ANALYSIS OF STRUCTURES

THE ANALYSIS OF STATICALLY-INDETERMINATE STRUCTURES BY THE DEFORMATION METHOD

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BY

M. SMOLIRA

Ph.D., A.M.I.C.E., D.I.C.

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PREFACE

The main purpose of this work is to give in handy form time-saving methods of analysing continuous frames with curved members and Vierendeel trusses both continuous and simply supported. Chapter I by way of introduction deals with rapid methods of analysis of continuous beams and frames with straight members. The method described is based on the actual deformations of elastic structures and on well-known fundamental relations between the stresses and strains, and requires the setting out and solving of simultaneous equations of equilibrium of angular deformations of the structure by expressing the conditions of compatibility of deformations of the members.

The method is "exact", that is it includes no more assumptions or limitations than those inherent in the elastic theory, and it provides a rapid solution of many problems which often present mathematical difficulties seemingly out of proportion to their importance. A wide range of problems concerned with elastic structures may successfully be solved by the method. Many of the equations given are applicable in a general case for unsymmetrical frames of any shape, with prismatic or non-prismatic members, and with any system of loading.

The method is not "automatic". It is particularly necessary to visualise the deformed shape of the structure, and this should be drawn with some exaggeration in order to set out properly the equations of equilibrium. Every figure and every operation has its meaning and its clear physical interpretation, thus helping the setting out of equations of equilibrium and avoiding errors. It should also be remembered that a correct solution of simultaneous equations is not proof of a correct result unless the equations are properly set out. No sign convention is necessary, moments and forces being assumed to act in the directions necessary to close the angular or linear gaps caused by the relaxation of the continuity of joints. The degree of indeterminacy is apparent by inspection in every case, and no formulae for this are given or are necessary. Sway is taken into account in the actual deformations of the structures, and therefore no separate treatment of sway is required. Little extra effort is required to take into account beams or frames of different spans and cross sections or with non-prismatic members. Deflections, although generally eliminated from the equations at early stages, can be easily calculated, as is shown in examples.

Although many equations and formulae are derived, it is not necessary to remember any of them or to refer to any set of equations given. In each case equations can be written immediately by simple reasoning, with the help of load functions and elastic constants for each beam or column. It is not proposed to supply ready-made formulae for various cases, and no effort has been made to present or tabulate them. However, in the course of the analyses some solutions are given in explicit form, and these may be compared with the results in books on the theory of structures. Numerical examples are given to illustrate the application of the theory.
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NOTATION.

A, B, C, etc., Joints.

\( A \), Area of \( \frac{m}{EI} \) diagram.

\( a, b \), Position of a concentrated load.

\( c \), Coefficients.

\( d \), Depth of beam.

\( E \), Modulus of elasticity.

\( h \), Height of columns.

\( H \), Horizontal forces in frames.

\( I \), Moments of inertia.

\( k \), Coefficients.

\( L, l \), Spans.

\( m \), Bending moments.

\[ n = \frac{I_1}{I_2}. \]

\( p \), The rise of curved beams; also pressure.

\( P \), Forces.

\( R \), Radius of circular beams.

\( s \), Length of inclined beams.

\( S \), Forces in struts.

\( S.c. \), Shearing condition.

\( V \), Shearing forces.

\( w \), Uniformly-distributed load.

\( W \), Concentrated load.

\( x, y \), Co-ordinates.

\( \bar{x}, \bar{z} \), Position of centre of gravity of \( \frac{m}{EI} \) diagram.

\( \alpha, \beta \), Angular deformations of beams due to unit bending moment.

\( \gamma, \delta \), Angular deformations of beams due to unit horizontal force.

\( \Delta \), Deflections of beams and frames.

\( \lambda \), Horizontal sway in frames and trusses.

\( \theta \), Load functions.

\( \phi \), Half central angle of circular beams.
CHAPTER I
CONTINUOUS BEAMS

General Case.—A continuous beam of any shape, and subjected to the action of any system of loading and settlement of the supports, may be conveniently analysed by considering the angular deformations of the beam. Statically-indeterminate bending moments can be found by setting out equations of equilibrium of angular deformations of the joints.

First assume that the beam is cut at the supports and thus relaxed from the effect of continuity, and calculate angular deformations $\theta$ of each beam due to the external system of loading (Fig. 1). The angular gaps at each support must now be closed by applying statically-indeterminate bending moments of such values that all gaps are closed at the same time. For example, at support B (Fig. 1b and 1c) the condition of equilibrium is

$$\sum \rho_b = \sum \theta_b \pm \sum \frac{A}{L}$$

or

$$m_a \theta_{ba} + m_b \theta_{ba} + m_b \theta_{be} + m_e \theta_{be} = (\theta_{ba} + \theta_{be}) \pm \left( \frac{A}{L_1} + \frac{A}{L_2} \right)$$

in which $\rho$ represents rotations of the beam due to statically-indeterminate bending moments, and $\alpha$ and $\beta$ are angular deformations of the beam due to unit bending moment at the point of its application and at the far end of the beam respectively (Figs. 2 and 3).
The values of $\alpha$ and $\beta$ depend only on the geometrical shape of the beam, and not on the system of loading. These values are referred to as the elastic constants of the beam. The values $\theta$ (Fig. 2) are load functions and, in a general case, may be calculated from the well-known relations

$$\varepsilon = \int_A^B \frac{m}{EI} \, dx; \quad \theta_a = \frac{\varepsilon}{L}; \quad \theta_b = \frac{\varepsilon}{L}; \quad \bar{x} = \frac{1}{\varepsilon} \int_A^B \frac{mx}{EI} \, dx; \quad \text{and} \quad \Delta_z = \int \frac{m}{EI} \, dx. \quad (2)$$

in which $\varepsilon$ is total angular deformation of the beam from A to B, $\bar{x}$ and $\tilde{z}$ define the centre of gravity of the $\frac{m}{EI}$ diagram, and $\Delta_z$ is the deflection at any point.

Generally the formulae are calculated from equations (2), but for more complicated loading or non-prismatic beams the summation method will probably lead to a quicker result than integration. This is illustrated by various examples. The elastic constants $\alpha$ and $\beta$ (Fig. 3) can also be calculated from equations (2). For prismatic beams, however, they have the well-known values

$$EI\varepsilon = \frac{L}{2}, \quad EI\alpha = \frac{L}{3}, \quad EI\beta = \frac{L}{6} \quad . \quad . \quad . \quad (3)$$

The following examples illustrate the procedure.

**Two Spans.**—Consider a two-span prismatic or non-prismatic beam with any system of loading. First, calculate the angular deformations $\theta_{ba}$ and $\theta_{bc}$ and the elastic constants $\alpha$ and $\beta$ from equations (2). The angular gap ($\theta_{ab} + \theta_{bc}$)
at support $B$ is now closed by applying a statically-indeterminate bending moment $m_b$ calculated from equation of equilibrium (4) at support $B$.

$$B, \ m_bx_{ba} + m_b\alpha_{bc} = \theta_{ba} + \theta_{bc} \dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\dot{\doc{
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For a concentrated load \( W \) on the second span \((\text{Fig. 6})\), equations \((4)\) become \((10a)\):

\[
\begin{align*}
E, & \quad \frac{L_1}{3EI_1}m_b + \frac{L_2}{3EI_2}m_b + \frac{L_3}{6EI_2}m_c = \frac{I}{6EI_2}m_o(L_2 + b) \\
C, & \quad \frac{L_2}{6EI_2}m_b + \frac{L_2}{3EI_2}m_c + \frac{L_3}{3EI_3}m_e = \frac{I}{6EI_2}m_c(L_2 + a)
\end{align*}
\]

in which \( \theta_{bc} = \frac{1}{6}m_o(L_2 + b) \) and \( \theta_{eb} = \frac{1}{6}m_o(L_2 + a) \)—see Appendix.

For \( L_1 = L_2 = L_3 \) and \( I_1 = I_2 = I_3 \):

\[
m_b = \frac{Wab}{15L^2}(2L + 5b) \text{ and } m_c = \frac{Wab}{15L^2}(7L - 5b)
\]  \(\text{ (10b)}\)

For \( a = b = \frac{L}{2} \):

\[
m_b = m_c = \frac{3}{4}\frac{WL}{6}.
\]  \(\text{ (10c)}\)

For a uniformly-distributed load on part of span BC \((\text{Fig. 7})\) statically-indeterminate bending moments \( m_b \) and \( m_c \) are calculated from equations \((11)\) of equilibrium.

\[
\begin{align*}
B, & \quad m_b\frac{L_1}{3EI_1} + m_b\frac{L_2}{3EI_2} + m_c\frac{L_3}{6EI_2} = \theta_{bc} \\
C, & \quad m_b\frac{L_2}{6EI_2} + m_c\frac{L_2}{3EI_2} + m_c\frac{L_3}{3EI_3} = \theta_{eb}
\end{align*}
\]

in which the load functions \( \theta_{bc} \) and \( \theta_{eb} \), calculated from equations \((2)\)—see also Appendix—are given in \((12)\).

\[
EI_2\theta_{bc} = \frac{wa^2}{24L^2}(2L_2 - a) \quad \text{and} \quad EI_2\theta_{eb} = \frac{wa^2}{24L^2}(2L_2^2 - a^2)
\]  \(\text{ (12)}\)

For \( L_1 = L_2 = L_3 \) and \( I_1 = I_2 = I_3 \):

\[
m_b = \frac{wa^2}{30L}(7L - 8a + \frac{5a^2}{2L}) \text{ and } m_c = \frac{wa^2}{30L}(2L + 2a - \frac{5a^2}{2L})
\]  \(\text{ (13a)}\)
CONTINUOUS BEAMS

For \( a = L \):

\[
m_b = m_c = \frac{wL^2}{20} \quad \quad \quad (13b)
\]

![Figure 8](image)

For uniformly-distributed loads of different magnitudes on each span (Fig. 8), equations (6) become (14):

\[
\begin{align*}
B, & \quad \frac{L_1}{3EI_1} m_b + \frac{L_2}{3EI_2} m_b + \frac{L_2}{6EI_2} m_c = \frac{w_1L_1^3}{24EI_1} + \frac{w_2L_1^3}{24EI_2} \\
C, & \quad \frac{L_2}{6EI_2} m_b + \frac{L_3}{3EI_3} m_c + \frac{L_3}{3EI_3} m_c = \frac{w_2L_2^3}{24EI_2} + \frac{w_3L_3^3}{24EI_3}
\end{align*}
\]

Four Spans.—In a similar way equations of equilibrium of angular deformations may be set out for a beam with any number of spans. For four spans of any shape with any system of loading the equations are as in (15).

\[
\begin{align*}
B, \quad m_b \alpha_{bc} + m_c \alpha_{bc} + m_d \beta_{bc} = \theta_{ba} + \theta_{bc} \\
C, \quad m_b \beta_{cd} + m_c \alpha_{cb} + m_d \alpha_{cd} + m_d \beta_{cd} = \theta_{bc} + \theta_{cd} \\
D, \quad m_c \beta_{dc} + m_d \alpha_{dc} + m_d \alpha_{de} = \theta_{dc} + \theta_{de}
\end{align*}
\]

![Figure 9](image)

For prismatic beams with a concentrated load \( W \) on the first span (Fig. 9) equation (15) becomes (16).

\[
\begin{align*}
B, & \quad \frac{L_1}{3EI_1} m_b + \frac{L_2}{3EI_2} m_b - \frac{L_2}{6EI_1} m_c = \frac{m_c}{6EI_2}(L_1 + b) \\
C, & \quad \frac{L_2}{3EI_2} m_c + \frac{L_3}{3EI_2} m_c - \frac{L_2}{6EI_2} m_b - \frac{L_3}{6EI_3} m_d = 0 \\
D, & \quad \frac{L_3}{3EI_3} m_d + \frac{L_4}{3EI_4} m_d - \frac{L_3}{6EI_3} m_c = 0
\end{align*}
\]

If \( L_1 = L_2 = L_3, \ I_1 = I_2 = I_3, \) and \( a = \frac{L}{2} \):

\[
m_b = \frac{5}{4\times8} WL, \quad m_c = \frac{3}{11\times8} WL \quad \text{and} \quad m_d = \frac{3}{4\times8} WL
\]

\[
(17)
\]
Equation of Three Moments.—The Equation of Three Moments can be obtained by setting out the equation of equilibrium of angular deformations for the support B of a continuous beam (Fig. 10) as follows.

\[ B, \frac{L_1}{6EI_1} m_a + \frac{L_1}{3EI_1} m_b + \frac{L_2}{3EI_2} m_b + \frac{L_2}{6EI_2} m_c = \theta_{ba} + \theta_{bc}, \]

or in the well-known form

\[ B, \frac{L_1}{I_1} m_a + 2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) m_b + \frac{L_2}{I_2} m_c = \frac{6\bar{x}_1}{I_1} \bar{A}_1 + \frac{6\bar{x}_2}{I_2} \bar{A}_2 \quad (18) \]

in which \( \bar{A} \) is the area and \( \bar{x} \) the centre of gravity of the free bending-moment diagrams.

Settlement of the Supports.

The influence of settlement of the supports may be analysed by the use of equations (1). This is illustrated by the following examples.

Three Spans Subjected to Settlement of an Intermediate Support (Fig. 11).—Denoting by \( \Delta \) the settlement of support B, the angular gaps at B and C in the beam relaxed from the effect of continuity are \( \left(\frac{\Delta}{L_1} + \frac{\Delta}{L_2}\right) \) and \( \frac{\Delta}{L_2} \) respectively. In a general case of non-prismatic beams the following equations of equilibrium for supports B and C may be set out:

(B) \( m_0\Delta_{ba} + m_0\Delta_{be} - m_c\beta_{bc} = \frac{\Delta}{L_1} + \frac{\Delta}{L_2} \); (C) \( m_c\Delta_{cb} + m_c\Delta_{cd} - m_b\beta_{bc} = \frac{\Delta}{L_2} \quad (19) \)
CONTINUOUS BEAMS

For prismatic beams these equations reduce to

\[
\begin{align*}
B, & \quad \frac{L_1}{3EI_1} m_b + \frac{L_2}{3EI_2} m_b - \frac{L_2}{6EI_2} m_c = \frac{A}{L_1} + \frac{A}{L_2} \\
C, & \quad \frac{L_2}{3EI_2} m_c + \frac{L_3}{3EI_3} m_c - \frac{L_2}{6EI_2} m_d = \frac{A}{L_2}
\end{align*}
\]

(20)

For \( L_1 = L_2 = L_3 \) and \( I_1 = I_2 = I_3 \):

\[
m_b = \frac{8AEI}{5L^2} ; \quad m_c = \frac{12AEI}{5L^2} .
\]

(21)

**Fig. 12.**

Three Spans Subjected to Settlement of an External Support.—In this case (Fig. 12) only one angular gap \( \frac{\Delta}{L_1} \) at support B is formed in the relaxed condition of the beam; the statically-indeterminate bending moments \( m_b \) and \( m_c \) are calculated from (22). For prismatic beams these equations become (23). If all spans are equal and the moment of inertia constant, from equations (23) we obtain (24).

\[
\begin{align*}
B, & \quad m_a x_{ba} + m_b x_{bc} - m_c x_{cd} = \frac{\Delta}{L_1} \\
C, & \quad m_c x_{cb} + m_d x_{cd} - m_b x_{cb} = 0
\end{align*}
\]

(22)

\[
\begin{align*}
B, & \quad \frac{L_1}{3EI_1} m_b + \frac{L_2}{3EI_2} m_b - \frac{L_2}{6EI_2} m_c = \frac{A}{L_1} \\
C, & \quad \frac{L_2}{3EI_2} m_c + \frac{L_3}{3EI_3} m_c - \frac{L_2}{6EI_2} m_d = 0
\end{align*}
\]

(23)

For \( L_1 = L_2 = L_3 \) and \( I_1 = I_2 = I_3 \),

\[
m_b = \frac{8AEI}{5L^2} \quad \text{and} \quad m_c = \frac{2AEI}{5L^2} .
\]

(24)

**Fig. 13.**

Three Spans Subjected to Settlement of Two Intermediate Supports (Fig. 13).—If supports B and C settle \( \Delta_1 \) and \( \Delta_2 \) respectively, the angular gaps...
formed at B and C are as in Fig. 13 and the equations of equilibrium are given in

\[ \begin{cases} 
B, & \frac{L_1}{3I_1} m_b + \frac{L_2}{3I_2} m_b + \frac{L_3}{6I_2} m_c = \frac{A_1}{L_1} E + \frac{A_1 - A_2}{L_2} E \\
C, & \frac{L_2}{6I_2} m_b + \frac{L_2}{3I_2} m_c + \frac{L_3}{3I_3} m_c = \frac{A_2}{L_2} E - \frac{A_1 - A_2}{L_2} E 
\end{cases} \]  

(25)

For \( L_1 = L_2 = L_3 \), and \( I_1 = I_2 = I_3 \), and \( A_1 = A_2 = \Delta \):

\[ m_b = m_c = \frac{6\Delta EI}{5L^2} \]  

(26)

**Three Spans with Fixed Ends Subjected to Settlement of Intermediate Supports in Opposite Directions.**—Fig. 14 shows the beam in its relaxed position, with angular gaps marked at each support. Four statically-indeterminate bending moments are calculated from (27).

\[ \begin{cases} 
A, & \frac{L_1}{3I_1} m_a - \frac{L_1}{6I_1} m_c = \frac{A_1}{L_1} E \\
B, & \frac{L_2}{3I_2} m_b + \frac{L_2}{6I_2} m_c - \frac{L_3}{6I_2} m_a - \frac{L_3}{6I_3} m_c = \frac{A_1}{L_3} E + \frac{A_1 + A_2}{L_2} E \\
C, & \frac{L_1}{3I_2} m_a - \frac{L_2}{3I_2} m_c - \frac{L_2}{6I_3} m_c = \frac{A_1}{L_2} E + \frac{A_1 + A_2}{L_3} E \\
D, & \frac{L_3}{3I_3} m_d - \frac{L_3}{6I_3} m_c = \frac{A_2}{L_3} E 
\end{cases} \]  

(27)

For \( L_1 = L_2 = L_3 \), \( I_1 = I_2 = I_3 \) and \( A_1 = A_2 = \Delta \):

\[ m_a = m_d = \frac{36\Delta EI}{5L^2} \] and \( m_b = m_c = \frac{42\Delta EI}{5L^2} \).  

(28)

**Three Spans with Intermediate Support Subjected to Angular Settlement \( \phi \) (Fig. 15).**—Two statically-indeterminate bending moments \( m_2 \) and \( m_3 \) are calculated from (29).

\[ \begin{cases} 
BC, & m_2 \frac{L_2}{3EI_2} = m_3 \frac{L_2}{6EI_2} = \phi \\
C, & m_3 \frac{L_2}{3EI_2} + m_2 \frac{L_3}{3EI_3} - m_2 \frac{L_3}{6EI_2} = 0 
\end{cases} \]  

(29)
Bending moment \( m_1 \) in the first span is statically determinate and is calculated from

\[
BA, \quad m_1 \frac{L_1}{3EI_1} = \phi \quad \ldots \quad (29a)
\]

For a beam with a constant moment of inertia,

\[
m_1 = \frac{3\phi EI}{L_1}; \quad m_2 = \frac{12(L_2 + L_3)\phi EI}{L(3L_2 + 4L_3)}; \quad m_3 = \frac{6\phi EI}{3L_2 + 4L_3}.
\]

For \( L_1 = L_2 = L_3 \):

\[
m_1 = \frac{3\phi EI}{L}; \quad m_2 = \frac{24\phi EI}{7L}; \quad m_3 = \frac{6\phi EI}{7L} \quad \ldots \quad (30)
\]

Three Spans: External Support Subjected to Angular Settlement (Fig. 16).—Statically-indeterminate bending moments \( m_a, m_b, \) and \( m_c \) are calculated from (31).

\[
\begin{align*}
A, & \quad \frac{L_1}{3EI_1} m_a - \frac{L_1}{6EI_1} m_b = \phi \\
B, & \quad m_b \left( \frac{L_1}{3EI_1} + \frac{L_2}{3EI_2} \right) - m_a \frac{L_1}{6EI_1} - m_c \frac{L_2}{6EI_2} = 0 \\
C, & \quad m_c \left( \frac{L_2}{3EI_2} + \frac{L_3}{3EI_3} \right) - m_b \frac{L_2}{6EI_2} = 0
\end{align*}
\]

If all spans are equal and the moment of inertia constant,

\[
m_a = \frac{5\phi EI}{13L}; \quad m_b = \frac{12\phi EI}{13L}; \quad m_c = \frac{3\phi EI}{13L}. \quad \ldots \quad (32)
\]

\textbf{Examples.}

\textbf{Example 1.}—As a numerical example consider the three-span beam (Fig. 17) with a concentrated load \( W \) of \( = 10,000 \) lb., and subject to a settlement of A.S.—B
support B of $\Delta_b = 1$ in. The elastic constants for spans AB and BC and the load functions $\theta_{ba}$ and $\theta_{eb}$ are shown in Tables I and II respectively.

**TABLE I.—BEAM AB.**

<table>
<thead>
<tr>
<th>Point</th>
<th>x ft</th>
<th>d ft</th>
<th>I ft$^2$</th>
<th>$m_{ba}/i$</th>
<th>$m_{a}/ds$</th>
<th>$m_{a}/X ds$</th>
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<tbody>
<tr>
<td>1</td>
<td>4.0</td>
<td>3.0</td>
<td>225</td>
<td>0.05</td>
<td>0.2666</td>
<td>1.07</td>
</tr>
<tr>
<td>2</td>
<td>12.0</td>
<td>3.0</td>
<td>225</td>
<td>0.15</td>
<td>0.8000</td>
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<td>1.1333</td>
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<tr>
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<td>28.0</td>
<td>3.0</td>
<td>248</td>
<td>0.35</td>
<td>1.6918</td>
<td>47.37</td>
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<tr>
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<td>3.0</td>
<td>245</td>
<td>0.45</td>
<td>1.1324</td>
<td>47.97</td>
</tr>
<tr>
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<td>44.0</td>
<td>3.0</td>
<td>249</td>
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<td>0.8800</td>
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<tr>
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<td>3.0</td>
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<td>0.5184</td>
<td>28.98</td>
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<td>483</td>
<td>0.75</td>
<td>0.3018</td>
<td>18.11</td>
</tr>
<tr>
<td>9</td>
<td>68.0</td>
<td>3.0</td>
<td>592</td>
<td>0.85</td>
<td>0.1725</td>
<td>11.73</td>
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<tr>
<td>10</td>
<td>76.0</td>
<td>3.0</td>
<td>1124</td>
<td>0.95</td>
<td>0.0101</td>
<td>7.77</td>
</tr>
</tbody>
</table>

$ds = 8$ ft.; $Ee = 7308.4$; $\bar{x} = \frac{229.020}{7308.4} = 31.34$ ft.

$E\alpha_{ba} = 7308.4 \times \frac{3x^3}{80} = 2863.1$; $E\beta_1 = 4445.3$.

$ds = 12$ ft.; $Ee = 8695.9$; $\bar{x} = \frac{475.060}{8695.9} = 54.63$ ft.; $\bar{x} = 65.37$ ft.;

$E\alpha_{be} = 8695.9 \times \frac{65.37}{120} = 4737.1$; $E\beta_2 = 3958.8$.

For $W = 10,000$ lb. when $a = 40$ ft., $Ee = 3,282,300$,

$\bar{x} = \frac{186,666,000}{3,282,300} = 56.87$, $E\theta_{be} = 1,555,500$, $E\theta_{eb} = 1,726,800$.

Statically-indeterminate bending moments $m_b$ and $m_e$ for the concentrated load are calculated from the following equations of equilibrium:

$$(2863.1 + 4737.1)m_b + 3958.8m_e = 1,555,500;$$

$$(2863.1 + 4737.1)m_b + 3958.8m_e = 1,726,800,$$

from which $m_b = 118,460$ ft.-lb. and $m_e = 165,500$ ft.-lb.
CONTINUOUS BEAMS

TABLE II.—BEAM BC.

<table>
<thead>
<tr>
<th>Point</th>
<th>bef</th>
<th>def</th>
<th>1 ft.</th>
<th>mbe = m</th>
<th>M</th>
<th>I kds</th>
<th>M kds</th>
<th>W = 10,000 lb</th>
<th>a = 400-c</th>
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<tr>
<td>11</td>
<td>60</td>
<td>1.085</td>
<td>95.09</td>
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<td>0.1199</td>
<td>0.72</td>
<td>0.47</td>
<td>52</td>
<td>310</td>
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<td>50</td>
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<td>50.00</td>
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<td>30</td>
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<td>0.7236</td>
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<td>36</td>
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<tr>
<td>Σ</td>
<td></td>
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<td>4756</td>
<td>32825</td>
<td>186656</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Bending moments \( m_b \) and \( m_e \) due to settlement of support B are calculated from equations (17):

\[
(2863.1 + 4737.1) m_b = 3958.8 m_e = \left( \frac{A}{80} + \frac{A}{120} \right) E
\]

\[
(2863.1 + 4737.1) m_e = 3958.8 m_b = \frac{A}{130} E
\]

from which \( m_b = 3.84\Delta E \) and \( m_e = 2.11\Delta E \).

For \( \Delta = 1 \) in. and \( E = 4,000,000 \) lb. per square inch, \( m_b = 184,320 \) ft.-lb. and \( m_e = 101,280 \) ft.-lb.

It is interesting to note that, for a beam of constant cross section 6 ft. by 1 ft. equations (17) become

\[
\left( \frac{80}{3} + \frac{120}{3} \right) m_b - \frac{120}{6} m_e = \left( \frac{\Delta}{80} + \frac{\Delta}{120} \right) EI ; \quad \left( \frac{80}{3} + \frac{120}{3} \right) m_e - \frac{120}{6} m_b = \frac{\Delta}{120} EI,
\]

from which \( m_b = 192,000 \) ft.-lb. and \( m_e = 120,000 \) ft.-lb.

Masts with Stays.

Masts with stays (Fig. 18) are subjected to the simultaneous action of external load and translations of joints. Consider a mast with two spans and a cantilever (Fig. 18), the stays, mast, and load being in the same plane. Assume that points B and C do not move, and calculate bending moments \( m_b' \) and \( m_e \) as in (33a).

\[
m_e = \frac{1}{2} w_0 a^2
\]

\[
\left( \frac{L_1}{3EI_1} + \frac{L_2}{3EI_2} \right) m_b' + \frac{L_2}{6EI_2} m_e = \frac{w_1 L_1^3}{24EI_1} + \frac{w_2 L_2^3}{24EI_2}
\]

Calculate the shearing forces, the reactions \( R_b \) and \( R_e \), and the tensile forces in the stays:

\[
T_1 = \frac{R_b'}{\cos \phi_1} \quad \text{and} \quad T_2 = \frac{R_e'}{\cos \phi_2}
\]
The elongations of the stays are
\[ \delta_1 = \frac{T_1 r_1}{E_w A_{w1}} \quad \text{and} \quad \delta_2 = \frac{T_2 r_2}{E_w A_{w2}} \quad \ldots \quad (33c) \]
in which \( r_1 \) and \( r_2 \) are the lengths of the stays, \( E_w \) the modulus of elasticity and \( A_w \) the area of the cross section of the stays. Neglecting the shortening of the post due to axial forces, the lateral translations of points B and C are
\[ \Delta_b = \frac{\delta_1}{\cos \phi_1} ; \quad \Delta_c = \frac{\delta_2}{\cos \phi_2} ; \quad \text{and} \quad \Delta = \Delta_c \frac{L_1}{L_1 + L_2} - \Delta_b. \]

**Fig. 18.**

**Fig. 19.**

Bending moment \( m_b'' \) due to the relative translation of points B and C is calculated from
\[ \left( \frac{L_1}{3E_c I_1} + \frac{L_2}{3E_c I_2} \right)m_b'' = \frac{\Delta}{L_1} + \frac{\Delta}{L_2} \quad \ldots \quad (33d) \]
in which \( E_c \) is the modulus of elasticity of the column.

The final bending moment at B is \( m_b = m_b' + m_b'' \).

**Example 2.**—Assume that the mast with wire stays shown in Fig. 19 is subjected to a uniformly-distributed load \( w \) of 100 lb. per ft. From equations (33a),
\[ m_c = \frac{1}{2} \times 100 \times 20^2 = 20,000 \text{ ft.-lb.} \]
\[ m_b' \left( \frac{50 \times 100}{3 \times 0.0045} + \frac{50 \times 100}{3 \times 0.00126} \right) - 20 \times \frac{50 \times 100}{6 \times 0.00126} = \frac{100 \times 50^3}{24} \left( \frac{1}{0.0045} + \frac{1}{0.00126} \right); \]
from which \( m_b' = 23,400 \text{ ft.-lb.} \).
The shearing forces in the post are

\[ V_{cd} = 20 \times 100 = 2000 \text{ lb.}; \quad V_{cb} = \frac{1}{5} \times 100 \times 50 - \frac{3400}{50} = 2440 \text{ lb.}; \]

\[ V_{be} = 2500 + 60 = 2560 \text{ lb.}; \quad V_{ba} = 2500 + \frac{23400}{50} = 2960 \text{ lb.}; \]

\[ V_{ab} = 2500 - 460 = 2040 \text{ lb.}; \quad R_e = 4440 \text{ lb.} \text{ and } R_b = 5520 \text{ lb.} \]

The forces in the wires, calculated from equations (33b), are

\[ T_1 = \frac{5520}{0.2335} = 23,600 \text{ lb. and } T_2 = \frac{4440}{0.2335} = 19,000 \text{ lb.,} \]

and the elongations of the wires are

\[ \delta_1 = \frac{23,600 \times 51.4 \times 12}{0.192 \times E_w} = \frac{75,815,000}{E_w} \text{ and } \delta_2 = \frac{19,000 \times 102.8 \times 12}{0.110 \times E_w} = \frac{213,075,000}{E_w}. \]

The lateral translations of B and C are

\[ \Delta_b = \frac{75,815,000}{0.2335E_w} = \frac{324,690,000}{E_w}, \quad \Delta_c = \frac{213,075,000}{0.2335E_w} = \frac{912,527,000}{E_w}, \]

and \( E_w \Delta = \frac{912,527,000}{100} - 324,690,000 = 131,574,000. \)

The bending moment \( m_b \) due to the relative translation of B and C is calculated from (33d):

\[ m_b'' \left( \frac{50}{3 \times 0.0045 E_w} + \frac{50}{3 \times 0.00126 E_w} \right) = \frac{131,574,000}{50E_w} \times 2, \]

from which \( m_b = \frac{69,100}{n} \) and, assuming that \( n = \frac{E}{E_w} = 10, \ m_b'' = 6900 \text{ ft.-lb.} \)

Finally, \( m_b = 23,400 + 6900 = 30,300 \text{ ft.-lb.} \)

**Stabilising Beams.**

Stabilising beams (Figs. 20a and b) are often used to support columns close to existing buildings. To analyse the stresses in continuous stabilising beams it is usual to assume that the load on the ground is uniformly distributed and that there is no differential settlement between adjoining bases. The latter may be treated separately as shown under "Settlement of Supports" (page 6).

