heads of the columns likewise move relatively to their foundation which is assumed to be sufficiently rigid not to be deformed by this movement. Therefore the columns are bent to the shape indicated by the broken lines but remain vertical at the head and base.

**Bending Moments.**—Since each column is of the same height \( l \) and deflects the same amount \( \Delta \), the bending moment \( M_x \) on any column is inversely proportional to its moment of inertia \( I_x \), because mathematically \( \Delta \) is the second integral of \( \frac{M}{I} \). The total horizontal force \( F_H \) on the structure acts as a shearing force on the columns. The bending moment on any column is the shearing force \( Q_x \) on that column multiplied by \( \frac{1}{I} \).
Fig. 68.—Typical Details of Load-bearing Wall (See pages 213 et seq.)
that is $M_x$ is $\frac{1}{2}Q_x l$. Since $M_x$ is proportional to $I_x$, $Q_x$ is likewise. If
the total number of columns supporting the superstructure is $N$, the shearing
force on any column is $\frac{F_H}{\sum I} I_x$. Therefore

$$M_x = \frac{F_H I_x}{2 \sum I}.$$  

(II.5a)

If all the columns are the same size or practically so, the shearing force
on each column is $\frac{F_H}{N}$ and the bending moment on each column is $\frac{F_H l}{2N}$.

**Additional Vertical Forces.**—In addition to the direct vertical load
on each column due to the weight of the superstructure there is, due to
$F_H$, an additional force acting downwardly on each leeward column and
upwardly on each windward column. Consider the group of $N$ (any number)
columns shown in Fig. 67. The total overturning moment on the group
at the level of the foundation is $F_H h$. The moment of inertia of the group
about the centroidal axis $X-X$ (at right-angles to the line of action of
$F_H$) is $\Sigma N x^2$, in which $N_x$ is the number of columns in a row at a distance
$x$ from the axis. The additional force $P_x$ on any column in a row at
distance $x$ from the axis is therefore given by the relation

$$P_x = \frac{F_H h x}{\left(\sum \frac{N x^2}{N_x}\right) N_x}.$$  

(II.5b)

The force $P_x$ is positive (downward) if the column is on the leeward side
of the centroidal axis and negative (upward) if on the windward side.

**Example No. 84.—Group of Columns.** A group of twelve columns
arranged in three rows of four as in Fig. 67b supports a coal bunker. The
total weight of the superstructure is 1800 tons when filled and 840 tons
when empty. The height from the foundation to the underside of the
bunker is 15 ft. The total wind pressure $F_H$ is 24,000 lb. and the centre
of action $h$ is 35 ft. above the foundation. Calculate the bending moments
and loads on the columns if each column in the outer row is 15 in. square
and in the central row 18 in. square.

The moments of inertia (concrete only) of the columns are $\frac{15^4}{12}$ and
$\frac{18^4}{12}$ respectively, and the sum $\sum I$ is therefore

$$8 \left(\frac{15^4}{12}\right) + 4 \left(\frac{18^4}{12}\right) = \frac{4[(2 \times 15^4) + 18^4]}{12}.$$  

Substitution in formula (II.5a) gives the bending moment on each exterior
column as,

$$\frac{24,000 \times (15 \text{ ft.} \times 12) \times \left(\frac{15^4}{12}\right)}{4[(2 \times 15^4) + 18^4]} = 266,000 \text{ in.-lb.}$$
Likewise the bending moment on each interior column is

\[ 266,000 \times \left( \frac{18}{\bar{x}} \right)^4 = 550,000 \text{ in.-lb.} \]

The additional load on each exterior column is calculated by substituting \( F_H = 24,000 \text{ lb.}, h = 35 \text{ ft.}, x = 12 \text{ ft.}, \) and \( N_x = 4 \) in formula (11.5b). The term \( \Sigma N_x x^2 \) is \( 2(4 \times 12^2) = 1152 \text{ ft.-column units} \); the central row of columns \((x = 0)\) does not affect this calculation as it is on the centroidal axis. Therefore \( P_x = \pm \frac{24,000 \times 35 \times 12}{1152 \times 4} = \pm 2188 \text{ lb.} \), say, 1 ton on each column, which is negligible compared with the least load due to the weight of the empty superstructure, which is about 35 tons on a corner column and about 140 tons on an inner central column.

Columns of Trestles and Towers.

A reinforced concrete trestle or an open framework supporting a tower (for example, a water tower), may comprise a number of columns braced together. An exact analysis would take into account the variation in size of the members and the slope of the junctions (or joints) which rotate slightly when the trestle or tower is deformed by horizontal load. The consequent complex calculation is not often worth while since again the horizontal force assumed is very approximate as it is generally the pressure of the wind on the pipes, tanks, cables, or the like which the trestle supports. The approximate method described in the following is sufficiently accurate. Referring to Fig. 69a, ABZY is a planar trestle comprising two columns AY and BZ stiffened by intermediate braces CD, EF, etc. At the top there may be a brace AB which supports the pipes, cables, or the like, or this member may form part of the bottom of the tank of a water tower; in the latter case two or more trestles would support the tank. At the bottom of each trestle there may be a brace YZ, or this member may be the foundation of the trestle or tower. In the diagram...
the legs are shown to be inclined, but generally this inclination is neglected except insofar as the shearing forces on the braces are affected.

Columns.—If the total horizontal force on one trestle is $F_H$ and is assumed to be applied at the level AB, and if $W$ is the total load and $W_D$ the total dead load supported by one trestle, the maximum and minimum loads on any one column are

$$\begin{align*}
P_{\text{max.}} &= \frac{W}{2} + \frac{F_H h}{l_n} \quad \text{(on the leeward column);} \\
P_{\text{min.}} &= \frac{W_D}{2} - \frac{F_H h}{l_n} \quad \text{(on the windward column).}
\end{align*}$$

(11.6a)

The assumed simplified deformation of the trestle under the influence of $F_H$ is shown in Fig. 69a. The shearing force on each column is $\frac{1}{2}F_H$, and the bending moments on the column immediately below junction A and B and above junction C and D are $\left(\frac{1}{2}F_H\right) \times \left(\frac{1}{2}h_1\right)$, that is $\frac{1}{4}F_H h_1$. Similarly below C and D and above E and F, the bending moments are $\frac{1}{4}F_H h_2$, and so on, so that generally

$$M_{\text{above junction}} = \frac{1}{4}F_H h_x; \quad M_{\text{below junction}} = \frac{1}{4}F_H h_{x+1}.$$  (11.6b)

Braces.—The bending moment on a brace is greatest at the junction with the column, where it is the sum of the bending moments on the column above and below the corresponding junction; for example the bending moment on the brace CD is $\frac{1}{4}F_H h_1 + \frac{1}{4}F_H h_2$, or in general

$$M_{\text{brace}} = \frac{1}{4}F_H (h_x + h_{x+1}).$$  (11.6c)

The bending moments on the top and bottom braces (or their equivalents) are $\frac{1}{4}F_H h_1$ and $\frac{1}{4}F_H h_2$ respectively. The shearing force on a brace is the rate of change of bending moment on the brace, that is for brace CD the bending moment varies from $\frac{1}{4}F_H (h_1 + h_2)$ at one end to a bending moment of the same magnitude but of opposite sign at the other end; therefore the rate of change of bending moment in the length $l_1$ is $\frac{F_H}{2l_1}(h_1 + h_2)$ and this shearing force is constant throughout the brace. A suitable arrangement of reinforcement at the junction of a brace and column is shown in Fig. 69b.

Example No. 85.—Braced Trestle. A trestle 45 ft. high supporting a conveyor gantry comprises two columns 15 ft. apart at the bottom and 10 ft. apart at the top. The total vertical load is 25 tons, of which 5 tons are live load. The horizontal wind force is 6 tons. Calculate the loads and bending moments on the columns and braces.

From formula (11.6a) the maximum and minimum loads on the columns are

$$\begin{align*}
P_{\text{max.}} &= \frac{25}{2} + \frac{6 \times 45}{15} = 12\frac{1}{2} + 18 = \text{say, 32 tons downwardly including the weight of the trestle.} \\
P_{\text{min.}} &= (\frac{1}{2} \times 20) - 18 = \text{minus 8 tons, say, 6} \frac{1}{2} \text{ tons net upwardly allowing for the weight of the trestle.}
\end{align*}$$
The shearing force on each column is \(\frac{1}{2} \times 6 = 3\) tons. If the vertical spacing of the braces is 15 ft., the bending moment on each column (just above and below a junction) is \(\frac{1}{2} \times 6 \times 15 = 22.5\) ft.-tons, that is 605,000 in.-lb. Therefore the columns should be designed to resist a bending moment of 605,000 in.-lb. combined with (i) a thrust of 32 tons, or (ii) a pull of 6\(\frac{1}{2}\) tons. Methods of design for case (i) are described in Chapter IV and for case (ii) in Chapter V.

The maximum bending moment on each intermediate brace is

\[2 \times 605,000 = 1,210,000\text{ in.-lb.}\]

and, assuming that the average length of the brace is 12 ft. 6 in., the maximum shearing force is

\[\frac{2 \times 1,210,000}{12.5\text{ ft.} \times 12} = 16,000\text{ lb.}\]

The top brace is subjected to a bending moment of 605,000 in.-lb., and the shearing force is therefore

\[\frac{2 \times 605,000}{10\text{ ft.} \times 12} = \text{say, 10,000 lb.}\]

The foundation of the trestle is subjected simultaneously to a vertical downward load of 32 tons on the leeward side, and \((12\frac{1}{2} - 18 + 1\frac{1}{2}) = -4\) tons upward load on the windward side; at the base of each column a horizontal force of 3 tons acts from left to right simultaneously with a clockwise moment of 605,000 in.-lb. if the wind blows from left to right; if the wind blows in the opposite direction, these effects are reversed. The design of the foundation would be similar to that in Example No. 93 on page 242.

**Load-bearing Walls.**

A wall supporting a vertical load acts as a column which is very wide in one direction compared with the dimension (the thickness) in the other direction, and in general should be designed similarly to a column as regards compressive stress and resistance to buckling. Some of the restrictions on the design of columns are relaxed in the design of walls. The principles in the following are based on the empirical recommendations in British Standard Codes No. 114 and No. 123 ("Dense Concrete Walls") and No. 111 ("Load-bearing Walls").

**Safe Load.**—The basis of the design of a wall subjected to a vertical load only is that, if the amounts of vertical and horizontal reinforcement are each not less than 0.2 per cent. of the cross-sectional area of the wall, and if the vertical bars are tied in to prevent them from buckling, the safe load is calculated in the same way as for a column; that is, formula (2.3) in page 34 is applicable. The safe load is reduced if the wall is thin in comparison with its height, in which case the wall is analogous to a slender column. The compressive stress permissible in the concrete may be increased, however, if the ratio of the height to the length of the wall exceeds \(1\frac{1}{4}\). The general formula for the safe load \(P_W\) lb. per linear foot of wall is therefore

\[P_W = R_L[12Dp_{oc}R_W + A_{se}(p_{se} - p_{oc}R_W)], \quad (11.7)\]

in which \(D\) is the thickness of the wall in inches; \(R_L\) is the reduction factor for a slender wall as given by formulae (2.5b) and (2.5c) on page 37;
$A_{se'}$ is the cross-sectional area of vertical reinforcement in 1 ft. length of wall; and $R_W$ is the factor for the increased stress in the concrete and is 1-2 if the ratio of story height to length is $\frac{1}{2}$ or less, 1-1 if the ratio is unity, and 1-0 if the ratio is $\frac{1}{2}$ or more (values for intermediate ratios may be interpolated). The length of the wall in this ratio is the overall length, or, if there are openings in the wall, it is the horizontal distance between adjacent openings. The reduction factor $R_L$ is applicable only if the effective height of the wall exceeds $15D$, the effective height being, in general, determined as described on page 38 for columns. If a wall is stiffened by transverse walls or substantial pilasters, the distance between which is less than the effective height, this distance may be substituted for the effective height in calculating the slenderness factor.

**Thickness.**—The thickness of the wall should be not less than 4 in. but, if the wall is exposed to the weather or is in contact with earth, the thickness should be not less than 6 in. In the case of a cavity wall the least thickness of each leaf should be 4 in. and the width of the cavity not less than 2 in. Party walls between two premises should be cavity walls.

**Fire Resistance.**—A reinforced concrete wall in a building is generally required to have a specified degree of fire-resistance. If the aggregates conform to the appropriate British Standard, and the cover of concrete over the reinforcement is not less than 1 in., a 4-in. solid wall is considered to have a resistance of two hours, which is that normally required. In buildings in which there is a special fire risk, four hours resistance may be required, and this resistance requires a wall not less than 7 in. thick. If fire resistance is not of great importance, the cover of concrete over the reinforcement need be only $\frac{3}{4}$ in. or $\frac{1}{2}$ in. over secondary reinforcement such as ties.

**Reinforcement.**—A method of tying in the vertical bars is to insert cramped horizontal bars as shown in Fig. 68 (pages 208 and 209). If the vertical bars are not tied in, their effect on the safe load should be ignored, that is the term containing $A_{se'}$ should be omitted from formula (11.7). Since the contribution of the vertical bars to the safe load is small, and satisfactory methods of tying them in are relatively expensive, it is generally more economical to ignore such reinforcement when calculating the safe load. In this case it is sometimes recommended that the horizontal reinforcement need be only 0-1 per cent, of the cross-sectional area of the wall, but generally at least 0-2 per cent. of vertical and horizontal reinforcement is advisable to resist the tensile forces produced by shrinking of the concrete as it sets and hardens. To be effective for this purpose the bars should be spaced not more than 18 in. apart, but 12 in. is preferable. The risk of cracking due to shrinking is greatest at the corners of openings in a wall and therefore additional bars of not less than $\frac{1}{2}$ in. diameter should be provided at each side of, and above and below, an opening and diagonally across each corner of the opening; all such trimmer
bars should extend at least 48 times the diameter of the bar beyond the corners of the opening. Since the effect of shrinking is accumulative, it is advisable to divide a long wall into parts not more than 50 ft. long, adjacent parts being separated by a narrow gap filled with a joint-filling material. Walls are sometimes constructed in panels between vertical columns and horizontal beams after the columns and beams have been constructed. Such panels should be tied into the horizontal and vertical supports by reinforcement which should be equivalent to not less than \( \frac{1}{4} \) in. bars at 12-in. centres. The connection should not be so rigid as to restrict entirely shrinking and other movement of the panel.

The details recommended in the foregoing, which are illustrated in Fig. 68, are in general based on experience, which in the case of walls is probably more valuable than calculation because a concrete wall has to resist the effects of such influences as moisture variations and changes of temperature, in addition to the compressive effect of the vertical load and the tensile effect of shrinking of the concrete. A reasonable degree of thermal insulation is also generally required. Concrete alone is a poor thermal insulator, but by lining the inside face of a solid or cavity concrete wall the thermal transmittance may be halved. To prevent moisture creeping up or down a wall a damp-proof course is necessary just above the level of the ground and at the roof; reference should be made to books on building construction. The effect of change of moisture content of a concrete wall is to cause the wall to expand or contract, and generally the reinforcement recommended in the foregoing is sufficient to resist such effects if the concrete is dense. If conditions of exposure are unusually severe, more reinforcement should be provided. Changes of temperature produce strains that are practically the same in concrete as in the steel embedded therein. If the temperature of the wall is the same throughout, the provision of gaps between parts of the wall, as described in the foregoing, alleviates the consequent stresses. If, however, the faces of the wall are liable to be subjected to different temperatures more complex stresses are set up and should be resisted as described in Volume II.