**Single Span** (Fig. 20c).—This type of beam is statically determinate, but it is considered here to facilitate calculations of continuous beams. The equivalent uniformly distributed load is

\[ w = \frac{1}{2} W \left( \frac{1}{a} + \frac{1}{l} \right). \]

Shearing forces are \( V_a = W; \ V_b = W \frac{a}{l}; \ V_c = W \left( 1 + \frac{a}{L} \right). \)

The bending moments between A and D are

\[ m_a = Wx - \frac{1}{2} wx^2, \ m_a = V_b(l - a), \text{ and } m_c = W_a - \frac{1}{2} wa^2. \]
The position of maximum bending moment is found from \( \frac{dm}{dx} = 0 \), from which
\[
x' = \frac{W}{w}, \quad \text{and the maximum bending moment is}
\]
\[
\max. \; m_0 = \frac{W^2}{2w} \quad \quad \quad \quad \quad \quad (35)
\]

**Two Spans.**—The statically-indeterminate bending moment \( m_0 \) for a two-span stabilising beam (*Fig. 21*) is calculated from the equation of equilibrium for

\[
\theta_{ba} = \frac{L_1}{3EI_1} + \frac{L_2}{3EI_2} = m_0 \quad \quad \quad \quad \quad \quad (36)
\]

in which \( EI_1 \theta_{ba} = \frac{wa^2}{12} \left( L_1 - \frac{a^2}{2L_1} \right) \)—see Appendix.
For \( L_1 = L_2 \) and \( I_1 = I_2 \):

\[
m_b = \frac{3}{2} \frac{\theta_{ba}}{L} \tag{37}
\]

**Example 3.**—Consider the beam in Fig. 22 for which \( W = 100 \) tons, \( L = 16 \text{ ft.} \), \( L = 24 \text{ ft.} \), and the base at A is 5 ft. by 8 ft.

\[
w = \frac{100}{2} \left( \frac{1}{2.5} + \frac{1}{13.5} \right) = 23.7 \text{ tons per foot.}
\]

Load on the ground, \( \frac{23.7}{8} = 2.96 \) tons per square foot. The load function \( \theta_{ba} = \frac{23.7 \times 5.0}{12} \left( 16 - \frac{5.0^2}{2 \times 16} \right) = 150.5 \), and the bending moment at B, from equation (36), is \( m_b \left( \frac{16}{3} + \frac{24}{3} \right) = 150.5 \), from which \( m_b = 11.30 \) ft.-tons. The shearing force at A is \( V_{ab} = 100 - \frac{11.30}{16.0} = 99.3 \) tons, and the bending moment on span AB is \( m_{ab} = \frac{99.3^2}{2 \times 23.7} = 208 \) ft.-tons.

**Three Spans.**—Statically-indeterminate bending moments \( m_b \) and \( m_c \) are calculated from equations of equilibrium (38).

\[
\begin{align*}
B, \quad & m_b \left( \frac{L_1}{3I_1} + \frac{L_2}{3I_2} \right) - m_c \frac{L_2}{6I_2} = \theta_{ba} \\
C, \quad & m_c \left( \frac{L_2}{3I_2} + \frac{L_3}{3I_3} \right) - m_b \frac{L_2}{6I_2} = 0
\end{align*}
\tag{38}
\]
For \( L_1 = L_2 = L_3 \):

\[
m_b = \frac{8}{5L} \theta_{ba} \quad \text{and} \quad m_c = \frac{2}{5L} \theta_{ba}.
\]  

(39)

in which \( \theta_{ba} = \frac{wa^2}{12} \left( \frac{L-a^2}{2L} \right) \).

**Continuous Beam on Elastic Foundation.**

Consider a continuous beam supported throughout its length on an elastic foundation and carrying known loads \( W_1, W_2, \) etc. (Fig. 24). Problems of this kind often occur when columns bear on a continuous footing. It is generally assumed that the pressure on the ground varies linearly between the points of application of the loads and that there is no differential settlement of columns. The latter, if known, can be taken into account as shown under "Settlement of Supports" (page 6). With these assumptions, the statically-indeterminate bending moments and the pressure on the ground at each point of application of the external load are unknown. The problem can best be illustrated by a numerical example.

**Example 4.**—Pressures \( \dot{p}' \) and \( \dot{p}'' \) at the ends of two cantilevers can be expressed in terms of pressures on the first and the last span respectively.

\[
\begin{align*}
\dot{p}' &= \dot{p}_a - \frac{a}{L}(\dot{p}_b - \dot{p}_a) = I \cdot 333\dot{p}_a - 0.333\dot{p}_b \\
\dot{p}'' &= \dot{p}_d - \frac{b}{L}(\dot{p}_e - \dot{p}_d) = I \cdot 333\dot{p}_d - 0.333\dot{p}_e
\end{align*}
\]  

(40)

Similarly, bending moments \( m_a \) and \( m_d \) can be expressed in terms of pressures in the first and the last span:

\[
m_a = \frac{a^2}{6}(\dot{p}_a + 2\dot{p}') = 9.76\dot{p}_a - 1.77\dot{p}_b; \quad m_d = \frac{b^2}{6}(\dot{p}_d + 2\dot{p}'') = 22\dot{p}_d - 4\dot{p}_e.
\]  

(41)

The angular deformations \( \theta \) due to the resistance of the ground can be calculated from equations (2)—see Appendix and equations (42).

\[
\begin{align*}
EI \theta_{ba} &= \frac{13}{45} (\dot{p}_a + \dot{p}_b) - m_a \frac{L}{\theta} = 14.08\dot{p}_a + 41.95\dot{p}_b \\
EI \theta_{bc} &= \frac{13}{45} (\dot{p}_b + \dot{p}_c) = 307.2\dot{p}_b + 269.0\dot{p}_c \\
EI \theta_{bd} &= \frac{13}{45} (\dot{p}_c + \dot{p}_d) = 269.0\dot{p}_b + 307.2\dot{p}_d \\
EI \theta_{cd} &= \frac{13}{45} (\dot{p}_d + \dot{p}_e) - m_d \frac{L^3}{\theta} = 130.26\dot{p}_b - 18.7\dot{p}_d
\end{align*}
\]  

(42)
CONTINUOUS BEAMS

The shearing forces in each beam expressed in terms of the resistance of the ground are given in (43).

\[
\begin{align*}
V_{aa'} & = \frac{1}{6} a \cdot P' + \frac{1}{2} a \cdot P = 4.7666 P_a - 0.666 P_b \\
V_{ab} & = \frac{1}{6} L_1 (2P_a + P_b) = 4P_a + 2P_b \\
V_{ba} & = \frac{1}{6} L_1 (P_a + 2P_b) = 2P_a + 4P_b \\
V_{bc} & = \frac{1}{6} L_2 (2P_b + P_c) = 8P_b + 4P_c \\
V_{cb} & = \frac{1}{6} L_2 (P_b + 2P_c) = 4P_b + 8P_c \\
V_{cd} & = \frac{1}{6} L_3 (2P_c + P_d) = 6P_c + 3P_d \\
V_{dc} & = \frac{1}{6} L_3 (P_c + 2P_d) = 3P_c + 6P_d \\
V_{dd'} & = \frac{1}{2} b \cdot P_d + \frac{1}{2} b \cdot P = 7P_d - P_c
\end{align*}
\]

Six equations of equilibrium can be set out by using four conditions of equilibrium of shearing forces and two conditions of angular deformations at points B and C, as in (44).

\[
\begin{align*}
\text{s.c.A... } V_{aa'} + V_{ab} - \frac{mb-ma}{L_1} &= W_1 \\
\text{s.c.B... } V_{ba} + V_{bc} + \frac{mb-ma}{L_1} + \frac{mb-mc}{L_2} &= W_2 \\
\text{s.c.C... } V_{cb} + V_{cd} - \frac{mb-mc}{L_2} + \frac{mc-md}{L_3} &= W_3 \\
\text{s.c.D... } V_{cd} + V_{dd'} - \frac{mc-md}{L_3} &= W_4 \\
\text{A BC... } \frac{L_1}{3} m_a + \frac{L_1}{3} m_b + \frac{L_2}{3} m_b + \frac{L_2}{6} m_c &= \Theta_{ba} + \Theta_{bc} \\
\text{B CD... } \frac{L_2}{6} m_b + \frac{L_2}{3} m_c + \frac{L_3}{3} m_c + \frac{L_3}{6} m_d &= \Theta_{cb} + \Theta_{cd}
\end{align*}
\]

Substituting values from (41), (42), and (43) in (44), six equations with four unknown pressures and two unknown bending moments are obtained as in (45),

\[
\begin{align*}
\text{s.c.A... } & 8.6666 P_a + 7333 P_b - \frac{1}{12} (mb - 9.76 P_a + 1.777 P_b) = 40 \\
\text{s.c.B... } & 2P_a + 12P_b + 4P_c + \frac{1}{12} (mb - 9.76 P_a + 1.777 P_b) + \frac{1}{24} (mb - mc) = 150 \\
\text{s.c.C... } & 4P_b + 14P_c + 3P_d - \frac{1}{24} (mc - 22P_d + 4R) = 100 \\
\text{s.c.D... } & 2P_c + 13P_d - \frac{1}{18} (mc - 22P_d + 4R) = 60 \\
\text{A BC... } & \frac{12}{6} (9.76 P_a - 1.777 P_b) + \frac{12}{3} m_b + \frac{24}{3} m_b + \frac{18}{6} m_b = 14.08 P_a + 3.915 P_b + 26.9 P_c \\
\text{B CD... } & \frac{24}{6} m_b + \frac{24}{3} m_c + \frac{18}{6} m_c + \frac{18}{6} (22P_d - 4R) = 26.9 P_b + 4.374 P_c - 18.7 R_d
\end{align*}
\]

from which \( P_a = 4.76 \) tons per foot; \( P_b = 9.62 \) tons per foot; \( P_c = 2.84 \) tons
per foot; \( P_d = 4.80 \) tons per foot; \( m_b = 200 \) ft.-tons; \( m_c = 190 \) ft.-tons. Also \( P' = 3.15 \) tons per foot; \( P'' = 5.44 \) tons per foot; \( m_a = 29.5 \) ft.-tons; \( m_d = 94.1 \) ft.-tons. The distribution of bending moments, shearing forces, and loads on the ground is shown in Fig. 25.

**Trussed Beams.**

A trussed beam (Fig. 26a) consists of a beam AB, a tie AEFB, and two struts, and can be analysed by the deformation method. As statically-indeterminate values it is usual to assume the forces in the struts, other forces being expressed in terms of these values.

For a beam with two struts symmetrically placed (Fig. 26a), the tension in the tie and the compression in the beam can be expressed in terms of force \( S \) in the strut, as follows: \( C = T = \frac{S}{\tan \phi} \); \( T_1 = \frac{S}{\sin \phi} \). It is assumed that the
tie is removed (Fig. 26b) and the deflection \( \Delta_w \) of the beam at C and D due to
the external system of loading is calculated from equations (2); for a uniformly-
distributed load the deflection becomes (see Appendix),

\[
\Delta_w = \frac{wa}{24EI_b} \left[ L^3 - a^2(2L - a) \right] \quad . \quad . \quad . \quad (46a)
\]

Similarly, the deflection \( \Delta_s \) of the beam due to a concentrated load \( S \) applied
upwardly at points C and D (Fig. 26c) is

\[
\Delta_s = \frac{Sa^2}{6EI_b} \left( 2a + 3b \right) \quad . \quad . \quad . \quad (46b)
\]

The shortening of the beam due to the axial force \( C \) (Fig. 26d) is

\[
\lambda = \frac{C}{E_bA_b} \frac{L}{2} = \frac{SL}{2 \tan \phi_e A_b} \quad . \quad . \quad . \quad (47a)
\]

and the vertical translation of point E (see Appendix) is

\[
\delta_1 = \frac{\lambda}{\tan \phi_e} = \frac{SL}{2 \tan^2 \phi_e A_b} \quad . \quad . \quad . \quad (47b)
\]

The extensions \( \delta' \) and \( \delta'' \) of the horizontal and the inclined parts of the tie
(Fig. 26e) are

\[
\delta' = \frac{T}{E_s A_s} \frac{b}{2} = \frac{Sb}{2 \tan \phi_e A_s} \quad . \quad . \quad . \quad (48a)
\]

\[
\delta'' = \frac{Tc}{E_s A_s} = \frac{Sc}{\sin \phi_e A_s} \quad . \quad . \quad . \quad (48b)
\]

and the vertical translation due to these extensions of the tie is

\[
\delta_s = \frac{\delta'}{\tan \phi_e} + \frac{\delta''}{\sin \phi_e} = \frac{Sb}{2 \tan \phi_e A_s} + \frac{Sc}{\sin^2 \phi_e A_s} \quad . \quad . \quad . \quad (48c)
\]

The statically-indeterminate force \( S \) can now be calculated from the equation
of equilibrium of vertical translations along the line CE (Fig. 26f):

\[
\Delta_w - \Delta_s = \delta_1 + \delta_s \quad . \quad . \quad . \quad (49a)
\]

and by substituting values from equations (46), (47) and (48),

\[
S \left[ \frac{na^2(2a + 3b)}{6} + \frac{nL}{2 \tan^2 \phi_e A_b} + \frac{b}{2 \tan^2 \phi_e A_s} + \frac{c}{\sin^2 \phi_e A_s} \right] = \frac{wan}{24} \left[ L^3 - a(2L - a) \right] \quad . \quad . \quad . \quad (49b)
\]

where \( n = \frac{E_s}{E_o} \).

When \( S \) is found, all bending moments, shearing forces, and axial forces
can be calculated from the equations of statics.

Bending moments at C and D are calculated from \( m_0 = m_4 = m_6^0 - m_6 \) in
which \( m_6 = Sa \), or, alternatively, from the equation of equilibrium of angular
deforations at C :

\[
C, \quad \frac{a}{3EI} + \frac{b}{3EI} + \frac{b}{6EI} = \frac{\Delta_s}{a} \quad . \quad . \quad . \quad (50)
\]

from which \( m_6 = \frac{6\Delta_s E_b I_b}{a(2a + 3b)} \).
Example 5.—As a numerical example, consider the trussed beam shown in Fig. 27, which is subjected to a uniformly-distributed load \( w = 1000 \, \text{lb. per foot} \).

![Fig. 27.](image)

From Fig. 27, \( \tan \phi = \frac{3.5}{12} = 0.292 \); \( \sin \phi = 0.2804 \); \( c = \frac{3.5}{0.2804} = 12.5 \, \text{ft.} \);

\[
I_b = \frac{0.75}{12} = 0.0625 \, \text{ft.}^4; \quad A_b = 0.75 \, \text{sq. ft.} \quad A_s = 0.0157 \, \text{sq. ft.} \quad n = 15.
\]

From equation (51b),

\[
S = \frac{15 \times 12^2 (24 + 3 \times 12)}{6} + \frac{15 \times 36 \times 0.0625}{2 \times 0.292^2 \times 0.75} + \frac{12 \times 0.0625}{2 \times 0.292^2 \times 0.75} + \frac{12.5 \times 0.0625}{0.2804^2 \times 0.0157} = \frac{1000 \times 12 \times 15}{24} [36^3 - 12^2 (72 - 12)],
\]

from which \( S = 12,700 \, \text{lb.} \).

Also \( T = C = \frac{12,700}{0.292} = 43,300 \, \text{lb.} \); \( T = \frac{12,700}{0.2804} = 45,400 \, \text{lb.} \);

and \( m_v = \frac{1}{3} \times 1000 \times 12 \times (36 - 12) - 12,700 \times 12 = 8400 \, \text{ft.-lb.} \).

Interconnected Beams.

In this section the deformation method is used to analyse various beams interconnected in plan or in elevation. As previously, all joints are freed from the effect of continuity and allowed to deflect to their final positions. All angular gaps are then indicated on a diagram of the beams and equations of equilibrium.

![Fig. 28.](image)
CONTINUOUS BEAMS

derived. In certain cases, torsional resistances of beams are also taken into account.

In Fig. 28 two beams are simply-supported and connected by ties, and the whole structure is allowed to deflect to its final position (as indicated by B', C', F', and G') by freeing all joints from the effects of continuity, as shown by dotted lines. The angular gaps are then indicated and the equations of equilibrium are set out as in (51).

\[
\begin{align*}
ABC & \left(m_1 \alpha_{bc} + m_1 \alpha_{bc} + m_1 \alpha_{bc} = \frac{A}{a} - \theta_1 - \theta_2 \right) \\
EFG & \left(m_2 \alpha_{ef} + m_2 \alpha_{ef} + m_2 \alpha_{ef} = \frac{A}{a} - \theta_3 \right) \\
& \left(\frac{m_1}{a} + \frac{m_2}{a} = \Sigma W_bf \right)
\end{align*}
\]

\[\text{Fig. 29.}\]

**Example 6.**—For the beams shown in Fig. 29, equations (51) become

\[
\begin{align*}
ABC & \left(\frac{12}{31} m_1 + \frac{16}{31} m_1 + \frac{16}{61} m_1 = \frac{A}{a} - \frac{1000 \times 12^3}{24 \times 1} - \frac{1000 \times 16^3}{24 \times 1} \right) \\
EFG & \left(\frac{12}{31} m_2 + \frac{16}{31} m_2 + \frac{16}{11} m_2 = \frac{A}{a} - \frac{2000 \times 12^3}{24 \times 1} - \frac{2000 \times 16^3}{24 \times 1} \right) \\
& \left(\frac{m_1}{12} + \frac{m_2}{12} = \frac{1}{2} \left(1000 + 2000\right) \left(12 + 16\right) \right)
\end{align*}
\]

from which \( m_1 = 433,500 \text{ ft.-lb.} \) and \( m_2 = 70,200 \text{ ft.-lb.} \). Also

\[ P = \frac{433,500}{12} - \frac{1}{6} \times 1000 \times 12 - \frac{1}{6} \times 1000 \times 16 = 22,100 \text{ lb.} \]

**Example 7.**—If one beam is continuous over four supports and another is simply-supported (Fig. 30), the equations for a general case with any symmetrical system of loading are as (53).

\[
\begin{align*}
ABC & \left(\frac{12}{31} m_1 + \frac{16}{21} m_1 = \frac{A}{a} - \frac{1000 \times 12^3}{24 \times 1} - \frac{1000 \times 16^3}{24 \times 1} \right) \\
EFG & \left(\frac{12}{31} m_2 + \frac{16}{21} m_2 = \frac{A}{a} - \frac{2000 \times 12^3}{24 \times 1} - \frac{2000 \times 16^3}{24 \times 1} \right) \\
& \left(\frac{2a}{31} m_3 + \frac{12}{31} m_3 - \frac{12}{61} m_3 = \frac{A}{a} + \frac{1000 \times 12^3}{24 \times 1} + \frac{1000 \times 20^3}{24 \times 1} \right) \\
& \left(\frac{m_1}{24} + \frac{m_2}{24} + \frac{m_3}{24} = \frac{12}{2} \times 1000 \left(12 + 16\right) + \frac{12}{2} \times 2000 \times 16 \right)
\end{align*}
\]
For a particular case of beams with the dimensions and loading shown in Fig. 30, equations (53) become (54), in which $\varepsilon_{bc} = \alpha_{bc} + \beta_{bc}$, and $\varepsilon_{f3} = \alpha_{f3} + \beta_{f3}$,

$$\begin{align*}
ABC: \quad & m_1 \alpha_{bc} + m_1 \varepsilon_{bc} = \frac{A}{a} - \theta_1 - \theta_2 \quad \vdots \\
EFG: \quad & m_2 \alpha_{fe} + m_2 \varepsilon_{f3} = \frac{A}{a} - \theta_3' - \theta_2' \quad \vdots \\
KAB: \quad & m_3 \alpha_{kb} + m_3 \varepsilon_{ob} - m_1 \beta_{ob} = \frac{A}{a} + \theta_1 + \theta_3 \quad \vdots \\
SxBF: \quad & \frac{m_1}{a} + \frac{m_2}{a} + \frac{m_3}{a} = \frac{1}{2} w_1 (a + b) + \frac{1}{2} w_2 b
\end{align*}$$

from which $m_1 = 128,000$ ft.-lb.; $m_2 = -4400$ ft.-lb.; $m_3 = 236,400$ ft.-lb.

Also $$P = \frac{128,000 + 236,400}{12} - \frac{1}{2} \times 1000(12 + 16) = 16,300 \text{ lb}.$$ 

The three simply-supported beams shown in plan in Fig. 31 are interconnected at B and E. This arrangement often occurs in staircases. The deflected shapes
of the beams freed from the effects of continuity and all angular gaps are as shown. Assuming that the supports offer no torsional resistance, the equations of equilibrium are as (55a), in which \( \varepsilon_{bc} = \alpha_{bc} + \beta_{bc} \), and \( \varepsilon_{ab} = \alpha_{ab} + \beta_{ab} \).

\[
\begin{align*}
ABC \cdots m_1 \alpha_{ba} + m_1 \alpha_{bc} &= \frac{A}{a} + \frac{A}{b} \\
6EB \cdots m_2 \alpha_{eb} + m_1 \alpha_{eb} &= \frac{A}{c} \\
Sc. B \cdots \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} &= W
\end{align*}
\]  

(55a)

Example 8.—If for the beams shown in Fig. 31 \( a = 9 \) ft., \( b = 15 \) ft., \( c = 6 \) ft., \( d = 6 \) ft., and \( W = 30,000 \) lb., equations (55a) become (55b).

\[
\begin{align*}
ABC \cdots & \frac{9}{3} m_1 + \frac{15}{3} m_1 = \frac{A}{a} EI_1 + \frac{A}{b} EI_1 \\
6EB \cdots & \frac{6}{3} m_2 + \frac{6}{3} m_2 = \frac{A}{c} EI_2 \\
Sc. B \cdots & \frac{m_q}{9} + \frac{m_q}{15} + \frac{m_q}{6} = 30,000
\end{align*}
\]  

(55b)

\[\text{Fig. 32.}\]

Example 9.—If the beams shown in Fig. 32 support a uniformly-distributed load \( w \) of 1000 lb. per foot (Fig. 32), the equations of equilibrium become (56),

\[
\begin{align*}
ABC \cdots & \frac{9}{3} m_1 + \frac{15}{3} m_1 = \frac{A}{a} + \frac{A}{b} - \Theta_1 - \Theta_2 \\
DBE \cdots & \frac{45}{3} m_2 + \frac{6}{3} m_2 + \frac{6}{3} m_2 = \frac{A}{c} - \Theta_3 - \Theta_4 \\
Sc. B \cdots & \frac{m_q}{9} + \frac{m_q}{15} + \frac{m_q}{6} = W
\end{align*}
\]  

(56)

in which \( W = \frac{1}{2} \times 1000 \times 9 + \frac{1}{2} \times 1000 \times 15 + \frac{1}{2} \times 1000 \times 4.5 + \frac{1}{2} \times 1000 \times 6 = 17,250 \) lb  
\( \Theta_1 = \frac{1}{24} \times 1000 \times 9^3 = 30,200; \Theta_2 = 3,800; \Theta_3 = 14,000; \Theta_4 = 9,000 \)

from which \( m_1 = -25,700 \) ft.-lb. and \( m_2 = 33,400 \) ft.-lb. Fig. 32 shows the distribution of the bending moments and shearing forces.

Fig. 33 shows two beams interconnected at B, with simple supports at A, C, D, and E, but capable of resisting torsional moments at these supports. Both beams are allowed to deflect to their final position at B′ when the joints
have been released from the continuity. The deflected shapes of the beams and all angular gaps are shown in Fig. 33, and the equations of equilibrium are as

\[
\begin{align*}
ABC & . . . . . \frac{m_1}{a} \alpha_{ba} + m_2 \alpha_{bc} = \frac{A}{a} + \frac{A}{b} - \theta_{ba} - \theta_{bc} \\
DBE & . . . . . \frac{m_3}{c} \alpha_{be} + m_4 \alpha_{bd} = \frac{A}{c} + \theta_{be} - \theta_{bd} \\
ABD & . . . . . \frac{c}{L_2} (m_1 - m_2) \tau_{bd} = \frac{A}{a} - \theta_{ba} \\
CBE & . . . . . \frac{c}{L_1} (m_4 - m_3) \tau_{bc} = \frac{A}{c} - \theta_{be} \\
S.C.B & . . . . . \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c} + \frac{m_4}{d} = W_B
\end{align*}
\]

in (57), in which \(\tau\) represents the torsional deformation of the beam BD due to unit twisting moment \(EI\tau_{ba} = kL\), and \(k\) is the torsional coefficient for the beam.

For rectangular beams, using St. Venant’s equation for the polar moment of inertia, \(k = \frac{EI}{GJ} = 0.65 \left[ 1 + \left( \frac{d}{b} \right)^4 \right]\), where \(G\) is the shear modulus, \(J\) the polar moment of inertia, and \(d\) is the greater and \(b\) the smaller cross-sectional dimensions of the beam. From this expression, for \(\frac{d}{b} = 2\), \(k = 3.25\).

**Example 10.**—For the interconnected beams shown in Fig. 34 equations (57)

\[
\begin{align*}
ABC & . . . . . \frac{6}{5} m_1 + \frac{12}{5} m_2 = \frac{A}{a} E I + \frac{A}{12} E I \\
DBE & . . . . . \frac{3}{5} m_4 + \frac{24}{5} m_3 = \frac{A}{4} E I + \frac{A}{21} E I \\
ABD & . . . . . \frac{6}{5} m_1 + (m_4 - m_3) \frac{2}{5} \tau_{bd} = \frac{A}{5} E I \\
CBE & . . . . . (m_4 - m_3) \frac{6}{10} \tau_{bc} - \frac{2}{5} \tau_{bd} = \frac{A}{21} E I \\
S.C & . . . . . \frac{m_1}{6} + \frac{m_2}{12} + \frac{m_3}{21} + \frac{m_4}{9} = 10,000
\end{align*}
\]

in (58).
become (58) from which $m_1 = 31,600$ ft.-lb., $m_2 = 28,900$ ft.-lb., $m_3 = 9200$ ft.-lb., and $m_4 = 17,000$ ft.-lb.

Beams AC and DE (Fig. 35) are interconnected at B and joined by perimeter beams which can resist the torsion from the interconnected beams; the equations of equilibrium are as in (59).

A.S.—C
\[ \begin{align*}
ABC &\quad \ldots \quad m_1 \alpha_{ba} + m_2 \alpha_{bc} = \frac{A}{a} + \frac{A}{b} \\
DBE &\quad \ldots \quad m_3 \alpha_{bd} + m_4 \alpha_{be} = \frac{A}{c} + \frac{A}{d} \\
ABE &\quad \ldots \quad m_1 \alpha_{ba} + \frac{C}{L_2} (m_1 - m_2) T_{be} + m_5 \alpha_{eg} = \frac{A}{a} \\
CBD &\quad \ldots \quad m_3 \alpha_{bd} + \frac{A}{L_1} (m_4 - m_3) T_{bc} + m_7 \alpha_{ch} = \frac{A}{c} \\
A &\quad \quad \ldots \quad m_8 \alpha_{af} - \left[ \frac{b}{L_1} (m_4 - m_3) - m_8 \right] \alpha_{ag} = 0 \\
C &\quad \quad \ldots \quad m_7 \alpha_{ch} - \left[ \frac{a}{L_1} (m_4 - m_3) - m_7 \right] \alpha_{ck} = 0 \\
D &\quad \quad \ldots \quad m_6 \alpha_{df} - \left[ \frac{d}{L_2} (m_1 - m_2) - m_6 \right] \alpha_{dh} = 0 \\
E &\quad \quad \ldots \quad m_5 \alpha_{eg} - \left[ \frac{c}{L_2} (m_1 - m_2) - m_5 \right] \alpha_{ek} = 0 \\
\text{s.c. B} &\quad \ldots \quad \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c} + \frac{m_4}{d} = W
\end{align*} \]

Example II.—For the beams shown in Fig. 36 equations (59) become (60),

![Fig. 36.]

from which \( m_1 = 34,211 \text{ ft.-lb.} \); \( m_2 = 31,372 \text{ ft.-lb.} \); \( m_3 = 13,590 \text{ ft.-lb.} \);
\( m_4 = 9343 \text{ ft.-lb.} \); \( m_5 = 1325 \text{ ft.-lb.} \); \( m_6 = 168 \text{ ft.-lb.} \); \( m_7 = 424 \text{ ft.-lb.} \); and
\( m_8 = 853 \text{ ft.-lb.} \).
CONTINUOUS BEAMS

\[
\begin{align*}
ABC & \ldots \frac{6}{3} m_1 + \frac{12}{3} m_2 = \frac{4}{6} EI + \frac{4}{12} EI \\
DBE & \ldots \frac{2}{3} m_3 + \frac{9}{3} m_4 = \frac{4}{21} EI + \frac{4}{7} EI \\
ABE & \ldots \frac{6}{3} m_1 + (m_1 - m_2) \frac{21}{36} x 3 \cdot 25 x 9 + \frac{6}{3} m_5 = \frac{4}{6} EI \\
CBD & \ldots \frac{2}{3} m_3 + (m_4 - m_3) \frac{6}{18} x 3 \cdot 25 x 12 + \frac{2}{3} m_f = \frac{4}{21} EI \\
A & \ldots \frac{2}{3} m_8 - \frac{9}{3} [(m_4 - m_3) \frac{12}{18} - m_8] = 0 \\
C & \ldots \frac{2}{3} m_7 - \frac{9}{3} [(m_4 - m_3) \frac{6}{18} - m_7] = 0 \\
D & \ldots \frac{6}{3} m_6 - \frac{12}{3} [(m_1 - m_2) \frac{9}{30} - m_6] = 0 \\
E & \ldots \frac{6}{3} m_5 - \frac{12}{3} [(m_1 - m_2) \frac{21}{30} - m_5] = 0 \\
S & \ldots \frac{m_1}{6} + \frac{m_2}{12} + \frac{m_3}{21} + \frac{m_4}{9} = 10,000
\end{align*}
\]

Beams on a Diagonal Grid.

The diagonal grid with beams interconnected at all joints (Fig. 37) has certain advantages by redistributing bending moments and shearing forces so
that the beams that would tend to deflect most are partially and elastically supported by other beams. It is assumed, as previously, that all beams meeting at a joint have equal deflections. Further, since the torsion is greatest near the corners of the grid it is usual to assume that only beams near the corners are subjected to torsion; the torsional resistance of other beams is neglected.

The analysis of this form of structure often requires a large number of simultaneous equations, and only with geometrical symmetry of the structure or by making approximate assumptions can the number of equations be reduced. An unsymmetrical system of loading can generally be simplified by adding the solutions of two equivalent systems, one with a symmetrical and one with an unsymmetrical system of loading. Except for a larger number of unknown bending moments, the analysis to be carried out is not more difficult than those of the previous examples.

The structure is allowed to deflect to its final position by freeing the joints from the effect of continuity, and all angular gaps are indicated. The equations of equilibrium required to close the gaps simultaneously are then derived.

For the grid shown in Fig. 37, all the angular gaps and bending moments are indicated and the equations of equilibrium are as in (61).

\[ \text{Fig. 38.} \]
**Example 12.**—For the diagonal grid shown in Fig. 38, which supports a load $W$ at each internal point of intersection, equations (61) become (62), from

\[
\begin{align*}
A, \quad & m_6 + m_4 \alpha_{ac} + \frac{1}{2} m_4 \alpha_{cb} \cos \phi + \frac{1}{2} m_4 \alpha_{ad} cos \phi - m_5 \beta_{ac} = \frac{AC}{L} \\
ACE, \quad & m_6 \alpha_{ce} - m_5 \alpha_{ca} + m_4 \beta_{ca} - m_7 \beta_{ce} = \frac{AE-AC}{L} - \frac{AC}{L} \\
DCA, \quad & m_5 \alpha_{ca} - m_4 \beta_{ca} + \frac{1}{3} (m_5 + m_4) T_{cd} = \frac{AC}{L} \\
CEF, \quad & m_7 \alpha_{ec} - m_7 \alpha_{ef} - m_6 \beta_{ec} = \frac{AE-AC}{L} - \frac{AE}{L} - \frac{DF}{L} \\
ABC, \quad & m_1 \alpha_{bc} + m_1 \alpha_{ba} \cos \phi - \frac{1}{2} m_6 \beta_{ba} \cos \phi - m_2 \beta_{bc} = \frac{AC}{L} \\
BCD, \quad & m_2 \alpha_{cb} + m_2 \alpha_{cd} - m_1 \beta_{cb} + m_3 \beta_{dc} = \frac{2AC}{L} \\
FDA, \quad & m_3 \alpha_{da} \cos \phi + m_3 \alpha_{df} - \frac{1}{2} m_4 \beta_{da} \cos \phi - m_3 \beta_{df} = \frac{DF}{L} \\
EF, \quad & m_8 \alpha_{fe} - m_8 \alpha_{fd} - m_7 \beta_{fe} - m_3 \beta_{fd} = \frac{AE}{L} - \frac{AE-DF}{L} \\
S_{c}, \quad & m_1 + m_3 + m_4 + m_6 + m_8 = \frac{AC}{L} \\
S_{e}, \quad & m_6 + m_7 + m_8 = \frac{AF}{L} \\
S_{f}, \quad & m_7 - m_8 = \frac{EF}{L} \\
\end{align*}
\]

\[
\begin{align*}
\frac{2l}{3} m_4 - \frac{l}{6} m_6 &= \frac{AC}{L} EI \\
ACE \quad & \frac{1}{3} m_6 - \frac{1}{3} m_5 + \frac{1}{6} m_4 - \frac{1}{6} m_7 = \frac{AE}{L} EI - \frac{2AC}{L} EI \\
DCA \quad & \frac{1}{3} m_5 - \frac{1}{3} m_6 + \frac{1}{2} (m_3 + m_4) \frac{L^2}{2} = \frac{AC}{L} EI \\
CEF \quad & \frac{1}{3} m_7 + \frac{1}{3} m_7 - \frac{L}{6} m_6 + \frac{1}{6} m_8 = \frac{2AE}{L} EI - \frac{AC}{L} EI - \frac{DF}{L} \\
ABC \quad & \frac{1}{3} m_1 + \frac{1}{3} m_1 - \frac{1}{2} m_6 + \frac{1}{6} m_7 = \frac{2AC}{L} EI \\
BCD \quad & \frac{1}{3} m_2 + \frac{1}{3} m_2 - \frac{1}{2} m_6 + \frac{1}{6} m_3 = \frac{2AC}{L} EI \\
FDA \quad & \frac{1}{3} m_3 + \frac{1}{3} m_3 - \frac{1}{2} m_6 + \frac{1}{6} m_8 = \frac{AE}{L} EI \\
EF \quad & \frac{1}{3} m_8 + \frac{1}{3} m_8 + \frac{1}{6} m_7 - \frac{1}{6} m_3 = \frac{2AE}{L} EI - \frac{AE}{L} EI \\
S_{c} \quad & m_1 + 2m_2 - m_3 + m_4 + m_5 - m_6 - m_7 = W L \\
S_{e} \quad & m_6 + 2m_7 - m_8 = \frac{1}{2} W L \\
S_{f} \quad & m_3 + 2m_8 - m_7 = \frac{1}{2} W L
\end{align*}
\]

which $m_1 = 0.4185 WL$; $m_2 = 0.512 WL$; $m_3 = 0.4140 WL$; $m_4 = 0.2843 WL$; $m_5 = 0.068 WL$; $m_6 = 0.0564 WL$; $m_7 = 0.3244 WL$; $m_8 = 0.2052 WL$. Also, $\Delta C = 0.1782 \frac{W L^3}{EI}$; $\Delta E = 0.3459 \frac{W L^3}{EI}$; and $\Delta F = 0.2181 \frac{W L^3}{EI}$.

Fig. 38 shows the bending-moment diagrams, the shearing forces, and the deflections in terms of $WL$, $W$, and $\frac{W L^3}{EI}$ respectively.
CHAPTER II

FRAMES

ELASTIC frames can be analysed by the deformation method in a similar way to continuous beams, except that the translations of the joints, or sway, must also be taken into account. To find the values of statically-indeterminate bending moments, consider that the structure is relaxed from continuity at the joints, apply the external system of loading, and calculate angular gaps at the joints, taking into account the translations of the joints. Finally, set out equations of equilibrium of angular deformations, with the help of elastic constants of beams and columns.

Portal Frames.

General Case.—Consider a portal frame (Fig. 39), with prismatic or non-prismatic members, and subjected to any system of loading. The deflected shape of the frame, in the condition relaxed from continuity at the joints, is shown in Fig. 39 by dotted lines. The values of $\Delta$ represent a final horizontal translation of the joints. The angular gaps at A and B are \( \left( \theta_a - \frac{\Delta}{h_1} \right) \) and \( \left( \theta_b + \frac{\Delta}{h_2} \right) \) respectively, $\theta$ representing load functions as defined by equations (2).

These angular gaps must now be closed by the bending moments $m_a$ and $m_b$, which, together with the sway $\Delta$, are calculated from two conditions of equilibrium of the joints A and B, and from one condition of shear. These are as in (63),

\[
\begin{align*}
\text{CAB,} & \quad m_a \alpha_{ab} + m_a \alpha_{ac} + m_b \beta = \theta_a - \frac{\Delta}{h_1} \\
\text{ABD,} & \quad m_b \alpha_{ba} + m_b \alpha_{bd} + m_a \beta = \theta_b + \frac{\Delta}{h_2} \\
\text{S.c.,} & \quad \frac{m_a}{h_1} = \frac{m_b}{h_2}
\end{align*}
\]

in which S.c. is the "shearing condition".

Fig. 39.
For prismatic beams and columns, with any system of loading, these equations reduce to (64).

\[
\begin{align*}
\text{CAB, } m_a & \frac{L}{3EI_b} + m_a \frac{h_1}{3EI_1} + m_b \frac{L}{6EI_b} = \theta_a - \frac{A}{h_1} \\
\text{ABD, } m_b & \frac{L}{3EI_b} + m_b \frac{h_2}{3EI_2} + m_a \frac{L}{6EI_b} = \theta_b + \frac{A}{h_2}
\end{align*}
\]

S.c, \( m_a = \frac{m_b}{h_2} \)

\( (64) \)

**Portal Frame with Concentrated Load.**—If a concentrated load \( W \) is placed on a beam as in Fig. 40, the load functions \( \theta_a \) and \( \theta_b \) are (see Appendix):

\[
EI_b \theta_a = \frac{Wab}{6L}(L + b), \quad EI_b \theta_b = \frac{Wab}{6L}(L + a).
\]

![Portal Frame with Concentrated Load](image)

Fig. 40.

Substituting these values in (64),

\[
m_a = \frac{Wab}{2L} \cdot \frac{h_1 I_b}{h_1 (L + b) + h_2 (L + a)}
\]

\( (65a) \)

\[
m_b = \frac{h_2 m_a}{h_1}
\]

\( (65b) \)

in which \( n_1 = \frac{I_b}{I_1} \) and \( n_2 = \frac{I_b}{I_2} \), and if \( h_1 = h_2 = h \) and \( I_1 = I_2 = I_o \),

\[
m_a = m_b = \frac{3abW}{2n(2h + 3L)}
\]

\( (66) \)

where \( n = \frac{I_b}{I_o} \).

It is interesting to note that the position of the load \( W \) that causes maximum sway can be calculated from the condition \( \frac{d \Delta}{da} = 0 \), from which \( a = \frac{L}{2} \left( 1 + \frac{\sqrt{3}}{3} \right) \).

**Portal Frame with Uniformly-distributed Load on Part of Beam** (Fig. 41).—The load functions \( \theta_a \) and \( \theta_b \), for a beam with uniformly-distributed
load on part of its length, are \( EI_b \theta_a = \frac{wa^2}{24L} (2L^2 - a^2) \) and \( EI_b \theta_b = \frac{wa^2}{24L} (2L^2 - a^2) \) (see Appendix).

Substituting these values in (64),

\[ m_a = \frac{w a^2 h_1}{8L} \cdot \frac{h_1(2L - a)^2 + h_2(2L^2 - a^2)}{(n_1 h_1^3 + n_2 h_2^3) + L(h_1^2 + h_1 h_2 + h_2^2)} \]  

\[ m_b = \frac{h_2 m_a}{h_1} \]  

in which \( n_1 = \frac{I_b}{I_{c1}}, \ n_2 = \frac{I_b}{I_{c2}}, \) and, if \( h_1 = h_2 = h \) and \( I_1 = I_2 = I_c \), and \( n = \frac{I_b}{I_c} \)

\[ m_a = m_b = \frac{w a^2}{4} \cdot \frac{3L - 2a}{2nh + 3L} \]  

For uniformly-distributed load extending over the whole beam

\[ m_a = m_b = \frac{wL^3}{4(2nh + 3L)} \]  

**Portal Frame with any Load Acting on a Column.**—If any horizontal load acts on a column AC (Fig. 42) the angular gaps at A and B are \( \frac{A}{h} - \theta_{ae} \) and \( \frac{A}{h} \) respectively, and the equations of equilibrium are as in (71).
A, \[ m_a \left( \frac{L}{3EI_b} + \frac{h}{3EI_1} \right) - m_b \frac{h}{6EI_b} = \frac{A}{h} - \theta_{ae} \]

B, \[ m_b \left( \frac{L}{3EI_b} + \frac{h}{3EI_2} \right) - m_a \frac{L}{6EI_b} = \frac{A}{h} \]

S.c., \[ \frac{m_a}{h} + \frac{m_b}{h} = W \frac{a}{h} \]

For a concentrated load \( W \) acting on a column as in Fig. 42, the load function \( \theta_{ae} \) is \( EI \theta_{ae} = \frac{Wab}{6h} (h + a) \). Substituting this value in (71),

\[ m_a = \frac{Wa}{2h} \frac{h^2 + 3hL + a^2}{3L + 2h} \quad m_b = \frac{Wa}{2h} \frac{3h^2 + 3hL - a^2}{3L + 2h} \]

and, if \( a = h \):

\[ m_a = m_b = \frac{Wh}{2} \]

![Diagram of a portal frame with a crane load](image)

**Portal Frame with Triangular Pressure on a Column AC.—** The load function \( \theta_{ae} \) for a triangular load (see Appendix) is

\[ EI_1 \theta_{ae} = \frac{7Wh_1^2}{4 \times 45} \]

Substituting this value in (71),

\[ m_a = \frac{Wh}{60} \frac{30L + (20n_1 - 7)h}{3L + 2n_1h} \frac{3L + 2n_2h}{3L + 2n_1h} \quad m_b = \frac{Wh}{3} - m_a \]

in which \( n_1 = \frac{I_b}{I_1} \) and \( n_2 = \frac{I_b}{I_2} \).

If \( I_b = I_1 = I_2 \):

\[ m_a = \frac{Wh}{20} \frac{10L + 9h}{3L + 2h} \quad m_b = \frac{Wh}{60} \frac{30L + 13h}{3L + 2h} \]

**Portal Frame with a Crane Load.—** If one column is eccentrically loaded the deflected shape of the frame in a relaxed condition is as indicated in Fig. 44. The load function \( \theta_{ae} \) (see Appendix) is \( EI \theta_{ae} = \frac{W_e}{6h^2} (a^3 + 3a^2b - 2b^3) \), and (64) becomes (77)
Fig. 44.

\[ \begin{align*}
A, & \quad m_a \left( \frac{L}{3EI_b} + \frac{h_1}{6EI_1} \right) - m_b \frac{L}{6EI_b} = \frac{\Delta}{h} - \frac{We}{6h^2EI_b} (a^3 + 3a^2b - 2b^3) \\
B, & \quad m_b \left( \frac{L}{3EI_b} + \frac{h}{3EI_2} \right) - m_a \frac{L}{6EI_b} = \frac{\Delta}{h}
\end{align*} \]

\[ \text{S.c.,} \quad \frac{m_a}{h} - \frac{m_a}{h} = \frac{m_b}{h} \]

in which \( m_a = We \). For \( I_b = I_1 = I_2 \),

\[ m_a = \frac{We}{2h(3L + 2h)} (h^2 + 3hL + 3b^2); \quad m_b = \frac{3We}{2h(3L + 2h)} (h^2 + hL - 3b^2). \quad (78) \]

Frames with Inclined Members.

Portal Frames with Inclined Beams.—These frames can be analysed in a similar way to the frames previously considered. The angular gaps, for any system of loading, are shown in Fig. 45, and the equations of equilibrium are in (79).

\[ \begin{align*}
A, & \quad m_a \left( \frac{s}{3EI_b} + \frac{h_1}{3EI_1} \right) + m_b \frac{s}{6EI_b} = \theta_{ab} - \frac{\Delta}{h_1} \\
B, & \quad m_b \left( \frac{s}{3EI_b} + \frac{h_2}{3EI_2} \right) + m_a \frac{s}{6EI_b} = \theta_{ba} + \frac{\Delta}{h_2}
\end{align*} \]

\[ \text{S.c.,} \quad \frac{m_a}{h_1} = \frac{m_b}{h_2} \]

(79)
For a uniformly-distributed load \( w \) and constant moments of inertia (79) become

\[
m_a = \frac{wL^3}{8} \frac{(h_1 + h_2)h_1}{h_1^3 + h_1 h_2 s + h_2 h_2 s + h_2^3 + h_2^3} ; \quad m_b = m_a \frac{h_2}{h_1} .
\]  

(80)

and, for a concentrated load \( W \) on a beam,

\[
m_a = \frac{Wabh_1}{2L} \frac{h_1(L + b) + h_2(L + a)}{h_1^3 + h_1 h_2 s + h_2 h_2 s + h_2^3 + h_2^3} ; \quad m_b = m_a \frac{h_2}{h_1} .
\]  

(81)

**Frames with Inclined Columns.**—In the case of a portal frame with inclined columns (Fig. 46), points A and B will move horizontally by \( \Delta \), and vertically by \( \lambda_a \) and \( \lambda_b \) respectively. Neglecting the shortening of the columns due to the axial loads, these vertical translations of the joints do not represent new unknowns, and can be expressed in terms of the horizontal sway \( \Delta \) as follows:

\[
\lambda_a = \frac{\Delta}{h_1} ; \quad \lambda_b = \frac{\Delta}{h_2} ; \quad \lambda = \lambda_a + \lambda_b.
\]

The deformed shape of the frame and the angular gaps at A and B are shown in Fig. 46. The equations of equilibrium can now be set out, for any system of loading and any shape of frame, as in (82).

\[
\begin{align*}
A, & \quad m_a \alpha_{ab} + m_a \alpha_{ac} + m_b \beta_{ab} = \theta_{ab} - \frac{\Delta}{h_1} \frac{\lambda}{L} \\
B, & \quad m_b \alpha_{ba} + m_b \alpha_{bd} + m_a \beta_{ba} = \theta_{ba} + \frac{\Delta}{h_2} \frac{\lambda}{L}
\end{align*}
\]

S.c., \( \frac{m_a}{h_1} = \frac{m_b}{h_2} \)

(82)

**Frame with Inclined Columns submitted to the Action of a Lateral Force.**—The deformed frame is as shown in Fig. 47, and the equations of equilibrium are as in (83),
A, \[ m_a x_{ab} + m_a x_{ab} - m_b \beta_{ab} = \frac{A}{h_1} + \frac{\lambda}{L} \]
B, \[ m_b x_{bd} + m_b x_{bd} - m_a \beta_{ba} = \frac{A}{h_2} + \frac{\lambda}{L} \]

S.c., \( \frac{m_a}{h_1} + \frac{m_b}{h_2} = W \)

in which the value of \( \lambda \) is as defined in the previous section.

**Fig. 47.**

**EXAMPLE 13.**—Consider the frame in Fig. 48. To make the analysis clearer, the influence of each load will be dealt with separately.

**Fig. 48.**

For a point load \( W_1 = 10,000 \text{ lb.} \) at a distance \( a = 6 \text{ ft.} \) from \( A \). The load functions are:

\[ \theta_a = \frac{10,000 \times 6 \times 14}{6 \times 20} (20 + 14) = 238,000; \]
\[ \theta_b = \frac{10,000 \times 6 \times 14}{6 \times 20} (20 + 6) = 182,000; \]
\[ \lambda = \frac{8A}{16} + \frac{4A}{10} = 0.9A. \]
From equation (82)

\[
\begin{align*}
A, & \quad m_a^{20} + m_a^{17.89} + m_b^{20} = 238,000 - \frac{A}{16} - \frac{0.9A}{20}; \\
B, & \quad m_b^{20} + m_b^{10.77} + m_a^{20} = 182,000 + \frac{A}{10} + \frac{0.9A}{20}; \\
S.c., & \quad \frac{m_a}{16} = \frac{m_b}{10},
\end{align*}
\]

from which \( m_a = 17,000 \) ft.-lb. and \( m_b = 10,620 \) ft.-lb. It is interesting to note that, if the columns were vertical, from equation (65a)

\[
m_a = \frac{10,000 \times 6 \times 14}{2 \times 20} - \frac{16 [16(20 + 14) + 10(20 + 6)]}{16^4 + 10^4 + 20(16^2 + 10 \times 16 + 10^2)} = 17,520 \text{ ft.-lb.}
\]

\[
m_b = 17.52 \times \frac{10,000}{16} = 10,950 \text{ ft.-lb.}
\]

and that if both the columns were of the same height \( (h_1 = h_2 = 13 \text{ ft.}) \),

\[
m_a = m_b = \frac{3 \times 6 \times 14 \times 10,000}{2(2 \times 13 + 3 \times 20)} = 14,650 \text{ ft.-lb.}
\]

For a horizontal force \( W_2 = 1000 \text{ lb.} \) acting at A (Fig. 48).—Equations (83) become

\[
\begin{align*}
A, & \quad m_a^{17.89} + m_a^{20} - m_b^{20} = \frac{A}{16} + \frac{0.9A}{20}; \\
B, & \quad m_b^{10.77} + m_b^{20} - m_a^{20} = \frac{A}{10} + \frac{0.9A}{20}; \\
S.c., & \quad \frac{m_a}{16} + \frac{m_b}{10} = 1,000,
\end{align*}
\]

from which \( m_a = 4980 \) ft.-lb. and \( m_b = 6880 \) ft.-lb.

For a uniformly-distributed load on part of the beam (Fig. 48). The load functions are

\[
\theta_{ab} = \frac{1,000 \times 8^2}{24 \times 20}(2 \times 20 - 8)^2 = 136,500 \text{ and } \theta_{ba} = \frac{1,000 \times 8^2}{24 \times 20}(2 \times 20^2 - 8^2) = 98,100
\]

and from (83)

\[
\begin{align*}
A, & \quad m_a^{20} + m_a^{17.89} + m_b^{20} = 136,500 - \frac{A}{16} - \frac{0.9A}{20}; \\
B, & \quad m_b^{20} + m_b^{10.77} + m_a^{20} = 98,100 + \frac{A}{10} + \frac{0.9A}{20}; \\
S.c., & \quad \frac{m_a}{16} = \frac{m_b}{10},
\end{align*}
\]

from which \( m_a = 9540 \) ft.-lb. and \( m_b = 5960 \) ft.-lb.
EXAMPLE 14.—If a closed rectangular frame is loaded as in Fig. 49 equations of equilibrium can be set out in general terms as in (84).

\[
\begin{align*}
A... m_a \alpha_{ab} + m_a \alpha_{dc} + m_b \beta_{ab} + m_c \beta_{dc} &= \theta_{ba} - \frac{A}{h} \\
B... m_b \alpha_{ba} + m_b \alpha_{bd} + m_a \beta_{ba} + m_d \beta_{bd} &= \theta_{ba} + \frac{A}{h} \\
C... m_c \alpha_{cb} + m_c \alpha_{cd} + m_a \beta_{ca} + m_d \beta_{cd} &= \theta_{cd} + \frac{A}{h} \\
D... m_d \alpha_{dc} + m_d \alpha_{db} + m_b \beta_{db} + m_c \beta_{dc} &= \theta_{dc} - \frac{A}{h} \\
S.c... m_a - m_e &= m_b - m_d
\end{align*}
\]

With load functions

\[
\begin{align*}
EI\theta_{ab} &= \frac{1000 \times 15^2}{24 \times 24} (2 \times 14 - 15)^2 = 425,400; \\
EI\theta_{ba} &= \frac{1000 \times 15^2}{24 \times 24} (2 \times 24^2 - 15^2) = 362,100.
\end{align*}
\]

\[
W_1 = 1000 \times 15 \times \frac{10.5}{12} = 13,120 \text{ lb.}, \quad W_2 = 1880 \text{ lb.}
\]

\[
\begin{align*}
EI\theta_{od} &= \frac{13,120 \times 6 \times 18}{6 \times 24} (24 + 18) + \frac{1880 \times 18 \times 6}{6 \times 24} (24 + 6) = 455,600. \\
EI\theta_{de} &= \frac{13,120 \times 6 \times 18}{6 \times 24} (24 + 6) + \frac{1880 \times 18 \times 6}{6 \times 24} (24 + 18) = 354,400.
\end{align*}
\]

\[
\begin{align*}
A... m_a \frac{24}{3} + \frac{10}{3} m_a + \frac{24}{6} m_b + \frac{10}{6} m_c &= 425,400 - \frac{1}{h} E1 \\
B... m_b \frac{24}{3} + \frac{10}{3} m_b + \frac{24}{6} m_a + \frac{10}{6} m_d &= 362,100 + \frac{1}{h} E1 \\
C... m_c \frac{24}{3} + \frac{10}{3} m_c + \frac{10}{6} m_a + \frac{24}{6} m_w &= 455,600 + \frac{1}{h} E1 \\
D... m_d \frac{24}{3} + \frac{10}{3} m_d + \frac{10}{6} m_b + \frac{24}{6} m_c &= 354,400 - \frac{1}{h} E1 \\
S.c... m_a - m_b - m_c - m_d &= 0
\end{align*}
\]
the equations of equilibrium are as in (84a), from which \( m_a = 27,650 \text{ ft.-lb.} \), \( m_b = 18,510 \text{ ft.-lb.} \), \( m_c = 28,470 \text{ ft.-lb.} \), and \( m_d = 19,330 \text{ ft.-lb.} \).

**Continuous Frames with Straight Members.**

**General Case.**—Continuous frames may be analysed in a similar way to single-span frames. An increased number of unknowns is accompanied by a similar increase in the number of conditions of equilibrium.

In an imaginary condition of the frame, with joints relaxed from the effect of continuity, every external joint with two members, as A or C in Fig. 50, provides one condition of equilibrium, while an internal joint (as B) with three members provides two conditions. The intermediate junction, such as B in Fig. 50, therefore has two angular gaps, that is between members ABC and ABE (or alternatively EBC). When all angular gaps are marked, taking sway into account, the equations of equilibrium can be set out in the usual way. This procedure is illustrated in the following examples.

For a two-span frame (Fig. 50), with any shape of members and any system of loading, the equations of equilibrium are as in (85)

\[
\begin{align*}
\text{DAB,} \quad & m_1\alpha_{ab} + m_1\alpha_{ad} + m_2\beta_{ab} = \theta_{ab} - \frac{A}{h} \\
\text{ABC,} \quad & m_2\beta_{ba} + m_2\alpha_{ba} + m_3\alpha_{bc} + m_4\beta_{bc} = \theta_{ba} + \theta_{bc} \\
\text{ABE,} \quad & m_3\beta_{ba} + m_3\alpha_{ba} + (m_2 - m_3)\alpha_{be} = \theta_{ba} + \frac{A}{h} \\
\text{BCF,} \quad & m_4\beta_{cb} + m_4\alpha_{cb} + m_4\alpha_{cf} = \theta_{cb} + \frac{A}{h} \\
\text{S.c.,} \quad & \frac{m_1}{h} = \frac{m_2 - m_3}{h} + \frac{m_4}{h}
\end{align*}
\]

When bending moments are calculated from (85) all the other values can be obtained from equations of statics.

**Two-span Frame with Uniformly-distributed Load on One Beam (Fig. 51).**—If both beams are of equal span and the frame has constant moments of inertia, equations (85) reduce to (86).
ANALYSIS OF STRUCTURES

\[ \begin{align*}
DAB, \quad m_1 \frac{L}{3} + m_2 \frac{h}{3} + m_3 \frac{L}{6} &= \frac{wL^3}{24} - \frac{A}{h} EI \\
ABC, \quad m_1 \frac{L}{6} + m_2 \frac{L}{3} + m_3 \frac{L}{3} - m_4 \frac{L}{6} &= \frac{wL^3}{24} \\
ABE, \quad m_2 \frac{L}{6} + m_3 \frac{L}{3} + (m_2 - m_3) \frac{h}{3} &= \frac{wL^3}{24} + \frac{A}{h} EI \\
BCF, \quad -m_3 \frac{L}{6} + m_4 \frac{L}{3} + m_4 \frac{h}{3} &= \frac{A}{h} EI \\
S.c., \quad m_1 &= m_2 - m_3 + m_4.
\end{align*} \] . . . (86)

The general solution of these equations gives (87)

\[ \begin{align*}
m_1 &= \frac{wL^3}{24} \cdot \frac{6L + 7h}{3L^2 + 7hL + 4h^2}, \quad m_2 = \frac{wL^3}{24} \cdot \frac{6L^2 + 13hL + 6h^2}{3L^2 + 7hL + 4h^2} \\
m_3 &= \frac{whL^3}{24} \cdot \frac{6h + 5L}{3L^2 + 7hL + 4h^2}, \quad m_4 = \frac{whL^3}{24} \cdot \frac{1}{3L^2 + 7hL + 4h^2} \\
\end{align*} \] . . . (87)

Two-span Frame with Horizontal Force \( W \).—The deflected shape of the frame, with joints relaxed from continuity, is as shown in Fig. 52, the angular

\[ \begin{align*}
DAB, \quad m_1 \alpha_{ab} + m_2 \alpha_{ad} - m_2 \beta_{ab} &= \frac{A}{h} \\
ABC, \quad -m_1 \beta_{ab} + m_2 \alpha_{bc} + m_3 \alpha_{bc} - m_4 \beta_{bc} &= 0 \\
ABE, \quad -m_1 \beta_{ab} + m_2 \alpha_{ba} + (m_2 + m_3) \alpha_{ba} &= \frac{A}{h} \\
BCF, \quad -m_3 \beta_{cb} + m_4 \alpha_{cb} + m_4 \alpha_{cf} &= \frac{A}{h} \\
S.c., \quad m_1 \frac{1}{h} + m_2 \frac{1}{h} + \frac{m_3}{h} &= W
\end{align*} \] . . . (88a)
gaps being \( \frac{A}{h} \) and zero. Equations (85) become (88a). If the members are prismatic, equation (88a) becomes (88b), from which, if both beams are of equal span, and the frame is of constant cross section, the solution of equation (88b) is

\[
m_1 = m_4 = \frac{W h}{12} \frac{3L + 4h}{L + h}; \quad \text{and} \quad m_2 = m_3 = \frac{W h}{12} \frac{3L + 2h}{L + h}.
\]  

(89)

**Three Spans.—** For a three-span frame loaded as shown in Fig. 53 the equations of equilibrium are as shown in (90).

\[
\begin{align*}
\text{ABC} & \ldots \frac{L_1}{311} m_1 + \frac{L_2}{312} m_2 - \frac{L_3}{612} m_3 = \frac{w L^3}{2411}, \\
\text{ABE} & \ldots \frac{L_1}{311} m_1 + \frac{h}{31h} (m_1 - m_2) = \frac{w L_1^3}{2411}, \\
\text{BCD} & \ldots \frac{L_2}{312} m_2 + \frac{L_3}{313} m_3 - \frac{L_3}{613} m_2 = 0 \\
\text{s.c.} & \ldots m_1 - m_2 = m_3 - m_4
\end{align*}
\]

(90)

If \( L_1 = L_2 = L_3 \):

\[
\begin{align*}
m_1 &= \frac{w L^2}{24} \frac{9L + 8h}{3L + 5h}; \\
m_2 &= \frac{3}{5} w L^2 - \frac{2}{3} m_1; \\
m_3 &= 2m_1 + 2m_2 - \frac{w L^4}{64}; \\
m_4 &= m_2 + m_3 - m_1
\end{align*}
\]

(91)

**Three Spans with Concentrated Load on Middle Span.—** The deflected shape of the frame and the angular gaps at B and C are shown in Fig. 54. The equations of equilibrium are as in (92), in which

\[
E I \theta_{bc} = \frac{W ab}{6L} (L + b) \quad \text{and} \quad E I \theta_{eb} = \frac{W ab}{6L} (L + a).
\]

A.S.—D
If all beams are of equal span, and the frame is of constant cross section,
\[
\begin{align*}
  m_2 &= \frac{Wab}{3L^2} \left( 3L^2 + 3bL + 2hL + 5bh \right) ; \\
  m_1 &= \frac{Wab}{6L^2} \left( 2L + 5b \right) - \frac{3}{2}m_2 ; \\
  m_3 &= \frac{Wab}{L^2} \left( L + b \right) - 2(m_1 + m_2) ; \\
  m_4 &= m_3 + m_1 - m_2
\end{align*}
\]

**General Case of Three Span.**—In a general case of three-span frames of any shape, and with any system of loading, seven equations of equilibrium are required to obtain the statically-indeterminate bending moments. These may be written as in (94a).

For prismatic beams and columns as shown in Fig. 55, equations (94a) reduce to (94b).

**Three Span with Inclined Columns.**—A continuous frame with inclined columns and with any system of loading is subjected to horizontal and vertical translations of the joints, which must be taken into account when setting out the equations of equilibrium. The vertical translations \( \lambda \) are expressed in
\[ EAB \ldots m_1 \alpha_{ab} + m_1 \alpha_{ac} + m_2 \beta_{ab} = \Theta_{ab} - \frac{A}{h} \]
\[ ABC \ldots m_2 \alpha_{ba} + m_2 \alpha_{bb} + m_3 \alpha_{bc} + m_4 \beta_{bc} = \Theta_{ba} + \Theta_{bc} \]
\[ ABF \ldots m_1 \beta_{ba} + m_2 \alpha_{ba} + (m_2 - m_3) \alpha_{bf} = \Theta_{ba} + \frac{A}{h} \]
\[ BCD \ldots m_3 \beta_{cb} + m_4 \alpha_{cb} + m_5 \alpha_{cd} + m_6 \beta_{cd} = \Theta_{cb} + \Theta_{cd} \]
\[ BCG \ldots m_3 \beta_{cb} + m_4 \alpha_{cb} + (m_4 - m_5) \alpha_{cg} = \Theta_{cb} + \frac{A}{h} \]
\[ CDK \ldots m_5 \beta_{dc} + m_6 \alpha_{dc} + m_6 \alpha_{dk} = \Theta_{dc} + \frac{A}{h} \]
\[ S.C. \] \[ \frac{m_1}{h} = \frac{m_2 + m_3}{h} + \frac{m_4 - m_5}{h} + \frac{m_6}{h} \]

\[ EAB \ldots \frac{L_1}{3E_1b_1} m_1 + \frac{h}{3E_1c_1} m_1 + \frac{L_2}{6E_1b_1} m_2 = \Theta_{ab} - \frac{A}{h} \]
\[ ABC \ldots \frac{L_1}{6E_1b_1} m_1 + \frac{L_1}{3E_1b_2} m_2 + \frac{L_2}{6E_1b_2} m_4 = \Theta_{ba} + \Theta_{bc} \]
\[ ABF \ldots \frac{L_1}{6E_1b_1} m_1 + \frac{L_1}{3E_1b_2} m_2 + \frac{h}{3E_1c_2} (m_2 - m_3) = \Theta_{ba} + \frac{A}{h} \]
\[ BCD \ldots \frac{L_3}{6E_1b_2} m_3 + \frac{L_3}{3E_1b_3} m_4 + \frac{L_3}{6E_1b_3} m_6 = \Theta_{cb} + \Theta_{cd} \]
\[ BCG \ldots \frac{L_3}{6E_1b_2} m_3 + \frac{L_3}{3E_1b_3} m_4 + \frac{h}{3E_1c_3} (m_4 - m_5) = \Theta_{cb} + \frac{A}{h} \]
\[ COK \ldots \frac{L_3}{6E_1b_3} m_5 + \frac{L_3}{6E_1b_3} m_6 + \frac{h}{3E_1c_3} m_6 = \Theta_{dc} + \frac{A}{h} \]
\[ S.C. \ldots \frac{m_1}{h} = \frac{m_2 - m_3}{h} + \frac{m_4 - m_5}{h} + \frac{m_6}{h} \]

\[ ABC \ldots \frac{L_1}{3E_1b_1} m_1 + \frac{L_1}{3E_1b_2} m_2 + \frac{L_1}{6E_1b_2} m_3 = \Theta_{ba} + \Theta_{bc} - \frac{A}{L} - \frac{\lambda_1}{L_1} \]
\[ EBC \ldots \frac{L_1}{3E_1b_1} m_1 + \frac{h}{3E_1c_1} (m_2 - m_1) + \frac{L_1}{6E_1b_1} m_3 = \Theta_{bc} - \frac{A}{L} - \frac{\lambda_1}{L_1} \]
\[ BCD \ldots \frac{L_3}{3E_1b_1} m_3 + \frac{L_3}{3E_1b_2} m_4 + \frac{L_3}{6E_1b_2} m_5 = \Theta_{cd} + \Theta_{cd} + \frac{\lambda_3}{L} + \frac{\lambda_3}{L_3} \]
\[ BCF \ldots \frac{L_3}{3E_1b_1} m_3 + \frac{h}{3E_1c_1} (m_3 - m_4) + \frac{L_3}{6E_1b_1} m_2 = \Theta_{bc} + \frac{A}{L} + \frac{\lambda_3}{L_3} \]
\[ S.C. \ldots \frac{m_2 - m_1}{r_1} \cos \phi_1 = \frac{m_3 - m_4}{r_2} \cos \phi_2 \]

**Fig. 56.**
terms of the horizontal translation $\Delta$, and therefore do not represent separate unknowns. (For the relation between $\lambda$ and $\Delta$ see Appendix.) For a frame with inclined columns (Fig. 56) with any system of loading the equations of equilibrium can be set out as in (95), in which

$$\lambda = \lambda_1 + \lambda_2; \quad \frac{\lambda_1}{h_1} = \Delta_1; \quad \frac{\lambda_2}{h_2} = \Delta_2; \quad \xi_1 = \sqrt{\Delta^2 + \lambda_1^2}; \quad \text{and} \quad \xi_2 = \sqrt{\Delta^2 + \lambda_2^2}.$$

**Single-span Frames with Curved Members.**

**General Case.**—Frames with curved members can be analysed by the deformation method if, in addition to the angular deformations, the displacements at joints due to the curvature of the members are taken into account. As previously, the structure is assumed to be relaxed from the effect of continuity by cutting it at suitable places, load functions and elastic constants are calculated, and equations of equilibrium of each point at which the imaginary cut has been introduced are set out.