Example No. 86.—Load-bearing Wall. Calculate the safe load on the 6-in. wall in Fig. 68 (pages 208 and 209) if the least amount of mild-steel reinforcement is provided and that the concrete is 1:2:4 ordinary quality (\( f_{ce} = 760 \) lb. per square inch). (Such a wall might replace the secondary edge beams in Fig. 41 on page 134.)

Minimum vertical and horizontal reinforcement:

\[
A_{se'} < \frac{0.2}{100} (12 \times 6) = 0.144 \text{ sq. in. per foot,}
\]

that is not less than 0.072 sq. in. near each face in both directions, which is provided by \( \frac{1}{2} \)-in. bars at 7\( \frac{1}{4} \)-in. centres (0.079 sq. in.); therefore

\[
A_{se'} = 0.158 \text{ sq. in.}
\]

Slenderness: \( l = 12 \) ft. (story-height); \( D = 6 \) in.; from formula (2.5b),

\[
R_L = 1.5 - \frac{12 \times 12}{30 \times 6} = 0.7.
\]

Ratio of story-height to length between
openings $= 12 \div 6 = 2$. Ratio of story-height to overall length of panel $= 12 \div 30 = 0.4; but ratio of story-height to length between openings $= 12 \div 6 = 2$; since this latter ratio exceeds $\frac{1}{2}$, $R_w = 1.0$.

**Safe load.**

(a) Assume that the vertical bars are not tied in and are therefore ineffective. By substitution in formula (11.7) but omitting the second term,

$$P_w = 0.7 \times 12 \times 6 \times 760 \times 1.0 = 38,300 \text{ lb. per foot}.$$  

(b) Assume that the vertical bars are tied in by means of crimped bars as shown in the alternative detail in Fig. 68. By substitution in the entire formula (11.7) with $p_{sc} = 18,000$ lb. per square inch,

$$P_w = 0.7[(12 \times 6 \times 760) + (0.158 \times 17,240)] = 40,208 \text{ lb. per foot}.$$  

It is seen from this example that the additional safe load due to taking the vertical bars into account is negligible. The extra cost of providing the crimped horizontal bars is therefore not worth while.

**Bending.**—The recommendations in the foregoing relate principally to walls in buildings subjected to vertical load only. If a wall is subjected to bending the effect of the vertical load and the bending moment should be combined in a manner similar to that described on page 198 for columns subjected to bending. The recommendations are in general applicable to the walls of containers, such as tanks, bunkers, silos, and the like (as described in Volume II) which may support vertical loads as well as being subjected to bending due to the pressure of the contained material. In particular, the recommendation regarding the stiffening effect of transverse walls is a valuable guide to the design of the walls of multiple-compartment containers.

**Surface Finish.**—An important feature of a concrete wall is the surface finish of which there are many varieties. The concrete may be exposed, in which case it may be left as it is after removal of the shuttering except for rubbing down to remove the more obvious blemishes, or it may be surfaced by grinding or toothing, or it may be brushed down or treated in other manners to expose the aggregate. Alternatively, concrete cast in place may be concealed by rendering, or by coverings of precast concrete slabs, masonry, or similar materials on the outer face, or by spraying, plastering, or lining with building board or the like on the inner face. If a finish such as fluting is used which produces indentations in the surface of the structural concrete, it is necessary to ensure that the required cover of concrete over the reinforcement is obtained at the deepest part of the indentation. Some building boards, plaster, and sprayed materials may be considered as increasing the fire-resistance of the wall and therefore a thinner wall may be satisfactory, and the minimum cover of 1 in. may be reduced to, say, $\frac{1}{6}$ in. Reference should be made to publications dealing specifically with surface finishes of concrete structures.
CHAPTER XII

FOUNDATIONS

The principal terms used in this chapter and relating to the loads and pressures imposed on the ground and the reaction of the ground to these forces are explained in the following.

Foundation.—If the load imposed on the ground by a structure be greater than the load which the ground can support safely, a base must be provided to spread the load over a larger area of ground or a form of construction must be provided which transfers the load to stronger ground at a greater depth. Such a base or other construction is the "foundation structure" and, in this chapter, the term "foundation" is used to mean solely this base or other construction. If the ground be capable of supporting the load without a base or other construction, the ground itself constitutes the foundation; for example, if a pier of a bridge, a dam, or a wall is built directly on sound rock it may not require an enlarged base, and the rock is the foundation.

Imposed Loads.—The load imposed on a foundation is the weight of the structure, or structural member, and the loads supported by the structure, or member, and is termed the "net imposed load" $P$. The total load $P_T$ imposed on the ground is the net imposed load $P$ plus the weight of the foundation $P_B$ and is termed the "total imposed load".

Imposed Pressures.—The "net imposed pressure" $\phi_0$ is the intensity of pressure on the ground due to the net imposed load. The "imposed pressure" $\phi$ is the intensity of total pressure on the ground due to the total imposed load, and, if the weight of unit area of the foundation is $P_B$, the imposed pressure is the sum of $\phi_0$ and $P_B$.

Reaction of the Ground.—The intensity of the total reaction of the ground to the imposed pressure is equal to the imposed pressure $\phi$. The "net reaction" is the part of the total reaction due to the net imposed pressure, that is it is the difference between the total reaction and the weight of unit area of the foundation, and is equal to the difference between $\phi$ and $P_B$, that is $\phi_0$.

Load-bearing Capacity.—The "safe load-bearing capacity" $q_a$ of the ground is the intensity of pressure that can be imposed on the ground without causing excessive settlement.
Design of Foundations.

The process of designing a foundation comprises first the calculation of the net imposed load, secondly the assessment of the load-carrying capacity of the ground, and then the design of a foundation which distributes the total imposed load in such a way that the imposed pressure does not exceed the safe load-bearing capacity of the ground. The results of an investigation of the ground and the determination of its physical properties (a study commonly called "soil mechanics") allied with experience are necessary to enable the selection of a stratum suitable for the foundation to bear upon to be made and the establishment of the safe load-bearing capacity*.

The physical properties of soils are not dealt with, since the purpose of this chapter is to consider solely the design of foundations on the assumption that the average intensity of pressure that may be imposed safely on the ground is known. This pressure, the depth at which the foundation stratum occurs, and the magnitude of the load to be carried determine the type of foundation structure. Formulae for the design of foundations are generally complex because of the large number of variable factors they contain. Therefore only basic formulae, such as those giving the pressure imposed on the ground and the simple principles of design, are given in this chapter. The examples illustrate the application of these principles.

Types of Foundations.

A foundation structure may be one of three main types, namely, a base supporting a single concentrated load, such as the base of a column or pier; a foundation supporting two or more concentrated loads, such as a common base for a row or group of columns; or a continuous base, such as the footing of a wall, supporting a load which is uniformly distributed or approximately so. The most common type of reinforced concrete base for one load is a simple pyramidal base, as in the diagram on page 3 and in Fig. 70, which is suitable for a wide range of loads and types of soils if the bearing stratum is near the surface. If the imposed load is small or the safe load-carrying capacity of the ground is great, a small non-reinforced base may be sufficient and would be cheaper. If the bearing stratum is at medium depth, say, within 15 ft. of the surface, it may be more economical to excavate a pit and construct a pier therein to transmit the load to the ground, but if the ground is waterlogged or if the bearing stratum is at a greater depth, piles driven or bored from the surface are generally more economical. In doubtful cases two or more designs should be prepared and the least costly selected. Combined bases are provided if the loads to be supported are so close together that

* See "Site Investigations" (B.S. Code No. 2001) and "Foundations" (Code No. 4, Inst. of Civil Engineers).
separate bases are impracticable owing to overlapping. When a row of closely-spaced columns or a wall are to be supported, a long narrow footing is provided if the bearing stratum is within 3 ft. or so of the surface. If deeper, the columns or wall are supported on a beam which is supported at intervals on piers founded on the bearing stratum or, if very deep, on one or more rows of piles. Simple single and combined foundations are dealt with in this chapter. For bridges, marine structures and other subaqueous work, special foundations are generally required and may be either cylinders, caissons, or more complex constructions.

Each of the foregoing types of simple foundations may be in one of three classes, namely, where the resultant line of action of the imposed load or the resultant of several loads is vertical and coincides with the

![Fig. 70.—Concentrically-loaded Bases.](image)

centroid of the bearing area; where the line of action of the imposed load or loads is vertical but is eccentric as regards the centroid of the bearing area; and where horizontal forces or moments or both act on the foundation with or without concentric or eccentric vertical loads. Where practicable a concentric foundation is preferable since the pressure is more or less uniformly distributed on the ground and may therefore be equal to the safe load-bearing capacity of the ground, thereby requiring a foundation of the smallest area. Eccentric loading may, however, be unavoidable, for example because restrictions of the site make it impracticable to extend the foundation equally in all directions, or because of different incidences of the live load. The pressure on the ground under an eccentrically-loaded foundation varies at different parts of the foundation and, since the greatest pressure should not exceed the safe pressure, the average pressure is less than the greatest permissible pressure; consequently the
area of the foundation may be greater than the area of a similar foundation supporting a concentric load of the same magnitude. Foundations are subjected to horizontal forces and moments in the case of frames when the bases are fixed. If the frame is hinged at the bases, the foundation is subjected to a horizontal force but no moment. In both cases vertical loads also are imposed on the foundation.

Separate Bases with Concentric Loads.

The most simple foundation is the base of a column, pier, or similar member imposing a concentrated load concentrically on the base. If the safe load-bearing capacity of the ground is \( q_a \) lb. per square foot and the load to be supported is \( P_T \) lb. including the weight of the base, the area of the base should be not less than \( \frac{P_T}{q_a} \) sq. ft. Alternatively, if the weight of the base is \( \rho_B \) lb. per square foot and the net imposed load is \( P \) lb., the area of the base should be not less than \( \frac{P}{q_a - \rho_B} \) sq. ft.

Square Base.—A square base is generally the cheapest. The width should be not less than \( \sqrt{\frac{P}{q_a - \rho_B}} \) ft. If the width is not more than two or three times the width of the column or pier, a non-reinforced base is generally cheaper, but if the width is greater a splayed reinforced concrete base as in Fig. 70a is preferable. The object of the pyramidal shape is to save concrete. The base cantilevers in all directions from the column and the bending moments are greatest under the column and reduce to zero towards the edges of the base; the thickness at the edges may therefore be less than at the centre. The critical planes for bending are at the sides of a reinforced concrete column or pier cast monolithically with the base, that is planes X—X in Fig. 70a. The bending moment at each of these four planes is the moment of the total net reaction of the ground to one side of the column or pier, the moment being calculated about the side of the column or pier. This bending moment acts in two directions mutually at right-angles and is resisted by two layers of reinforcement near the bottom of the base. The bars in each layer are also mutually at right-angles and, as stated on page 116, the bars should have sufficient bond-length from the section of greatest stress, that is the side of the column, to the end of the bar or to the beginning of the hook if one is provided. The bars in each layer may be spaced uniformly across the width of the base.

The base must be sufficiently thick to provide the compressive resistance necessary to resist the bending moment; this resistance should be provided by the central rectangular block, the width of which is generally 2 in., or more, greater than the width of the column. The foundation should also be thick enough to provide resistance to the shearing forces. The critical
planes for shearing resistance are at a distance equal to the effective depth of the base from the side of the column or pier, that is planes S—S in Fig. 70a. The thickness of a pyramidal base at S—S and the length of the planes are obtained from the geometry of the base. The total shearing force on the four planes S—S is the total net reaction of the ground on the base minus the net reaction on the area within the square bounded by the planes S—S. The shearing stress is the total shearing force divided by the effective area of the planes, and should not exceed the shearing stress permissible without reinforcement to resist the shearing force. Such reinforcement is uneconomical, except in large bases, but if required it is generally provided conveniently by bending up some of the bars from the bottom of the base. If reinforcement is provided to resist the whole of the shearing force, the shearing stress may be increased but should be not more than four times the stress permissible without reinforcement.

So long as the thickness of the base is sufficient to provide the necessary resistance to bending and shearing, the base may be splayed from the greatest thickness required at the middle to 6 in. or more, for large bases, or 3 in., for small bases, at the edges. To provide a clean surface on which to lay the reinforcement, a layer of lean concrete, commonly known as "blinding", should be provided on the bottom of the excavation; the layer should be 2 in. or 3 in. thick. The cover of structural concrete below the reinforcement is additional to this layer. The "blinding" may generally be 1:3:6 concrete, but if the safe load-bearing capacity of the ground is great, as in the case of rock, the concrete should be stronger so that the permissible direct compressive stress is not less than the safe load-bearing capacity of the ground.

Throughout the design of a foundation checks should be made on the units of the various factors because the safe load-bearing capacity of the ground is generally expressed in tons per square foot, the stresses in the concrete and reinforcement in pounds per square inch, the sizes of the foundation in feet, and thicknesses and the like in inches. The different units are clarified in the examples which follow.

Example No. 87a.—Square Base. Design a square concentrically-loaded base to support a column 24 in. square which imposes a load $P$ of 500,000 lb. on the base. The safe load-bearing capacity of the ground is 2 tons per square foot ($\sigma_a = 4480$ lb. per square foot).

Assuming that the weight $p_B$ of the base is 300 lb. per square foot, the width should be not less than $\sqrt{\frac{500,000}{4480 - 300}}$ ft., say, 11 ft. For a base 11 ft. square, the net reaction of the ground is $\frac{500,000}{11 \times 11} = 4140$ lb. per square foot. The area of the base beyond one side of the column is $(\frac{11}{2} - \frac{2}{2}) \times 11 = 49\cdot5$ sq. ft., and the centroid of this area is 2.25 ft. from the side of the column. The total bending moment across the width
of the base is therefore \((4140 \times 49.5 \times 2.25)\frac{12}{184} = 5,540,000\text{ in.-lb}\). If \(p_{st} = 20,000\text{ lb}\) and \(p_{eb} = 1000\text{ lb per square inch}\), \(Q_e = 184\). If the top of the base is 30 in. square, the effective depth \(d_1\) should be not less than \(\sqrt{\frac{5,540,000}{184 \times 30}} = 31.6\text{ in.}\). If the thickness of the base is 36 in. and \(\delta = 1\frac{1}{4}\text{ in. (cover)} + \frac{1}{2}\text{ in. (half the diameter of a bar) = say, 2 in.}, \delta = 34\text{ in.}\). The lever-arm is approximately \(\frac{1}{4} \times 34 = \text{say, 30 in.}, and the cross-sectional area of reinforcement required is \(\frac{5,540,000}{30 \times 20,000} = 9.23\text{ sq. in.}\),

which is provided by ten \(1\frac{1}{2}\)-in. bars in each of two layers arranged as in Fig. 70a.

The minimum bond-length required is \(41\frac{1}{2} \times 1\frac{1}{2} = 47\text{ in.}\). The bond-length provided (from the side of the column to the beginning of the hook)

\[\frac{\text{11 ft.}}{2} - \frac{2 \text{ ft.}}{2} = \text{say, 6 in. = 48 in.},\]

which is sufficient especially as a semi-circular hook is provided. To allow space for this hook, the thickness \(d_{min}\) at the edges of the base must be not less than 9 in.

From the dimensions so determined, the length of the critical shearing planes can be calculated, that is the length of each is 24 in. (width of column) + 68 in. (= \(2a_1\)) = 92 in. The thickness at these planes is \(\frac{1}{2}(11 \times 12) - \frac{1}{2} \times 92\) = 36 in. + 9 = 19 in. The effective depth at these planes is 19 - 2 = 17 in. The total shearing force on four planes is \(4140\left[11^3 - \left(\frac{92}{12}\right)^3\right] = 258,000\text{ lb.}\). The shearing stress is

\[
\frac{258,000}{(4 \times 92)(\frac{11}{4} \times 17)} = 46\text{ lb per square inch},\] which is less than the shearing stress permissible without reinforcement.