The load functions for a curved beam of any shape and cross section (Fig. 57) consist of angular deformations $\theta_a$ and $\theta_b$, and also of a linear deformation $\Delta_o$.

[Diagram of a curved beam with notation]

These, similarly to equation (2), are calculated from the well-known equations

$$\varepsilon = \int_A^B \frac{m ds}{EI}; \quad \bar{\varepsilon} = \int_A^B \frac{m ds}{EI}; \quad \bar{\varepsilon} = L - \bar{\varepsilon}; \quad \theta_a = \frac{\bar{x}}{L}; \quad \theta_b = \frac{\bar{x}}{L}; \quad \Delta_o = \int_A^B \frac{m y d s}{EI}$$

(96)

in which $\varepsilon$ represents the total angular deformation of a curved beam from A to B, $\theta_a$ and $\theta_b$ are angular deformations at A and B respectively, $\bar{x}$ and $\bar{\varepsilon}$ define the centre of gravity of the $\frac{m}{EI}$ diagram, $\Delta_o$ is the total horizontal deformation of a curved beam, and $x$ and $y$ are ordinates of any point on a beam. Some more complicated cases are calculated from equations (96) by the method of summation. Load functions for several cases of curved beams with various systems of loading are given in the Appendix.

Consider a pitched beam (Fig. 58) subjected to a uniformly-distributed load. From (96),

$$EI\theta_a = EI\theta_b = \int_A^B \frac{m ds}{2} = \frac{1}{2} wL^2 s; \quad EI\Delta_o = \int_A^B \frac{m y ds}{2} = \frac{5}{48} wL^2 \phi s .$$

(97)

in which $s$ is the length of the inclined portion and $\phi$ is the rise of the beam.

Equations (96) may also be used to calculate the elastic constants of curved
beams. Unit bending moment at the end A (Fig. 59a) will produce the angular deformations \( \alpha_{ab} \) and \( \beta \), and also linear deformation \( \Delta_{ab} \). Unsymmetrical curved beams will have different values of \( \alpha \) and \( \Delta \) when a unit bending moment is applied at the other end (Fig. 59b), the values of \( \beta \) being the same according to Clerk Maxwell’s Reciprocal Theorem. The angular deformations at A and B due to a unit horizontal force applied along AB (Fig. 59c) are denoted by \( \gamma \) and \( \delta \) respectively, and the horizontal translation of the joints by \( \Delta^h \). It should be noted that, from the Reciprocal Theorem, \( \gamma \) and \( \delta \) will be numerically equal to the corresponding values of \( \Delta^m \).

As an example of elastic constants of a curved beam, consider a symmetrical pitched beam (Fig. 60). Elastic constants, \( \alpha \), \( \beta \), and \( \Delta_m \) for the unit bending moment, calculated from (96), are
\[ EI_\varepsilon = \int_A^B m_\cdot ds = s; \quad \bar{x} = \frac{L}{3}; \quad EI_\alpha = \frac{8}{5} s; \quad EI_\beta = \frac{4}{3} s; \quad EI_\Delta^m = \frac{4}{3} p s \]  
(98)

and the elastic constants due to unit horizontal force are

\[ EI_\gamma = EI_\delta = \frac{4}{3} p s; \quad EI_\Delta^h = \frac{8}{3} p^2 s \]  
(99)

It should be noted that, although in the following chapters frames with pitched roofs are generally shown, the analysis applies to frames of any shape.

Symmetrical Frame with Symmetrical Loading.—With the load functions and elastic constants calculated from (96), statically-indeterminate bending moments can now be obtained by setting out the equations of equilibrium of angular deformations of the joints. For a symmetrical frame of any shape with symmetrical loading (Fig. 61) the angular gaps at A and B are \( \left( \theta_o + \frac{1}{2} \cdot \frac{\Delta_o}{h} \right) \).

The equation of equilibrium for joint A can be set out as follows:

\[ \text{CAB}, \quad m_\alpha \alpha + m_\beta + m_\alpha \alpha + m_\alpha \alpha = \frac{m_\alpha \alpha}{h} + H_\gamma + \frac{1}{2} H_\Delta^h = \theta_o + \frac{1}{2} \cdot \frac{\Delta_o}{h} \]  
(100)

It should be noted that, in setting out the equations of equilibrium, each figure has its geometrical meaning, which should be well understood and visualised to avoid error. For equation (100) these are

- \( m_\alpha \alpha \), rotation of beam AB due to bending moment \( m_\alpha \) at A.
- \( m_\beta \), rotation of beam AB due to \( m_\beta \) at B.
- \( m_\alpha \alpha \), rotation of column AC due to \( m_\alpha \) at A.
- \( \frac{\Delta^m}{h} \), rotation of beam AB due to horizontal movement of joint A when \( m_\alpha \) is at A.
- \( H_\gamma \), rotation of beam AB due to horizontal force at A.
- \( \frac{1}{2} H_\Delta^h \), force H is applied at A.

For a symmetrical pitched frame (Fig. 61) with uniformly-distributed load, the load functions from (97) are \( EI_\beta \beta = \frac{1}{12} w L^2 s \) and \( EI_\Delta \Delta = \frac{5}{8} w L^2 p s \), and the elastic constants from (98) are \( EI_\alpha \alpha = \frac{8}{5} s, \quad EI_\beta \beta = \frac{4}{3} s, \quad EI_\Delta^m = \frac{4}{3} p s \) and \( EI_\Delta^h = \frac{4}{3} p^2 s \). Substituting these values in (100),

\[ \frac{2s}{3 L_b} + \frac{s}{3 L_c} + \frac{h}{3 L_c} + \frac{h}{2 L_b} + \frac{h}{2 L_b} + \frac{h}{3 L_b} = \frac{w L^2 s}{12 L_b} + \frac{5}{96} \frac{w L^2 p s}{h L_b} \]  
(101)

\[ \text{Fig. 61} \]
and, if \( I_b = I_c \),

\[
m_a = \frac{wL^2}{32} \frac{hs(8h + 5\delta)}{h^3 + 3h^2s + 3hps + \delta^2s} .
\]

(102)

**General Case of Frame with Curved Member.**—In a general case of a frame with a curved member, of any shape and with any system of loading (Fig. 62), the angular gaps at A and B are \( (\theta_a + \frac{\Delta a}{h}) \) and \( (\theta_b + \frac{\Delta_0 - \Delta a}{h}) \), in which \( \Delta a \) represents the final horizontal translation of joint A; other symbols are as defined previously. It should be noted that point A' in Fig. 62 represents

![Diagram](image)

Fig. 62.

the final position of joint A, while point B’ will move under the action of moments \( m_a \) and \( m_b \) and force \( H \) before reaching its final position at \( B'' \). The values of \( m_a, m_b, \) and \( \Delta a \) are calculated for two conditions of angular deformations at A and B, and for one condition of shear, as follows:

\[
\begin{align*}
\text{CAB, } m_a\alpha_{ab} + m_b\beta + m_a\alpha_{ao} + H\gamma &= \theta_a + \frac{\Delta a}{h_1} \\
\text{ABD, } m_a\beta + m_a\alpha_{ba} + m_b\alpha_{bo} + m_b\alpha_{bd} + \frac{m_{mb}}{h_1} + H\delta + H\frac{\Delta a}{h_2} &= \theta_b + \frac{\Delta_0 - \Delta a}{h_2}
\end{align*}
\]

(103)

S.c., \( \frac{m_a}{h_1} = \frac{m_b}{h_2} \).

For a pitched frame with a concentrated load \( W \) the elastic constants are given by equation (98), and equations (103) become (104).

\[
\begin{align*}
m_a\left(\frac{2s}{3I_b} + \frac{s}{3I_b} + \frac{h}{3I_c} + \frac{\delta s}{2hI_b}\right) &= \frac{Wabs(L + b)}{3L^2I_b} + \frac{\Delta a}{h}E \\
m_a\left(\frac{2s}{3I_b} + \frac{s}{3I_b} + \frac{\delta s}{2hI_b} + \frac{h}{3I_c} + \frac{\delta s}{2hI_b} + \frac{\delta s}{2hI_b} + \frac{2\delta^2s}{3h^2I_b}\right)
\end{align*}
\]

(104)

\[
= \frac{Wabs(L + a)}{3L^2I_b} + \frac{Wabs(3L^2 - 4a^2) - \Delta a}{6h^2I_b}E
\]

from which, for \( I_b = I_c \), \( m_a = m_b = \frac{Wahs}{4L^2} \frac{6bhL + \delta(3L^2 - 4a^2)}{h^3 + 3h^2s + 3hps + \delta^2s} \).

(105a)

and, for \( a = b = \frac{L}{2} \), \( m_a = \frac{WHLs}{8} \frac{3h + 2\delta}{h^3 + 3h^2s + 3hps + \delta^2s} \).

(105b)
Frame with Lateral Load (Fig. 63).—If a frame of any shape is subjected to any lateral load acting on a column, equations (103) require only slight adjustment as follows:

\[
\begin{align*}
\text{CAB, } m_a x_{ab} + m_a x_{ae} - m_b \beta - H \delta &= \frac{\Delta a}{h} \\
\text{ABD, } m_b x_{ba} + m_b x_{bd} + m_b \frac{\Delta d}{h} + H \delta + H A_{h} - m_a \beta - m_a \frac{\Delta a}{h} &= \frac{\Delta a}{h} \\
\text{S.c., } \frac{m_a + m_b}{h} &= W_a
\end{align*}
\]  

(106)

For a symmetrical pitched frame with a horizontal force acting at A (Fig. 63), equations (106) become

\[
\begin{align*}
\text{CAB, } &\quad m_a \frac{2s}{3l_b} + m_a \frac{h}{3l_c} - m_b \frac{s}{3l_b} - m_b \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} \\
\text{ABD, } &\quad m_b \frac{2s}{3l_b} + m_b \frac{h}{3l_c} + m_b \frac{\delta s}{2hl_b} + m_b \frac{\delta s}{2hl_b} = \frac{2\beta s}{3h^2 l_b} - m_a \frac{s}{3l_b} - m_a \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} \\
\text{S.c., } &\quad m_a + m_b = W h
\end{align*}
\]  

(107)

from which, if \( I_c = I_b \),

\[
m_b = \frac{W h^2}{4} \frac{2h^2 + 6hs + 3ps}{h^3 + 3h^2 s + 3hps + \beta s} ; \quad m_a = \frac{W h^2}{4} \frac{2h^3 + 6h^2 s + 9hps + 4\beta s}{h^3 + 3h^2 s + 3hps + \beta s}
\]  

(108)

Frame with Pitched Roof and Uniformly-distributed Load on a Column.—If a uniformly-distributed load acts on a column AC (Fig. 64), equations (106) become (109), from which, if \( I_c = I_b \), equation (110) is obtained.

\[
\begin{align*}
\text{CAB, } &\quad m_a \frac{2s}{3l_b} + m_a \frac{h}{3l_c} - m_b \frac{s}{3l_b} - m_b \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} - \frac{W l^3}{24 l_b} \\
\text{ABD, } &\quad m_b \frac{2s}{3l_b} + m_b \frac{h}{3l_c} + m_b \frac{\delta s}{2hl_b} + m_b \frac{\delta s}{2hl_b} = \frac{2\beta s}{3h^2 l_b} - m_a \frac{s}{3l_b} - m_a \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} \\
\text{S.c., } &\quad m_a + m_b = \frac{W h}{2} \frac{l}{h}
\end{align*}
\]  

(109)

\[
\begin{align*}
\text{CAB, } &\quad m_a \frac{2s}{3l_b} + m_a \frac{h}{3l_c} - m_b \frac{s}{3l_b} - m_b \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} - \frac{W l^3}{24 l_b} \\
\text{ABD, } &\quad m_b \frac{2s}{3l_b} + m_b \frac{h}{3l_c} + m_b \frac{\delta s}{2hl_b} + m_b \frac{\delta s}{2hl_b} = \frac{2\beta s}{3h^2 l_b} - m_a \frac{s}{3l_b} - m_a \frac{\delta s}{2hl_b} = \frac{\Delta a}{E} \\
\text{S.c., } &\quad m_a + m_b = \frac{W h}{2} \frac{l}{h}
\end{align*}
\]  

(110)

\[
\begin{align*}
m_a &= \frac{W h}{16} \frac{11h^3 + 36h^2 s + 42hps + 16\beta s}{h^3 + 3h^2 s + 3hps + \beta s} \\
m_b &= \frac{W h}{16} \frac{5h^2 + 6s(2h + \beta)}{h^3 + 3h^2 s + 3hps + \beta s}
\end{align*}
\]  

(110)
Frame with Horizontal Load acting on Curved Member.—If a frame with a curved member is submitted to the action of any horizontal load acting on a beam (Fig. 65), equations (106) become (111).

\[
\begin{align*}
\text{CAB} & \quad m_a \alpha_{v_a} + m_a \alpha_{v_c} - m_b \beta - H_d \gamma = - \left( \Theta_b + \Delta \frac{\alpha}{h} \right) \\
\text{CBD} & \quad m_b \alpha_{v_b} + m_b \alpha_{v_d} + m_b \beta \frac{h}{2h_1} + H_d \delta + H_d \Delta \frac{h}{n} - m_{a+\beta} - m_{a+\Delta} \frac{\alpha}{h} = \Theta_b - \Delta \frac{\alpha}{h} - \Delta \frac{\alpha}{h} \\
\text{SC} & \quad m_a + m_b = W, h
\end{align*}
\]

For a symmetrical frame with a pitched roof and with a uniformly-distributed load acting on the inclined portion of a beam (Fig. 65) the load functions are

\[
\begin{align*}
\text{CAB} & \quad \left( \frac{2s}{3l_1} + \frac{h}{2l_1} \right) m_a - \left( \frac{s}{3l_1} + \frac{ps}{2h_1} \right) m_b = \frac{5Wps}{16I_b} - \Delta \frac{\alpha}{h} E \\
\text{CBD} & \quad \left( \frac{2s}{3l_1} + \frac{h}{2l_1} + \frac{ps}{2h_1} + \frac{ps}{2h_1} + \frac{2ps}{3l_1} \right) m_a - \left( \frac{s}{3l_1} + \frac{ps}{2h_1} \right) m_b = \frac{17Wps}{48I_b} + \frac{11Wp^2s}{24h_1} - \Delta \frac{\alpha}{h} E \\
\text{SC} & \quad m_a + m_b = W, h
\end{align*}
\]

(see Appendix) \( EI_{\theta_a} = \frac{1}{18} Wp^3, EI_{\theta_b} = \frac{1}{18} Wp^3, EI_{\alpha} = \frac{1}{16} Wp^2s, \) and equations (111) become (111a), from which, if \( I_b = I_c, \)

\[
\begin{align*}
m_a & = \frac{Wh^3}{16} \left( 8h^3 + 24h^2s + 28hps + 11p^2s \right) \\
m_b & = \frac{Wh^3}{16} \left( 8h^3 + 24h^2s + 20hps + 5p^2s \right)
\end{align*}
\]
Frame with a Crane Load (Fig. 66).—If a frame with a curved member is subjected to a crane load, the angular gaps at A and B are \( \left( \frac{\Delta a}{h} - \theta_{ae} \right) \) and \( \frac{\Delta a}{h} \) respectively, and equations (106) require only slight adjustment:

\[
\begin{align*}
\text{CAB, } & m_a \alpha_{ab} + m_a \alpha_{ae} - m_b \beta - H_d \gamma = \frac{\Delta a}{h} - \theta_{ae} \\
\text{ABD, } & m_b \alpha_{bd} + m_c \alpha_{cd} + \frac{m_b^2}{h} + H_d \delta + H_a \frac{\Delta h}{h} - m_a \beta - ma \frac{\Delta a}{h} = \frac{\Delta a}{h} \\
\text{S.C., } & \frac{m_0}{h} - \frac{m_a}{h} = \frac{m_b}{h} 
\end{align*}
\]

in which \( m_0 = W_e \).

For a symmetrical pitched frame (Fig. 66),

\[
\begin{align*}
\text{CAB, } & m_a \frac{2s}{3l} + m_a \frac{h}{3l} - m_b \frac{s}{3l} - m_b \frac{\beta s}{2hl} = \frac{\Delta E}{h} - \theta_{ae} \\
\text{ABD, } & m_b \frac{2s}{3l} + m_b \frac{h}{3l} + m_b \frac{\beta s}{2hl} + m_b \frac{\beta s}{2hl} + m_b \frac{2 \beta^2 s}{3 h^2 l} - m_a \frac{s}{3l} - m_a \frac{\beta s}{2hl} = \frac{\Delta a}{h} \\
\text{S.C., } & m_a + m_b = W_e
\end{align*}
\]

in which

\[
EI \theta_{ae} = \frac{W_e}{6h^2} (a^3 + 3a^2b - 2b^3).
\]

Example 15.—As a numerical example, consider the frame with various systems of loading shown in Fig. 67. To make the analysis clear, statically-
indeterminate bending moments are calculated separately for each load. The elastic constants are:

\[ EI\alpha = \frac{3}{8} \times 23.324 = 15.55; \quad EI\beta = \frac{1}{8} \times 23.324 = 7.775; \]
\[ EI\Delta^m = EI\gamma = EI\delta = \frac{1}{12} \times 12 \times 23.324 = 139.92; \]
\[ EI\Delta^h = \frac{5}{48} \times 12^2 \times 23.324 = 2240. \]

*Uniformly-distributed Load on the Whole Span.*—The load functions (see Appendix) are:

\[ EI\theta_e = \frac{1000}{12} \times 40 \times 40 \times 23.324 = 3,110,000; \]
\[ EI\Delta_e = \frac{5}{48} \times 40 \times 40 \times 23.324 \times 12 = 46,600,000. \]

From equation (100),

\[ m_a \left(15.55 + 7.775 + \frac{16}{3} + \frac{139.92}{16} + \frac{139.92}{16} + \frac{1}{2} \times \frac{2240}{16^2} \right) \]
\[ = 3,110,000 + \frac{1}{2} \times \frac{46,600,000}{16}, \]

from which \( m_a = 92,200 \) ft.-lb. Alternatively, from (102),

\[ m_a = \frac{1000 \times 40^2}{32} \]
\[ \times \frac{23.324 \times 16 (8 \times 16 + 5 \times 12)}{16^3 + 3 \times 16^2 \times 23.324 + 3 \times 16 \times 23.324 \times 12 + 12^2 \times 23.324} = 92,200 \text{ ft.-lb.} \]

*Uniformly-distributed Load on Part of Span.*—The load functions are

\[ EI\theta_a = \frac{1000 \times 15^2 \times 23.324}{12 \times 40^2} (80 - 15)^2 = 1,154,810; \]
\[ EI\theta_b = \frac{1000 \times 15^2 \times 23.324}{12 \times 40^2} (2 \times 40^2 - 15^2) = 813,150; \]
\[ EI\Delta_e = \frac{1000 \times 15^2 \times 12 \times 23.324}{12 \times 40^2} (3 \times 40^2 - 2 \times 15^2) = 14,267,730. \]

From equations (103),

\[ m_a \left(15.55 + 7.775 + \frac{139.92}{16} + \frac{16}{3} \right) = 1,154,810 + \frac{\Delta a_{EI}}{h}. \]
\[ m_a \left(15.55 + 7.775 + \frac{139.92}{16} + \frac{139.92}{16} + \frac{16}{3} + \frac{139.92}{16} + \frac{2240}{16^2} \right) \]
\[ = 813,150 + \frac{14,267,730}{16} - \frac{\Delta a_{EI}}{h}, \]

from which \( m_a = 28,300 \) ft.-lb.

*Concentrated Load of 10,000 lb. at (a) of 10 ft. from A.*—The load functions (see Appendix) are,

\[ EI\theta_a = \frac{10,000 \times 10 \times 30 \times 23.324 \times 70}{3 \times 40^2} = 1,020,000; \]

\[ 4020? \]
ANALYSIS OF STRUCTURES

\[ EI\theta_b = \frac{10,000 \times 10 \times 30 \times 23.324 \times 50}{3 \times 40^2} = 728,000 \]

\[ EI\Delta_b = \frac{10,000 \times 10 \times 23.324 \times 12(3 \times 40^2 - 4 \times 10^2)}{6 \times 40^2} = 12,850,000. \]

From (103),

\[ m_a \left( 15.55 + 7.775 + \frac{139.92}{16} + \frac{16}{3} \right) = 1,020,000 + \frac{\Delta a}{h} EI, \]

\[ m_a \left( 15.55 + 7.775 + \frac{139.92}{16} + \frac{139.92}{16} + \frac{16}{3} + \frac{139.92}{16} + \frac{2240}{16^2} \right) = 728,000 + \frac{12,850,000}{16} - \frac{\Delta a}{h} EI, \]

from which \( m_a = 25,200 \) ft.-lb.

**Horizontal Force of 1000 lb. at point A.**—From (106),

\[ 15.55m_a + \frac{16}{3} m_a - 7.775m_b - \frac{139.92}{16} m_b = \frac{\Delta a}{h} EI, \]

\[ 15.55m_b + \frac{16}{3} m_b + \frac{139.92}{16} m_b + \frac{139.92}{16} m_b + \frac{2240}{16^2} m_b - 7.775m_a - \frac{139.92}{16} m_a = \frac{\Delta a}{h} EI, \]

\[ m_a + m_b = 1000 \times 16, \]

from which \( m_a = 10,080 \) ft.-lb., and \( m_b = 5920 \) ft.-lb.

**Uniformly-distributed Pressure on Column AC.**—The load function \( \theta_{ab} \) is

\[ EI\theta_{ab} = \frac{1}{2} \times 1000 \times 16^3 = 170,500, \]

and, from (106),

\[ 15.55m_a + \frac{16}{3} m_a - 7.775m_b - \frac{139.92}{16} m_b = \frac{\Delta a}{h} EI - 170,500, \]

\[ 15.55m_b + \frac{16}{3} m_b + \frac{139.92}{16} m_b + \frac{139.92}{16} m_b - 7.775m_a - \frac{139.92}{16} m_a \]

\[ + \frac{139.92}{16} m_b + \frac{2240}{16^2} m_b = \frac{\Delta a}{h} EI, \]

and \[ m_a + m_b = \frac{1}{2} \times 1000 \times 16^2, \]

from which \( m_a = 79,000 \) ft.-lb., and \( m_b = 49,000 \) ft.-lb.

**Wind Pressure on Inclined Portion A to apex.**—The load functions (see Appendix) are

\[ EI\theta_a = \frac{5}{16} \times 1000 \times 12^2 \times 23.324 = 1,049,580, \]

\[ EI\theta_b = \frac{17}{48} \times 1000 \times 12^2 \times 23.324 = 1,189,522, \]

\[ EI\Delta_b = \frac{11}{24} \times 1000 \times 12^3 \times 23.324 = 18,472,474, \]

and the equations of equilibrium are

\[ 15.55m_a + \frac{16}{3} m_a - 7.775m_b - \frac{139.92}{16} m_b = -\left(1,049,580 + \frac{\Delta a}{h} EI\right), \]
\[ 15.55m_a + \frac{16}{3}m_b + \frac{139.92}{16}m_b - 7.775m_a - \frac{139.92}{16}m_a \]
\[ + \frac{139.92}{16}m_b + \frac{2240}{16^2}m_b = 1,189,522 + \frac{18,472,474}{16} - \frac{\Delta a}{h}EI, \]

\[ m_a + m_b = 1000 \times 12 \times 16, \]
from which \( m_a = 87,200 \) ft.-lb., and \( m_b = 104,800 \) ft.-lb.

**Crane Load.**—The load function (see Appendix) is

\[ EI\theta_{ab} = -\frac{20,000 \times 2}{6 \times 16^2} (4^3 + 3 \times 4^2 \times 12 - 2 \times 12^3) = 73,180, \]
and the equations of equilibrium are

\[ 15.55m_a + \frac{16}{3}m_a - 7.775m_b - \frac{139.92}{16}m_b = \frac{\Delta}{h}EI - 73.180, \]
\[ 15.55m_b + \frac{16}{3}m_b + \frac{139.92}{16}m_b - 7.775m_a - \frac{139.92}{16}m_a \]
\[ + \frac{139.92}{16}m_b + \frac{2240}{16^2}m_b = \frac{\Delta}{h}EI, \]
and \( m_a + m_b = 20,000 \times 2, \)
from which \( m_a = 24,470 \) ft.-lb., and \( m_b = 15,530 \) ft.-lb.

**Example 16.**—Consider a frame with non-prismatic members (Fig. 68). The frame is divided into eighteen equal segments, and the elastic constants and load functions are calculated from (96) by the summation method; the calculations are given in Table III for beam AB and in Table IV for the columns.

For \( m_a = 1, \)

\[ E\varepsilon = 14.24 \times 4.4 = 62.66; \quad \bar{\varepsilon} = \frac{265.62}{14.24} = 18.66; \quad \bar{\varepsilon} = 25.14. \]

\[ E\alpha = 62.66 \times 25.14 = 35.80; \quad E\beta = 26.86. \]

\[ E\Delta^m = 138.86 \times 4.4 = 611. \]

For \( H = 1: \quad E\gamma = E\delta = 611; \quad E\Delta^h = 1,454.59 \times 4.4 \times 2 = 12,800. \]
### TABLE III.

<table>
<thead>
<tr>
<th>Point</th>
<th>d (ft)</th>
<th>X (ft)</th>
<th>y (ft)</th>
<th>I (ft&lt;sup&gt;4&lt;/sup&gt;)</th>
<th>Dead weight of frame</th>
<th>N = 20,000 lb at a = 13.2 ft</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.80</td>
<td>2.20</td>
<td>1.65</td>
<td>1.824</td>
<td>224</td>
<td>12.25</td>
</tr>
<tr>
<td>6</td>
<td>2.40</td>
<td>6.60</td>
<td>4.60</td>
<td>1.522</td>
<td>64.0</td>
<td>55.56</td>
</tr>
<tr>
<td>7</td>
<td>1.70</td>
<td>11.00</td>
<td>7.25</td>
<td>0.409</td>
<td>99.0</td>
<td>24.20</td>
</tr>
<tr>
<td>8</td>
<td>1.40</td>
<td>15.40</td>
<td>9.90</td>
<td>0.288</td>
<td>130.0</td>
<td>56.76</td>
</tr>
<tr>
<td>9</td>
<td>1.25</td>
<td>19.80</td>
<td>12.00</td>
<td>0.150</td>
<td>162.0</td>
<td>87.17</td>
</tr>
<tr>
<td>10</td>
<td>1.25</td>
<td>24.20</td>
<td>12.00</td>
<td>0.163</td>
<td>142.0</td>
<td>88.54</td>
</tr>
<tr>
<td>11</td>
<td>1.40</td>
<td>28.60</td>
<td>9.90</td>
<td>0.229</td>
<td>178.0</td>
<td>32.00</td>
</tr>
<tr>
<td>12</td>
<td>1.70</td>
<td>33.00</td>
<td>7.25</td>
<td>0.049</td>
<td>129.0</td>
<td>31.75</td>
</tr>
<tr>
<td>13</td>
<td>2.40</td>
<td>37.40</td>
<td>4.60</td>
<td>1.152</td>
<td>27.50</td>
<td>10.00</td>
</tr>
<tr>
<td>14</td>
<td>2.80</td>
<td>41.80</td>
<td>1.65</td>
<td>1.824</td>
<td>242.0</td>
<td>10.00</td>
</tr>
</tbody>
</table>

Σ — — — — — 174.87 | 81.04 | — 2,842.89 | 51,045 | 28,8853

### TABLE IV.

<table>
<thead>
<tr>
<th>Point</th>
<th>X (ft)</th>
<th>d (ft)</th>
<th>l (ft&lt;sup&gt;4&lt;/sup&gt;)</th>
<th>m&lt;sub&gt;_x&lt;/sub&gt;</th>
<th>m&lt;sub&gt;_x&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.0</td>
<td>3.6</td>
<td>3.887</td>
<td>0.872</td>
<td>5.204</td>
</tr>
<tr>
<td>3</td>
<td>6.0</td>
<td>3.6</td>
<td>22.50</td>
<td>0.620</td>
<td>1.655</td>
</tr>
<tr>
<td>2</td>
<td>10.0</td>
<td>2.4</td>
<td>113.2</td>
<td>0.370</td>
<td>3.200</td>
</tr>
<tr>
<td>1</td>
<td>14.0</td>
<td>1.8</td>
<td>8486</td>
<td>0.115</td>
<td>3.310</td>
</tr>
</tbody>
</table>

Σ — — — 10.56 | 8.60 |

---

For dead load of frame:  
\[ E \theta_o = 1,748,710 \times 4.4 = 7,694,300; \]
\[ E \Delta_o = 18,104,700 \times 4.4 \times 2 = 159,321,000. \]

For \( W = 20,000 \) lb at \( a = 13.2 \) ft.:  
\[ E \varepsilon = 2,862,000 \times 4.4 = 12,596,000; \]
\[ \bar{x} = \frac{57,104}{2,862.8} = 19.95 \text{ ft}; \]
\[
\bar{z} = 24.05 \text{ ft.}; \quad E\theta_a = 12,596,000 \times 24.05 = 6,885,000; \\
E\theta_b = 5,711,000; \quad E\Delta_a = 28,985,500 \times 4.4 = 127,536,000.
\]

\[
ds = 4 \text{ ft.}; \quad \bar{x} = \frac{8.608}{1.056} = 8.15 \text{ ft.}; \quad \bar{z} = 7.85 \text{ ft.}; \quad E\alpha_{ae} = 1.056 \times 7.85 = 4 \times 2.07.
\]

**Uniformly-distributed Load.**—From equation (100),

\[
m_a \left(35.8 + 26.86 + 2.07 + \frac{6\Pi}{16} + \frac{6\Pi}{16} + \frac{12,800}{2 \times 16} \right) = 7,694,300 + \frac{159,321,000}{2 \times 16},
\]

from which \( m_a = 76,060 \) ft.-lb.

**Concentrated load of 20,000 lb. at \((a) = 13.2 \text{ ft.}.**—From equations (103),

\[
m_a \left(35.8 + 28.86 + 2.07 + \frac{6\Pi}{16} \right) = 6,885,000 + \frac{\Delta a}{h} E,
\]

\[
m_a \left(35.8 + 26.86 + 2.07 + \frac{6\Pi}{16} + \frac{6\Pi}{16} + \frac{6\Pi}{16} + \frac{12,800}{16^2} \right) = 5,711,000 + \frac{127,536,000}{16} - \frac{\Delta a}{h} E,
\]

from which \( m_a = 61,911,000 \) ft.-lb.

**Example 17.**—The load functions for a north-light frame (Fig. 69) are calculated as follows. For a uniformly-distributed load \( w = 1000 \) lb. per ft.,

\[
EI\bar{e} = \frac{1000 \times 12^2}{12 \times 30} \left[3 \times 6 \times 13.42 + 26.84(30 + 3 \times 6) \right] = 2,190,000
\]

\[
\bar{e} = \frac{4 \times 6^2 \times 13.42 + 26.84(8 \times 30^2 - 11 \times 24 \times 30 + 4 \times 24^2)}{2[3 \times 6 \times 13.42 + 26.84(30 + 3 \times 6)]} = 18.
\]

\[
EI\theta_a = \frac{2,190,000 \times 12}{30} = 876,000; \quad EI\theta_b = 1,314,000.
\]

\[
EI\Delta_0 = \frac{1000 \times 24^2 \times 12}{24 \times 30} \left[4 \times 6 \times 13.42 + 26.84(5 \times 30 - 4 \times 24) \right] = 17,000,000.
\]

The elastic constants are:

For \( m_{ab} = 1, \)

\[
EIe = \frac{1}{2 \times 30} \left[13.42(30 + 24) + 26.84 \times 24 \right] = 22.81.
\]

![Fig. 69.](image-url)
\[
\begin{align*}
\ddot{x} &= \frac{1}{3} \times \frac{13.42 \times 6(30 + 48) + 26.84 \times 24(30 + 12)}{13.42(30 + 24) + 26.84 \times 24} = 8.12 \text{ ft.} \\
EI\alpha_{ab} &= 22.81 \times \frac{21.88}{30} = 16.64; \quad EI\beta = 6.17. \\
EI\Delta^{ma} &= \frac{12}{6 \times 30} \left[13.42(30 + 48) + 2 \times 26.84 \times 24\right] = 155.67.
\end{align*}
\]
For \(m_{ba} = 1,\)
\[
\begin{align*}
EI\varepsilon &= \frac{1}{2 \times 30} \left[26.84(30 + 6) + 13.42 \times 6\right] = 17.44. \\
\ddot{z} &= \frac{1}{3} \times \frac{26.84 \times 24(30 + 12) + 13.42 \times 6 \times (30 + 48)}{26.84(30 + 6) + 13.42 \times 6} = 10.62. \\
EI\alpha_{ba} &= 11.27; \quad EI\beta = 6.17. \\
EI\Delta^{mb} &= \frac{12}{6 \times 30} \left[26.84(30 + 12) + 2 \times 13.42 \times 6\right] = 85.8.
\end{align*}
\]
For \(H = 1, \) \(EI\gamma = \Delta^{ma} = 155.67; \) \(EI\delta = \Delta^{mb} = 85.8; \)
\[
\begin{align*}
EI\Delta^h &= \frac{12^2}{3} (13.42 + 26.84) = 1932.
\end{align*}
\]
Substituting these values in (103),
\[
\begin{align*}
\text{CAB, } m_a \left(16.64 + 6.17 + \frac{12}{3} + \frac{155.67}{12} \right) &= 876,000 + \frac{\Delta a}{h} EI. \\
\text{ABD, } m_a \left(11.27 + 6.17 + \frac{12}{3} + \frac{155.67}{12} + \frac{85.8}{12} + \frac{85.8}{12} + \frac{1932}{12^3} \right) \\
&= 1,314,000 + \frac{17,000,000}{12} - \frac{\Delta a}{h} EI
\end{align*}
\]
from which \(m_a = 35.620 \text{ ft.-lb.}\)

**Example 18.**—The load functions for the frame in *Fig. 70* are calculated as follows (see Appendix) for a uniformly-distributed load.

\[
EI\varepsilon = \frac{40^3}{I^2} + \frac{1}{8} \times 1000 \times 24 \times 16 \times 12 = 7,633,000;
\]
\[ \ddot{x} = 20 \times \frac{\bar{r} + \bar{a} \times 0.6^2 \times 0.4 \times 0.3}{1 + 6 \times 0.6 \times 0.4 \times 0.3} = 21.2 \text{ ft.}; \]

\[ EI\theta_a = 7,633,000 \times \frac{18.8}{40} = 3,590,000; \quad EI\theta_b = 4,043,000; \]

\[ EI\Delta_e = \frac{1}{12} \times 1000 \times 24 \times 16 \times 12^2 + 12 \times 3,590,000 = 56,920,000; \]

and the elastic constants are:

For \( m_{ab} = 1 \): \( EI\varepsilon = 20 + \frac{16 \times 12}{40} + 12 = 36.8. \)

\[ \ddot{x} = \frac{40^3 + 6 \times 24 \times 16 \times 12}{3(40^2 + 2 \times 16 \times 12 + 2 \times 40 \times 12)} = 10.4 \text{ ft.} \]

\[ EI\varepsilon_{ab} = 36.8 \times \frac{29.6}{40} = 27.2. \quad EI\beta = 9.6. \]

\[ EI\Delta^{ma} = 12(24 + 12)\left(1 - \frac{24}{80}\right) = 302.5. \]

For \( m_{ba} = 1 \): \( EI\varepsilon = 20 + \frac{24 \times 12}{40} = 27.2. \)

\[ \ddot{x} = \frac{40^3 + 6 \times 24 \times 16 \times 12}{3(40^2 + 2 \times 24 \times 12)} = 14.1. \]

\[ EI\varepsilon_{ba} = 27.2 \times \frac{25.9}{40} = 17.6. EI\beta = 9.6. \]

\[ EI\Delta^{mb} = \frac{24 \times 12}{80}(24 + 12) = 129.5. \]

For \( H = 1 \):

\[ EI\gamma = 302.5; \quad EI\delta = 129.5; \quad EI\Delta^a = 127^2(24 + \frac{3}{8} \times 12) = 4608. \]

By substituting these values in equations (103),

\[ m_a \left(27.2 + 9.6 + \frac{16}{3} + \frac{302.5}{16}\right) = 3,590,000 + \frac{\Delta a}{h} EI. \]

\[ m_a \left(9.60 + \frac{302.5}{16} + 17.6 + \frac{16}{3} + \frac{129.5}{16} + \frac{129.5}{16} + \frac{4608}{16^3}\right) \]

\[ = 43,000 + \frac{56,920,000}{16} - \frac{\Delta a}{h} EI, \]

from which \( m_a = 76,200 \text{ ft.-lb.}, \) and \( m_e = 133,500 \text{ ft.-lb.} \)

**Frames with Ties.**—Frames with curved beams and joined with ties (Fig. 71) can be analysed in a similar way to frames previously discussed except that, in addition to the equations of angular deformations, the equations of horizontal translations must be taken into account. In a statically-determinate condition, with joints A and B relaxed from the effect of continuity, the tie is assumed to be detached at B. The angular and linear gaps formed in this condition are as shown in Fig. 71. Statically-indeterminate bending moments and

A.S.—E
force $P$ in the tie are calculated from two equations of equilibrium of angular deformations at A and B, and from one equation of linear deformation at B. These, for any shape of frame and any system of loading, are set out as follows:

$$\begin{align*}
\text{CAB, } & m_a \alpha_{ab} + m_a \alpha_{ac} + m_b \beta + (H + P) \gamma = \theta_a + \frac{\Delta a}{h} \\
\text{ABD, } & m_a \alpha_{ba} + m_a \alpha_{bd} + m_a \beta + (H + P) \delta = \theta_b + \frac{\Delta t - \Delta a}{h} \\
\text{B, } & m_a \Delta a + m_b \Delta b + (H + P) \Delta h = \Delta_0 - \Delta t
\end{align*}$$

$$\Delta t = \frac{PL}{E_s A_s},$$
in which $\Delta t$ is the elongation of the tie under force $P$, $A_s$ and $E_s$ are the area of cross section of the tie and its modulus of elasticity respectively, and other symbols are as defined previously.

For a symmetrical pitched roof with any system of loading, equations (115) become

$$\begin{align*}
\text{CAB, } & m_a \left( \frac{2s}{3E_c I_1} + \frac{h}{3E_c I_2} + \frac{s}{3E_c I_1} + \frac{ps}{2hE_c I_1} \right) + P \frac{ps}{2E_c I_1} = \frac{\theta_a}{E_c I_1} + \frac{\Delta a}{h} \\
\text{ABD, } & m_a \left( \frac{s}{3E_c I_1} + \frac{2s}{3E_c I_2} + \frac{h}{3E_c I_2} + \frac{ps}{2hE_c I_1} \right) + P \frac{ps}{2E_c I_1} = \frac{\theta_b}{E_c I_1} + \frac{\Delta t - \Delta a}{h} \\
\text{B, } & m_a \left( \frac{ps}{2hE_c I_1} + \frac{ps}{2hE_c I_1} + \frac{2p^2s}{3h^2E_c I_1} \right) + P \frac{2p^2s}{3hE_c I_1} = \frac{\Delta}{h} - \frac{\Delta t}{h}
\end{align*}$$

Or, eliminating $\Delta a$,

$$\begin{align*}
m_a \left( 2s + \frac{2h}{3} I_1 + \frac{ps}{h} \right) + P \left( \frac{ps - \frac{LI_1}{nA_s h}}{n} \right) & = \theta_a + \theta_b \\
m_a \left( \frac{ps}{h} + \frac{2p^2s}{3h^2} \right) + P \left( \frac{2p^2s}{3h} + \frac{LI_1}{nA_s h} \right) & = \frac{\Delta_0}{h}
\end{align*}$$

in which $m = \frac{E_s}{E_c}$.

**Example 19.**—As a numerical example, consider a pitched frame (Fig. 72) with uniformly-distributed load on part of the beam, for which $I = 0.666$ ft.\(^4\), $A_s = 0.307$ sq. in. (0.00216 sq. ft.), and $m = 15$. The load functions, calculated
in example 15, are \( EI\theta_a = 1,154,810 \); \( EI\theta_b = 813,150 \); \( EI\Delta_0 = 14,267,730 \), and (116) become

\[
m_a \left( 2 \times 23.324 + \frac{2 \times 16}{3} + \frac{12 \times 23.324}{16} \right) \\
+ P \left( \frac{12 \times 23.324 - 40 \times 0.666}{15 \times 0.00216 \times 16} \right) = 1,154,810 + 813,150.
\]

\[
m_a \left( \frac{12 \times 23.324}{16} + \frac{2 \times 12^2 \times 23.324}{3 \times 16^2} \right) \\
+ P \left( \frac{2 \times 12^2 \times 23.324}{3 \times 16} + \frac{40 \times 0.666}{15 \times 0.00216 \times 16} \right) = \frac{14,267,730}{16},
\]

from which \( m_a = 20,777 \) ft.-lb. and \( P = 1811 \) lb.

**Influence of Change of Temperature and Settlement of Supports on Frames with Curved Members.**

**Change of Temperature.**—The influence of change of temperature on frames with curved members, of any shape (Fig. 73), can be calculated from the general equations (103), which require only slight modification as follows:

\[
\text{CAB, } m_{ab} + m_{ac} + m_{b} + H\gamma = \frac{\Delta a}{h_1}.
\]

\[
\text{ABC, } m_{ab} + m_{bd} + m_{a} + \frac{m_{ab}}{h_2} + \frac{m_{bd}}{h_2} + H\delta + \frac{H\Delta h}{h_2} = \pm \frac{\Delta t - \Delta a}{h_2}
\]

S.c., \( \frac{m_a}{h_1} = \frac{m_b}{h_2} \).

**Fig. 73.**
In (117) $\Delta t$ is the total deformation of the curved member along the line AB due to change of temperature, and other symbols are as defined previously.

For symmetrical frames of any shape, equations (117) reduce to a form similar to (100):

$$A, \ ma \alpha_{ab} + ma \alpha_{ac} + ma \beta + ma \frac{A^m}{h} + Hy + \frac{1}{2}h \frac{AH}{h} = \pm \frac{\Delta t}{2h} \quad (118)$$

For symmetrical pitched frames, equation (118) becomes

$$A, \ ma \left( \frac{2s}{3EI_b} + \frac{h}{3EI_c} + \frac{s}{3EI_b} + \frac{p_s}{2hEI_b} + \frac{p_s}{2hEI_b} + \frac{p_s^2}{3h^2EI_b} \right) = \pm \frac{\Delta t}{2h}.$$  

Substituting in this equation,

$$\Delta t = 2 \int_A^c a_4 T \ ds \cos \phi = a_4 TL \quad (119)$$

we obtain, for $I_b = I_c, \ ma = \frac{3h \Delta t EI}{h^3 + 3h^2s + 3hps + p_s^2}. \quad (120)$

in which $\alpha_t$ is the coefficient of thermal expansion, $T$ is the change of temperature, $\phi$ the angle between the inclined member of the frame and a horizontal, and other symbols are as defined elsewhere.

**Horizontal Movement of Supports.**—The influence of horizontal movement of supports can be analysed in a similar way. The angular gaps at A and B (Fig. 74) are $\frac{\Delta H}{h}$ and zero respectively, and the equation of equilibrium becomes

$$A, \ ma \alpha_{ab} + ma \beta + ma \alpha_{ac} + ma \frac{A^m}{h} + Hy + H_\frac{A^h}{h} = \frac{\Delta H}{h} \quad (121)$$

For a symmetrical pitched roof,

$$ma \left( \frac{2s}{3EI_b} + \frac{s}{3EI_c} + \frac{h}{3EI_b} + \frac{p_s}{2hEI_b} + \frac{p_s}{2hEI_b} + \frac{p_s^2}{3h^2EI_b} \right) = \frac{\Delta H}{h},$$

from which, for $I_b = I_c,$

$$ma = \frac{3h \Delta H EI}{h^3 + 3h^2s + 3hps + p_s^2} \quad (122)$$

**Example 20.**—Consider a frame shown in Fig. 75, when subjected to change of temperature. Assume $T = 50$ deg. F., $\alpha = 0.000006$ per $1$ deg. F., $E = 4,000,000$ lb. per square inch, and $I_b = 0.666$ ft. $4$. From equation (118),

$$ma \left( 15.55 + \frac{16}{3} + 7.775 + \frac{139.92}{16} + \frac{139.92}{16} + \frac{2240}{2 \times 16^2} \right) = \pm \frac{\Delta t}{16},$$
from which \( m_a = \pm 0.001237 \Delta t EI \). Alternatively, from (120),

\[
\frac{3 \times 16 \Delta t EI}{16 + 3 \times 16^2 \times 23.324 + 3 \times 16 \times 12 \times 23.324 + 12^2 \times 23.324} = \pm 0.001237 \Delta t EI.
\]

By substituting values for \( \Delta t \), \( E \), and \( I \),

\[
m_a = \pm 0.001237 \times 0.0000006 \times 50^\circ \times 40 \text{ ft.} \times 576,000 \times 0.666 = \pm 5700 \text{ ft.-lb.}
\]

**Frames with Circular Members**

**General Case.**—In analysing frames with circular members of constant cross sections, the elastic constants and, in some cases, also the load functions can be expressed in an explicit form, and coefficients prepared for various central angles. For uniformly-distributed load, for example, as in Fig. 76, the load functions calculated from equations (96) are

\[
EI\theta_a = EI\theta_b = \int_0^\phi mds = \frac{1}{2}w R^3 \left( \phi \sin^2 \phi + \frac{1}{4} \sin 2\phi - \frac{\phi}{2} \right) = \frac{1}{2}wc_1 R^3 \quad (123)
\]

\[
EI\Delta_o = \int_A^B myds = wR^4 \left[ \frac{3}{8} \sin^3 \phi + \phi \cos \phi \left( \frac{1}{2} - \sin^2 \phi \right) - \frac{1}{4} \sin 2\phi \cos \phi \right] = wc_2 R^4. \quad (124)
\]

in which

\[
c_1 = \phi \sin^2 \phi + \frac{1}{4} \sin 2\phi - \frac{\phi}{2},
\]

\[
c_2 = \frac{3}{8} \sin^3 \phi + \phi \cos \phi \left( \frac{1}{2} - \sin^2 \phi \right) - \frac{1}{4} \sin 2\phi \cos \phi. \quad (125)
\]

\( R \) is the radius of the central line of a circular member, and \( \phi \) is half the central angle. The coefficients \( c_1 \) and \( c_2 \) for various values of central angles \( \phi \) are shown in Fig. 80 (see also Appendix).

In a similar way, the load functions for a concentrated load at the crown (Fig. 77) can be expressed as follows:

\[
EI\theta_o = m_o R \phi \frac{\sin \phi + \cos \phi - 1}{\sin \phi} \quad . \quad (126)
\]

\[
EI\Delta_o = m_o R^2 \phi \frac{\sin 2\phi + \cos^2 \phi + 1 - 2 \sin^2 \phi - 2 \cos \phi}{\sin \phi} \quad . \quad (127)
\]

For a semi-circle, \( \phi = \frac{\pi}{2} \); \( EI\theta_o = m_o R \left( \frac{\pi}{2} - 1 \right) \); \( EI\Delta_o = m_o R^2 \). \quad (128)

For more complicated systems of loading, however, equations (96) are used with integration replaced by summation.
The elastic constants for a circular beam can also be expressed in explicit form, and, for unit bending moment, these are as follows (Fig. 78):

\[ EI_e = \int_A^B m \cdot ds = \phi . R; \quad EI_\alpha = \frac{3}{2} \phi R; \quad EI_\beta = \frac{1}{2} \phi R. \quad \ldots \quad (129) \]

\[ EI_{\Delta m} = \int_A^B m \cdot y \cdot ds = R^2 (\sin \phi - \phi \cdot \cos \phi) = c_3 \cdot R^2. \quad \ldots \quad (130) \]

Similarly, the elastic constants for a unit horizontal force (Fig. 79) are

\[ EI_{\gamma} = EI_\delta = R^2 (\sin \phi - \phi \cdot \cos \phi) = c_3 \cdot R^2 \quad \ldots \quad (131) \]

\[ EI_{\Delta h} = \int_A^B \text{my}ds = R^3 \left( \phi \cos^2 \phi - \frac{1}{4} \sin 2\phi + \frac{\phi}{2} \right) = c_4 R^3, \text{ in which } c_3 = \sin \phi - \phi \cos \phi. \]

\[ c_4 = \phi \cos^2 \phi - \frac{1}{2} \sin 2\phi + \frac{\phi}{2}. \quad \ldots \quad (132) \]

The values of coefficients \( c_3 \) and \( c_4 \), for various half central angles \( \phi \), are shown in Fig. 80. It can be seen from Fig. 80 that, for small central angles \( \phi \), the translations of joints \( \Delta_o \), \( \Delta_m \), and \( \Delta_h \) are also small; the effect of curvature is therefore negligible and the beam may be treated as straight.

**Example 21.**—As a numerical example, consider a frame with a circular member and with loading as shown in Fig. 81. To make the analysis clear, each load is considered separately. The radius and half-central angle are

\[ R = \frac{L^2}{8\phi} + \frac{\phi}{2} = \frac{60^2}{8 \times 12} + \frac{12}{2} = 43.5 \text{ ft}. \]
\[ \sin \phi = \frac{30}{43.5} = 0.6900, \text{ and } \phi = 43.38'' (0.7615 \text{ rad}). \]

From Fig. 8o \( c_1 = 0.2316, c_2 = 0.0496, c_3 = 0.1389, \text{ and } c_4 = 0.0306, \) and the elastic constants, calculated from equations (129) to (131), are

\[ EI\alpha = \frac{3}{8} \times 0.7615 \times 43.5 = 22.1; \quad EI\beta = \frac{1}{8} \times 0.7615 \times 43.5 = 11.05; \]
\[ EI\Delta_m = EI\gamma = 0.1389 \times 43.5^2 = 262.5; \quad EI\Delta_a = 0.0306 \times 43.5^3 = 2510. \]

For Uniformly-distributed Load \( w = 1000 \text{ lb. per foot}: \)

\[ EI\theta_0 = \frac{1}{6} \times 1000 \times 0.2316 \times 43.5^3 = 9.531,830; \]
\[ EI\Delta_a = 1000 \times 0.0496 \times 43.5^4 = 177,598,250, \]

**Diagram of Coefficients**

For U.D.L.

\[ EI\theta_0 = \frac{1}{6} \pi c_3 R^3 \]
\[ EI\Delta_a = \omega c_2 R^4 \]

For \( m=1 \),

\[ EI\alpha = \frac{3}{8} \phi R, \quad EI\beta = \frac{1}{8} \phi R \]
\[ EI\Delta_m = c_3 R^2 \]

For \( H=1 \),

\[ EI\gamma = EI\delta = c_3 R^2 \]
\[ EI\Delta_a = c_4 R^2 \]

![Fig. 8o.](image-url)
and, from (96),

\[ m_a \left[ 22.10 + 11.05 + \frac{16}{3} + \frac{262.5}{16} + \frac{262.5}{16} + \frac{2510}{2 \times 16^2} \right] = 9.531,830 + \frac{177,598,250}{2 \times 16} \]

from which \( m_a = 197,930 \) ft.-lb.

For Concentrated Load \( W = 10,000 \) lb. at the Crown: From equations (126) to (128),

\[ EI\theta_\alpha = \frac{43.5}{2} \times \frac{10,000 \times 30 \times 0.7615 \times 0.6900 + 0.7238 - 1}{0.6900} = 2356.5 \]

\[ EI\Delta_\alpha = \frac{10,000 \times 30}{2} \times \frac{43.5^2}{0.6900} (0.7615 \times 0.9989 + 0.7238 + 1 - 2 \times 0.6900^2 - 2 \times 0.7238) = 47,411,000. \]

Substituting these values in (23),

\[ m_a \left( 22.10 + 11.05 + \frac{16}{3} + \frac{262.5}{16} + \frac{262.5}{16} + \frac{2510}{2 \times 16^2} \right) = 2,356,500 + \frac{47,411,000}{2 \times 16} \]

from which \( m_a = 50,370 \) ft.-lb.

For Concentrated Load \( W = 10,000 \) lb. at 12 ft. from support: The load functions are calculated from (96) by the method of summation as shown in Table V.

\[ \Delta x = 5.5212 \text{ ft.} \]

\[ EIe = 558,000 \times 5.5212 = 3,080,830. \]

\[ \bar{x} = \frac{13,182,940}{558,000} = 23.62 \text{ ft.}; \bar{z} = 36.38 \text{ ft.} \]

\[ EI\theta_a = 3,080,830 \times \frac{36.38}{60} = 1,868,090. \]

\[ EI\theta_b = 1,212,740. \]

\[ EI\Delta_b = 5,190,400 \times 5.5212 = 28,657,240. \]

Substituting these values in equations (103),

\[ m_a \left( 22.1 + 11.05 + \frac{16}{3} + \frac{262.5}{16} \right) = 1,868,090 + \frac{\Delta a}{h}EI. \]
### TABLE V.

<table>
<thead>
<tr>
<th>Point</th>
<th>x ft</th>
<th>y ft</th>
<th>(m_0) ft-lb</th>
<th>(m_{0, x}) ft-lb</th>
<th>(m_{0, y}) ft-lb</th>
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<tbody>
<tr>
<td>1</td>
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\[
m_a \left( \frac{22.1 + 11.05}{3} + \frac{16}{16} \right) + 26.25 \left( \frac{26.25}{16} + \frac{26.25}{16} + \frac{2510}{16} \right) = 1,212,740 + \frac{28,657,240}{16} - \frac{\Delta a}{EI},
\]

from which \( m_a = 31,970 \) ft-lb.

For Wind Pressure \( W = 1000 \) lb. at top of column: From equations (166),

\[
22\cdot m_a + \frac{16}{3} m_a - 11\cdot 05 m_b - \frac{262.5}{16} m_a = \frac{\Delta a}{EI}.
\]

\[
22\cdot m_b + \frac{16}{3} m_b - 11\cdot 05 m_a + \frac{262.5}{16} m_b - \frac{262.5}{16} m_a + \frac{262.5}{16} m_a + \frac{2510}{16} m_a = \frac{\Delta a}{EI}.
\]

\( m_a + m_b = 1000 \times 16 \), from which \( m_a = 13,076.4 \) ft-lb. and \( m_b = 2923.6 \) ft-lb.

Example 22.—A frame with semi-circular beam, and loaded as shown in Fig. 82, is now analysed. For \( \phi = 90 \) deg. \((1.5708 \text{ rad})\), from Table XV (see Appendix), \( c_1 = 0.7854, c_2 = 0.6667, c_3 = 1, c_4 = 0.7854 \), and the elastic constants are:

\[
EI_x = \frac{3}{2} = 1.5708 \times 12 = 12.56; \quad EI_y = \frac{3}{2} = 1.5708 \times 12 = 6.28;
\]

\[
EI_d^m = EI_d^h = 1 \times 12^2 = 144; \quad EI_d^h = 0.7854 \times 12^2 = 1360.
\]

Uniformly-distributed Load on the Beam.—The load functions are

\[
EI_0 = \frac{3}{2} \times 1000 \times 0.7854 \times 12^2 = 680,000;
\]

\[
EI_d^h = 1000 \times 0.6667 \times 12^2 = 13,840,000;
\]

and, from equation (100),

\[
m_a \left( 12.56 + 6.28 + \frac{24}{3} + \frac{144}{24} + \frac{144}{24} + \frac{1360}{2 \times 24^2} \right) = 680,000 + \frac{13,840,000}{2 \times 24},
\]

from which \( m_a = 24,200 \) ft-lb.
Concentrated Load \( W_1 = 10,000 \text{ lb. at the Crown.} \) — The load functions, from (126) to (128), are

\[
EI \theta_0 = 60,000 \times 12 \times 0.5728 = 412,420; \quad EI \Delta_0 = 60,000 \times 12^2 = 8,640,000.
\]

From (100),

\[
m_a \left( 12.56 + 6.28 + \frac{24}{3} + \frac{114}{24} + \frac{114}{24} + \frac{1360}{2 \times 24^2} \right)
\]

\[
= 412,420 + \frac{8,640,000}{2 \times 24},
\]

from which \( m_a = 14,800 \text{ ft.-lb}. \)

Concentrated Load \( W = 10,000 \text{ lb. at } a = 6 \text{ ft. from A.} \) — The load functions are calculated by the summation method as follows (see Table VI).

\[ ds = 3.14 \text{ ft.} \quad EI \varepsilon = 201,100 \times 3.14 = 632,000, \quad \bar{x} = \frac{1.956,950}{201,100} = 9.72 \text{ ft.} \]

<table>
<thead>
<tr>
<th>Point</th>
<th>( x ) (ft)</th>
<th>( y ) (ft)</th>
<th>( m_a ) (lbs)</th>
<th>( m_a \cdot x )</th>
<th>( m_a \cdot y )</th>
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FRAMES

\[ EI\theta_a = 632,000 \frac{14 \cdot 28}{24} = 376,000, \quad EI\theta_b = 256,000. \]

\[ EI\Delta_0 = 2,048,000 \times 3 \cdot 14 = 6,420,000. \]

From (103),

\[
m_a \left( 12 \cdot 56 + 6 \cdot 28 + \frac{24}{3} + \frac{144}{24} \right) = 376,000 + \frac{\Delta a}{h} EI
\]

\[
m_a \left( 6 \cdot 28 + \frac{144}{24} + 12 \cdot 56 + \frac{24}{3} + \frac{144}{24} + \frac{144}{24} + \frac{1,360}{24^2} \right)
\]

\[ = 256,000 + \frac{6,420,000}{24} - \frac{\Delta a}{h} EI, \]

from which \( m_a = 11,220 \) ft.-lb.

**Horizontal Force** \( W = 1000 \) lb. at A.—From (106),

**CAB**, \( m_a \left( 12 \cdot 56 + \frac{24}{3} \right) - m_b \left( 6 \cdot 28 + \frac{144}{24} \right) = \frac{\Delta b}{h} EI, \)

**ABD**, \( m_b \left( 12 \cdot 56 + \frac{24}{3} + \frac{144}{24} + \frac{144}{24} + \frac{1360}{24^2} \right) - m_a \left( 6 \cdot 28 + \frac{144}{24} \right) = \frac{\Delta b}{h} EI, \)

S.c., \( m_a + m_b = Wh = 24,000, \)

from which \( m_a = 14,500 \) ft.-lb., and \( m_b = 9500 \) ft.-lb.

**Uniformly-distributed Pressure on Column AC.**—From equations (106),

**CAB**, \( m_a \left( 12 \cdot 56 + \frac{24}{3} \right) - m_b \left( 6 \cdot 28 + \frac{144}{24} \right) = \frac{\Delta a}{h} EI - \frac{1000 \times 24^3}{24}, \)

**ABD**, \( m_b \left( 12 \cdot 56 + \frac{24}{3} + \frac{144}{24} + \frac{144}{24} + \frac{1360}{24^2} \right) - m_a \left( 6 \cdot 28 + \frac{144}{24} \right) = \frac{\Delta a}{h} EI, \)

S.c., \( m_a + m_b = \frac{1}{2} \times 1000 \times 24 \times 24, \)

from which \( m_a = 162,640 \) ft.-lb., and \( m_b = 125,360 \) ft.-lb.

**Example 23.**—A frame with an elliptical beam (Fig. 83) is analysed in a similar way. The load functions and the elastic constants are calculated by the method of summation as shown in Tables VII and VIII.

For \( m_a = 1 \):

\[ EI\varepsilon = 5 \cdot 981 \times 4 \cdot 083 = 24 \cdot 42, \quad \bar{x} = \frac{69 \cdot 302}{5 \cdot 981} = 11 \cdot 587 \text{ ft.} \]
### ANALYSIS OF STRUCTURES

#### TABLE VII.

<table>
<thead>
<tr>
<th>Point</th>
<th>x(ft)</th>
<th>y(ft)</th>
<th>$m_a = \frac{W_{\infty} \times 1000 \text{ lb/ft}}{u-d.l. \times \text{in}^2}$</th>
</tr>
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<td>$m_{x}$</td>
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</table>

\[ EI_a = 24.42 \frac{28.413}{40} = 17.63, \quad EI_B = 6.79. \]

\[ EI A^m = 42.679 \times 4.083 = 174.26. \]

For \( H = 1 \): \[ EI_y = 174.26, \quad EI A^h = 350.28 \times 4.083 = 1430.2. \]

For Uniformly-distributed Load:

\[ EI \theta_0 = 756,000 \times 4.083 = 3,086,750; \]

\[ EI A_0 = 6,505,200 \times 4.083 \times 2 = 53,121,460. \]

From equation (100),

\[ m_a \left( 17.63 + 6.79 + \frac{12}{3} + \frac{174.26}{12} + \frac{174.26}{12} + \frac{1430.2}{12^2} \right) = 3,086,750 + \frac{53,121,460}{2 \times 12}, \]

from which \( m_a = 83,610 \) ft.-lb.