**Oblong Base.**—If restriction of space prohibits the provision of a square base of adequate area, an oblong base similar to that in Fig. 70b may be suitable. The area \(BL\) should be not less than \(\frac{Pr}{q_0}\) sq. ft. The bending moments, the compressive resistance, or the effective depth, and the cross-sectional area of reinforcement required in each direction are calculated in the same way as for a square base, but in this case the bending moments about the two axes are not equal. The bars in the direction of the length of the base may be spaced uniformly across the width \(B\), but those in the direction of the width should be spaced more closely under the column or pier than towards the ends of the base. The length of each critical plane \(S-S\) for resistance to shearing force is derived in the same way as for a square base but, if the base is pyramidal, the depths of the planes are not equal. A method of considering the shearing resistance is to assume that the shearing force due to the net reaction of the ground on the trapezoidal area cross-hatched from right to left in Fig. 70b is resisted by the shearing on plane \(S_1-S_2\) and that on the trapezoidal area cross-hatched from left to right is resisted by the shearing on plane \(S_3-S_4\).
Example No. 87b.—Oblong Base. Design a concentrically-loaded base for the conditions in Example No. 87a, if the width $B$ of the base is not to exceed 5 ft.

As the area of the base is not less than $\frac{500,000}{4480 - 350} = 120$ sq. ft., the length $L$ must be not less than $\frac{120}{5} = 24$ ft. The net reaction of the ground is $\frac{500,000}{24 \times 5} = 4170$ lb. per square foot. In the direction of the length, the area of the base beyond the face of the column is

$$[(\frac{3}{4} \times 24) - (\frac{3}{1} \times 2)]5 = 55$$ sq. ft.

and the bending moment across the width $B$ is therefore

$$(4170 \times 55 \times 5.5)12 = 15,150,000 \text{ in.-lb.}$$

In the direction of the breadth of the base, the area beyond the side of the column is $$[(\frac{3}{4} \times 5) - (\frac{3}{1} \times 2)]24 = 36$$ sq. ft., and the bending moment across the length $L$ is $$(4170 \times 36 \times 0.75)12 = 1,350,000 \text{ in.-lb.}$$ If the base is splayed in the direction $L$ only (Fig. 70b) the effective depth at the middle must be not less than $\sqrt{\frac{15,150,000}{154 \times 60}} = 37$ in., that is the thickness can be, say, 42 in., for which $d_t = 40$ in. and $l_a = 35$ in. Therefore the cross-sectional area of the longitudinal reinforcement should be not less than $\frac{15,150,000}{20,000 \times 35} = 21.6$ sq. in., which is provided by eighteen 14-in. bars spaced at 3-in. centres across the 5-ft. width of the base. All the bars need not extend the full length of the base; alternate bars may be staggered as in the case of bars a and b in Fig. 70b. The cross-sectional area of the transverse bars should be not less than

$$\frac{1,350,000}{20,000 \times (\text{say}) 34} = 2.0 \text{ sq. in.,}$$

which is provided by seven $\frac{3}{8}$-in. bars grouped at 12-in. centres under the column; similar bars spaced at, say, 18-in. centres should be provided between this group and the ends of the base similar to the arrangement in the longitudinal section in Fig. 70b.

The critical planes for shearing stress are at a distance of $$[(\frac{3}{4} \times 24) + 40] \text{ in.} = 4 \text{ ft. 4 in.}$$ from the centre of the column and extend right across this narrow base. The critical shearing force is therefore $4170(7 \text{ ft. 8 in. } \times 5 \text{ ft.}) = 160,000$ lb. If the top of the base is 36 in. wide in the direction $L$ and the thickness at the edge is 9 in., the thickness at the critical shearing plane is

$$\left(\frac{7 \text{ ft. 8 in.}}{10 \text{ ft. 6 in.}}\right)33 \text{ in.} + 9 \text{ in.} = 33 \text{ in.,}$$

and the effective depth at this plane is about 31 in. The shearing stress is therefore $\frac{160,000}{(\frac{3}{4} \times 31) \times 60}$ which is about 100 lb. per square inch, so that reinforcement is not required to resist the shearing force; reinforcement could be provided by bending up some of the longitudinal bars as shown in the longitudinal section in Fig. 70b.

Alternative Method.—The calculation of the bending moment on
square and oblong bases in the foregoing are based on the recommendations in B.S. Code No. 114, but an alternative method which is commonly adopted and is less conservative is to consider the base as comprising four trapezoidal leaves cantilevering from under the column or pier. Each leaf is bounded by one side of the column, the adjacent edge of the base, and the two corresponding diagonals. The bending moment on each leaf at the face of the column or pier is the product of the net reaction of the ground on a trapezoidal area and the distance from the face of the column or pier to the centroid of the trapezium. Since each leaf cannot deform freely as a cantilever because of interaction with adjacent leaves, the bending moment is likely to be less than that calculated on the assumption of free cantileverage, and is therefore very much less than the bending moments calculated in accordance with the recommendations of the B.S. Code.

**Foundation for a Steel Stanchion or Precast Column.**—The methods of design and examples in the foregoing relate to bases where the column or pier is constructed monolithically with the base. If this is not so, as in the case of a reinforced concrete foundation for a steel stanchion or a precast concrete column, the greatest bending moment occurs under the centre of the stanchion or column and not at the side of the member supported by the base.

**Example No. 88.—Foundation for a Steel Stanchion.** Design the concentrically-loaded base in Example No. 87a on the assumption that the column is a steel stanchion with a base-plate 3 ft. square.

The bending moment across each central axis of a concrete base 11 ft. square is $\frac{500,000}{2} \left( \frac{11}{4} - \frac{3}{4} \right) r_2 = 6,000,000$ in.-lb., and, by a calculation similar to that in Example No. 87a, a thickness of 36 in. and ten $\frac{1}{2}$-in. bars in each direction are seen to be satisfactory. Likewise, by comparison, the shearing resistance is sufficient.

**Circular and Polygonal Bases.**—The bending moments calculated in the preceding examples may be slightly excessive, since it is assumed that the net reaction of the ground is uniform under the entire base. It is likely that, especially for bases on sand or gravel, the pressure on the soil varies from a minimum at the edges to a maximum at the centre, although for practical purposes a uniform distribution of pressure is assumed. An example where it is permissible and economical to allow for non-uniform distribution of pressure is the base of a tank or other steel superstructure which is not monolithic with the base (as a concrete superstructure would be) and which imposes a concentric load, assumed to be uniformly distributed, on a circular area of diameter $D$. In this case, a circular or polygonal base is provided (Fig. 71). Such a base may be analysed as an elastic plate but, because it is usually thick compared with its diameter, it is questionable whether such an analysis is applicable.
The practical method described in the example which follows avoids the complex mathematics of the theory of plates but gives comparable results.

**Example No. 89.—Circular Base.** Design a base to support a load of 3000 tons applied uniformly on an area of 36 ft. diameter, if the safe load-carrying capacity of the ground is $1\frac{1}{4}$ tons per square foot.

The intensity of the applied load is $\frac{P}{\pi D^2} = \frac{3000}{1020} = \text{say, 3 tons per square foot}$. Assuming that the base weighs about $\frac{1}{2}$ ton per square foot, its area must be not less than $\frac{3000}{1\frac{1}{4} - \frac{1}{2}} = 2570$ sq. ft., but to allow for variations of the pressure on the ground, more than this area should be provided. As a first trial, increase the area by two-thirds, that is 4300 sq. ft., which requires a circular base of 74 ft. diameter. Therefore $D = 36$ ft., $B = 74$ ft. and $P = 3000$ tons. The base will tend to deform to a saucer-shape and as a result the ground under the middle of the base will be more compressed than at the edge. If, however, the base be extremely rigid there will be no such variation of pressure and the intensity of the reaction of the ground uniformly distributed over the entire area of the base

$$\left(\frac{\pi B^2}{4} = 4300 \text{ sq. ft.}\right)$$

would be $\frac{3000}{4300} = 0.7$ ton per square foot as indicated by the line (a) in Fig. 71. If the base be flexible the reaction would be greatest under the middle and, in the extreme case, there would be no reaction at the edge, that is the variation would be as indicated by the line (b) in Fig. 71. If the variation is assumed to be parabolic, the volume of the paraboloid of reaction of the ground must equal $P$ and therefore the greatest intensity of this reaction is $\frac{2 \times 3000}{4300} = 1.4$ tons per square foot. The actual distribution is probably about the mean of these two extremes, and it is not unreasonable to assume a parabolic distribution as indicated by the line (c) in Fig. 71 for which the greatest intensity of reaction is $\frac{1}{2}(0.7 + 1.4) = 1.05$ tons per square foot.

The total volume of this bulb of pressure must equal $P$, and therefore the intensity of the net reaction $p_*$ at the edge is calculated from the expression

$$4300p_* + \frac{1}{2}(1.05 - p_*)4300 = 3000 \text{ tons;}$$

therefore $p_* = 0.35$ tons per square foot. The imposed pressure of 1.05 tons per square foot plus the weight of the base does not exceed the safe load-bearing capacity of the ground; therefore the size of the base is satisfactory.

To avoid complex arithmetic, a block of reaction equivalent to that represented by (c) and comprising two superimposed cylinders can be assumed as indicated by (d) in Fig. 71. If the equivalent net intensity of reaction at the edge is assumed to be 0.5 ton per square foot, the equivalent net intensity of pressure under the middle must be $1\frac{1}{4}$ tons per square foot since the total net reaction must equal 3000 tons. Therefore, for the purpose of design, the forces on the base are a net reaction of 0.5 ton per square foot under the entire area and a net imposed pressure of

$$3 - 1\frac{1}{4} + 0.5 = 2.17 \text{ tons per square foot}$$
Part of Fig. 71. (See facing page.)
REINFORCEMENT IN QUADRANT

Fig. 71—Concentrically-loaded Circular Base. (See also facing page.)

Peripheral Bars in 40 ft. lengths with 3 ft. laps (no hooks).

2 No. Bars f at edge (with hook at outer end)

Additional \( \frac{1}{4} \) Bars (f) at 24" c.c.
(with hooks at outer end)

Additional \( \frac{1}{4} \) Bars (f) at 24" c.c.
(with hooks at outer end)

Similar bars in other direction.

B = 37 FT.

46 No. \( \frac{1}{8} \)" Bars (a, b, c, d, e)
in bottom in both directions.
Bars 40 ft. long (straight—no hooks).
Arranged in groups of 5 No. each with ends of adjacent bars staggered 5 ft.

\( E = 37 \) FT.

24" c.c., 12" c.c., 9" c.c.
over the loaded area. The total bending moment across a diameter is the difference between the moments due to this reaction and pressure acting on semi-circular areas as in Fig. 71. The centroids of each area are at distances \( \frac{2B}{3\pi} = 15.7 \) ft. and \( \frac{2D}{3\pi} = 7.65 \) ft. respectively. The total bending moment is

\[
\left( \frac{1}{8} \times \frac{1}{8} \times 4300 \times 15.7 \right) - \left( 2.17 \times \frac{1}{8} \times 1020 \times 7.65 \right) = 8400 \text{ ft.-tons},
\]

but is not uniformly distributed along the diameter; it is greatest at the centre and decreases to zero at the terminals of the diameter. If the variation is parabolic the maximum bending moment at the centre is

\[
\frac{8400}{74} \times 1\frac{1}{8} = 170 \text{ ft.-tons per foot width},
\]

that is, 4,575,000 in.-lb. per foot width. This bending moment acts radially in all directions and is resisted by reinforcement in the form of a grid of bars mutually at right-angles.

The effective depth required at the centre of the base is

\[
\sqrt[\pi]{\frac{4,575,000}{184 \times 12}} = 45.5 \text{ in.}
\]

therefore a thickness of 4 ft. is sufficient, but, to reduce the amount of reinforcement, the thickness could be 5 ft. under the loaded area sloped down to 2 ft. at the edges. The reinforcement required near the bottom is \( \frac{4,575,000}{\frac{4}{9} \times 57 \times 20,000} = 4.6 \text{ sq. in. per foot}, \)

which is provided by 1\( \frac{1}{4} \)-in. bars at 3-in. centres (in two directions at right-angles) at the middle, but the spacing may be increased successively to, say, 2 ft. at the edge. The reinforcement in one quadrant is shown in Fig. 71.

The shearing force is greatest around the periphery of the loaded area and the total force is the net reaction of the ground on the annulus outside this area, that is

\[
0.5(4300 - 1020) = 1640 \text{ tons},
\]

or

\[
\frac{1640 \times 2240}{\pi \times 36} = 32,500 \text{ lb. per foot of periphery}.
\]

Therefore the shearing stress is \( \frac{32,500}{12 \times \frac{4}{9} \times 57} = 54 \text{ lb. per square inch} \).

If the load on the base were applied on a narrow annular ring of mean diameter 36 ft. instead of being spread over the entire area of 36 ft. diameter, the calculation would be exactly as for the latter distribution, except that the bending moment would be less because the centroid of the loaded area would be at \( \frac{D}{\pi} = 11.5 \) ft. from a diameter instead of 7.65 ft.

**Pressures Imposed on Ground by Eccentric Load.**

If the load on a symmetrical base is eccentric, the ground pressure is not uniform but varies from a maximum at the edge nearer to the load to a minimum at the opposite edge as shown in Fig. 72a. For equilibrium it is necessary for the total reaction to be equal to the imposed load and for the centre of gravity of the reaction of the ground to be on the line of action of the load, that is at a distance \( e \) from the centre-line of the base.
Considering a base of length \( L \) and width \( B \), the total imposed pressure \( \frac{1}{2}BL( \phi_{\text{max.}} + \phi_{\text{min.}} ) \) must equal \( P_T \). The centre of gravity of the trapezoidal block of reaction is such that \( X \) is

\[
\frac{\frac{1}{2}B^2\phi_{\text{min.}} + \frac{1}{2}B^2(\phi_{\text{max.}} - \phi_{\text{min.}})}{\frac{1}{2}B(\phi_{\text{max.}} + \phi_{\text{min.}})},
\]

that is \( X \), which must be equal to

\[
\frac{1}{2}B - e, \quad \text{is} \quad \frac{B(2\phi_{\text{min.}} + \phi_{\text{max.}})}{3(\phi_{\text{min.}} + \phi_{\text{max.}})}.
\]

Combining the expressions for \( P_T \) and \( X \) and reducing gives

\[
\phi_{\text{max.}} = \frac{P_T}{BL}(1 + \frac{6e}{B}), \quad \text{and} \quad \phi_{\text{min.}} = \frac{P_T}{BL}(1 - \frac{6e}{B}).
\]

These expressions may be derived also by considering the interaction between the loaded base and the reaction of the ground as a combination of a direct load \( P_T \) and a moment \( P_Te \). The maximum and minimum imposed pressures (using the comparable stress formulae on page 79) are

\[
\phi_{\text{max.}} = \frac{P_T}{BL} + \frac{P_T e}{Z} \quad \text{and} \quad \phi_{\text{min.}} = \frac{P_T}{BL} - \frac{P_T e}{Z}.
\]

Since the section modulus \( Z \) is \( \frac{LB^2}{6} \), substitution gives

\[
\phi_{\text{max.}} = \frac{P_T}{BL}(1 + \frac{6e}{B}) \quad \text{and} \quad \phi_{\text{min.}} = \frac{P_T}{BL}(1 - \frac{6e}{B}). \quad (12.1a)
\]

It is obvious that if \( e \) exceeds \( \frac{1}{6}B \), \( \phi_{\text{min.}} \) would be negative, that is tensile; since a tensile stress cannot be induced in the ground, formula (12.1a) ceases to apply if \( e \) exceeds \( \frac{1}{6}B \). Conditions then are as shown in Fig. 72b, and the analysis may be made by the first method of deriving formula (12.1a). The total imposed load \( P_T \) is equal to \( \frac{1}{2}P_{\text{max.}}xL \); \( X \) is \( \frac{1}{2}x \) and also

---

**Fig. 72**—Pressures due to Eccentric Loading.
BASIC REINFORCED CONCRETE DESIGN

must be equal to \( \frac{1}{2}B - \varepsilon \). Therefore \( P_T \) is equal to \( \frac{4P_T}{3L} \left( \frac{1}{2}B - \varepsilon \right) \); hence

\[
\hat{p}_{\text{max}} = \frac{4P_T}{3L(B - 2\varepsilon)}.
\]  

(12.1b)

It is preferable to design bases so that where possible \( \varepsilon \) does not exceed \( \frac{1}{2}B \), thereby avoiding the anomalous condition of there being a part of the base not bearing on the ground.

In the common theories in the foregoing it is assumed that there is rectilinear variation of pressure, but it is unlikely that this is the case with materials of the nature of soils which form bearing strata. A rectilinear distribution gives a more adverse result than parabolic or other probable forms of distribution and is therefore on the safe side.

Separate Bases with Eccentric Load.

A separate base with an eccentric load occurs mainly if a column, or other origin of the load, is near the boundary of a site or other obstruction which prevents extension of the base symmetrically in all directions about the centre of the load. Unequal distribution of pressure, such as results from an eccentric load, is also caused if the base is subjected to a moment or horizontal force. Therefore these conditions can be considered together, especially as they commonly occur simultaneously as on the foundation of a frame or of a trestle or gantry. Consider the general case in Fig. 73a in which the vertical load \( P \) acts at a distance \( \varepsilon_P \) from the centre-line of the base, and a horizontal force \( F_H \) causes a clockwise moment \( F_Hd' \) to act on the base of thickness \( d' \), which is also acted upon by an external moment \( M \) (assumed to be positive if it acts clockwise). The total clockwise (positive) moment on the base is therefore \( P\varepsilon_P + F_Hd' + M \) and the total vertical load is \( P + P_B \) if \( P_B \) is the total weight of the base. Trial dimensions, and therefore the weight, of the base must be assumed. The resultant eccentricity is

\[
\frac{P\varepsilon_P + F_Hd' + M}{P + P_B},
\]

which should be as small as possible in order to reduce the non-uniformity of pressure; this may be done by moving the base (relative to the line of action of \( P \)) so that \( \varepsilon_P \) may even be negative in the extreme case, the optimum condition then being that the counter-clockwise moment \( P\varepsilon_P \) is equal to the clockwise moment \( F_Hd' + M \).

The same rules apply, with obvious modifications, if \( F_H \) or \( M \) or both act in directions contrary to those assumed in the foregoing expressions, in which case they are negative. If the restrictions are such that the base cannot be moved to a more favourable position, the expression for the resultant eccentricity applies directly with only such modifications as are required due to any other actions being negative. In the case of foundations for columns, trestles, and the like subjected to the effects of wind,
nothing is gained by moving the base, since the moment due to the horizontal shearing force $F_H$ and the moment $M$ are both clockwise when the wind blows from left to right and are both anti-clockwise when it blows from the opposite direction. In this case, the base is generally placed centrally under the load (that is $\epsilon_P = 0$) but account must be taken

of the variable magnitude of $P$, which is less when on the windward side than when on the leeward side of the structure.

Having determined the resultant eccentricity, substitution in formula (12.1a), or (12.1b) if unavoidable, gives the minimum and maximum pressures. If the column or other superstructure is cast monolithically with the base, the critical planes for the bending moments are $X-X$, $Y-Y$, and $Z-Z$ (Fig. 73b). The intensity of the net reaction of the ground $p_x$ at plane $X-X$ is $p_1 - \frac{x}{B}(p_{\text{max.}} - p_{\text{min.}})$, in which

$$p_1 = p_{\text{max.}} - p_b.$$  

Similarly at plane $Y-Y$, the intensity of the net reaction $p_y$ is

$$p_2 + \frac{y}{B}(p_{\text{max.}} - p_{\text{min.}}).$$
in which \( p_a = p_{\text{min}} - p_b \). The bending moment at each plane is, as in the case of a concentrically-loaded base, the product of the total net reaction on the rectangular area outside the plane and the distance from the plane to the centroid of this area.

The four critical planes for resistance to shearing are also as shown in Fig. 73b. The total net reaction on each of the four shaded areas is the shearing force to be resisted at the corresponding plane. The trapezoidal shape of the area, the non-rectangular shape of the plane, and the variation of reaction under each area make exact calculation of the shearing stress on each plane complex and not worth while in the case of ordinary bases of proportions similar to that in Fig. 73b. The approximate greatest shearing stress can be estimated as follows. The greatest shearing force is that on the plane on the left-hand side of plane \( Y - Y \). The loaded area is slightly less than \((y - d_y)L\) and the average intensity of reaction is slightly less than the mean of \( p_y \) and \( p_z \). Therefore the shearing force \( Q_{\text{approx.}} \) is less than \( \frac{1}{2}L(p_y + p_z)(y - d_y) \). Since the area of the vertical plane is less than \( d_0 L \), the shearing stress is approximately \( \frac{Q_{\text{approx.}}}{\frac{1}{8}d_0'L} \); the units in this expression must be consistent. So long as this stress is substantially less than the permissible stress, the method of calculation is satisfactory.

The foregoing methods of determining the bending moments and shearing stresses are illustrated in Example No. 90. If an important base should differ considerably in shape and proportions from that in Fig. 73b, or if the distribution of pressure is unlike that in the diagram, it is preferable to calculate the shearing stress more accurately. If the principles are understood, the only complexity is the arithmetical work in applying them.

Example No. 90.—Eccentrically-loaded Base for a Column. Design a base similar to that in Fig. 73b if the imposed load is 250,000 lb., the size \( d_0 \) of the square column is 24 in., and \( x \) must not exceed 3 ft. The column imposes on the base an anti-clockwise moment \( M \) of 82,000 ft.-lb., and a horizontal force \( F_H \) of 10,000 lb. at the top of the base and acting from right to left (see Fig. 73a). The safe load-bearing capacity of the ground is 2 tons per square foot.

Assume as a first trial that the width \( L \) of the base is 9 ft., the breadth \( B \) is 12 ft., and the thickness \( d' \) at the middle of the base is 3 ft. Assuming that the average weight of the base is 300 lb. per square foot, the total weight \( P_B \) of the base is \( 9 \times 12 \times 300 = 32,400 \) lb., and \( \varepsilon_p \) is

\[
\left(\frac{1}{3} \times 12\right) - \left[3 + \left(\frac{1}{3} \times 2\right)\right] = 2 \text{ ft.}
\]

The resultant eccentricity is

\[
\frac{(250,000 \times 2) - (10,000 \times 3) - 82,000}{250,000 + 32,400} = 1.34 \text{ ft.,}
\]

which is less than \( \frac{12}{6} \) and indicates that formula (12.1a) is applicable.
Therefore
\[
\frac{\Delta \rho_{\text{max}}}{\rho_{\text{min}}} = \frac{282,400}{12} \times 9 \left(1 \pm \frac{6 \times 1.34}{12}\right)
\]

= 4380 lb. and 865 lb. per square foot respectively.

Since \( \rho_{\text{max}} \) does not exceed the safe load-bearing capacity, the size of the base is satisfactory; if the greatest imposed pressure had been greater or much less than 4480 lb. per square foot it would be necessary to alter the size of the base.

To determine the maximum bending moment, which by inspection appears to be the moment at plane \( Y-Y \), calculate
\[
\rho_x = 865 - 300 = 565 \text{ lb. per square foot}
\]
and, since \( y = 12 -(3 + 2) = 7 \text{ ft.} \)
\[
\rho_y = 565 + \frac{1}{2}(4380 - 865) = 2615 \text{ lb. per square foot.}
\]

Therefore
\[
M_Y = [(565 \times 7 \times \frac{1}{2} \times 7) + (2615 - 565)(\frac{1}{2} \times 7)]9 \times 12 = 3,300,000 \text{ in.-lb.}
\]

If the top of the base is 30 in. square (\( b = 30 \text{ in.} \)), the effective depth (\( d_1 \text{ in.} \)) must be not less than \( \sqrt{\frac{3,300,000}{184 \times 30}} = 24.5 \text{ in.} \); therefore a base 36 in. thick (\( d_1 = \text{say, 34 in.} \)) is satisfactory.

\[
A_{st} = \frac{3,300,000}{20,000 \times \frac{1}{4} \times 34} = 5.55 \text{ sq. in., which is provided by twelve}
\]
\( \frac{1}{4} \)-in. bars spaced at 9-in. centres in the direction of dimension \( B \). It is necessary to check that this reinforcement is sufficient at plane \( X-X \), for which \( \rho_1 = 4380 - 300 = 4080 \text{ lb. per square foot, and} \)
\[
\rho_x = 4080 -(\frac{1}{4} \times 2050) = 3568 \text{ lb. per square foot.}
\]
\[
M_X = [(3568 \times 3 \times \frac{1}{2} \times 3) + (4080 - 3568)(\frac{1}{4} \times 3) \times \left(\frac{1}{2} \times 3\right)]9 \times 12 = 1,890,000 \text{ in.-lb.}
\]

Therefore the bars provided at plane \( Y-Y \) are sufficient to resist the bending moment on the other side of the column and some curtailment is possible.

To calculate the transverse bending moment, it is sufficient to consider the average intensity of net reaction of the ground acting at plane \( Z-Z \), that is \( \frac{1}{2}(4080 + 565) = 2323 \text{ lb. per square foot acting on an area 12 ft. long and} \)
\( (\frac{1}{4} \times 9) - (\frac{1}{4} \times 2) = 3 \text{ ft. 6 in. wide.} \)
\[
M_Z = 2323 \times 3.5 \times \left(\frac{1}{4} \times 3.5\right) \times 12 \times 12 = 2,050,000 \text{ in.-lb.}
\]
\[
A_{st} = \frac{2,050,000}{20,000 \times \frac{1}{6} \times 33} = 3.55 \text{ sq. in., which is provided by nine} \frac{1}{6} \text{-in.}
\]
bars grouped at 6-in. centres under the column, with similar bars at say, 12-in. centres between this group and the edges of the foundation.

Consider the resistance to shearing at the plane to the left of \( Y-Y \).
\( Q_{\text{approx.}} \) is less than \( (\frac{1}{4} \times 9)(2615 + 565) \left(7 - \frac{34}{12}\right) = 59,600 \text{ lb.} \) If the base slopes down to 9 in. at the edges, \( d_0 \) (by calculation or graphical construction) is about 18 in. The shearing stress is therefore

\[
\frac{59,600}{\frac{1}{8} \times 18 \times (9 \times 12)} = 35 \text{ lb. per square inch.}
\]

The type of eccentrically-loaded base in Fig. 73b is not economical if
the applied moments or eccentricity of the load are great. In this case
the base should, if possible, be linked to an adjacent base to form a pair
of tied bases or a balanced foundation as described on pages 240 and 238
respectively. In some structures it is possible to select a position of the
base relative to the column so that a resultant eccentricity is avoided or
reduced to the least amount. An example is the base for a column of a
rectangular frame as in Fig. 73c. The actions on the base are the vertical
load \( P \) which is assumed to act on the centre-line of the column, a moment \( M \)
which acts counter-clockwise if the column is the left-hand column of the
frame, and a horizontal force \( F_H \) acting at the top of the base. For there
to be no resultant eccentricity, the algebraic sum of the moments, that is
\( M + F_H d' - P e_P \), should be equal to zero, that is, \( e_P \) should be equal
to \( \frac{M + F_H d'}{P} \), in which case there is uniform pressure on the ground. If
this optimum condition can be obtained as regards the primary actions
\( M, F_H, \) and \( P \) it is likely that under other conditions, say, when wind on
the structure of which the frame forms a part is taken into account, an
exact balance will not be obtained and there will be some eccentricity
and therefore a non-uniform pressure on the ground.

**Polygonal Chimney Foundation.**—The foundation of a chimney is
an example of a separate base that cannot be linked to adjacent bases or be
arranged in such a manner that the resultant eccentricity is eliminated.
The foundation supports the shaft directly or a plinth to which the flues
are connected. It may be circular on plan, although it is more often
polygonal, for instance octagonal (Fig. 74), thereby simplifying construction
compared with a circular base. The vertical load is constant and is due
to the weight of the chimney shaft (which may be of brick, concrete, or
steel), a plinth of concrete or brickwork, the refractory lining, and the
flues. The moment \( M \) and horizontal force \( F_H \) are due to wind on the
chimney and act at the top of the base. The total moment at the bottom
of a base of thickness \( d' \) is therefore \( M + F_H d' \). If the base is octagonal
and the width across the flats is \( B \), the area of the base is \( 0.828B^2 \) and the
least section modulus, which is that about an axis through the corners,
is \( 0.102B^3 \). If the weight of the base is \( P_B \), the pressures on the ground are

\[
\text{On leeward side: } p_{\text{max}} = \frac{P + P_B}{0.828B^2} \pm \frac{M + F_H d'}{0.102B^2}.
\]

\[
\text{On windward side: } p_{\text{min}} = \frac{p_{\text{max}}}{0.828B^2}.
\]

\[ (12.1c) \]

The size of the base is best determined by trial and must comply with
the conditions that \( p_{\text{max}} \) must not exceed the safe bearing pressure and
\( p_{\text{min}} \) should be positive, that is \( \frac{M + F_H d'}{0.102B^3} \) should not exceed \( \frac{P + P_B}{0.828B^2} \).

For this condition \( B \) should be not less than \( 8.12 \cdot \frac{M + F_H d'}{P + P_B} \), or the weight
of the base \( P_B \) should be not less than \( \frac{8.12}{B} (M + F_H d') - P \). The chimney
and base must also be stable, that is the moment $M + F_{Hd'}$ tending to overturn the base about one edge must be counteracted by the total downward force $P + P_B$ acting at the centre, that is $(P + P_B)(\frac{1}{2}B)$ must be not less than $M + F_{Hd'}$ multiplied by a factor of safety, say, $1\frac{1}{4}$ or more. Stability is assured if the minimum weight $P_B$ given in the fore-

![Diagram](image)

Fig. 74.—Octagonal Base for Chimney.

...going is provided. (Similar expressions are readily derived for bases of other shapes.)

The complexity, or otherwise, of the calculation of the bending moments and shearing forces on a polygonal base depends on the assumptions and approximations made in the computation. The simple method described in the following gives reasonable results. The assumptions are: (i) that the base is monolithic with the plinth (or shaft) and that the critical bending moment and shearing force occur at the vertical plane at the face
of the plinth or shaft (this is not necessarily true for a base supporting a steel or brick chimney); (ii) that the ratio of the projection of the base beyond the plinth or shaft to the thickness of the base is small (say, five or less) so that the base is so stiff that deformation due to bending is negligible and therefore the pressures obtained from formula (12.10) are not reduced because of flexibility; (iii) that the ratio of the width of the base to the width of the plinth does not exceed about two (for such proportions the bending moments are very similar to those obtained by the exceedingly complex analysis of a circular elastic plate); (iv) that a polygonal base of eight or more sides is considered to be equivalent (for the purpose of calculating bending moments and shearing force only) to a circular base of diameter equal to that of the inscribed circle; (v) that the critical plane for shearing forces is at a distance from the plinth (or shaft) equal to the effective depth of the base (this is also true for the base of a steel or brick chimney).