#### TABLE VIII.

<table>
<thead>
<tr>
<th>Point</th>
<th>x(ft)</th>
<th>y(ft)</th>
<th>$W_{e} \text{ 1000 lb. at its green}$</th>
<th>$W_{e} \text{ 1000 lb. at 0.25&quot; from A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$M_n (\text{in})$</td>
<td>$M_y (\text{in})$</td>
</tr>
<tr>
<td>1</td>
<td>0.37</td>
<td>1.98</td>
<td>0.20</td>
<td>0.39</td>
</tr>
<tr>
<td>2</td>
<td>2.80</td>
<td>5.20</td>
<td>1.40</td>
<td>7.28</td>
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<tr>
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<td>5.25</td>
<td>7.30</td>
<td>3.10</td>
<td>22.65</td>
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<tr>
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<td>10.30</td>
<td>8.45</td>
<td>5.05</td>
<td>43.60</td>
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<tr>
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<td>14.00</td>
<td>9.55</td>
<td>7.00</td>
<td>66.90</td>
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<td>37.20</td>
<td>5.20</td>
<td>0.65</td>
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<td>12</td>
<td>41.03</td>
<td>1.98</td>
<td>0.10</td>
<td>39.68</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>25.75</td>
<td>230.82</td>
<td>39.83</td>
<td>647.05</td>
</tr>
</tbody>
</table>
Concentrated Load \( W = 10,000 \text{ lb. at the Crown.} \)

\[ EI \theta_a = 25,750 \times 4.083 = 105,200. \quad EI \Delta_a = 230,820 \times 4.083 \times 2 = 1,886,000. \]

Substituting these values in (100),

\[ m_a \left(17.63 + 6.79 + \frac{12}{3} + \frac{174.26}{12} + \frac{174.26}{12} + \frac{1430.2}{2 \times 12^2} \right) = 105,200 + \frac{1,886,000}{2 \times 12}, \]

from which \( m_a = 2950 \text{ ft.-lb.} \).

For \( W = 10,000 \text{ lb. at } a = 10 \text{ ft. (Table VIII).} \)

\[ EI \epsilon = 39.330 \times 4.083 = 160,580; \quad \bar{x} = \frac{647.050}{39.330} = 16.45 \text{ ft.}; \]

\[ EI \theta_a = 160,580 \times \frac{23.55}{40} = 94,540; \quad EI \theta_b = 66,040; \]

\[ EI \Delta_a = 341,030 \times 4.083 = 1,392,430. \]

Substituting these values in (103),

\[ m_a \left(17.63 + 6.79 + \frac{12}{3} + \frac{174.26}{12} \right) = 94.540 + \frac{\Delta a}{h} EI, \]

\[ m_a \left(17.63 + 6.79 + \frac{12}{3} + \frac{174.26}{12} + \frac{174.26}{12} + \frac{174.26}{12} + \frac{1430.2}{12^2} \right) \]

\[ = 66.04 + \frac{1,392,430}{12} - \frac{\Delta a}{h} EI, \]

from which \( m_a = 2220 \text{ ft.-lb.} \).

Horizontal Load \( W = 1000 \text{ lb. at the Springing.} \)

From equations (106),

\[ m_a \left(17.63 + \frac{12}{3} \right) - m_b \left(6.79 + \frac{174.26}{12} \right) = \frac{\Delta b}{h} EI, \]

\[ m_b \left(17.63 + \frac{12}{3} + \frac{174.26}{12} + \frac{174.26}{12} + \frac{1430.2}{12^2} \right) - m_a \left(6.79 + \frac{174.26}{12} \right) = \frac{\Delta b}{h} EI, \]

\[ m_a + m_b = 1000 \times 12; \quad \text{from which} \]

\[ m_a = 7,841,800 \text{ ft.-lb.}, \quad \text{and } m_b = 4,138,200 \text{ ft.-lb.}. \]

Fixed Ends with Curved Members.—Fixed-end frames with curved members are analysed in a similar way to single-span frames with curved members, except that additional equations of equilibrium are required for the fixed supports. For a frame of any shape, and with any load on column AC (Fig. 84), the equations of equilibrium are in (133).

\[
\begin{align*}
\mathbf{c} & \begin{pmatrix}
\alpha_{ab} + m_a \alpha_{ac} - m_b \beta_{ab} - m_c \beta_{ac} - H_d \cdot \gamma = \frac{\Delta a}{h} \\
A \mathbf{D} \end{pmatrix}, \quad m_a \alpha_{cd} - m_b \beta_{cd} - m_c \beta_{dc} - m_d \alpha_{ap} - m_a \alpha_{ab} - m_b \beta_{ab} - m_c \beta_{ac} + \frac{m_b}{h} \Delta b + H_d \cdot S + H_d \cdot \frac{\Delta h}{h} = \frac{\Delta a}{h} \\
\mathbf{c} & \begin{pmatrix}
m_c \alpha_{ca} - m_a \beta_{ca} = \frac{\Delta a}{h} \\
D \end{pmatrix}, \quad m_b \alpha_{db} - m_d \beta_{db} = \frac{\Delta a}{h} \\
\mathbf{j} & \begin{pmatrix}
m_a + m_b + m_c + m_d = W_a \cdot h \\
\end{pmatrix}
\end{align*}
\]

\[ \text{(133)} \]
EXAMPLE 24.—Consider a frame with fixed ends (Fig. 85). The elastic constants (see Example 22) are

\[ EI_\alpha = 12.56, \quad EI_\beta = 6.28, \quad EI_\Delta^m = 144, \quad \text{and } EI_\Delta^h = 1360. \]

By substituting these values in (133) we obtain (133a), from which

\[ m_a = 6800 \text{ ft.-lb.}; \quad m_b = 2650 \text{ ft.-lb.}; \quad m_c = 8060 \text{ ft.-lb.}; \quad \text{and } m_d = 6410 \text{ ft.-lb.} \]

\[
\begin{align*}
\text{CAB} & \quad 12.56 m_a + \frac{24}{3} m_a - 6.28 m_b - \frac{24}{6} m_c - \frac{144}{24} (m_b + m_d) = \frac{\Delta a}{h} EI \\
\text{ABD} & \quad 12.56 m_b + \frac{24}{3} m_b - 6.28 m_a - \frac{24}{6} m_d - \frac{144}{24} m_a + \frac{144}{24} m_b + \\
& \quad + \frac{144}{24} (m_b + m_d) + \frac{1360}{24} (m_b + m_d) = \frac{\Delta a}{h} EI \\
\text{c} & \quad \ldots \frac{24}{3} m_c - \frac{24}{6} m_a = \frac{\Delta a}{h} EI \\
\text{D} & \quad \ldots \frac{24}{3} m_d - \frac{24}{6} m_b = \frac{\Delta a}{h} EI \\
\text{S.c} & \quad m_a + m_b + m_c + m_d = 1000 \times 24
\end{align*}
\]

Thin Elastic Rings.

Thin elastic rings of any shape submitted to the action of any system of loading may conveniently be analysed by the deformation method. As before, imaginary cuts are introduced at any convenient points and the statically-indeterminate bending moments and forces required to close the angular and linear gaps are applied. The equations of equilibrium are set out in the usual manner.

Circular Ring submitted to the Action of Concentrated Load (Fig. 86). —With imaginary cuts introduced at A and B, the deformed shape of the ring is as shown by dotted lines. The elastic constants and load function \( \theta_0 \) are calculated from equations (129) to (131). For \( \phi = 90 \text{ deg.} \left( \frac{\pi}{2} \right) \),

\[
\begin{align*}
EI_\alpha &= \frac{\pi R}{3}; \quad EI_\beta = \frac{\pi R}{6}; \quad EI_\Delta^m = EI_\gamma = R^2; \quad EI_\Delta^h = \frac{\pi R^3}{4} \\
EI \theta_0 &= \frac{W}{2} \cdot \gamma = \frac{WR^2}{2}; \quad EI \Delta_0 = \frac{W}{2} \cdot \Delta^h = \frac{\pi WR^3}{8}.
\end{align*}
\]
The equation of equilibrium \((100)\) reduces to \(m_a(\alpha + \beta) = \theta_o\), or, substituting values of \(\alpha\), \(\beta\), and \(\theta\), \(m_a\left(\frac{\pi R}{3} + \frac{\pi R}{6}\right) = \frac{WR^2}{2}\), from which
\[
m_a = \frac{WR}{\pi} \quad . \quad . \quad . \quad . \tag{134}
\]
and the bending moment at \(C\) is
\[
m_o = m^o_c - m_a = \frac{WR}{2} - \frac{WR}{\pi} = \frac{WR}{2}\left(1 - \frac{2}{\pi}\right) \quad . \quad . \quad . \tag{135}
\]
The bending moment at any point \(E\) on the ring is
\[
m_s = \frac{WR}{2}\left(\cos \phi - \frac{2}{\pi}\right) \quad . \quad . \quad . \tag{136}
\]
from which, for \(m_s = 0\), \(\phi = 50\ \text{deg. 28 min.}\) The deflection at \(A\) is calculated from \(\Delta_a = \Delta_o - 2m_a\Delta^m\), from which
\[
EI\Delta_a = -WR^3\left(\frac{2}{\pi} - \frac{\pi}{8}\right) \quad . \quad . \quad . \tag{137}
\]

**Circular Ring with a Tie submitted to the Action of Two Concentrated Loads** (Fig. 87).—Now introduce imaginary cuts at \(C\) and \(D\) and calculate the load functions \(\theta_o\) and \(\Delta_o\) from \((128)\) or \((135)\): \(EI\theta_o = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right)\), \(EI\Delta_o = \frac{WR^2}{2}\)
and the equations of equilibrium for joint \(C\) are
\[
\text{CAD}, \quad m_o\alpha + m_o\beta + \frac{1}{2}P\gamma = \theta_o; \quad \text{C}, \quad m_o\Delta^m + \frac{1}{2}P\Delta_h^m = \frac{\Delta_o}{2} \quad . \quad . \quad . \tag{138}
\]
in which \(P\) is the force in the tie.

From equations \((138)\), \(m_o = \frac{WR}{2}\left(\frac{\pi + 2}{\pi + 4}\right) \quad . \quad . \quad . \quad . \tag{139}\)
ANALYSIS OF STRUCTURES

\[ P = \frac{W}{\frac{1}{4} + \frac{\pi}{4}} \quad \quad (140) \]

and the bending moment at \( A \) is

\[ m_a = m_0 - m_e - PR = \frac{3WR}{\pi + 4} \quad \quad (141) \]

**Circular Ring Submitted to the Action of Three Equally-spaced Concentrated Loads.**—The imaginary cuts are at the points of application of the loads, and the deflected shape is as shown by dotted lines (Fig. 88). Each third of a circle is in the same static conditions, and is submitted to the action of

\[ W' = \frac{W}{2} \cos \phi = \frac{W\sqrt{3}}{4}, \quad \phi = 60^\circ \left(\frac{\pi}{3}\right). \]

The elastic constants, calculated from (123) to (131), are

\[ EI\alpha = \frac{2\pi R}{9}, \quad EI\beta = \frac{\pi R}{9}, \quad \text{and} \]

\[ EI\gamma = R^2(\sin \phi - \phi \cos \phi) = \frac{1}{2} \left(\sqrt{3} - \frac{\pi}{3}\right)R^2. \]

Similarly,

\[ EI\theta_o = W'\gamma = \frac{W\sqrt{3}}{4} \cdot \frac{1}{2} \left(\sqrt{3} - \frac{\pi}{3}\right)R^2. \]

Substituting these values in \((\alpha + \beta)m_b = \theta_o:\)

\[ \frac{\pi R}{3}m_b = \frac{W\sqrt{3}}{4} \cdot \frac{1}{2} \left(\sqrt{3} - \frac{\pi}{3}\right)R^2; \]

from which

\[ m_b = \frac{WR}{8} \left(\frac{9}{\pi} - \sqrt{3}\right) \quad \quad \quad (142) \]

**Circular Ring with a Concentrated Load and a Uniformly-distributed Load.**—If imaginary cuts are made at points \( A \) and \( B \) the top and bottom semi-circles will have different angular and linear deformations (Fig. 89). These are, for the top semi-circle, \( \phi = \frac{\pi}{2}, \quad W = 2Rw, \)

\[ EI\theta_t = \frac{1}{4}W \cdot 2R \cdot R \left(\frac{\pi}{2} - 1\right) = \frac{1}{8}WR^2(\pi - 2); \quad EI\Delta_1^o = \frac{1}{2} \cdot \frac{1}{4}W \cdot 2R \cdot R^2 = \frac{WR^3}{4}; \]
and for the bottom semi-circle,
\[ EI\theta_2 = \frac{1}{2}wR^3 \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi wR^3}{8}; \quad EI\Delta_2^o = \frac{1}{2}wR^4 \cdot \frac{3}{8} = \frac{wR^4}{3}. \]

The equations of equilibrium reduce to
\[ \text{ACD}, \quad 2m_a(\alpha + \beta) = \theta_1 + \theta_2; \quad A, \quad 2H \cdot \Delta^\alpha = \Delta_1^o - \Delta_2^o \quad \text{. (I43)} \]
or, substituting values of elastic constants and load functions,
\[ \text{ACD}, \quad \frac{\pi R}{2} m_a = \frac{1}{2}wR^3(\pi - 2) + \frac{\pi}{8}wR^3; \quad A, \quad \frac{\pi R^3}{2} H = \frac{1}{2}wR^4 - \frac{1}{2}wR^4; \]
from which \( m_a = wR^2 \left( \frac{5}{8} - \frac{1}{\pi} \right) \);
\[ H = \frac{wR}{3\pi} \quad \text{. . . . . . .} \quad \text{(I44)} \]

**Fig. 89.**

It is interesting to note that equations (I43) are independent from each other. This agrees with Clerk Maxwell’s Reciprocal Theorem and is due to the geometrical symmetry of the structure.

**Closed Ring on Two Supports.**—A closed ring supported on two points, and submitted to the action of any system of loading, can be analysed if imaginary

\[
\begin{align*}
A_{\ldots}, \quad m_1: \alpha \Delta_{ac} + m_1 \cdot \alpha_{ab} + m_3 \cdot \beta_{ac} - m_z \cdot \beta_{ab} - V_{ac} \cdot \delta + m_1 \cdot \frac{m^a}{L} + m_3 \cdot \frac{m^d}{L} - V_{ac} \cdot \frac{A^b}{L} = \\
= \theta_{ac} + \frac{\lambda}{2R} \cdot \Delta^a_{ab} - \Delta^b_{ab} - \theta_{bd} \\
B_{\ldots}, \quad m_2: \alpha \Delta_{bd} + m_2 \cdot \alpha_{bd} - m_1 \cdot \beta_{bd} - V_{bd} \cdot \delta + m_2 \cdot \frac{m^b}{L} - m_3 \cdot \frac{m^d}{L} - V_{bd} \cdot \frac{A^b}{L} = \\
= \theta_{bd} + \frac{\lambda}{2R} \cdot \Delta^a_{bd} - \Delta^b_{bd} - \theta_{bd} \\
C_{\ldots}, \quad m_3: \alpha \Delta_{ca} + m_3 \cdot \alpha_{cd} + m_1 \cdot \beta_{ac} + m_2 \cdot \beta_{bd} - V_{ac} \cdot \delta = \theta_{ca} - \frac{\lambda}{2R} \\
D_{\ldots}, \quad m_2 \cdot \alpha \Delta_{bd} + m_4 \cdot \alpha_{dc} + m_3 \cdot \beta_{bd} - V_{bd} \cdot \delta = \theta_{bd} + \frac{\lambda}{2R} \\
S, \ldots, \quad \frac{m_3 - m_1}{2R} = \frac{m_4 + m_2}{2R} \\
\end{align*}
\]

A.S.—F
cuts are introduced at A, B, C, and D. The angular gaps are as shown in Fig. 90, and the equations of equilibrium can be set out as in (145) in which

\[ V_{ac} = V_{bd} = \frac{m_1 + m_2}{L}. \]

![Figure 90](image1)

Example 25.—Consider a closed ring with two semicircular ends (Fig. 91), under a concentrated load of 10,000 lb. at a = 10 ft. from A. The elastic constants and load functions are:

For semicircle AC (see Example 22):

\[ EI\alpha = 12.56; \quad EI\beta = 6.28; \quad EI\Delta^m = EI\gamma = 144; \quad \Delta^h = 1360; \]

\[ EI\theta_{ac} = 7500 \times 12^2 = 1,080,000; \quad EI\Delta^m_{ac} = 7500 \times \frac{\pi}{4} 12^3 = 10,200,000. \]

For semicircle BD:

\[ EI\theta_{bd} = 2500 \times 12^2 = 360,000; \quad EI\Delta^m_{bd} = 2500 \times \frac{\pi}{4} 12^3 = 3,400,000. \]

For Beam AB:

\[ EI\theta_{ab} = \frac{10,000 \times 10 \times 30(40 + 30)}{6 \times 40} = 875,000; \]

\[ EI\theta_{ba} = \frac{10,000 \times 10 \times 30(40 + 10)}{6 \times 40} = 625,000. \]
Substituting these values in equations (145) we get (145a), from which 
\[ m_1 = 13,417 \text{ ft.-lb., } m_2 = 24,733 \text{ ft.-lb., } m_3 = 42,279 \text{ ft.-lb., and } m_4 = 4129 \text{ ft.-lb.} \]

\[
\begin{align*}
\text{A...} & \cdot 12.56 m_1 + \frac{14.4}{3} m_4 + 6.28 m_3 - \frac{40}{3} m_2 - \frac{14.4}{40} (m_1 + m_2) + \frac{14.4}{40} m_3 - \\
& - \frac{14.4}{40} (m_1 + m_2) = 1,080,000 - 875,000 + \frac{6,800,000}{40} + \frac{\lambda}{2R} EI \\
\text{B...} & \cdot 12.56 m_2 + \frac{14.4}{3} m_3 - 6.28 m_4 - \frac{40}{3} m_1 - \frac{14.4}{40} m_1 + \frac{14.4}{40} m_2 - \frac{14.4}{40} m_4 \\
& - \frac{14.4}{40} (m_1 + m_2) = 625000 - 700000 + \frac{6,800,000}{40} + \frac{\lambda}{2R} EI \\
\text{C...} & \cdot 12.56 m_3 + \frac{14.4}{3} m_4 + 6.28 m_1 - \frac{40}{3} m_3 - \frac{14.4}{40} (m_1 + m_2) = 1,080,000 - \frac{\lambda}{2R} EI \\
\text{D...} & \cdot 12.56 m_4 + \frac{14.4}{3} m_1 - 6.28 m_2 + \frac{40}{3} m_3 + \frac{14.4}{40} (m_1 + m_2) = 360,000 + \frac{\lambda}{2R} EI \\
\text{and...} & \quad m_1 + m_2 - m_3 + m_4 = 0
\end{align*}
\]

Continuous Beam supported on Elastic Circular Ring (Fig. 92).—
Statically-indeterminate bending moments are calculated from the condition of equality of deflections of the beam and the ring at B. Denoting by \( \Delta_b \) the actual deflection at B, the bending moment \( m_b \) on the beam, for any shape of beam and any system of loading, can be expressed in terms of this deflection by the use of equations (1), as follows:

\[
(\alpha_{ba} + \alpha_{be})m_b = \theta_{ba} + \theta_{be} - \left( \frac{\Delta}{L_1} + \frac{\Delta}{L_2} \right),
\]

from which \( EI_b \Delta_b = f(c, P) \),
in which \( P \) is the unknown force of the beam at B and \( c \) represents beam constants, which depend on the system of loading and the geometrical shape of the beam.

Similarly, the deflection of the ring \( \Delta_r \) at B can be expressed in terms of \( P \).

From equation (137), \( EI_r \Delta_r = PR^3 \left( \frac{2}{\pi} - \frac{\pi}{8} \right) \). Equating \( \Delta_r = \Delta_b \), the force \( P \) is found. All other values can now be calculated in terms of this force.

Example 26.—For a two-span beam bearing on an elastic ring (Fig. 93), from equations (1),

\[
B, \left( \frac{20}{3} + \frac{30}{3} \right) m_b = \frac{10,000 \times 8 \times 12}{6 \times 20} (20 + 8) - \left( \frac{1}{20} + \frac{3}{20} \right) \Delta_b EI_b
\]

from which \( m_b = 13,440 - \frac{\Delta_b EI_b}{200} \) and \( P = \frac{10,000 \times 8}{20} + \frac{13,440}{20} - \frac{\Delta_b EI_b}{200 \times 20} \),

from which \( EI_b \Delta_b = 4000(4672 - P) \).

From equation (137), \( EI_r \Delta_r = P \times 5^3 \left( \frac{2}{\pi} - \frac{\pi}{8} \right) = 30.4875P \). Equating \( \Delta_b = \Delta_r, 4000(4672 - P) = 30.4875P \frac{I_b}{I_r} \), from which \( P = 4082 \text{ lb.} \).
The bending moments and deflections can now be calculated in terms of $P$ as follows.

$EI\Delta_b = 4000(4672 - 4082) = 2,360,000$. $m_b = 13,440 - \frac{2,360,000}{200} = 1640$ ft.-lb.

$m_r = \frac{5 \times 4.082}{\pi} = 6500$ ft.-lb.; $m_f = \frac{1}{2} \times 4.082\left(1 - \frac{2}{\pi}\right) \times 5 = 3710$ ft.-lb.

Assuming $E = 4,000,000$ lb. per square inch,

$\Delta_b = \Delta_r = \frac{2,360,000 \times 12}{576,000 \times 0.666} = 0.074$ in.

**Example 27.**—If two elastic rings, elliptical and circular, are pin-joined at A and B (Fig. 94) and submitted to the action of a concentrated load at A, statically-indeterminate bending moments and forces can be calculated from the condition of equality of the deflections at A. It should be noted, however, that the ellipse and circle, when fully deflected, must remain in contact only at points A and B, otherwise the principle of superposition does not apply, and the result may be wrong.
Imaginary cuts are introduced in the ellipse at A and B (Fig. 94), and the elastic constants and the load functions calculated from (96) by the method of summations. These are:

\[ EI_r \alpha = 13 \cdot 65, \quad EI_r \beta = 3 \cdot 48, \quad EI_r A_m = 143 \cdot 08, \quad EI_r A_h = 1545 \cdot 22; \]

\[ EI_r \theta_e = \frac{W}{2} \gamma = \frac{1}{2} \times 143 \cdot 08 W_e = 71 \cdot 54 W_e; \]

\[ EI_r A_n = \frac{W}{2} A_h = \frac{1}{2} \times 1545 \cdot 22 W_e = 772 \cdot 61 W_e; \]

in which \( W_e \) is the load carried by the elliptical ring.

The bending moment at A in the ellipse is

\[ m_a = \frac{\theta}{\alpha + \beta} = \frac{71 \cdot 54 W_e}{17 \cdot 12} = 4 \cdot 175 W_e, \]

and the deflection at A in the ellipse is

\[ EI_r A' = A - A^m m_a = 772 \cdot 61 W_e - 143 \cdot 08 \times 4 \cdot 175 W_e = 175 \cdot 25 W_e. \]

Similarly the deflection at A in the circle is, from (137),

\[ EI_c A'_c = W_e R^3 \left( \frac{\pi}{8} - \frac{2}{\pi} \right) = 304 \cdot 88 - 30 \cdot 488 W_e, \]

in which \( W_c \) is the load carried by the circular ring.

Equating \( EI_r A'_a = EI_c A'_a \): \( W_e = 2581 \cdot 3 \) lb., and \( W_c = 7418 \cdot 7 \) lb.

The bending moments on the ellipse and on the circle can be calculated in terms of these forces as follows.

Ellipse: \( m'_a = 4 \cdot 175 \times 2581 \cdot 3 = 10,780 \) ft.-lb.

\( m'_a = \frac{1}{2} \times 2581 \cdot 3 \times 15 = 10,780 = 8600 \) ft.-lb.
Circle: \[ m_c^c = \frac{5}{\pi} \times 7418.7 = 11,807 \text{ ft.-lb.} \]

\[ m_c^e = \frac{1}{2} \times 7418.7 \times (1 - \frac{2}{\pi}) = 6740 \text{ ft.-lb.} \]

Assuming that the modulus of elasticity of the ellipse and circle \( E \) is 4,000,000 lb. per square inch, the deflection at A is

\[ \Delta_a^e = \Delta_a^c = \frac{175.25 \times 2581.3 \times 12}{0.0833 \times 576,000} = 0.113 \text{ in.} \]
CHAPTER III
CONTINUOUS FRAMES WITH CURVED MEMBERS

Continuous frames with curved members can be analysed in a similar way to single-span frames discussed in Chapter II, except that generally, in addition to angular gaps, linear gaps must be closed simultaneously by statically-indeterminate bending moments and forces. Only in the case of two-span frames may the equations for linear gaps be avoided if it is assumed that joint B is at its final position at B' (Fig. 95), while joints A and C have moved to A' and C' in a statically-determinate condition and will move finally to A'' and C'' when statically-indeterminate bending moments and forces are applied.

Two-span Frames with Curved Members.—Following this procedure for frames with two spans, only translation \(\Delta b\) of joint B will enter into equations of equilibrium, and no equations for linear gaps will be required. The translations \(\Delta a\) and \(\Delta c\) of joints A and C, if required, can be calculated separately.

In a general case of two-span frames, of any shape and with any system of loading, the angular gaps at A, B, and C, in the condition relaxed from the effect of continuity of joints, are as shown in Fig. 95, in which \(\Delta^h_1\) and \(\Delta^h_2\) are horizontal translations of joints in the first and second span in a statically-determinate case. Other symbols are as defined previously. The equations of equilibrium for the joints A, B and C can now be set out (146).

\[
\begin{align*}
DA.B. \quad & m_1 \alpha_{ab} + m_1 \alpha_{ad} + m_2 \beta_1 + m_2 \frac{\Delta^m_1}{h} + m_2 \frac{\Delta^m_2}{h} + H_d \cdot \gamma_1 + H_d \cdot \frac{\Delta^h}{h} = \theta_{ab} + \frac{\Delta^h - \Delta b}{h} \\
ABC. \quad & m_1 \beta_1 + m_2 \alpha_{ba} + m_3 \alpha_{bc} + m_2 \beta_2 + H_d \cdot \delta_1 + H_f \cdot \gamma_2 = \theta_{ba} + \theta_{bc} \\
ABE. \quad & m_1 \beta_1 + m_2 \alpha_{ba} + (m_2 - m_3) \alpha_{bc} + H_d \cdot \delta_1 = \theta_{ba} + \frac{\Delta b}{h} \\
BCF. \quad & m_3 \beta_2 + m_4 \alpha_{cb} + m_4 \alpha_{cf} + m_3 \frac{\Delta^m_2}{h} + m_4 \frac{\Delta^m_2}{h} + H_f \cdot \delta_2 + H_f \cdot \frac{\Delta^h_2}{h} = \theta_{cb} + \frac{\Delta^h_2 - \Delta b}{h} \\
S.c. \quad m_4 = (m_2 - m_3) + m_4
\end{align*}
\]

(146)
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For symmetrical frames and symmetrical systems of loading, only the first two equations are required, in which case \( \Delta b = 0 \).

When bending moments are calculated from (I46), the translations of joints A and C, if required, can be obtained from (I46a).

\[
\begin{align*}
\Delta a &= \Delta^a_0 - \Delta b - m_1 \Delta^m a_1 - m_2 \Delta^m b_1 - H_d \Delta^h_1 \\
\Delta c &= \Delta^c_2 + \Delta b - m_4 \Delta^m c - m_5 \Delta^m b - H_f \Delta^h_2
\end{align*}
\]  

(I46a)

If one span only is loaded (Fig. 96) equations (I46) require slight alteration as in (I47).

\[
\begin{align*}
DAB &\ldots m_1 \alpha_{ab} + m_1 \alpha_{ab} + m_1 \Delta^m a_1 + m_2 \beta_1 + m_2 \Delta^m b_1 + H_d \gamma_1 + H_d \frac{\Delta^h h}{h} = \theta_{ab} + \frac{\Delta^f - \Delta b}{h} \\
ABC &\ldots m_1 \beta_1 + m_2 \alpha_{ab} + m_2 \alpha_{bc} + m_3 \beta_2 + H_d \gamma_2 + H_f \gamma_2 = \theta_{ba} \\
ABE &\ldots m_1 \beta_1 + m_2 \alpha_{ba} + m_2 + m_3 \alpha_{bc} + H_d \gamma_1 = \theta_{ba} + \Delta b \\
BCF &\ldots m_4 \alpha_{ef} + m_4 \alpha_{cf} - m_3 \beta_2 - m_3 \Delta^m c_2 + m_4 \Delta^m e_3 + H_f \delta_2 + H_f \Delta^h h = \Delta b \\
S. c. &\ldots m_1 = m_2 + m_3 + m_4
\end{align*}
\]  

(I47)

Similarly, if a frame is submitted to the action of lateral forces on column AD (Fig. 97), equations (I46) become (I48).

Fig. 96.