Consider a sector of a base of width $B$ supporting a cylindrical plinth or shaft of diameter $D$. If the width of the sector at the edge of the plinth is unity ($1 \text{ ft.}$), the width at the edge of the base is $\frac{B}{D}$ ft. as shown in Fig. 74b in which also the variation of the maximum intensity of the net reaction of the ground is shown. The pressure $p_D$ is

$$p_0 + \frac{D}{B}(p_D - p_0).$$

The total reaction of the ground $P_s$ on the projecting length $\frac{1}{4}(B - D)$ is represented by the volume of the block of pressure under this length. The block comprises an upper trapezoidal block of uniform depth $p_0$, which represents the pressure due to the vertical load, and a lower truncated pyramidal block of depth varying from $(p_0 - p_0)$ at the edge of the base to $(p_D - p_0)$ at the edge of the plinth and representing the increase in pressure due to the moment. Consideration of the shape of the combined block indicates that the distance $X$ from the critical plane to the centre of gravity of the block is between half and two-thirds of the projection, say, about $\frac{2}{3} \times \frac{1}{2}(B - D)$. The bending moment on unit length at the face of the plinth or shaft is $P_s X$. The compressive resistance to the bending moment determines the thickness of the base, which can be sloped downwardly to the edge. The reinforcement to resist the bending moment can be arranged radially as shown in Fig. 74d, or as a rectangular grid.

The shearing stress can also be considered in two parts. The shearing force on the surface of a cylinder of diameter $D + 2d_1'$ due to the vertical load $P$ is the load on the annulus of width $\frac{1}{4}(B - D - 2d_1')$ and internal radius $\frac{1}{2}D + d_1'$, that is $P - \frac{1}{4}\pi p_0 (D + 2d_1')^2$. On a vertical plane $1 \text{ ft.}$ wide at this radius, the corresponding shearing force is

$$\frac{P}{\pi(D + 2d_1')} - \frac{p_0}{4}(D + 2d_1').$$
The additional shearing force per foot due to the moment \( M + F_H d' \) is represented by the volume of a truncated pyramid of length 
\[
\frac{1}{4}(B - D - 2d'_1),
\]
of width varying from \( \frac{B}{D} \) to \( \frac{B}{D + 2d'_1} \), and of depth varying from \( (p_b - p_0) \) to \( p_b \), which is \( \frac{D + 2d'_1}{B}(p_b - p_0) \). The total shearing force on unit width is the sum of these two component shearing forces. The effective depth of the plane at which this shearing acts is not less than \( d'_0 \), which can be calculated from the geometry of a right cross-section of the base.

The application of the principles described in the foregoing is shown in the example which follows.

**Example No. 91.—Chimney Foundation.** Design a base for a chimney the data for which are given in Fig. 74c, namely, diameter \( D \) of plinth (monolithic with the base) = 22 ft.; weight \( P \) of plinth, shaft, lining, etc. = 1250 tons. Moment \( M \) due to wind = 6000 ft.-tons. Horizontal force \( F_H \) (at top of base) due to wind = 50 tons. Safe load-bearing capacity of ground = \( 1\frac{1}{2} \) tons per square foot.

Assume as a first trial an octagonal base of thickness \( d' = 5 \) ft. \( (d'_1 = 57 \text{ in.}) \) and distance \( B \) across the flats = 48 ft. If the base is splayed downwards to 1 ft. at the edges, the weight \( P_B \) is about 510 tons. The area of the base is \( 0.828 \times 48^2 = 1900 \) sq. ft. Substitution in formula (12.1c) gives

\[
\frac{p_{\text{max}}}{p_{\text{min}}} = \frac{1250 + 510}{1900} \left[ 1 \pm \frac{6000 + (50 \times 5)}{0.102 \times 48^3} \right] = 2240 \left[ \frac{3315}{835} \right] \text{lb. per square foot.}
\]

The factor of safety against overturning is

\[
\frac{(1250 + 510)(\frac{1}{2} \times 48)}{6000 + (50 \times 5)} = 6\frac{1}{2}.
\]

Also \( P_B = \frac{510 \times 2240}{1900} = 600 \) lb. per square foot;

\[
p_0 = \frac{1250 \times 2240}{1900} = 1470 \text{ lb. per square foot;}
\]

\[
p_b = 3315 - 600 = 2715 \text{ lb. per square foot;}
\]

and \( p_D = 1470 + \frac{22}{48}(2715 - 1470) = 2040 \) lb. per square foot.

The total reaction \( P_s \) represented by the volume of the pressure blocks is:

Trapezoidal block: \( \frac{1470}{2} \left( 1 + \frac{48}{22} \right) \left( \frac{48 - 22}{2} \right) = 30,600 \) lb.

Pyramidal block:

\[
x \left[ (2715 - 1470)\frac{48}{22} + (2040 - 1470) + \sqrt{2720 \times 570} \right] = 19,650 \text{ lb.}
\]

\[
P_s = 50,250 \text{ lb.}
\]

Arm of moment = \( \frac{1}{8} \left( \frac{48 - 22}{2} \right) \) approximately = 8 ft. approximately.
Bending moment = 50,250 × 8 × 12 = approximately 4,800,000 in.-lb.

\[ d_1 < \sqrt{\frac{4,800,000}{184 \times 12}} = 47 \text{ in.}; \text{ therefore } d' = 5 \text{ ft. is satisfactory.} \]

\[ A_{sl} = \frac{4,800,000}{20,000 \times \frac{3}{8} \times 57} = 4.8 \text{ sq. in. per foot; provide three } \frac{1}{2} \text{-in. bars radially in each foot of periphery below the edge of the plinth. It is usual to provide a mesh in the top of the base under the plinth to resist stresses due to temperature; } 1 \text{-in. bars at } 12 \text{-in. centres are satisfactory for this purpose (about 0.1 per cent.).} \]

Shearing force: \( D + 2d_1' = 22 + 9.5 = 31.5 \text{ ft.} \)

\[ \rho_s = \frac{31.5}{48} (2715 - 1470) = 817 \text{ lb. per square foot.} \]

Due to \( P: \)

\[ \frac{1250 \times 2240}{31.5 \pi} - \left( \frac{1470}{4} \times 31.5 \right) = 16,700 \text{ lb.} \]

Due to moment:

\[ \frac{48 - 31.5}{6} \times \left( \frac{(2715 - 1470)48}{22} + \left( 817 \times \frac{48}{31.5} \right) + \sqrt{2720 \times 1245} \right) = 16,000 \]

Total = 32,700 lb.

By construction, \( d_0 = 40 \text{ in. approximately; } q = \frac{32,700}{72 \times \frac{3}{8} \times 40} = 13 \text{ lb. per square inch.} \)

**Interconnected Bases with Eccentric Loads.**

**Balanced Foundations.**—If a base is subjected to an eccentric load and there is an adjacent base to which it can be connected, the two bases can be designed as a combined balanced foundation imposing uniform pressure on the ground. Consider that base \( A \) in Fig. 75a is subjected to a load \( P_1 \) which may vary from \( P_1(\text{min.}) \) to \( P_1(\text{max.}) \) and acts off-centre at a distance \( \varepsilon \). Base \( B \) is subjected to a concentric load \( P_2 \) which may vary from \( P_2(\text{min.}) \) to \( P_2(\text{max.}) \). The two bases, the weights of which are \( P_A \) and \( P_B \) respectively, are connected by a beam of weight \( P_b \). Taking moments about the centre of base \( A \), the upward force \( R \) at base \( B \) due to the eccentricity of \( P_1 \) on \( A \) is \( \frac{P_A \varepsilon}{L} \). There is an equal additional downward force on foundation \( A \). Therefore the loads \( R_A \) and \( R_B \) on the ground are

\[
\begin{align*}
R_A(\text{max.}) &= P_1(\text{max.}) + P_A + \frac{1}{2}P_b + R_{\text{max.}} \\
R_A(\text{min.}) &= P_1(\text{min.}) + P_A + \frac{1}{2}P_b + R_{\text{min.}} \\
R_B(\text{max.}) &= P_2(\text{max.}) + P_B + \frac{1}{2}P_b - R_{\text{min.}} \\
R_B(\text{min.}) &= P_2(\text{min.}) + P_B + \frac{1}{2}P_b - R_{\text{max.}}
\end{align*}
\]

\[
\left\{ \begin{array}{l}
\text{in which} \\
R_{\text{max.}} = \frac{\varepsilon P_1(\text{max.})}{L} \\
R_{\text{min.}} = \frac{\varepsilon P_1(\text{min.})}{L}
\end{array} \right.
\]

Base \( A \) must be designed for a concentric load \( R_A(\text{max.}) \) and base \( B \) for
a concentric load $R_B^{(\text{max})}$. To ensure that there is sufficient load at B to counterbalance the upward force $R_{\text{max}}$, $R_B^{(\text{min})}$ must be positive and should be great enough to ensure that $P_2^{(\text{min})} + P_B + \frac{1}{2}P_b$ exceeds $R_{\text{max}}$ by sufficient margin to ensure a factor of safety of, say, $1\frac{1}{2}$. The load $R_{A^{(\text{min})}}$ is not critical since it is always positive and is less than $R_A^{(\text{max})}$. The bending moment on the beam is the cantilever moment produced by $R$, and the greatest bending moment therefore occurs at section X—X and

Fig. 75.—Balanced Foundations.

is $IR_{\text{max}}$, if the counter-effect of the weight of the beam is neglected. Likewise the greatest shearing force on the beam is equal to $R_{\text{max}}$.

**Example No. 92.—Balanced Foundation.** Design a balanced foundation for the two columns in Fig. 75b, the data being:

$P_1^{(\text{max})} = 300,000$ lb.; $P_1^{(\text{min})} = 200,000$ lb.;
$P_2^{(\text{max})} = 500,000$ lb.; $P_2^{(\text{min})} = 350,000$ lb.

The distance $(L + \varepsilon)$ between the centres of the columns is $20$ ft. The distance between the centre-line of the column on base A and the edge of the base must not exceed $2$ ft. $6$ in. The safe load-bearing capacity of the ground is $2$ tons per square foot.

Assuming that the weight of the bases is $300$ lb. per square foot, the net intensity of imposed pressure must not exceed $4180$ lb. per square foot.

Therefore the area of base A must be not less than $\frac{300,000}{4180} = 72$ sq. ft., omitting the effect of $R_{\text{max}}$. As a first trial assume a base A $9$ ft. square; then $\varepsilon = 2$ ft. and $L = 18$ ft. By substitution in formula (12.2),

$$R_{\text{max}} = \frac{2 \times 300,000}{18} = 33,300 \text{ lb.}; \quad R_{\text{min}} = \frac{2 \times 200,000}{18} = 22,200 \text{ lb.}$$

The area of base B should be not less than $\frac{500,000}{4180} = 120$ sq. ft. As a first trial assume a base B $11$ ft. square. The net length $l_b$ of the beam is then $8$ ft. Assume that the beam is $24$ in. wide and $36$ in. deep; the weight $P_b$ is then $7200$ lb.
Base A.—

\[ P_A = 9^2 \times 300 = 24,300 \text{ lb.} \]
\[ R_{A(\text{max.})} = 300,000 + 24,300 + \left( \frac{1}{4} \times 7200 \right) + 33,300 = 361,200 \text{ lb.} \]

Pressure imposed on the ground \( \frac{361,200}{9^2 \times 2240} = 2 \text{ tons per square foot.} \)

Base B.—

\[ P_B = 11^2 \times 300 = 36,300 \text{ lb.} \]
\[ R_{B(\text{max.})} = 500,000 + 36,300 + 3600 - 22,200 = 517,700 \text{ lb.} \]

Pressure imposed on the ground \( \frac{517,700}{11^2 \times 2240} = 1.91 \text{ tons per square foot.} \)

By inspection, \( R_{B(\text{min.})} \) is positive.

Beam. \( l = 13 \text{ ft.} \ 6 \text{ in.; } l_b = 8 \text{ ft.; } P_b = 7200 \text{ lb.} \) Neglecting the net pressures imposed on the ground under the bases (which are practically self-counteracting), the maximum bending moment on the beam, allowing for the weight of the beam, is
\[ (33,300 \times 13.5 \times 12) - (7200 \times \frac{1}{4} \times 8 \times 12) = 5,054,000 \text{ in.-lb.} \]

\[ d_1 = \sqrt{\frac{5,054,000}{184 \times 24}} = 33.8 \text{ in.; therefore } d = 36 \text{ in.} \ (d_1 = 34 \text{ in.}) \text{ is satisfactory.} \]

\[ A_n = \frac{5,054,000}{20,000 \times 0.86 \times 34} = 8.65 \text{ sq. in., say, five } \frac{3}{4}-\text{in. bars near the top at the junction with base A; since the bending moment decreases rapidly towards base B, the bars may be curtailed, but two should extend into and across base B.} \]

Shearing force \( = 33,300 - \left( \frac{1}{4} \times 7200 \right) = 29,700 \text{ lb.;} \)

\[ q = \frac{29,700}{24 \times 0.86 \times 34} = 42 \text{ lb. per square inch.} \]

The thickness of the bases and the reinforcement near the bottom are determined in the same way as for a concentrically-loaded base.

**Tied Bases.**—The analysis of two connected bases (*Fig. 76*) is similar to, but more complex than, that of a simple balanced foundation because such bases are generally provided as the foundation of a trestle supporting a gantry or similar structure which imposes on the foundation moments and horizontal forces in addition to vertical loads. Since some of the actions on the foundation are due mainly to wind, they may act in either direction. For this reason, each base is generally placed centrally under the column or other member imposing the load. The tie is designed to relieve each base of the moment so that the base can be designed as concentrically loaded. Any tendency for a base to tilt due to the eccentric actions causes a corresponding deformation of the tie-beam, which is therefore made strong enough to resist this tendency. The bending of the tie-beam is as shown in *Fig. 76a*, the maximum bending moments (excluding the effect of the weight of the beam) being \( M + F_Hd' \) at base A and also at base B. The shearing force \( Q \) on the tie-beam, due to this bending moment,
is constant and is equal to the rate of change of the bending moment; it is assumed that the tie-beam is flexible compared with the rigid bases.

The resultant loads on the bases, with the actions as shown in Fig. 76a, are a concentric load $P_1$ minus $Q$ on base $A$, and a concentric load $P_2$ plus $Q$ on base $B$; that is, if $P_b$ is the total weight of the tie-beam,

$$Q = (M + F_Hd')^2$$

$$R_{A_{(\text{min})}} = P_{1(\text{min})} + P_A + \frac{1}{3}P_b - Q$$

$$R_{B_{(\text{max})}} = P_{2(\text{max})} + P_B + \frac{1}{3}P_b + Q$$

\[\begin{align*}
\end{align*}\]
The loads $P_1$ and $P_2$ are variable, and when $P_2$ (which is assumed to be on the leeward side) is greatest, the load $P_1$ may be negative, that is $P_1$ may be an upward pull on the windward side. It is therefore essential that the weight of each base be sufficiently great to ensure that $R_{A_{\text{min.}}}$ and $R_{B_{\text{min.}}}$ are positive. The area of each base must be sufficient to distribute the greatest load on to the ground without exceeding the permissible bearing pressure.

**Example No. 93.—Foundation for a Trestle.** Design a foundation for the steel trestle of a conveyor gantry the legs of which are 20 ft. apart at ground level. The actions imposed on each base are a vertical load of 10 tons due to the dead load, and 18 tons vertically, 4 tons horizontally, and a moment of 30 ft.-tons due to wind. When the wind blows from left to right these actions are as in Fig. 76b, that is

$$P_{1_{\text{min.}}} = +10 - 18 = -8 \text{ tons;}$$

$$P_{2_{\text{max.}}} = +10 + 18 = +28 \text{ tons;}$$

$$F_H = +4 \text{ tons;}$$

and $M = 30 \text{ ft.-tons (clockwise).}$ These actions will be reversed if the wind blows from right to left, so both bases must be the same. The safe load-bearing capacity of the ground $q_a$ is $1\frac{1}{4}$ tons per square foot.

Determine trial sizes. To ensure that $R_{A_{\text{min.}}}$ is positive, the weight of each base must exceed $P_{1_{\text{min.}}} + Q$; say, 16 tons. The area of each base must be about $\frac{P_{2_{\text{max.}}} + P_B + Q}{q_a}$, that is $28 + 16 + Q \over 1\frac{1}{4}$ sq. ft., say, 7 ft. square. The thickness $d'$ is therefore about $\frac{16 \times 2240}{7^2 \times 150}$ ft., say, 5 ft.

Assume that the tie-beam is 24 in. deep and 12 in. wide. With these dimensions, $P_A = P_B = \frac{7^2 \times 5 \times 150}{2240} = 16.4$ tons; $l = 20 - 7 = 13$ ft.; and $P_b = 13 \times 2 \times 1 \times 150 \div 2240 = 1.74$ tons.

By substitution in formulae (12.3),

$$Q = \frac{2[30 + (4 \times 5)]}{13} = 7.7 \text{ tons;}$$

$$R_{A_{\text{min.}}} = -8 + 16.4 + 0.87 - 7.7 = +1.57 \text{ tons, which being positive is satisfactory;}$$

$$R_{B_{\text{max.}}} = +28 + 16.4 + 0.87 + 7.7 = 52.97 \text{ tons.}$$

The maximum pressure imposed on the ground is $\frac{52.97}{49} = 1.08$ tons per square foot, which does not exceed the safe pressure.

The bending moment on the beam is $30 + (4 \times 5) = 50$ ft.-tons, say, 1,500,000 in.-lb. including the effect of the weight of the beam. With $A_{st} = A_{sc}$, the effective depth must be not less than

$$\sqrt{\frac{1,500,000}{406 \times 12}} = 17.5 \text{ in.;}$$

therefore $d' = 2$ ft. ($d_1 = 22$ in., $I_a = 20$ in. approximately) is satisfactory, and $A_{sc}$ can be less than $A_{st}$ which should be about

$$\frac{1,500,000}{20,000 \times 20} = 3.75 \text{ sq. in.;}$$

this area is provided by four 1$\frac{1}{4}$-in. bars in the top of the beam; provide also four 1$\frac{1}{4}$-in. bars at the bottom (as shown in Fig. 76b) to resist the reverse
bending moment and to allow for the wind blowing in the opposite direction.

The shearing stress is \( \frac{(7.7 + 0.87) \times 2240}{20 \times 12} = 80 \) lb. per square inch; therefore provide nominal binders, say, \( \frac{3}{8} \)-in. binders at 12-in. centres.

Since the bases are so deep compared with their breadth, they can be constructed as plain-concrete bases without reinforcement. The bolts securing the steel base-plates of the trestles must extend to the bottom of each base, as in Fig. 76b, to ensure that the whole weight of the base counteracts the uplift.

**Combined Foundations with Concentric Load.**

When two or more loads are so close together that separate square bases would overlap, it is generally more convenient and economical to provide one base arranged so that its centroid coincides with the centre of gravity of the loads, as by so doing the pressure is imposed uniformly on the ground.

**Rectangular Base.**—Consider a rectangular base carrying two loads \( P_1 \) and \( P_2 \) (Fig. 77a). The conditions are that (i) the area \( BL \) of the foundation of weight \( P_B \) must be not less than \( \frac{P_1 + P_2 + P_B}{q_a} \), and (ii) for coincidence of centroids, the distance \( x \) of, say, \( P_2 \) from one end of the base must comply with the relation that \( P_2x + P_1(l + x) = \frac{1}{2}B(P_1 + P_2) \). If a value for the width \( L \) is assumed, the length \( B \) can be determined from condition (i) and \( x \) from condition (ii). If \( B \) or \( x \) or both dimensions are unsuitable, another value of \( L \) must be selected.

The base acts as an inverted beam spanning longitudinally between the points of application of \( P_1 \) and \( P_2 \) and cantilevered beyond these points. The net reaction of the ground \( p_0 \) is \( P_1 + P_2 \) \( \frac{BL}{\sqrt{B}} \), and the cantilever bending moments are \( \frac{1}{2}p_0y^2L \) at \( P_1 \) and \( \frac{1}{2}p_0x^2L \) at \( P_2 \). The maximum ordinate of the free bending moment at the midpoint between \( P_1 \) and \( P_2 \) is \( \frac{1}{2}p_0l^2L \), and the resultant diagram is as in Fig. 77a. If \( M_1 \) and \( M_2 \) are unequal it is better to sketch this diagram to scale to determine the maximum bending moment, rather than to calculate it from formulae. If \( l \) is small it is probable that there will be no reversal of bending moment as illustrated. The entire width of the base is available to resist these bending moments and for small bases a rectangular cross-section is suitable. For wide and deep bases an inverted tee-section or a ribbed base as in Fig. 77a is economical.

Transverse bending occurs due to the action of distributing the concentrated loads on to the entire area of the ground under the base. The total transverse bending moment due to any concentrated load \( P \) is \( \frac{1}{4}P \times \frac{1}{4}L \times \frac{1}{4} \), that is \( \frac{1}{4}PL \). In a base of rectangular cross-section reinforcement to resist this cantilever moment is provided by a band of transverse bars near the bottom under the column (or other load) and the
reinforcement resisting longitudinal bending can then be distributed over the entire width of the base. If a ribbed base is provided the longitudinal reinforcement is concentrated in the width of the rib, and the transverse reinforcement is distributed uniformly throughout the length of the base, the transverse bending moment on unit length being \( \frac{1}{2} \rho_0 L^2 \).

![Diagram of a rectangular combined base with longitudinal and transverse bending moment diagrams, critical planes for shearing resistance, and cross section of ribbed base with transverse reinforcement.]

Fig. 77.—Rectangular Combined Base.

The critical planes for resistance to shearing force on a base of rectangular cross-section are at a distance \( d_1 \) from the sides of the column or other loaded area. The total shearing force on these planes is the load \( P \) minus the total reaction of the ground on the square (or rectangle) bounded by the four planes. If the dimensions \( L \) or \( x \) are so small that the square (or rectangle) cannot be completed, or if a ribbed base is provided, the
shearing forces should be calculated and dealt with as for an ordinary double-cantilevered beam.

**Example No. 94.—Concentrically-loaded Base to Support Two Loads.** Design a rectangular base to support loads of 30 tons ($P_1$) and 50 tons ($P_2$) spaced at 10-ft. (l) centres. The loaded area in each case is 1 ft. 6 in. square. The safe load-bearing capacity of the ground is 1 ton per square foot.

Assuming that the weight of the base is not more than 240 lb. per square foot, the net load-bearing capacity of the ground available to carry the loads is 2000 lb. per square foot and the area of the base must be not less than \[
\frac{(30 + 50) \times 2240}{2000} = 90 \text{ sq. ft.}
\]

If the width $L$ of the base is 5 ft., the length $B$ is \[
\frac{90}{5} = 18 \text{ ft.}
\]

For coincidence of centroids, \[
5x + 30(10 + x) = \frac{1}{2} \times 18(30 + 50),
\]
that is $x = 5\cdot25$ ft., and therefore $y = 18 - (10 + 5\cdot25) = 2\cdot75$ ft., which are suitable dimensions.

**Under $P_1$:** \[
M_1 = \frac{1}{2} \times 2000 \times 2\cdot75^2 \times 5 \times 12 = 453,000 \text{ in.-lb.}
\]

**Under $P_2$:** \[
M_2 = \frac{1}{2} \times 2000 \times 5\cdot25^2 \times 5 \times 12 = 1,650,000 \text{ in.-lb.}
\]

Free bending moment \[
\frac{2000 \times 10^2 \times 5 \times 12}{8} = 1,500,000 \text{ in.-lb.}
\]

If a bending-moment diagram similar to that in Fig. 77a is drawn, it is shown that the maximum bending moment at a plane between the loads is about 450,000 in.-lb. and produces tension at the top of the base.

For a base of rectangular cross-section,

\[
d_1 = \sqrt{\frac{1,650,000}{184 \times 60}} = 12\cdot2 \text{ in., say } d' = 12\cdot25 \text{ ft.}
\]

For a ribbed base with $b_r = 12$ in.,

\[
d_1 = \sqrt{\frac{1,650,000}{184 \times 12}} = 27\cdot4 \text{ in., say } d' = 2\cdot5 \text{ ft.}
\]

A ribbed base requires more shuttering but contains less concrete and reinforcement, and is likely to be the more economical. Therefore provide a ribbed base as in Fig. 77b with $d_1 = 28$ in. and $t_a = 24$ in. approximately.

**Under $P_2$:** \[
A_{st} = \frac{1,650,000}{20,000 \times 24} = 3\cdot44 \text{ sq. in.; provide four } 1\frac{3}{4} \text{-in. bars.}
\]

**Under $P_1$:** \[
A_{st} = \frac{453,000}{20,000 \times 24} = 0\cdot95 \text{ sq. in.; provide two } 1\frac{3}{4} \text{-in. bars.}
\]

The amount of reinforcement required near the top between the loads is about the same, and may be provided conveniently as shown in Fig. 77b.

The maximum shearing force occurs at the left-hand side of $P_2$ and is

\[
(50 \times 2240) - \left(5\cdot25 + \frac{1\cdot5}{2}\right)(2000 \times 5) = 52,000 \text{ lb.;}
\]

\[
q = \frac{52,000}{12 \times 24} = 180 \text{ lb. per square inch.}
\]

The shearing force at the right-hand side of $P_2$ is

\[
[5\cdot25 - (\frac{1}{2} \times 1\cdot5)](2000 \times 5) = 45,000 \text{ lb.;}
\]

\[
q = 156 \text{ lb. per square inch.}
\]

Reinforcement to resist the shearing forces
is required on both sides of $P_3$ and can be provided conveniently by bending up two $14$-in. bars from the bottom at 45 deg. in a double system as shown in Fig. 77b. The shearing stresses at $P_1$ are less than 100 lb. per square foot, as can be shown by a calculation similar to the foregoing.

The transverse bending moment is

$$2000 \times \left( \frac{1}{2} \times 2.5^3 \right) \times 12 = 75,000 \text{ in.-lb.}$$

If the maximum thickness of the flange is 12 in.,

$$a_1 = 10\frac{1}{4} \text{ in.} \quad \text{and} \quad l_a = 9\frac{3}{4} \text{ in.};$$

$$A_{st} = \frac{75,000}{20,000 \times 9\frac{3}{4}} = 0.4 \text{ sq. in.}; \text{ provide } \frac{1}{4}\text{-in. bars at 6-in. centres.}$$

Multiple-rectangle Base.—If the loads $P_1$ and $P_3$ differ considerably in magnitude it may be impossible to provide a rectangular base of suitable dimensions $B$ and $L$, in which case a non-rectangular base must be provided. One form of base is as shown in Fig. 78a, where two adjoining rectangles are provided such that the centroid of each rectangle is under the respective load. The areas $B_1L_1$ and $B_2L_2$ are proportional to $P_1$ and $P_3$, respectively, and are such that the safe pressure on the ground is not exceeded. It is necessary to assume one of the dimensions $B_1$, $B_3$, $L_1$, or $L_3$ (site conditions may dictate one of them); the remainder can be determined by simple calculation since $l$ is known. The subsequent calculation of the bending moments and shearing forces and the provision of resistance to these moments and forces proceed as for a rectangular base with $L_1$ and $L_2$ substituted for $L$ as necessary.

Trapezoidal Base.—Another, and more suitable, form of non-rectangular base is a trapezoidal as in Fig. 78b, the centroid of which is at the centre of gravity of the two loads. Such a base may be of rectangular cross-section or it may be ribbed. The design data are derived from the geometrical properties of a trapezium as described in the example which follows.

Example No. 95.—Trapezoidal Base. Design a trapezoidal base for the same loads and conditions as in Example No. 94, such that the dimensions $x$ and $y$ (Fig. 78b) are equal.

The known data are $l = 10 \text{ ft.}$,

$$B = (10 + 2y) \text{ ft.}, \quad \text{and} \quad \frac{1}{3}(L_1 + L_2)B = 90 \text{ sq. ft.}$$

For coincidence of centroids,

$$X = \frac{30y + 50(10 + y)}{80} = \frac{B}{3} \left( \frac{L_1 + 2L_2}{L_1 + L_2} \right).$$

It is necessary to assume one of the dimensions $y$, $L_1$, or $L_2$; in this example, assume $y = 3 \text{ ft.}$ Then $B = 16 \text{ ft.}; L_1 + L_2 = \frac{2 \times 90}{16} = 11.25 \text{ ft.};$ therefore $L_2 = 11.25 - L_1$, and

$$\left( \frac{30 \times 3}{80} \right) + \left( \frac{50 \times 13}{3} \right) = \frac{16}{3} \left( \frac{22.5 - L_1}{11.25} \right),$$

from which $L_1 = 3 \text{ ft. 6 in.}$, and $L_2 = 7 \text{ ft. 9 in.}$ The dimensions on plan are therefore as in Fig. 77c. Also

$$L_{11} = 3.5 + \frac{1}{8}(7.75 - 3.5) = 3.5 + 0.8 = 4.3 \text{ ft.};$$

$$L_{22} = 7.75 - 0.8 = 6.95 \text{ ft.}$$

The longitudinal bending moments are
Fig. 78.—Non-rectangular Combined Bases.

Under $P_1$: $M_1 = 2000\left[\left(\frac{1}{3} \times 3.5 \times 3\right) + \left(\frac{1}{3} \times 0.8 \times 3\right)\right] \times 12$

$= 435,600$ in.-lb. $= 435,600 \div 4.3 = 101,000$ in.-lb. per foot.

Under $P_2$: $M_2 = 2000\left[\left(\frac{1}{3} \times 6.95 \times 3\right) + \left(\frac{1}{3} \times 3^2 \times 2\right)\right] \times 12$

$= 865,800$ in.-lb. $= 865,800 \div 6.95 = 124,500$ in.-lb. per foot.

The maximum free bending moment between the loads is about

$2000\left[\frac{1}{3}(4.3 + 6.95)\right] \times 12 = 1,687,500$ in.-lb.

The net bending moment (from a diagram or approximate calculation) is
about 1,050,000 in.-lb. (tension at the top of base), that is
\[
\frac{1,050,000}{5.5 \text{ approx.}} = 191,000 \text{ in.-lb per foot.}
\]

If base is of rectangular cross-section, \( d_1 = \sqrt{\frac{191,000}{184 \times 12}} = 9.3 \text{ in.} \); provide a base 12 in. thick; \( d_1 = 10.4 \text{ in.} \) approximately and \( l_4 \) is about 9 in. The cross-sectional areas of longitudinal reinforcement are:

Under \( P_1 \): \( A_{st} = \frac{435,600}{20,000 \times 9} = 2.42 \text{ sq. in.} \) Provide six \( \frac{3}{4} \)-in. bars near the bottom throughout the length.

Under \( P_2 \): \( A_{st} = \frac{865,800}{20,000 \times 9} = 4.82 \text{ sq. in.} \) Provide additional five \( \frac{3}{4} \)-in. bars near the bottom under the load.