Fig. 97.
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\[
\begin{align*}
\Delta_{AB} &= m_1 \alpha_{AB} + m_2 \alpha_{AD} - m_2 \beta_1 + m_1 \frac{\alpha_{AE}}{h} - m_2 \frac{\beta_2}{h} + H_d \gamma_1 + H_d \delta_1 = \frac{\Delta_{AB}}{h} - \theta_{BA} \\
\Delta_{AE} &= m_2 \alpha_{AE} - \left( m_2 + m_3 \right) \alpha_{AE} - m_1 \beta_1 - H_d \gamma_2 = \frac{\Delta_{AE}}{h} \\
\Delta_{BC} &= m_3 \alpha_{BC} + \left( m_2 + m_3 \right) \alpha_{BC} - m_3 \beta_2 - H_f \gamma_2 = \frac{\Delta_{BC}}{h} \\
\Delta_{CF} &= m_4 \alpha_{CF} - m_3 \beta_3 - m_2 \alpha_{CF} - m_4 \frac{\alpha_{CF}}{h} - H_f \gamma_3 + H_f \delta_3 = \frac{\Delta_{CF}}{h} \\
\text{S.C.} &= m_1 + m_2 + m_3 + m_4 = W_A h
\end{align*}
\]

\( (148) \)

Example 28.—Statically-indeterminate bending moments for the frame in Fig. 98 may be calculated either from \((146)\) if both spans are considered to be loaded at the same time, or from \((147)\) if each span is loaded separately. The elastic constants and load functions are calculated as follows:

Span 1 (Example 15):

\[
\begin{align*}
EI_x &= 15.55; \quad EI_y = 7.775; \\
EI_{Ax} &= 139.92; \quad EI_{Ah} = 2240; \\
EI_{Ab} &= 1,020,000; \quad EI_{bA} = 728,000; \quad EI_{\theta B} = 12,850,000.
\end{align*}
\]

Span 2:

\[
\begin{align*}
EI_x &= \frac{3}{10} \times 32.312 = 21.54; \quad EI_y = 10.77. \\
EI_{Ax} &= \frac{1}{10} \times 16 \times 32.312 = 258.5. \\
EI_{Ah} &= \frac{3}{10} \times 16^2 \times 32.312 = 5514.53. \\
EI_{ba} &= \frac{1000 \times 20^2 \times 32.312}{6 \times 60} \left(3 \times 60 - 2 \times 20\right) = 5,026,300. \\
\bar{x} &= \frac{2 \times 60^2 - 20^2}{2 \left(3 \times 60 - 2 \times 20\right)} = 24.28 \text{ ft.; } \bar{y} = 35.72. \\
EI_{\theta_{ba}} &= 2,992,300; \quad EI_{\theta_{ab}} = 2,034,000. \\
EI_{\theta_B} &= \frac{1000 \times 20^2 \times 16 \times 32.312}{12 \times 60^2} \left(3 \times 60^2 - 2 \times 20^2\right) \\
&= 47,869,600.
\end{align*}
\]

For a concentrated load \( W = 10,000 \) lb. on the first span, from \((147)\) we get
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\((149a)\), from which \(m_1 = 28,943 \text{ ft.-lb.}, m_2 = 18,000 \text{ ft.-lb.}, m_3 = 6697 \text{ ft.-lb.},\) and \(m_4 = 4239 \text{ ft.-lb.}\)

\[
\begin{align*}
\text{DAB...} & \quad 15.55 m_1 + \frac{16}{3} m_1 + 7.75 m_2 + \frac{139.92}{16} m_3 + \frac{139.92}{16} m_4 + \frac{2240}{16} m_5 + \frac{1285000}{16} \frac{db}{h} E1 \\
\text{ABC...} & \quad 7.75 m_1 + 15.55 m_2 - \frac{16}{3} (m_2 + m_3) + \frac{139.92}{16} m_4 = 728,000 + \frac{Ab}{h} E1 \\
\text{ABE...} & \quad 7.75 m_1 + 15.55 m_2 + \frac{16}{3} (m_2 + m_3) + \frac{139.92}{16} m_4 = 728,000 + \frac{Ab}{h} E1 \\
\text{BCF...} & \quad 21.54 m_4 + \frac{16}{3} m_5 - 10.77 m_3 - \frac{258.5}{16} m_3 + \frac{258.5}{16} m_4 + \frac{5514.55}{16^2} m_5 = \frac{Ab}{h} E1 \\
\text{S.c...} & \quad m_1 = m_2 + m_3 + m_4
\end{align*}
\]

For a uniformly-distributed load on part of the second span we have \((149b)\)

\[
\begin{align*}
\text{DAB...} & \quad 21.54 m_4 + \frac{16}{3} m_5 + \frac{258.5}{16} m_3 + \frac{258.5}{16} m_4 + \frac{258.5}{16} m_5 + \frac{5504.43}{16^2} m_6 = 203,400 + \frac{47,886.86}{h} \frac{db}{E1} \\
\text{ABC...} & \quad 10.77 m_4 + 21.54 m_2 + 15.55 m_3 - 7.75 m_2 + \frac{258.5}{16} m_3 + \frac{139.92}{16} m_4 = 2992,300 \\
\text{ABE...} & \quad 10.77 m_1 + 21.54 m_2 + \frac{16}{3} (m_2 - m_3) + \frac{258.5}{16} m_5 = 2992,300 + \frac{Ab}{h} E1 \\
\text{BCF...} & \quad 15.55 m_5 + \frac{16}{3} m_6 - 7.75 m_3 - \frac{139.92}{16} m_3 + \frac{139.92}{16} m_4 + \frac{2240}{16} m_5 = \frac{Ab}{h} E1 \\
\text{S.c...} & \quad m_1 = m_2 - m_3 + m_4
\end{align*}
\]

from which \(m_1 = 4801 \text{ ft.-lb.}, m_2 = 26,913 \text{ ft.-lb.}, m_3 = 65,295 \text{ ft.-lb.},\) and \(m_4 = 43,183 \text{ ft.-lb.}\)

**EXAMPLE 29.**—The load functions and elastic constants for a single-span frame from **Example 16** are:

\[
\begin{align*}
\text{Beam:} & \quad E1\alpha = 35.8; \quad E1\beta = 26.86; \quad E1\Delta^m = E1\gamma = 611; \quad E1\Delta^h = 12,800. \\
& \quad E1\theta_{ab} = 6,885,000; \quad E1\theta_{ba} = 5,711,000; \quad E1\Delta^e = 127,536,000. \\
\text{Column:} & \quad E1\alpha_c = 2.07.
\end{align*}
\]

For the two-span frame shown in **Fig. 99**, from equations \((147)\) we derive

**Fig. 99.**

\[
\begin{align*}
\text{DAB...} & \quad 3580 m_1 + 207 m_2 + 2686 m_3 + \frac{611}{16} m_1 + \frac{611}{16} m_2 + \frac{611}{16} m_3 + \frac{12800}{16^2} m_4 + \frac{12753600}{16} \frac{db}{h} E1 \\
\text{ABC...} & \quad 2686 m_1 + 3580 m_2 - 3580 m_3 + 2686 m_2 + \frac{611}{16} m_1 + \frac{611}{16} m_2 + \frac{5711000}{h} \\
\text{ABE...} & \quad 2686 m_1 + 3580 m_2 + 207 (m_2 + m_3) + \frac{611}{16} m_1 + \frac{5711000}{h} + \frac{Ab}{h} E1 \\
\text{BCF...} & \quad 3580 m_4 + 207 m_4 - 2686 m_3 - \frac{611}{16} m_3 + \frac{611}{16} m_4 + \frac{12800}{16^2} m_4 + \frac{12753600}{16} \frac{db}{h} E1 \\
\text{S.c...} & \quad m_1 = m_2 + m_3 + m_4
\end{align*}
\]
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Fig. 100.

(147c), from which \( m_1 = 80,573 \) ft.-lb., \( m_2 = 19,920 \) ft.-lb., \( m_3 = 41,832 \) ft.-lb., and \( m_4 = 18,820 \) ft.-lb.

**EXAMPLE 30.**—For the elliptical frame in Fig. 100, the load functions and elastic constants (see Example 23) are

\[
EI_x = 17,63; \quad EI_y = 6,79; \quad EI_{Am} = EI_{Ay} = 174,26; \quad EI_{Ah} = 1430,2.
\]

For \( W = 10,000 \) lb.: \( EI_{th} = 945,400 \); \( EI_{be} = 660,400 \); \( EI_{Ab} = 13,924,300 \).

For uniformly-distributed load: \( EI_{bd} = 3,086,750 \); \( EI_{Ah} = 53,121,460 \).

From equations (147) we obtain (149d and 149e). It should be noted that the left-hand side of these equations remains constant for both systems of loading, but the right-hand side varies according to the values of the load functions. From these equations:

For \( W = 10,000 \): \( m_1 = 6440 \) ft.-lb.; \( m_2 = 9310 \) ft.-lb.; \( m_3 = 12,250 \) ft.-lb.; \( m_4 = 27,990 \) ft.-lb.

For u.d.l.: \( m_1 = 39,890 \) ft.-lb.; \( m_2 = 76,280 \) ft.-lb.; \( m_3 = 50,090 \) ft.-lb.; \( m_4 = 166,260 \) ft.-lb.
EXAMPLE 31.—For the frame shown in Fig. 101, submitted to the action of a concentrated horizontal force, the elastic constants and load functions (see Example 22), are

\[
EI\alpha = 12.56; \quad EI\beta = 6.28; \quad EI\Delta^m = EI\gamma = 144; \quad EI\Delta^h = 1360.
\]

\[
EI\theta_{ab} = EI\theta_{ba} = W\gamma = 10,000 \times 144 = 1,440,000.
\]

\[
EI\Delta^o = W\Delta^h = 10,000 \times 1360 = 13,600,000,
\]

and from equations (148) we derive (149f), from which

\[
m_1 = 92,560 \text{ ft.-lb.}, \quad m_2 = 5130 \text{ ft.-lb.}, \quad m_3 = 83,410 \text{ ft.-lb.}, \quad \text{and} \quad m_4 = 58,890 \text{ ft.-lb.}
\]

\[
DAB. \ldots 12.56 m_1 + \frac{2h}{3} m_2 - 6.28 m_2 + \frac{h}{24} m_1 - \frac{h}{24} m_2 + \frac{1360}{24} m_4 = \frac{1360}{24} h E I
\]

\[
ABE. \ldots 12.56 m_2 + \frac{2h}{3} (m_2 + m_3) - 6.28 m_3 - \frac{144}{24} m_1 = \frac{Ab}{h} E I - 1440000
\]

\[
EBC. \ldots 12.56 m_3 + \frac{2h}{3} (m_2 + m_3) - 6.28 m_4 - \frac{144}{24} m_4 = \frac{Ab}{h} E I
\]

\[
BCF. \ldots 12.56 m_4 + \frac{2h}{3} m_4 - 6.28 m_3 + \frac{144}{24} m_4 - \frac{144}{24} m_3 + \frac{1360}{24} m_4 = \frac{Ab}{h} E I
\]

\[
\text{S.C.} \ldots m_1 + m_2 + m_3 + m_4 = 24 \times 10,000
\]

EXAMPLE 32.—For the frame shown in Fig. 102, the elastic constants and load functions (see Example 18) are

For \( m_{ab} = 1 \):

\[
EI\alpha_{ab} = 27.2; \quad EI\beta = 9.6; \quad EI\Delta^m = 302.5;
\]

For \( m_{ba} = 1 \):

\[
EI\alpha_{ba} = 17.6; \quad EI\beta = 9.6; \quad EI\Delta^m = 129.5.
\]

For \( H = 1 \):

\[
EI\gamma = 302.5; \quad EI\delta = 129.5; \quad EI\Delta^h = 4608;
\]

For u.d.l.:

\[
EI\theta_{ab} = 359000; \quad EI\theta_{ba} = 4043000; \quad EI\Delta^o = 569200000;
\]

and from equations (147) we derive (149g), from which

\[
m_1 = 51,282 \text{ ft.-lb.}, \quad m_2 = 123,811 \text{ ft.-lb.}, \quad m_3 = 118,726 \text{ ft.-lb.}, \quad \text{and} \quad m_4 = 46,197 \text{ ft.-lb.}.
Continuous Frames of Three Spans with Curved Members.

When analysing three-span frames with curved members (Fig. 103), at least one linear gap will have to be considered in addition to the usual angular gaps at each joint. If it is assumed that joint B is at its final position at B', and column CG is at its final position at C'G, points A', C', and D', for frames in a statically-determinate condition, will move finally to A", C", and D" when statically-indeterminate bending moments and forces are applied. At joint C there will be a linear gap of \((\Delta \theta - \Delta b - \Delta c)\), which must be closed simultaneously with all angular gaps. For any shape of frame, and any system of loading, the equations of equilibrium can be set out as in (150).

\[
\begin{align*}
EAB... & \quad m_1 \alpha_{ab} + m_1 \alpha_{def} + m_2 \beta_1 + m_2 \Delta^m_{h} + m_1 \Delta^m_{h} + h \tau_1 + He \Delta^h_{h} = \theta_{ab} + \Delta^\theta + \Delta^b \\
ABC... & \quad m_1 \beta_1 + m_2 \alpha_{ab} + m_3 \alpha_{bc} + m_4 \beta_2 + He \cdot S_1 + (He + Hf) \cdot \gamma_2 = \theta_{ab} + \theta_{bc} \\
FBC... & \quad m_3 \alpha_{bc} + (m_3 - m_2) \alpha_{bf} + m_4 \beta_2 + (He + Hf) \gamma_2 = \theta_{bc} + \Delta^b \\
C... & \quad m_3 \Delta^m_{h} + m_4 \Delta^m_{c} + (He + Hf) \Delta^h_{h} = \Delta^\theta + \Delta b - \Delta c \\
BCD... & \quad m_3 \beta_2 + m_4 \alpha_{cb} + m_5 \alpha_{cd} + m_6 \beta_3 + (He + Hr) \Delta^h_{h} = \theta_{bc} + \theta_{cd} \\
BCG... & \quad m_3 \beta_2 + m_4 \alpha_{cb} + (m_4 - m_5) \alpha_{cg} + (He + Hf) \Delta^h_{h} = \theta_{bc} + \theta_{cd} \\
CDK... & \quad m_5 \alpha_{dc} + m_6 \alpha_{dk} + m_5 \beta_3 + m_6 \alpha_{dc} + m_6 \alpha_{dk} + m_6 \Delta^h_{h} + (He + Hr) \Delta^h_{h} = \theta_{dc} + \Delta^h_{h} + \Delta^c \\&c... & \quad m_1 + m_3 = m_4 + m_5 + m_6 \\
\end{align*}
\]

(149g)
For symmetrical frames with symmetrical loading the first four equations only are required. If the first span only is loaded (Fig. 104), equations (150) require slight alteration as in (150a).

If the second span only is loaded (Fig. 105), equations (150a) become (150b)

\[
\begin{align*}
EAB \ldots & m_1 \alpha_{ab} + m_1 \alpha_{ae} + m_2 \beta_1 + m_2 \frac{\Delta^b}{h} + m_1 \frac{\Delta^a}{h} + H_e \gamma_e + H_e \frac{\Delta_h}{h} = \theta_{ab} + \frac{\Delta_i}{h} - \frac{\Delta^b}{h} \\
ABC \ldots & m_1 \beta_1 + m_2 \alpha_{ba} = m_3 \alpha_{bc} + m_4 \beta_2 + H_e \delta_1 + (H_e - H_f) \gamma_2 = \theta_{ba} \\
FBC \ldots & m_3 \alpha_{bc} + (m_3 + m_2) \alpha_{cf} - m_4 \beta_2 - (H_e - H_f) \gamma_2 = \frac{\Delta^b}{h} \\
C \ldots & m_4 \Delta_2^m - m_3 \Delta_2^m + (H_e - H_f) \Delta_2^h = \Delta b - \Delta c \\
BDC \ldots & m_4 \alpha_{cb} - m_3 \beta_2 - m_4 \alpha_{cd} + m_5 \beta_3 + (H_e - H_f) \delta_2 + H_k \gamma_3 = 0 \\
BCG \ldots & m_4 \alpha_{cb} - m_3 \beta_2 + (m_4 + m_5) \alpha_{cg} + (H_e - H_f) \delta_2 = \frac{\Delta c}{h} \\
CDK \ldots & m_6 \alpha_{dc} + m_6 \alpha_{dk} - m_5 \beta_3 - m_6 \frac{\Delta^c}{h} + m_6 \frac{\Delta^d}{h} + H_k \delta_3 + H_k \frac{\gamma_3}{h} = \frac{\Delta c}{h} \\
S.C. \ldots & m_1 = m_2 + m_3 + m_4 + m_5 + m_6
\end{align*}
\]
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EAB... \( m_1. \alpha_{ab} + m_1. \alpha_{dc} - m_2. \beta_i - m_2. \frac{\Delta_i}{h} + m_1. \frac{\Delta_i}{h} + H_e. \gamma_1 + H_e. \frac{\Delta_i}{h} = \frac{Ab}{h} \)

ABC... \( m_1. \beta_i - m_2. \alpha_{ba} + m_3. \alpha_{bc} + m_4. \beta_2 + H_e. \delta_1 + (H_e + H_f). \gamma_2 = \theta_{bc} \)

FBC... \( m_3. \alpha_{bc} + (m_2 - m_3). \alpha_{br} + m_4. \beta_2 + (H_e + H_f). \gamma_2 = \theta_{bc} + \frac{Ab}{h} \)

C... \( m_3. \Delta_i + m_4. \Delta_2 + H_e. H_f. \Delta_2 = \Delta_2 - \Delta_b - \Delta_c \)

BCD... \( m_3. \beta_2 + m_4. \alpha_{cb} - m_5. \alpha_{cd} + m_6. \beta_3 + (H_e + H_f). \delta_2 + H_k. \gamma_3 = \theta_{cb} \)

BCG... \( m_3. \beta_2 + m_4. \alpha_{cb} + (m_4 + m_5). \alpha_{cg} + (H_e + H_f). \delta_2 = \theta_{cb} + \frac{Ac}{h} \)

CDK... \( m_6. \alpha_{dc} + m_6. \alpha_{dk} - m_5. \beta_3 - m_5. \frac{\Delta_3}{h} + m_6. \Delta_3 + H_k. \delta_3 + H_k. \frac{\Delta_3}{h} = \frac{Ac}{h} \)

S.C... \( m_1 + m_2 + m_3 = m_4 + m_5 + m_6. \)

Multiple-span Frames with Curved Members.

In a similar way, equations of equilibrium may be set out for frames with any number of spans. An increase in the number of statically-indeterminate bending moments is accompanied by a similar increase in the number of equations. For symmetrical frames and loading, however, the number of equations of equilibrium can be reduced by half.

For a four-span symmetrical frame with symmetrical loading (Fig. 106), the equations are (151a).

If the second and third spans are loaded symmetrically the equations are (151a).

\[
\begin{align*}
DAB... & \quad m_1. \alpha_{ab} + m_1. \alpha_{cd} + m_2. \beta_1 + H_d. \gamma_1 = \theta_{ab} + \frac{\Delta a}{h} \\
ABC... & \quad m_1. \beta_1 + m_2. \alpha_{ba} + m_3. \alpha_{bc} + m_4. \beta_2 + H_d. \delta_1 + (H_d + H_e). \gamma_2 = \theta_{ba} + \theta_{bc} \\
EBC... & \quad m_3. \alpha_{bc} + (m_3 - m_2). \alpha_{be} + m_4. \beta_2 + m_5. \Delta_2 + m_4. \frac{\Delta_2}{h} + \\
& \quad + (H_d + H_e). \gamma_2 + (H_d + H_e). \frac{\Delta_2}{h} = \theta_{bc} + \frac{\Delta_2}{h} \\
B... & \quad m_1. \Delta_1 + m_2. \Delta_1 + m_3. \Delta_2 + m_4. \Delta_2 + H_d. \Delta_2 + (H_d + H_e). \Delta_2 = \frac{\Delta_1}{h} + \frac{\Delta_2}{h} - \Delta_2 \\
CB... & \quad m_6. \alpha_{cb} + m_3. \beta_2 + (H_d + H_e). \delta_2 = \theta_{cb}
\end{align*}
\]

Fig. 106.
\[
\begin{align*}
\text{DAB} & \ldots m_1 \alpha_{ab} + m_1 \alpha_{ad} - m_2 \beta_1 + H_d \gamma_1 = \frac{\Delta^2}{h} \\
\text{ABC} & \ldots m_1 \beta_1 - m_2 \alpha_{ba} + m_3 \alpha_{bc} + m_4 \beta_2 + H_d \delta_1 + (H_d + H_e) \gamma_2 = \Theta_{bc} \\
\text{EBC} & \ldots m_3 \alpha_{bc} + (m_2 + m_3) \alpha_{be} + m_5 \beta_2 + m_4 \delta_2 + m_4 \frac{\Delta^2}{h} + \\
& + (H_d + H_e) \gamma_2 + (H_d + H_e) \Delta_2^h = \Theta_{bc} + \Delta^3 \frac{h}{h} \\
\text{B} & \ldots m_1 \Delta_1^m - m_2 \Delta_1^0 + m_3 \Delta_2^m + m_4 \Delta_2^0 + H_d \Delta_1^h + (H_d + H_e) \Delta_2^h = \Delta_2^0 - \Delta^4 \frac{h}{h} \\
\text{CB} & \ldots m_4 \alpha_{cb} + m_3 \beta_2 + (H_d + H_e) \delta_2 = \Theta_{cb}
\end{align*}
\]

For symmetrical frames with unsymmetrical loading (Fig. 107a), the number of simultaneous equations may also be reduced by half if the actual loading is replaced by two equivalent systems of loading, namely, one symmetrical (Fig. 107b), and one unsymmetrical (Fig. 107c). The final result is obtained by combining the results of these two equivalent cases.

For a four-span symmetrical frame with an unsymmetrical load on the second span only (Fig. 107a), the two equivalent loadings are as shown in Figs. 107b and 107c. Equations (151b) apply to a symmetrical system of loading. For unsymmetrical loading the equations of equilibrium are

\[
\begin{align*}
\text{DAB} & \ldots m_1 \alpha_{ab} + m_1 \alpha_{ad} - m_2 \beta_1 + H_d \gamma_1 = \frac{\Delta^2}{h} \\
\text{ABC} & \ldots m_1 \beta_1 - m_2 \alpha_{ba} + m_3 \alpha_{bc} + m_4 \beta_2 + H_d \delta_1 + (H_d + H_e) \gamma_2 = \Theta_{bc} \\
\text{EBC} & \ldots m_3 \alpha_{bc} + (m_2 + m_3) \alpha_{be} + m_5 \beta_2 + m_4 \delta_2 + m_4 \frac{\Delta^2}{h} + \\
& + (H_d + H_e) \delta_2 + (H_d + H_e) \Delta_2^h = \Theta_{bc} + \Delta^3 \frac{h}{h} \\
\text{B} & \ldots m_1 \Delta_1^m - m_2 \Delta_1^0 + m_3 \Delta_2^m + m_4 \Delta_2^0 + H_d \Delta_1^h + (H_d + H_e) \Delta_2^h = \Delta_2^0 - \Delta^4 \frac{h}{h} \\
\text{BCF} & \ldots m_4 \alpha_{cb} + 2m_4 \alpha_{cf} + m_3 \beta_2 + (H_d + H_e) \delta_2 = \Theta_{cb}
\end{align*}
\]

**Example 33.**—The elastic constants and load functions for the frame shown in Fig. 108 are:

For spans 1 and 3 (see Example 22):

\[EI_x \alpha = 12.56; \quad EI_y \beta = 6.28; \quad EI_y \Delta^m = EI_y \Delta^h = 144; \quad EI_x \Delta^h = 1360.\]

For span 2 (see Example 21):

\[EI_x \alpha = 22.1; \quad EI_y \beta = 11.05; \quad EI_y \Delta^m = EI_y \Delta^h = 262.5; \quad EI_x \Delta^h = 2510.\]

For a uniformly-distributed load on the second span,

\[EI_x \beta_0 = 9.531,830; \quad EI_y \Delta^0 = 177,598,250.\]

and from the first four equations (150) we derive (152b), from which

\[m_1 = 6085 \text{ ft.-lb.}, \quad m_2 = 299 \text{ ft.-lb.}, \quad \text{and } m_3 = 188,391 \text{ ft.-lb.}\]
CONTINUOUS FRAMES WITH CURVED MEMBERS

For a concentrated load $W = 10,000$ lb. at $a = 12$ ft. from B (see Example 21), the load functions are

$$
\begin{align*}
\mathcal{E}A\mathcal{B}... & 12.56 m_1 + \frac{16}{3} m_2 + \frac{16}{10} m_3 + \frac{16}{15} m_4 + \frac{16}{15} m_5 + \frac{16}{15} m_6 - 6.28 m_2 - \frac{16}{16} m_6 = \frac{\Delta b}{h} E \\
\mathcal{F}B\mathcal{C}... & \frac{22.1}{5} m_3 + \frac{16}{3} (m_2 + m_3) + \frac{16}{10} m_4 + \frac{262.5}{16} (m_3 + m_2 + m_4) = \frac{9531.830}{5} + \frac{\Delta b}{h} E \\
\mathcal{A}B\mathcal{C}... & 628 m_6 - 12.56 m_7 - \frac{22.1}{5} m_3 + \frac{16}{10} m_4 + \frac{262.5}{16} (m_3 + m_2 + m_4) = \frac{9531.830}{5} \\
\mathcal{C}... & \frac{262.5}{16} m_3 + \frac{262.5}{16} m_3 + \frac{262.5}{16} (m_2 + m_3) = \frac{177599.250}{16} - \frac{\Delta b}{h} E
\end{align*}
$$

(152b)

$EI_2 \theta_6 = 1,868,090; \ EI_2 \theta_6 = 1,212,740; \ EI_2 \Delta_6 = 28,657,240;$
and from equations (151) we derive (151a), from which $m_1 = 2687$ ft.-lb., $m_2 = 7633$ ft.-lb., $m_3 = 41,805$ ft.-lb., $m_4 = 18,889$ ft.-lb., $m_5 = 7908$ ft.-lb., and $m_6 = 4687$ ft.-lb.

A.S.-G
EXAMPLE 34.—The elastic constants and load functions for the frame in Fig. 109 (see Example 23) are

\[ EI_A = 17.63; \quad EI_B = 6.79; \quad EI_{D} = EI_{F} = 174.26; \quad EI_{A} = 430.2. \]

For a uniformly-distributed load on the second span \( EI_{b} = 3,086,750; \)
\( EI_{D} = 53,121,460; \) and from (150) we derive (152d), from which

\[ m_{1} = 30,331 \text{ ft.-lb.}, \quad m_{2} = 52,656 \text{ ft.-lb.}, \quad \text{and} \quad m_{3} = 55,569 \text{ ft.-lb.}. \]

\[ E A B...17.63 m_{1} + \frac{12}{12} m_{2} - \frac{12}{12} m_{2} + \frac{12}{12} m_{1} + \frac{12}{12} m_{1} = \frac{\Delta b}{h} \]
\[ A B C...6.79 m_{1} - 17.63 m_{2} - 17.63 m_{3} + 17.63 m_{2} + \frac{12}{12} m_{3} + \frac{12}{12} m_{3} + \frac{12}{12} m_{3} + \frac{12}{12} m_{3} = 3,086,750 \]
\[ F B C...17.63 m_{5} + \frac{12}{12} (m_{2} m_{3}) + 6.79 m_{3} + \frac{12}{12} (m_{1} + m_{2} + m_{3}) = 3,086,750 + \frac{\Delta b}{h} \]
\[ C...17.63 m_{2} + \frac{12}{12} (m_{2} + m_{3}) + \frac{12}{12} m_{3} + \frac{12}{12} m_{3} + \frac{12}{12} m_{3} = \frac{53,121,460}{12} - \frac{2 \Delta b}{h} \]

For a concentrated load \( W = 10,000 \text{ lb.}, \)
\[ EI_{b} = 945,400; \quad EI_{b} = 660,400; \quad \text{and} \quad EI_{D} = 13,924,300; \]
and from (150b) we derive (152c), from which

\[ m_{1} = 5451 \text{ ft.-lb.}, \quad m_{2} = 8259 \text{ ft.-lb.}, \quad m_{3} = 22,964 \text{ ft.-lb.}, \]
\[ m_{4} = 5506 \text{ ft.-lb.}, \quad m_{5} = 20,162 \text{ ft.-lb.}, \quad \text{and} \quad m_{6} = 11,006 \text{ ft.-lb.}. \]
CONTINUOUS FRAMES WITH CURVED MEMBERS

\[ E A B \ldots 17.63 m_1 + \frac{12}{3} m_1 - 6.79 m_2 - \frac{174.26}{12} m_2 + \frac{174.26}{12} m_3 + \frac{1630.2}{12} m_1 = \frac{Ab}{E} \]

\[ A B C \ldots 17.63 m_3 + \frac{12}{3} (m_2 + m_3) + 6.79 m_4 + \frac{174.26}{12} (m_1 + m_2 + m_3) = 945.40 + \frac{Ab}{E} \]

\[ F B C \ldots 6.79 m_1 - 17.63 m_2 + 17.63 m_3 + 6.79 m_4 + \frac{174.26}{12} m_1 + \frac{174.26}{12} m_2 + \frac{174.26}{12} (m_1 + m_2 + m_3) = 945.40 \]

\[ C \ldots 6.79 m_3 + 17.63 m_4 + \frac{12}{3} (m_4 + m_5) + \frac{174.26}{12} (m_1 + m_2 + m_3) = 660.40 + \frac{Ac}{E} \]

\[ B C D \ldots 6.79 m_3 + 17.63 m_4 + 17.63 m_5 + 6.79 m_6 + \frac{174.26}{12} (m_1 + m_2 + m_3) + \frac{174.26}{12} m_6 = 660.40 \]

\[ B C G \ldots \frac{174.26}{12} m_3 + \frac{174.26}{12} m_4 + \frac{1630.2}{12} (m_1 + m_2 + m_3) = \frac{13.924}{12} \frac{ab}{h} - \frac{Ac}{E} \]

\[ C D K \ldots 17.63 m_6 + \frac{12}{3} m_6 - 6.79 m_5 - \frac{174.26}{12} m_5 + \frac{174.26}{12} m_6 + \frac{1630.2}{12} m_5 + \frac{1630.2}{12} m_6 = \frac{Ac}{E} \]

\[ S c. \ldots m_1 + m_2 + m_3 - m_4 - m_5 - m_6 = 0 \]

\[ W = 20,000 \text{ lb.} \]

**Fig. 110.**

**Example 35.**—The elastic constants and the load functions for the frame shown in Fig. 110 are:

For spans 1 and 3: \( E \alpha = \frac{2 \times 14.42}{3 \times 0.282} = 34.04 \); \( E \beta = 17.02 \);

\[ E \Delta m = E y = \frac{8 \times 14.42}{2 \times 0.282} = 204.5 \]; \( E \Delta k = \frac{2 \times 8 \times 14.42}{3 \times 0.282} = 2180. \)

\[ E A B \ldots 34.04 m_1 + \frac{12}{3} m_1 - 17.02 m_2 + \frac{204.5}{12} m_2 + \frac{204.5}{12} m_1 + \frac{1240}{12} m_3 = \frac{Ab}{E} \]

\[ A B C \ldots 17.02 m_1 - 34.04 m_2 + 3580 m_5 + 2666 m_5 + \frac{204.5}{12} m_1 + \frac{640}{12} (m_1 + m_2 + m_3) = 6.885,000 \]

\[ E B C \ldots 3580 m_5 + 204.5 (m_1 + m_2 + m_3) + 2666 m_5 + \frac{640}{12} (m_1 + m_2 + m_3) = 6.885,000 + \frac{Ab}{E} \]

\[ C \ldots \frac{640}{12} m_3 + \frac{640}{12} m_5 + \frac{1240}{12} (m_1 + m_2 + m_3) = \frac{1240}{12} \frac{ab}{E} - \frac{Ac}{E} \frac{ac}{E} \]

\[ B C D \ldots 26.86 m_5 + 3580 m_5 + 34.04 m_5 + 17.02 m_5 + \frac{640}{12} (m_1 + m_2 + m_3) + \frac{204.5}{12} m_5 = 5.711,000 \]

\[ B C G \ldots 26.86 m_5 + 3580 m_5 + 204.5 (m_1 + m_2 + m_3) + \frac{640}{12} (m_1 + m_2 + m_3) = 5.711,000 + \frac{Ac}{12} \]

\[ C D K \ldots 34.04 m_5 + \frac{12}{3} m_6 - 17.02 m_5 - \frac{204.5}{12} m_5 + \frac{204.5}{12} m_5 + \frac{204.5}{12} m_6 + \frac{1240}{12} m_6 = \frac{Ac}{12} \]

\[ S c. \ldots \frac{m_1}{12} + \frac{m_5}{12} = \frac{m_4 + m_7}{16} + \frac{m_6}{12} \]
For span 2 (see Example 16):

\[ E\alpha = 35.8 \; ; \; E\beta = 26.86 \; ; \; E\Delta^m = E\gamma = 611 \; ; \; E\Delta^b = 12,800. \]

For column BF: \( E\alpha = 2.07. \)

For \( W = 20,000 \) lb.:

\[ E\alpha_s = 6,885,000 \; ; \; E\theta_o = 5,711,000 \; ; \; E}\Delta_o = 127,536,000. \]

Substituting these values in (150b) we obtain (152f), from which

\[ m_1 = -13,053 \text{ ft.-lb.} \; ; \; m_2 = -25,734 \text{ ft.-lb.} \; ; \; m_3 = +90,951 \text{ ft.-lb.} \; ;
\]
\[ m_4 = +44,877 \text{ ft.-lb.} \; ; \; m_5 = +23,580 \text{ ft.-lb.} \; ; \; m_6 = +15,483 \text{ ft.-lb.} \]

**Example 36.**—The elastic constants and load functions for the frame shown in Fig. 111 are:

For \( m_{ab} = 1 \):

\[ EI\alpha_{ab} = 16.64 \; ; \; EI\beta = 6.17 \; ; \; EI\Delta^m = 155.67. \]

For \( m_{ba} = 1 \):

\[ EI\alpha_{ba} = 11.27 \; ; \; EI\beta = 6.17 \; ; \; EI\Delta^m = 85.8. \]

For \( H = 1 \):

\[ EI\gamma = 155.67 \; ; \; EI\delta = 85.8 \; ; \; EI\Delta^b = 1932. \]

For a uniformly distributed load:

\[ EI\theta_{ab} = 876,000 \; ; \; EI\theta_{ba} = 1,314,000 \; , \; \text{and} \; EI\Delta^o = 17,000,000, \; \text{and from (150)} \]

\[
\begin{align*}
E A B & : 16.64 m_1 + \frac{12}{5} m_2 + \frac{617}{12} m_3 + \frac{858}{12} m_4 + \frac{155.67}{12} m_5 + \frac{1932}{12} m_6 = 876,000 + \frac{\Delta b}{h} E I \\
A B C & : 617 m_1 + 1127 m_2 + 16.64 m_3 + 617 m_4 + \frac{858}{12} m_5 + \frac{155.67}{12} (m_5 + m_3 - m_2) = 1314,000 + 876,000 \\
F B C & : 16.64 m_3 + \frac{12}{5} (m_3 - m_2) + 617 m_4 + \frac{155.67}{12} (m_4 + m_3 - m_2) = 1314,000 + \frac{\Delta b}{h} E I \\
c & : \frac{155.67}{12} m_3 + \frac{858}{12} m_4 + \frac{1932}{12} (m_4 + m_3 - m_2) + \frac{17,000}{12} - \frac{\Delta b}{h} E I = \frac{\Delta c}{h} E I \\
B C D & : 617 m_3 + 1127 m_4 + 16.64 m_5 + 617 m_6 + \frac{858}{12} (m_5 + m_3 - m_2) + \frac{155.67}{12} m_6 = 1314,000 + 876,000 \\
B C G & : 617 m_3 + 1127 m_4 + \frac{12}{5} (m_4 - m_3) + \frac{858}{12} (m_4 + m_3 - m_2) = 1314,000 + \frac{\Delta b}{h} E I \\
C D K & : 1127 m_6 + \frac{12}{5} m_6 + 617 m_6 + \frac{155.67}{12} m_5 + \frac{858}{12} m_6 + \frac{1932}{12} m_6 = 1314,000 + \frac{\Delta c}{h} E I \\
s.c & : m_1 - m_2 + m_3 = m_4 - m_5 + m_6
\end{align*}
\]

**Fig. 111.**

we obtain (152g), from which \( m_1 = 40,559 \) ft.-lb., \( m_2 = 11,293 \) ft.-lb., \( m_3 = 32,191 \) ft.-lb., \( m_4 = 30,732 \) ft.-lb., \( m_5 = 17,286 \) ft.-lb., and \( m_6 = 48,010 \) ft.-lb.