Between loads: \( A_{st} = \frac{1,050,000}{20,000 \times 9} = 5.83 \text{ sq. in.} \) Provide nine \( \frac{3}{4} \)-in. bars throughout the length near the top of the base plus five \( \frac{3}{4} \)-in. bars bent up from the bottom and arranged as in Fig. 77c.

The transverse bending moments and reinforcement (in bottom: \( d_1 = 9\frac{3}{4} \text{ in.}, l_4 = 8\frac{1}{2} \text{ in.} \)) are:

Under \( P_1 \): \[
\frac{30 \times 2240 \times 4.3}{8} \times 12 = 434,000 \text{ in.-lb.} ;
\]
\( A_{st} = 2.55 \text{ sq. in.; provide, say, three } 1\frac{1}{2} \text{-in. bars.} \)

Under \( P_2 \): \[
\frac{50 \times 2240 \times 6.95}{8} = 1,170,000 \text{ in.-lb.} ;
\]
\( A_{st} = 6.88 \text{ sq. in.; provide, say, seven } 1\frac{1}{2} \text{-in. bars.} \)

The maximum shearing force occurs on the left-hand side of load \( P_2 \) and is
\[
(50 \times 2240) - 2000(6.95 - 7.25)\frac{1}{2} \times 3 = 69,400 \text{ lb.} ;
\]
\[
q = \frac{69,400}{9 \times 6.95 \times 12} = 93 \text{ lb. per square inch.}
\]

**Combined Foundations with Eccentric Loads.**

Owing to site restrictions it may be not possible to provide a base of such dimensions and in such a position that the centroid coincides with the centre of gravity of two loads. The consequent eccentricity results in non-uniform pressure on the ground. Other circumstances may also result in a combined foundation being eccentrically loaded. For example, the loads \( P_1 \) and \( P_2 \) may be not constant; that is their magnitudes may vary from, say, dead load only to dead load plus live load. The percentage variation due to live load may be the same for each load, in which case, if the base is loaded concentrically by the dead load it will also be loaded concentrically under conditions of the dead load plus the live load. This
coincidence may not always occur, in which case the greatest and least loads must be considered separately.

It is a sound rule to design foundations so that they are loaded concentrically under the most common loading. An office or residential building is rarely loaded to the full extent of the assumed imposed load, and the most common loading condition is dead load only; therefore the foundations should be concentrically loaded under dead load or, if this is impracticable, the eccentricity under dead load should be as small as possible.

In the case of a pair of columns, the most adverse conditions of loading should be considered, for example, dead load only on one column and dead load plus the greatest amount of live load on the other; this condition produces the greatest variation of pressure on the ground and may also result in the greatest intensity of pressure on the ground. The condition of greatest total load (dead load plus live load) on each column should also be considered, since this condition produces the greatest pressure on the ground and sometimes the greatest intensity of pressure.

The dead load on the foundations of storage structures such as warehouses, elevated reservoirs, bunkers, silos, and the like is generally only a small fraction of the total load plus live load, and, since such structures are generally fully loaded, the foundations should be designed primarily for concentric loading (or little eccentricity) when they are fully loaded. The condition of dead load only should also be investigated, as should also the intermediate condition such as, in the case of a pair of columns, dead load only on one column and dead load plus the greatest live load on the other column, or as much of the live load as is possible with no live load on the companion column.

When the effects of wind form a considerable proportion of the load on a foundation (the bases of trestles supporting gantries are an example), each base should be loaded concentrically (or have little eccentricity) when the effects of wind are omitted, but the intensity of pressure on the ground is usually greatest when the effect of wind is taken into account. The effects of the wind are an increased thrust on the leeward column, a corresponding reduction of load on the windward column, and a horizontal force and moment at the bottom of each column. These effects can act in either one direction or the opposite direction depending on the direction of the wind; therefore it is not possible to avoid each base being loaded eccentrically, because if it is designed so that the pressure on the ground is uniform when the wind is in one direction, eccentricity will occur when the wind is in the contrary direction; see Example No. 93.

The general case of a rectangular base supporting two loads which load the base eccentrically is shown in Fig. 79. The method of design is similar to that for the foundations considered previously but modified to allow for the non-uniform pressure on the ground as explained in the example which follows.
Example No. 96.—Rectangular Combined Base Loaded Eccentrically. Reconsider the design of the rectangular base in Example No. 94 (Fig. 77b) if the load $P_1$ is only 20 tons and the load $P_2$ is 30 tons as a result of the omission of all or part of the live load.

![Diagram of rectangular base subjected to eccentric loads]

Fig. 79.—Rectangular Base Subjected to Eccentric Loads.

The weight of the base (at 150 lb. per cubic foot of concrete) is about 15,000 lb. (6.7 tons) or 170 lb. per square foot.

Centre of gravity of loads: \( 20 \times 275 = 550 \) ft.-tons.  
\( 30 \times 1275 = 382.5 \) " "  
\( 6.7 \times 90 = 603 \) " "  
\( 56.7 \) tons \( 497.8 \div 56.7 = 8.8 \) ft. from the left-hand end of the base. Therefore \( \delta = (\frac{1}{8} \times 18) - 8.8 = 0.2 \) ft.

\[ P_{\text{max}} = \left[ \frac{56.7}{5 \times 18} + \frac{6 \times 56.7 \times 0.2}{5 \times 18^2} \right] 2240 = 1505 \text{ lb. (maximum)} \]  
\[ P_{\text{min}} \geq \left[ \frac{56.7}{5 \times 18} - \frac{6 \times 56.7 \times 0.2}{5 \times 18^2} \right] 2240 = 1315 \text{ lb. (minimum)} \text{ per square foot.} \]

Therefore the net reaction of the ground varies from 1335 lb. per square foot at the left-hand end to 1145 lb. per square foot at the right-hand end. Since the pressures are low and do not vary greatly, it is sufficient to calculate the longitudinal bending moments as follows.

Under $P_1$: \[ M_1 = \frac{1}{2} \times 1335 \times 2.75^2 \times 5 \times 12 = 304,000 \text{ in.-lb.} \]

Under $P_2$: \[ M_2 = \frac{1}{2} \times 1145 \times 5.25^2 \times 5 \times 12 = 950,000 \text{ in.-lb.} \]

Between the loads:

Free bending moment \( = \left( \frac{1335+1145}{2} \right) \left( \frac{10^4 \times 5}{8} \right) \times 12 = 930,000 \text{ in.-lb.} \)

The maximum net bending moment (from a diagram) is about 300,000 in.-lb.

Since all the longitudinal bending moments due to the smaller eccentric loads are less than those for the greater concentric loads, the reinforcement provided in Fig. 77b is satisfactory. Similarly the reinforcement to resist the transverse bending moments and shearing forces is satisfactory.
Strip Foundations.

Where a line of columns or other loaded members is to be supported and independent square or rectangular bases would overlap or be close together, it is cheaper to provide a strip foundation extending under all the members. In doubtful cases it is worth while making rough designs of separate bases and a strip foundation, compare the quantities and costs, and select the cheaper. If the columns support a structure or machine which would be affected adversely should one column settle more than another, a combined foundation is essential.

Concentric Load.—If possible, a strip foundation should be arranged to be loaded concentrically, that is it should impose a uniform pressure on the ground either under the dead load for buildings or under the total load for storage structures. The area of such a foundation (Fig. 80a) is determined by dividing the total load $\Sigma P$ by the net permissible pressure on the ground, that is the safe permissible pressure minus the assumed weight of the foundation. For the condition of dead load only, the pressure on the ground must be sufficiently less than the safe load-bearing capacity to allow for the increase in pressure when the most adverse condition of live load occurs. For uniform distribution, the centroid of the foundation should coincide with the centre of gravity of the loads. If the uniform width $L$ of the foundation is assumed, the length $B$ can be calculated. The position of the foundation relative to the loads is arranged so that the relation $\frac{B}{2} = x + \frac{\Sigma P h}{\Sigma P}$ is satisfied. Adjustment of $B$ and $L$ may produce suitable dimensions, but, if it fails to give satisfactory proportions for a foundation of uniform width, a foundation of varying width may give a more ready coincidence of centroid and centre of gravity.

Eccentric Load.—If a satisfactory foundation cannot be provided by any of the foregoing methods, an eccentrically-loaded foundation must be provided. A foundation which is loaded concentrically for one condition of loading may be loaded eccentrically for other conditions. The pressure on the ground should be calculated for the most probable adverse condition of live load; that is by considering the live load to act on a group of columns at one end of the foundation only; the consequent eccentric loading may probably result in the greatest intensity of pressure. One or more conditions should be examined, but the assumed incidences of live load should be realistic and not some condition that is possible but unlikely or that may be transitory.

The pressures on the ground under an eccentrically-loaded strip foundation of uniform width are given theoretically by an adaptation of formula (12.1a), namely,

$$\begin{align*}
\bar{p}_{\text{max}} & = \frac{\Sigma P}{BL} \left[ I \pm \frac{6}{B} \left( \frac{B}{2} - \bar{X} - x \right) \right] + \bar{p}_b, \\
\bar{p}_{\text{min}} & = \frac{\Sigma P}{BL} \left[ I - \frac{6}{B} \left( \frac{B}{2} - \bar{X} - x \right) \right] + \bar{p}_b.
\end{align*}$$

\text{(12.4)}
in which \( X = \frac{\Sigma P_i}{x} \). The longitudinal bending moment at any cross-section of the foundation, whether it be loaded concentrically or eccentrically, is the algebraical summation of the moments of the loads and pressures on one side of the cross-section; that is the longitudinal bending moment at plane \( X-X \) (Fig. 80a) is the clockwise moment of the total

**Fig. 80.—Strip Bases.**

net reaction of the ground on the strip to the left of the plane deducted from the anti-clockwise moments of the loads to the left of the plane. The difference between this total net reaction and the loads to the left is the shearing force at the plane. These determinations may be done analytically. In the case of an eccentrically-loaded foundation it is generally simpler to use a semi-graphical method by considering two
systems of loads acting on the equivalent of a freely-supported beam of span $B$ as in Fig. 8ob. One system comprises the net reaction of the ground which results in a parabolic bending-moment diagram representing deformation which is concave downwards. The other system comprises the concentrated loads which produce a rectilinear diagram representing deformation which is concave upwards. The combination of the two diagrams, that is the subtraction of one from the other, gives the net longitudinal bending moment, which may produce tension throughout the bottom of the foundation or compression in some parts and tension in other parts of the bottom. Care must be taken with either method, since both involve small differences between large quantities; if the latter quantities are not sufficiently accurate the results may be unrealistic. The transverse bending moments are similar to those on a base supporting two loads.

In the foregoing analyses it is assumed that the strip foundation is rigid, that is its elastic deformations are so slight that they have no effect upon the resultant deformation of the ground. This fact may be true if the base is short compared with its thickness. If a base is slightly flexible the ground immediately under the loaded parts of the base is subjected to the greatest pressures, since, in order to distribute load to the ground between the more intensively-loaded parts, the base acts as a beam and will tend to deflect upwards between the loads. Therefore the ground is compressed less under the intermediate areas than under the intensively-loaded areas. If the pressure under the latter areas tends to become excessive, the ground under these areas tends to subside more and in doing so tends to carry the base with it; this settlement of the base, however, is prevented or at least restricted by the ground under the intermediate areas which therefore tend to carry more of the load and, in the limit, the pressure tends to become more or less uniform throughout, which is the condition for which the base is designed, and which produces the greatest longitudinal bending moments.

The assumption of a rigid base, although safe, may be too conservative in the case of a long base supporting many loads. A base supporting, say, up to five loads spaced fairly closely can be considered reasonably to be a rigid base. If a base supports more than, say, five concentrated loads it is not reasonable to assume (as is the case with a strictly rigid design) that the loads imposed at one end can affect appreciably the pressure on the ground at the other end, as is assumed in the ordinary theory of eccentric loads. Practical methods of providing a foundation for a long line of variable loads are either to provide a number of independent bases each supporting a group of loads or, if a strip foundation is provided, to design it in successive stages (Fig. 8oc), say, for loads $(1)$ to $(5)$, loads $(4)$ to $(8)$, and so on up to loads $(n - 4)$ to $(n)$. The pressure diagrams and the widths of the base can be "eased out" by eye to
overcome differences at the transition stages at loads (4) to (5), (7) to (8), etc.

If a strip foundation is subjected to horizontal loads or external moments acting at right-angles to its breadth, tie-beams connected to a foundation supporting an adjacent line of loads should be provided; the bending moments and shearing forces are calculated in a manner similar to that for tied bases (page 240).

**Wall Footings.**

The foundation for a wall generally comprises a continuous strip of concrete, commonly termed a "footing". The simplest condition is for

![Diagram](image)

**Fig. 81.—Wall Footings.**

the wall to impose on the footing a uniform vertical load of \( w \) lb. per foot length of wall and for there to be no horizontal forces or moments. There are two cases (Fig. 81a and b) to be considered, namely, (a) a reinforced concrete wall monolithic with a concrete footing and (b) a brick or masonry wall on a concrete footing. In both cases the width \( B \) of the footing must be not less than \( w \) divided by the net safe load-bearing capacity of the ground. The maximum transverse bending moment occurs at the centre of the footing and is expressed by the difference of \( \frac{wB}{24} \) acting, say,
clockwise, and \( \frac{wD}{2} \) acting therefore counter-clockwise, in which \( D \) is the thickness of the wall. This bending moment is the critical moment on a footing under an independent wall. If the wall is monolithic with the footing the critical bending moment is at the plane at the face of the wall and is the bending moment on the upward-acting cantilever of length \( \frac{1}{2}B - \frac{1}{2}D \). There is no longitudinal bending moment if the load imposed by the wall is uniform. The critical planes for resistance to shearing force are at a distance \( d_1 \) from the face of the wall, the shearing force being the net reaction on the part of the footing beyond these planes; or, more severely, the critical planes are at the faces of the wall.

**Example No. 97.—Wall Footing.** Design a concentrically-loaded footing for a 9-in. wall imposing a load of \( 3\frac{1}{2} \) tons (7850 lb.) per linear foot of wall if the safe load-bearing capacity of the ground is 1 ton per square foot, (a) if the wall is of reinforced concrete cast monolithically with the footing (Fig. 81a); and (b) if the wall is of brickwork (Fig. 81b).

Width of footing: \( B = \frac{3\frac{1}{2} \times 2240}{2000} \) approximately, say, 4 ft.

Net reaction of ground \( = \frac{7850}{4} = 1963 \) lb. per square foot.

(a) Bending moment at face of wall:
\[
\frac{1}{2}(B - D) = \frac{1}{2}(4 \text{ ft.} - 9 \text{ in.}) = 1 \text{ ft.} \quad 7\frac{1}{2} \text{ in.}
\]
\[
M = \frac{1}{2} \times 1963 \times 1 \cdot 625^2 \times 12 = 31,000 \text{ in.-lb.}
\]
If the footing is 9 in. thick, \( d_1 = 7\frac{1}{2} \text{ in.} \), \( l_a = 6\frac{1}{2} \text{ in.} \), and the transverse reinforcement required is \( \frac{31,000}{20,000 \times 6\frac{1}{2}} = 0.24 \) sq. in.; provide \( \frac{1}{2} \)-in. bars at 9-in. centres.

The longitudinal reinforcement is nominal and should be about 0.15 per cent. of \( (9 \times 48) \) sq. in., that is, 0.65 sq. in.; provide six \( \frac{1}{2} \)-in. bars. The reinforcement is arranged as in Fig. 81a.

The shearing force at the critical plane is
\[
1963(1 \text{ ft.} \quad 7\frac{1}{2} \text{ in.} - 7\frac{1}{2} \text{ in.}) = 1963 \text{ lb.;}
\]
\[
q = \frac{1963}{12 \times 6\frac{1}{2}} = 25 \text{ lb. per square inch.}
\]

(b) Bending moment at centre of footing:
\[
M = \left( 1963 \times \frac{4}{2} \times \frac{4}{4} \right) 12 - \left( \frac{7850}{2} \times \frac{9}{4} \text{ in.} \right) = 38,350 \text{ in.-lb.}
\]

Transverse reinforcement: \( A_{st} = \frac{38,350}{20,000 \times 6\frac{1}{2}} = 0.295 \) sq. in.; provide \( \frac{1}{2} \)-in. bars at 8-in. centres; otherwise the reinforcement is as in design (a).

If a wall imposes a horizontal force or a moment transversely to the length of the footing, the design of the footing is similar to cases of eccentric load dealt with previously. If there is an adjacent parallel footing the two can be tied together at intervals, thereby relieving both footings of eccentricity. If this method is impracticable, the footing must be designed to resist all the external actions without imposing excessive
pressure on the ground; this problem occurs commonly in retaining walls
and tanks and is dealt with in Chapter XVIII, Vol. II.

Foundation Rafts.

A base supporting three or more columns not in a straight line may vary
from a simple rectangular slab supporting four columns to an extensive
raft supporting an entire building. As for all combined bases, coincidence
of the centroid g of the base (Fig. 82) and the centre of gravity G of the
total load is desirable but may not be attainable.

Pressures Imposed on the Ground.—In the general analysis which
follows, the pressures imposed on the ground under a rectangular raft of
known dimensions and subjected to specified loads \( P_1 \) to \( P_n \) are evaluated.
Taking two adjacent sides as abscissae, the centre of gravity \( G \) of the loads
is established by \( \bar{X} = \frac{\sum^n P_x}{\sum^n P} \) and \( \bar{Y} = \frac{\sum^n P_y}{\sum^n P} \). The co-ordinates of the
centroid of the raft are \( x = \frac{1}{4}B \), \( y = \frac{1}{2}L \). The eccentricities of the loads
are \( e_x = \bar{X} - \frac{1}{4}B \) and \( e_y = \bar{Y} - \frac{1}{2}L \), either of which may be positive (as
drawn) or negative. The net reactions of the ground at each corner are
therefore as follows:

At O: \( p_0 = \frac{\Sigma P}{BL} \left[ x - 6(\bar{X} - \frac{1}{4}B) - 6(\bar{Y} - \frac{1}{2}L) \right] \)
At X: \( p_x = \frac{\Sigma P}{BL} \left[ x + 6(\bar{X} - \frac{1}{4}B) - 6(\bar{Y} - \frac{1}{2}L) \right] \)
At Y: \( p_y = \frac{\Sigma P}{BL} \left[ x - 6(\bar{X} - \frac{1}{4}B) + 6(\bar{Y} - \frac{1}{2}L) \right] \)
At Z: \( p_z = \frac{\Sigma P}{BL} \left[ x + 6(\bar{X} - \frac{1}{4}B) + 6(\bar{Y} - \frac{1}{2}L) \right] \)

One of these pressures is the greatest; for example if \( \bar{X} = \frac{1}{4}B \) and \( \bar{Y} = \frac{1}{2}L \)
are both positive, \( p_z \) is the greatest. The maximum pressure imposed on
the ground is the greatest of \( p_0 \), \( p_x \), \( p_y \) and \( p_z \) plus the weight \( p_b \) of unit
area of the raft. The entire area of ground below the raft should be
subjected to pressure, that is \( p_b \) plus the smallest of \( p_0 \), \( p_x \), \( p_y \) and \( p_z \) must
be positive, and the dimensions of the raft should be such that give this
condition.

In the foregoing analysis it is assumed that the raft is rigid and that
the variation of the reaction of the ground is rectilinear, both assumptions
being safe. Analyses in which the variable compression of the ground is
taken into account are complex but in the case of a large raft examine-
tation of the design on these lines may be necessary.* In practice some
approximations must be made since the several variable factors may make
the calculations, even for a small raft, so complex and long that the time

* See "Raft Foundations (Soil-line Method)" by Prof. A. L. L. Baker.
taken to design the raft may be out of proportion to the doubtful accuracy of the result. A first approximation is to assume that the net reaction of the ground is \( \Sigma P \over BL \) throughout the area of the raft, and is reasonable if \( P_0, P_x, P_y \) and \( P_z \) due to the maximum probable loads do not differ by more than, say, 25 per cent. (If the variation is greater, the raft can be divided into rectangular areas over each of which the net reaction of the ground is reasonably uniform.)

**Bending Moments and Shearing Forces.**—Algebraical analysis to determine the bending moments and shearing forces induced on the raft in distributing the concentrated loads evenly on the ground is not worth

![Diagram of Raft Foundations](image)

Fig. 82.—Raft Foundations.

while since it is less cumbersome to deal with actual numerical problems stage by stage so long as the basic principles are clear. These principles are that there must be sufficient safe moment of resistance and resistance to shearing force at any vertical plane to counteract the total bending moment and shearing force acting at the plane. The total shearing force is the difference between the total net reaction of the ground acting upwards on one side of the plane and the total imposed loads acting downwards on the same side of the plane. The total bending moment is the difference of the moments of these reactions and loads. The means by which the bending moments and shearing forces are resisted depends on the type of raft.
Types of Rafts.—The type of raft depends mainly on the size. Small rafts may be simple solid reinforced concrete slabs (Fig. 83a) of uniform thickness, or possibly thickened at the edges if the raft is near the surface. Such rafts are suitable for up to, say, six loads spaced closely, and the resistance is simply that of a solid slab. If the loads are more than about 10 ft. apart a solid slab might be excessively thick, and it may be cheaper to provide a thinner slab with ribs (Fig. 83b) extending between the positions of the loads. Such a raft is designed as an inverted floor; the members imposing the concentrated loads represent columns supporting a floor, and the ribs represent tee-beams; the load imposed on the floor is represented by the net reaction of the ground. The slab spans between the ribs which span between the members imposing the loads. A variant of an analogous floor is to design a raft as an inverted beamless floor,

![Diagram](image-url)

Fig. 83.—Types of Foundation Rafts.

and either this design or a ribbed raft is suitable for a large raft foundation of almost any area. The greater the spacing of the loads the greater the thickness of the slab or the depth of the ribs, so that for large rafts with loads widely spaced cellular construction (Fig. 83c), comprising a top and a bottom slab with ribs between, is generally cheaper, since the weight of the raft is less. The smaller the weight of a raft the better, since rafts are provided generally on sites where the ground is soft and as little as possible of the load-bearing capacity of the ground should be absorbed in carrying the raft itself. Cellular construction provides a level top surface which may be essential in a building, and provides a stiff light foundation comprising thin top and bottom slabs which act as compression flanges of the ribs and thin vertical webs which need be only wide enough to resist shearing forces. The webs intersect at the positions where the loads are imposed and at such positions a pier may be required. Examples of the design of rafts are given in "Design and Construction of Foundations" by G. P. Manning, and other books.
The foregoing types of foundation structures are mainly suitable if the bearing stratum is within a few feet of the surface of the ground. When the bearing stratum is, say, more than 5 ft. but less than 15 ft. below the surface, foundation piers are suitable. Such piers (Fig. 84) are concrete columns with little, if any, reinforcement. The shaft of the pier is smaller than the base but must be large enough to support the imposed load without buckling; the ratio of net height to width should not exceed six.

Since the excavation must be wide enough to enable men to work therein, say, not less than 3 ft. square, the use of a smaller shaft, although it would reduce the quantity of concrete and shuttering, would not generally reduce the amount of digging and timbering. Rules for the design of piers are therefore as follows.

Each pier should support as much load from the superstructure as practicable so as to require the least number of piers. Piers should not be subjected to moments or eccentric loads, since, because of their height, the effect of these actions are amplified; these actions should be resisted where possible by tie-beams connecting the tops of adjacent piers. The base of the pier should be so proportioned that it will transfer the total load, including the weight of the pier and any earth refilled into the hole, to the bearing stratum without exceeding the safe load-bearing capacity of the ground. The shaft can be of plain concrete slightly less rich in cement than ordinary reinforced concrete, say, 1 : 3 : 6, if the least width is not less than one-sixth of the height. The costs of an ordinary reinforced
concrete column extending from ground level to a suitable base and of a piled foundation should be investigated before deciding to provide a plain concrete pier. Piers may be an economical form of foundation structure for buildings when the bearing stratum is a few feet below the ground floor. In this case the walls are carried on beams at ground level and the beams are supported on the piers, which should be placed under columns or other concentrations of load. If the ground above the bearing stratum is very weak the ground floor may be carried on the wall beams, but if possible the ground floor should be separate from the beams and supported directly on the ground.

Example No. 98.—Foundation Pier. Design a foundation pier to carry a load of 120,000 lb. if the bearing stratum is 15 ft. below the surface of the ground and the safe load-bearing capacity of the stratum is 4 tons per square foot.

Assume the weight of the base, shaft, and earth on the base to be 15,000 lb. The total weight on the ground is 135,000 lb. and the area of the base must be \( \frac{135,000}{4 \times 2240} = 15 \) sq. ft., say, 4 ft. square and 2 ft. thick. The size of the shaft of net height, say, 11 ft. 6 in. = \( \frac{11.5}{6} \) ft., say, 2 ft. square. The compressive stress in the shaft is \( \frac{125,000 \text{ (approx.)}}{24^2} = 218 \) lb. per square inch; the permissible stress in \( 1:3:6 \) plain concrete is 310 lb. per square inch. Provide nominal reinforcement as in Fig. 84.

Foundation Piles.

When the bearing stratum is several feet below the surface a suitable foundation is provided by piles, which may be of concrete, steel, or timber. Concrete piles may be either precast piles of reinforced or prestressed concrete and are driven by a hammer, or cast in place in the ground in cavities formed by boring or by driving a tube. The sizes of piles may vary from small augered piles 8 in. diameter and 5 ft. long and capable of carrying only a few tons to large precast piles 18 in. square and 80 ft. long capable of supporting more than 50 tons, or larger special piles carrying several hundreds of tons. The principles of the design of a piled foundation are the same whatever the type, although the methods of construction differ considerably. The first step is to decide what type of pile is most suitable for the case being considered, and to assess the safe load on such a pile when used on the site and for the structure under consideration; these are major engineering decisions which can be based only on experience; summarising may be misleading.*

If the load to be carried is \( P \) tons and the weight of the foundation

* See "Introduction to Deep Foundations" by D. H. Lee.
structure (called the pile-cap), is $P_B$ tons, and if the safe load on one pile is $W$ tons (in addition to the weight of the pile), the number $N$ of piles required is $\frac{P + P_B}{W}$. Alternatively the load $W$ on each pile is $\frac{P + P_B}{N}$.

The purpose of the pile-cap is to distribute the load equally among the piles. A group of $N$ piles must therefore be arranged under the pile-cap so that the centroid of the group is directly under the imposed load. Exceptions to this rule may occur if the superstructure imposes a moment or an eccentric load on the group of piles, as may be the case for a chimney, water-tower, or similar structure.

**Design of Pile-caps.**—Pile-caps have to resist the shearing forces and bending moments due to the transference of the load from the column to the piles. The thickness must also be sufficient to provide sufficient bond length for the vertical bars in the column or other superstructure and for the bars projecting from the piles. A pile-cap is therefore generally at least 2 ft. thick and, since the distance between adjacent piles is generally 2 ft. 6 in. to 3 ft. 6 in., it acts as a deep beam. A method of designing

![Diagram of Pile-cap for Six Piles](image)

Fig. 86.—Pile-cap for Six Piles.

pile-caps is to consider the dispersion of the load as in Fig. 85a which shows the forces on a pile-cap for two piles. The tension in the reinforcement is the horizontal component of the inclined thrust, that is $\frac{T}{W} = \frac{l}{2d_1}$.

Therefore the cross-sectional area of the reinforcement must be $\frac{T}{P_{st}}$, that is $\frac{Pl}{4d_1P_{st}}$. Similar expressions can be derived for pile-caps for other numbers of piles. Consider a group of six piles supporting an axially-loaded column. The piles can be arranged hexagonally, or rectangularly similar to the piles in Fig. 86, which shows the details of a typical pile-cap for six piles. The load $W$ on each pile (assuming that the group is loaded concentrically) is $\frac{P + P_B}{6}$. The total cross-sectional area of longitudinal reinforcement required is $\frac{3W}{P_{st}} \left( \frac{l}{d_1} \right)$ and of transverse reinforcement $\frac{3W}{P_{st}} \left( \frac{l}{2d_1} \right)$.
Alternatively a pile-cap can be considered as an inverted cantilever, in which case the longitudinal bending moment on a pile-cap for six piles arranged as in Fig. 86 is $2Wl$ and the total transverse bending moment is $(\frac{1}{4})l(3W)$. The total area of longitudinal and transverse reinforcement is $2W \left( \frac{1}{4}p_W \left( \frac{l}{d_1} \right) \right)$ and $3W \left( \frac{1}{4}p_W \left( \frac{l}{2d_1} \right) \right)$ respectively, if the lever arm is assumed to be $\frac{3}{8}d_1$.

Note that $l$ and $d_1$ must be in the same units.

If there is only one pile or two piles, or if all the piles in a group are in a straight line, means must be taken to prevent movement sideways of the top of the pile, and for this purpose a tie-beam should be provided at right-angles to the line of piles or tie-beams should be provided in two directions mutually at right-angles in the case of a single pile. The tie-beams should be connected to an adjacent pile-cap.

**Loads on Piles in a Group subjected to a Moment.**—If a group of $N$ vertical piles is subjected to a moment $M$ as well as a vertical load $P$, the load supported by each pile may differ. Referring to Fig. 85b, if $h_1$, $h_2$, ..., $h_{n-1}$ and $h_n$ are the distances of piles $W_1$, $W_2$, etc., from the centre of gravity of the group, the load $W$ on any pile is $\frac{P + P_B}{N} \pm \frac{Mh_n}{\sum_{i=1}^{n} h_i^2}$.

The positive sign is applicable if the pile is on that side of the centre of gravity on which the load tends to be increased by the moment $M$, and the negative sign is applicable if it is on the other side. If the load $P$ is at a distance $\pm \theta$ from the centre of gravity of the group, $M = \pm Ps$.

**Example No. 99.—Pile-cap for Six Piles.** Design a pile-cap to carry a column subjected to an axial load of 210 tons and a moment of 350,000 ft.-lb. which can act clockwise or anti-clockwise. The safe load on a pile is 50 tons.

The number of piles must exceed \( \frac{471,000 \text{ lb.}}{50 \times 2240} \), which exceeds four; therefore provide six piles. For a piled foundation subjected to a moment the rectangular arrangement in Fig. 85b is better than an hexagonal group if the longitudinal axis of the group is in the plane of the bending moment on the column. The dimensions of the pile-cap are as in Fig. 86 and the weight $P_B$ is therefore about 32,000 lb.; $P + P_B = 503,000$ lb. The greatest load on each outer pile is

$$\frac{503,000}{6} + \frac{350,000 \times 3.5}{4 \times 3.5^2} = 83,800 + 25,000 = 108,800 \text{ lb.}, = 48.6 \text{ tons}.$$ 

The least load on each outer pile is $83,800 - 25,000 = 58,800 \text{ lb.} = 26$ tons. The load on each of the two central piles is $83,800 \text{ lb.} = 37$ tons.

Considering the pile-cap as an inverted cantilever, the reinforcement in Fig. 86 can be justified.
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