**Example 37.**—The elastic constants and load functions for the frame in Fig. 112 are (see Appendix):

For \( m_{ab} = 1 \):

\[ EI\varepsilon = \frac{1}{2}(2 \times 12 + 32.32) = 28.16; \]
\[ \frac{x}{\delta} = \frac{1}{3} \cdot \frac{30 \times 32.32}{2 \times 12 + 32.32} = 5.74 \text{ ft.} ; \quad z = 24.26 \text{ ft.} \]

\[ EI\alpha = 28.16 \cdot \frac{24.26}{30} = 22.79 ; \quad EI\beta = 5.37 ; \]

\[ EI\Delta_m = \frac{1}{6} \left( 3 \times 12 + 2 \times 32.32 \right) = 201.28. \]

For \( m_{ba} = 1 \): \( EI\varepsilon = \frac{1}{3} \times 32.32 = 10.79 ; \quad \varepsilon = \frac{3}{5} \times 30 = 20 \text{ ft.} ; \]

\[ EI\alpha = 16 \cdot 16 \frac{30}{5} = 10.79 ; \quad EI\beta = 5.37 ; \quad EI\Delta_m = \frac{1}{6} \times 12 \times 32.32 = 64.64. \]

For \( H = 1 \): \( EI\gamma = 201.28 ; \quad EI\delta = 64.64 ; \quad EI\Delta^h = \frac{12^2}{3} \left( 12 + 32.32 \right) = 2127.36. \]

For u.d.l.: \( EI\theta_a = EI\theta_b = \frac{1000 \times 30^3 \times 32.32}{24} = 1,212,000, \]

\[ EI\Delta_o = \frac{1000 \times 30^3 \times 32.32 \times 12}{24} = 14,550,000. \]

From (150) we obtain (152h), from which

\[
\begin{align*}
m_1 &= 28.289 \text{ ft.-lb.} ; \quad m_2 = 14.330 \text{ ft.-lb.} ; \quad m_3 = 35.303 \text{ ft.-lb.} ; \\
m_4 &= 42.233 \text{ ft.-lb.} ; \quad m_5 = 30.291 \text{ ft.-lb.} ; \quad m_6 = 37.220 \text{ ft.-lb.}
\end{align*}
\]
EXAMPLE 38.—The elastic constants and load functions for the frame shown in Fig. 113 calculated by the method of summation are:

For \( m_{ab} = 1 \):
\[ E I \alpha = 16.01; \quad E I \beta = 6.10; \quad E I \Delta^m = 171.78. \]

For \( m_{ba} = 1 \):
\[ E I \alpha = 12.91; \quad E I \beta = 6.10; \quad E I \Delta^m = 128.39. \]

For \( H = 1 \):
\[ E I \gamma = 171.78; \quad E I \delta = 128.39; \quad E I \Delta^h = 2729.57. \]

For u.d.l.:
\[ E I \theta_a = 1,205,210; \quad E I \theta_b = 1,190,840; \quad E I \Delta^o = 21,518,910. \]

From (150) we obtain (152), from which

\[ m_1 = 43,919 \text{ ft.-lb.}, \quad m_2 = 24,050 \text{ ft.-lb.}, \quad m_3 = 32,011 \text{ ft.-lb.}, \]
\[ m_4 = 10,973 \text{ ft.-lb.}, \quad m_5 = 6883 \text{ ft.-lb.}, \quad \text{and } m_6 = 47,789 \text{ ft.-lb.}. \]
CONTINUOUS FRAMES WITH CURVED MEMBERS

Example 39.—The elastic constants and load functions for the frame shown in Fig. 114 are calculated by the method of summations (see Tables IX, X and XI).

\[
\sin \phi = \frac{50}{80} = 0.625; \quad \phi = 38^\circ 41' \cdot (0.67516); \quad ds = \frac{80 \times 0.67516}{6} = 9 \text{ ft.}
\]

From Table IX \( m_a = 1 \):

\[
Ee = 4.5019 \times 9 = 40.52; \quad \bar{x} = \frac{188.007}{4.5019} = 41.76 \text{ ft.;}
\]

\[
Ea = 23.6; \quad E\beta = 16.92; \quad E\Delta^m = 66.305 \times 9 = 596.74.
\]

For \( H = 1 \):

\[
E\gamma = E\delta = 596.74; \quad E\Delta^a = 930.66 \times 9 = 8375.9.
\]

### TABLE IX.

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<th>Pt.</th>
<th>( x ) ft</th>
<th>( y ) ft</th>
<th>( a ) ft</th>
<th>( I ) ft(^4)</th>
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| \( \Sigma \) | | | | | | | | | 45019 \( \text{88007} \) \( \text{68345} \) \( \text{93006} \)

### TABLE X.

<table>
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<th>Pt.</th>
<th>( m_a ) (1000 lb. ft.)</th>
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<th>( m_1 \cdot y )</th>
<th>( m_a ) (1000 lb. ft.)</th>
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### ANALYSIS OF STRUCTURES

#### TABLE XI.

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<th>Pt.</th>
<th>y ft</th>
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<th>I ft$^4$</th>
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From Table XI, for $m_{as} = 1$:

\[ E\varepsilon = 5.645; \quad \bar{y} = \frac{77.38}{5.645} = 13.71 \text{ ft.}; \quad E\alpha_0 = 5.645 \times \frac{16.29}{30} = 3.0652 \]

From Table X, for $W = 10,000 \text{ lb. at } a = 15 \text{ ft. from B}$:

\[ E\varepsilon = 649.500 \times 9 = 5,845,500; \quad \bar{x} = \frac{27,342,700}{649,500} = 42.1 \text{ ft.} \]

\[ E\theta_b = 3,384,500; \quad E\theta_e = 2,461,000; \quad E\Delta_0 = 9,764,600 \times 9 = 87,881,900. \]

From Table X, for u.d.l. $w = 1,000 \text{ lb. per foot}$:

\[ E\theta_0 = 4,495,370 \times 9 = 40,458,000; \quad E\Delta_0 = 68,926,000 \times 9 \times 2 = 1,240,668,000. \]

\[
\begin{align*}
\text{EAB...} & \quad 2360m_1 + 506m_1 + \frac{59674}{30}m_1 + \frac{59674}{30}m_1 + \frac{83759}{30}m_1 \cdot 16.92m_2 = \frac{59674}{30}m_2 + \frac{904458,000 + \Delta b}{h} F \\
\text{FBC...} & \quad 2360m_3 + 306(m_1 + m_3) + 16.92m_3 + \frac{59674}{30}(m_1 + m_2 + m_3) = 40458000 + \Delta b F \\
\text{ABC...} & \quad 16.92m_1 - 2360m_2 + 2360m_3 + 16.92m_3 + \frac{59674}{30}m_1 + \frac{59674}{30}(m_1 + m_2 + m_3) = 40458000 \quad \text{(152k)} \\
\text{C...} & \quad \frac{59674}{30}m_3 + \frac{59674}{30}m_3 + \frac{83759}{30}(m_1 + m_2 + m_3) = \frac{1,240,668,000 - 2\Delta b}{h} E \\
\end{align*}
\]

\[
\begin{align*}
\text{EAB...} & \quad 2360m_1 + 506m_1 + \frac{59674}{30}m_1 - 16.92m_2 = \frac{59674}{30}m_1 + \frac{59674}{30}m_1 + \frac{83759}{30}m_1 \cdot 16.92m_2 = \frac{59674}{30}m_2 + \frac{904458,000 + \Delta b}{h} E \\
\text{EAB...} & \quad 16.92m_1 - 2360m_2 + 2360m_3 + 16.92m_3 + \frac{59674}{30}m_1 + \frac{59674}{30}(m_1 + m_2 + m_3) = 3384500 \quad \text{(152l)} \\
\text{FBC...} & \quad 2360m_3 + 306(m_1 + m_3) + 16.92m_2 + \frac{59674}{30}(m_1 + m_2 + m_3) = 3384500 + \Delta b E \\
\text{C...} & \quad \frac{59674}{30}m_3 + \frac{59674}{30}m_3 + \frac{83759}{30}(m_1 + m_2 + m_3) = \frac{83759}{30} + \Delta b E = \frac{83759}{30}E - \frac{AC}{h} E \\
\text{BCD...} & \quad 16.92m_3 + 2360m_4 + 2360m_5 + 16.92m_5 + \frac{59674}{30}(m_1 + m_2 + m_3) + \frac{59674}{30}m_5 = 2,461,000 \quad \text{(152l)} \\
\text{BCG...} & \quad 16.92m_3 + 2360m_4 + 306(m_1 + m_5) + \frac{59674}{30}(m_1 + m_2 + m_3) = 2,461,000 + \frac{AC}{h} E \\
\text{CDK...} & \quad 2360m_5 + 306m_6 - 16.92m_6 = \frac{59674}{30}m_6 + \frac{59674}{30}m_6 + \frac{59674}{30}m_6 + \frac{83759}{30}m_6 = \frac{AC}{h} E \\
\end{align*}
\]

s.c. $m_1 + m_2 + m_3 = m_4 + m_5 + m_6$
CONTINUOUS FRAMES WITH CURVED MEMBERS

For a uniformly-distributed load, from (150) we obtain (152b), from which
\[ m_1 = 776,320 \text{ ft.-lb.}, \quad m_2 = 326,570 \text{ ft.-lb.}, \quad m_3 = 375,670 \text{ ft.-lb.} \]

For a concentrated load, from (150b) we obtain (152d), from which
\[ m_1 = 2560 \text{ ft.-lb.}, \quad m_2 = 1160 \text{ ft.-lb.}, \quad m_3 = 69,870 \text{ ft.-lb.}, \quad m_4 = 9550 \text{ ft.-lb.}, \quad m_5 = 38,140 \text{ ft.-lb.}, \quad m_6 = 25,900 \text{ ft.-lb.} \]

**Fig. 115.**

**EXAMPLE 40.**—The elastic constants and load functions for the frame in Fig. 115 are calculated as follows (see Appendix).

For \( m_{ab} = 1 \):
\[ EI\varepsilon = 12\cdot07 + 6 = 18\cdot07 ; \]
\[ \varepsilon = \frac{18 \times 12\cdot07(90 - 18) + 30^3 - 6 \times 9^3 	imes 30 + 4 \times 9^3}{90(12\cdot07 + 12)} = 9.52 \text{ ft.} ; \quad \tilde{z} = 20.48 \text{ ft.} ; \]
\[ EI\alpha = 18\cdot07\frac{20.48}{30} = 21.33 ; \quad EI\beta = 5.74 ; \quad EI\Delta^m = \frac{1}{2} \times 8(12\cdot07 + 12) = 96.28. \]

For \( H = 1 \):
\[ EI\gamma = 96.28, \quad EI\Delta^h = 8\left(\frac{1}{2} \times 12\cdot07 + 12\right) = 1283. \]

For \( W = 10,000 \text{ lb.} \):
\[ EI\theta_o = 10,000 \times 9\left(\frac{1}{2} \times 12\cdot07 + 12\right) = 1,623,150. \]
\[ EI\Delta_o = 2 \times 10,000 \times 9 \times 8\left(\frac{1}{2} \times 12\cdot07 + 12\right) = 23,073,000. \]

Substituting these values in (151a), we obtain (152m), from which
\[ m_1 = 80,674 \text{ ft.-lb.}; \quad m_2 = 19,900 \text{ ft.-lb.}; \quad m_3 = 46,801 \text{ ft.-lb.}; \quad m_4 = 25,850 \text{ ft.-lb.}. \]

It is interesting to note that for single-span frames similar to those in Fig. 109, from (100),
\[ m_a \left(12\cdot33 + 5\cdot74 + \frac{10}{3} + \frac{96\cdot28}{10} + \frac{96\cdot28}{10} + \frac{1283}{2 \times 10^3} \right) = 1,623,150 + \frac{23,073,000}{2 \times 10}, \]
from which \( m_a = 59,000 \text{ ft.-lb.} \).

\[ \Delta A B \ldots 12\cdot33 m_1 + \frac{10}{3} m_2 + 5\cdot74 m_2 + \frac{96\cdot28}{10} m_1 = 1,623,150 + \frac{\Delta A}{h} E I \]

\[ A B C \ldots 5\cdot74 m_1 + 12\cdot33 m_2 + 12\cdot33 m_3 + 5\cdot74 m_4 + \frac{96\cdot28}{10} m_1 + \frac{96\cdot28}{10} m_1 = 2,623,150 \]

\[ E B C \ldots 12\cdot33 m_3 + \frac{10}{3}(m_3 - m_2) + 5\cdot74 m_4 + \frac{96\cdot28}{10} m_3 + \frac{96\cdot28}{10} m_4 + \frac{96\cdot28}{10} m_1 + \frac{1283}{10^2} (m_1 + m_3 - m_2) + \frac{1283}{10^2} (m_1 + m_3 - m_2) + \frac{1283}{10^2} (m_1 + m_3 - m_2) = 23,073,000 + \frac{\Delta A}{h} E I \]

\[ B \ldots \frac{96\cdot28}{10} m_1 + \frac{96\cdot28}{10} m_2 + \frac{96\cdot28}{10} m_3 + \frac{96\cdot28}{10} m_4 + \frac{1283}{10^2} m_1 + \frac{1783}{10^2} m_2 + \frac{1783}{10^2} m_3 + \frac{1783}{10^2} (m_1 + m_3 - m_2) = 23,073,000 + \frac{\Delta A}{h} E I \]

\[ C B \ldots 12\cdot33 m_4 + 5\cdot74 m_3 + \frac{96\cdot28}{10} (m_1 + m_3 - m_2) = 1,623,150 \]
EXAMPLE 41.—The elastic constants and load functions for the frame in Fig. 116 are calculated in Examples 15 and 28. They are:
For span 1: \( EI\alpha = 15.55;\) \( EI\beta = 7.775;\) \( EI\Delta^m = EI\gamma = 139.92;\) \( EI\Delta^h = 2240.\)
For span 2: \( EI\alpha = 21.54;\) \( EI\beta = 10.77;\) \( EI\Delta^m = EI\gamma = 258.5;\) \( EI\Delta^h = 5514.53.\)

\[
EI\theta_0 = \frac{1000}{12} \times 60^2 \times 32.312 = 9,700,000.
\]

\[
EI\Delta_0 = \frac{5}{48} \times 1000 \times 60^2 \times 16 \times 32.312 = 193,800,000.
\]

From equations (151b) we obtain (152n), from which

\[
m_1 = 35,991 \text{ ft.-lb.}, \quad m_2 = 36,254 \text{ ft.-lb.}, \quad m_3 = 174,826 \text{ ft.-lb.}, \quad m_4 = 177,592 \text{ ft.-lb.}
\]

\[
D A B \ldots 1555m_1 + \frac{16}{5} m_1 - 7.775m_2 + \frac{139.92}{16} m_1 = \frac{\Delta h}{h} E I
\]

\[
A B C \ldots 7775m_1 - 1555m_2 + 2154m_3 + 10.77m_4 + \frac{139.92}{16} m_1 + \frac{258.5}{16} (m_1 + m_2 + m_3) = 9,700,000
\]

\[
E B C \ldots 2154m_3 + \frac{16}{5} (m_2 + m_3) + 10.77m_4 + \frac{258.5}{16} m_4 + \frac{258.5}{16} (m_1 + m_2 + m_3)
\]

\[
+ \frac{5.514+5.53}{16} (m_1 + m_2 + m_3) = 193,800,000
\]

\[
B \ldots 139.92m_1 - \frac{139.92}{16} m_2 + \frac{258.5}{16} m_3 + \frac{258.5}{16} m_4 + \frac{2240}{16} m_4 + \frac{5514+53}{16} (m_1 + m_2 + m_3) = \frac{193,800,000 \Delta h}{h} E I
\]

\[
C B \ldots 2154m_4 + 10.77m_3 + \frac{258.5}{16} (m_1 + m_2 + m_3) = 9,700,000
\]

EXAMPLE 42.—A symmetrical frame with unsymmetrical loading, as in Fig. 117, may be analysed by considering two equivalent systems of loading, one symmetrical and the other unsymmetrical. Bending moments due to the symmetrical system are calculated in Example 41. Bending moments due to unsymmetrical loading are derived from equations (152a) and are as in (152p), from which

\[
m_1 = 46,638 \text{ ft.-lb.}; \quad m_2 = 52,318 \text{ ft.-lb.}; \quad m_3 = 189,456 \text{ ft.-lb.}; \quad m_4 = 93,148 \text{ ft.-lb.}
\]

The final result is obtained by combining the results of the two systems (Fig. 118).
CONTINUOUS FRAMES WITH CURVED MEMBERS

a) Symmetrical case

b) Unsymmetrical case

c) Final result

Fig. 118.

\[
\begin{align*}
DAB & \ldots \frac{15.55}{5} m_1 + \frac{16}{5} m_1 - 7775m_2 + \frac{139.92}{16} m_1 = \frac{\Delta \theta}{h} EI \\
ABC & \ldots 7775m_1 - 15.55m_2 + 21.54m_3 + 1077m_4 + \frac{139.92}{16} m_1 + \frac{2585}{16} (m_1 + m_2 + m_3) = 9,700,000 \\
BCE & \ldots 21.54m_3 + \frac{16}{5} (m_3 + m_4) + 1077m_4 + \frac{2585}{16} m_3 + \frac{2585}{16} m_4 + \frac{2885}{16} (m_1 + m_2 + m_3) + \\
& \quad + \frac{5514.53}{16} (m_1 + m_2 + m_3) = \frac{193,800,000}{16} \\
B & \ldots \frac{139.92}{16} m_1 - \frac{139.92}{16} m_2 + \frac{2585}{16} m_3 + \frac{2585}{16} m_4 + \frac{2240}{16} m_1 + \frac{5514.53}{16} m_4 (m_1 + m_2 + m_4) = \frac{193,800,000}{16} - \frac{\Delta \theta}{h} EI \\
BCF & \ldots 21.54m_4 + \frac{2885}{16} m_4 + 1077m_3 + \frac{2585}{16} (m_1 + m_2 + m_3) = 9,700,000
\end{align*}
\]

\[(152\theta)\]

Influence of Change of Temperature on Continuous Frames with Curved Members.

Equations (146), (150), and (151a), with slight modification, can be used in calculating thermal stresses in continuous frames with curved members.

Two-span Frames (Fig. 119).—For two-span frames the angular gaps and the deflected shape are shown in Fig. 119 by dotted lines, and the equations of equilibrium are:

\[
\begin{align*}
DAB & \ldots m_1 \alpha_{ab} + m_1 \alpha_{ad} + m_1 \frac{\Delta \theta}{h} - m_2 \beta_1 - m_2 \frac{\Delta \theta}{h} + H_d \theta_1 + H_d \frac{\Delta \theta}{h} = \frac{\Delta \theta}{h} + \frac{\Delta \theta}{h} \\
ABC & \ldots m_1 \beta_1 - m_2 \alpha_{ba} - m_3 \alpha_{bc} + m_4 \beta_2 + H_d \delta_1 + H_f \delta_2 = 0 \\
ABE & \ldots m_1 \beta_1 + m_2 \alpha_{ba} + (m_3 - m_2) \alpha_{be} - H_d \delta_1 = \frac{\Delta \theta}{h} \\
BCF & \ldots m_1 \alpha_{cb} + m_4 \alpha_{cf} - m_3 \beta_2 + m_4 \frac{\Delta \theta}{h} - m_3 \frac{\Delta \theta}{h} + H_f \delta_2 + H_f \frac{\Delta \theta}{h} = \frac{\Delta \theta}{h} - \frac{\Delta \theta}{h} \\
\text{S. c.} & \ldots m_1 + (m_3 - m_2) = m_4
\end{align*}
\]

\[(153)\]

in which \(\Delta \theta = \alpha_\theta TL\), \(\alpha_\theta\) is the coefficient of thermal expansion, \(T\) is the change of temperature, and other symbols are as defined previously.
Three-span Symmetrical Frames (Fig. 120).—Similarly, thermal stresses in three-span symmetrical frames are calculated from the following equations of equilibrium:

\[
\begin{align*}
\Delta A B C & : m_1 \alpha_{da} + m_2 \alpha_{de} - m_2 \beta_i - m_2 \frac{\Delta t}{h} + m_1 \frac{\Delta t}{h} + H_e \delta_i + H_h \frac{\Delta t}{h} = \Delta \theta + \frac{\Delta \theta}{h} \\
A B C & : -m_2 \beta_i + m_2 \alpha_{da} + m_3 \alpha_{be} + m_3 \beta_2 - H_e \delta_i - (H_e + H_h) \gamma_2 = 0 \\
A B F & : m_2 \alpha_{ba} + (m_2 - m_3) \alpha_{bf} - m_2 \beta_i - H_e \delta_i = \frac{\Delta \theta}{h} \\
B & : -m_3 \alpha_{da} + (H_e + H_h) \frac{\Delta t}{h} = \frac{\Delta \theta}{h} - \Delta b
\end{align*}
\]

\[ (154) \]

**Example 43.**—As a numerical example, consider the frame shown in Fig. 121 subjected to change of temperature. Assuming that \( T = 50 \) deg. F., \( \alpha = 0.000006 \), and \( E = 4,000,000 \) lb. per square inch (576 \( \times 10^6 \) lb. per square foot), the values of \( \Delta t \) are calculated as follows:

\[
\begin{align*}
EI\Delta t_1 &= \pm 0.000006 \times 50 \times 40 \times 576,000 \times 0.666 = \pm 4607.5. \\
EI\Delta t_2 &= \pm 0.000006 \times 50 \times 60 \times 576,000 \times 0.666 = \pm 6911.5.
\end{align*}
\]

The elastic constants (see Example 28) are:

Span 1: \( EIa = 15.55 \); \( EI\beta = 7.775 \); \( EI\Delta = 139.92 \); \( EI\Delta h = 2240. \)

Span 2: \( EIa = 21.54 \); \( EI\beta = 10.77 \); \( EI\Delta = 258.5 \); \( EI\Delta h = 5514.53. \)

Substituting these values in equations (153), we obtain (154a), from which

\[
m_1 = 10,409 \text{ ft.-lb.}, \quad m_2 = 11,944 \text{ ft.-lb.}, \quad m_3 = 10,243 \text{ ft.-lb.}, \quad m_4 = 8709 \text{ ft.-lb.}
\]
CONTINUOUS FRAMES WITH CURVED MEMBERS

\[
\begin{align*}
\text{DAB} & : 1555 m_1 + \frac{16}{3} m_2 - 7775 m_2 + \frac{139.92}{16} m_2 - \frac{139.92}{16} m_2 + \frac{2240}{16} m_2 = \frac{46075}{16} + \frac{\Delta b}{h} EI \\
\text{ABC} & : 7775 m_1 + 1555 m_2 - 2154 m_3 + 1077 m_4 + \frac{139.92}{16} m_3 + \frac{2585}{16} m_4 = 0 \\
\text{ABE} & : -7775 m_1 + 1555 m_2 + \frac{16}{3} (m_2 - m_2) - \frac{139.92}{16} m_1 = \frac{\Delta b}{h} EI \\
\text{BCF} & : 2154 m_3 + \frac{16}{3} m_3 - 1077 m_3 + \frac{2585}{16} m_4 - \frac{2585}{16} m_4 + \frac{5514.53}{16} m_4 = \frac{6918.5}{16} - \frac{\Delta b}{h} EI \\
\text{S. C.} & : m_1 - m_2 + m_3 - m_4 = 0
\end{align*}
\]

Equations (154a)

The thermal stresses in the three-span frame shown in Fig. 122 are calculated from equations (154). The values of $\Delta t$ and the elastic constants are:

Span 1:

\[EI\Delta t_1 = 46075; \quad EI\alpha = 15.55; \quad EI\beta = 7.775; \quad EI\Delta^m = 139.92; \quad EI\Delta^h = 2240.\]

Span 2:

\[EI\alpha = 21.54; \quad EI\beta = 10.77; \quad EI\Delta^m = 258.5; \quad EI\Delta^h = 5514.53; \quad EI\Delta t_2 = 6918.5\]

and the equations of equilibrium are as in (155a), from which $m_1 = 16,100$ ft.-lb., $m_2 = 21,050$ ft.-lb., and $m_3 = 11,110$ ft.-lb.

**Fig. 121.**

**Fig. 122.**

**Fig. 123.**

**Example 44.**—The thermal stresses in the frame in Fig. 123 are calculated from equations (154). The elastic constants are:

- **Beam:** $E\alpha = 23.60; \quad E\beta = 16.92; \quad E\Delta^m = E\gamma = 596.74; \quad E\Delta^h = 8375.9.$
- **Column:** $E\alpha = 3.06.$
Assuming that $T = 50$ deg. F., $\alpha = 0.000006$, and $E = 4,000,000$ lb. per square inch,

$$EI\Delta t = 0.000006 \times 50 \times 100 \times 576,000 = 17,300,$$

and the equations of equilibrium are as in (155b), from which

$$\begin{align*}
E A B \ldots & 15.55 \frac{m_1}{13} + \frac{m_2}{16} + \frac{m_3}{16} - 7.775 m_2 + \frac{159.93}{16} m_2 - \frac{159.93}{16} m_3 + \frac{259.3}{16} (m_1 + m_2 - m_3) = 0 \\
A B C \ldots & -7.775 m_1 + 15.55 m_2 + 21.54 m_3 + 10.77 m_3 - \frac{159.93}{16} m_3 - \frac{259.3}{16} (m_1 + m_2 - m_3) = 0 \\
B \ldots & -\frac{259.3}{16} m_3 + \frac{55.14}{16} \left( m_1 + m_2 - m_3 \right) = \frac{6.91.5}{2 \times 16} - \frac{Ab}{h} EI \\
E A B \ldots & 23.60 m_1 + 3.06 m_2 + \frac{596.74}{30} m_1 + \frac{596.74}{30} m_2 + \frac{596.74}{30} m_3 + \frac{8375.9}{30} m_1 = \frac{17.300}{30} + \frac{Ab}{h} E \\
A B C \ldots & -16.92 m_1 + 23.60 m_2 + 23.60 m_3 + 16.92 m_3 - \frac{596.74}{30} m_1 - \frac{596.74}{30} (m_1 + m_3 - m_2) = 0 \\
A B F \ldots & 23.60 m_2 + 3.06 (m_3 - m_2) - 16.92 m_1 - \frac{596.74}{30} m_1 = \frac{Ab}{h} E \\
B \ldots & -\frac{596.74}{30} m_3 + \frac{8375.9}{2 \times 30} \left( m_1 + m_3 - m_2 \right) = \frac{17.300}{2 \times 30} - \frac{Ab}{h} E
\end{align*}$$

$$m_1 = 39,091 \text{ ft.-lb.}, \quad m_2 = 68,660 \text{ ft.-lb.}, \quad \text{and } m_3 = 37,309 \text{ ft.-lb.}$$
CHAPTER IV

VIERENDEEL TRUSSES

In this chapter Vierendeel trusses are analysed, that is trusses composed of rectangular or trapezoidal panels without diagonal members, with the end connections of all members rigid, and designed to resist bending moments. Deformations due to axial forces are generally neglected, except for a separate investigation of their magnitude in a numerical example.

In the case of symmetrical trusses, with loads only at the joints, bending moments and shearing forces on the upper and lower chords are equal, and the points of contraflexure of vertical members are at mid-height. The equations of equilibrium are therefore set out for one chord only. In the case of unsymmetrical trusses the points of contraflexure of vertical members are not at mid-height and, generally, it is necessary to set out equations of equilibrium for the joints at the upper and lower chords. This, however, may be avoided by assuming that the bending moments on each corresponding upper and lower member are in the ratio of the stiffness of the members. This is illustrated by a numerical example. Vierendeel trusses with unsymmetrical or inclined chords, viaduct trestles, trusses with solid side or central portions, and continuous Vierendeel trusses are also discussed, and numerical examples illustrate the procedure. When bending moments are calculated from equations of equilibrium, all shearing and axial forces are calculated from conditions of statics.

Two-bay Vierendeel Trusses.—Assume that joints A, B, C, and E (Fig. 124)

[Diagram showing a Vierendeel truss with labeled parts]

are at their final positions of A', B', C', and E', $\lambda$ being the horizontal sway and $\Delta$ the vertical deflection of joint B. The deformed shape of the truss and the angular gaps are shown in Fig. 124. Statically-indeterminate bending moments $m_1, m_2, m_3, m_4$, and deflections $\lambda$ and $\Delta$, are calculated from four conditions of equilibrium of angular deformations and from two conditions of shear as in equation (156). The third equation is set out for members between joints ABE, but it may be replaced by a similar equation for members between joints EBC.
If all members are prismatic, equations (156) reduce to (157), and if all members have the same cross section and length the solution of (157) may be reduced to (158).

**Example 45.** For the truss in Fig. 124, and from equations (157), we get
VIERENDEEL TRUSSES

DAB \ldots \frac{10}{3} m_1 + \frac{8}{3} m_1 - \frac{6}{7} m_1 - \frac{10}{6} m_2 = \frac{A}{L_1} E_1 - \frac{\lambda}{n} E_1

ABC \ldots \frac{10}{3} m_2 + \frac{10}{3} m_2 - \frac{10}{6} m_1 - \frac{10}{6} m_4 = \frac{A}{L_1} E_1 + \frac{A}{L_2} E_1

ABE \ldots \frac{10}{3} m_2 + (m_2 - m_3) \frac{8}{3} - (m_2 - m_3) \frac{8}{3} - \frac{10}{6} m_1 = \frac{A}{L_1} E_1 - \frac{\lambda}{n} E_1

BCF \ldots \frac{10}{3} m_4 + \frac{8}{3} m_4 - \frac{8}{3} m_4 - \frac{10}{6} m_3 = \frac{A}{L_2} E_1 + \frac{\lambda}{n} E_1

\text{So.} \ldots \frac{m_1 + m_2}{10} + \frac{m_3 + m_4}{14} = \frac{1}{2} \times 12,000

\text{So.} \ldots m_1 + m_2 - m_3 = m_4

(158a), from which \( m_1 = 17,230 \text{ ft} \cdot \text{lb} \), \( m_2 = 17,770 \text{ ft} \cdot \text{lb} \), \( m_3 = 20,340 \text{ ft} \cdot \text{lb} \), and \( m_4 = 14,660 \text{ ft} \cdot \text{lb} \). Fig. 125 shows the distribution of these bending moments, the shearing forces, and the axial forces.

![Fig. 126.](image)

The two-bay truss with columns (Fig. 126) may be analysed in a similar way. For symmetrical loading the equations of equilibrium are as in (159), in which \( W_{ed} \) is the total external load along the line EB.

\[
\begin{align*}
\text{DAB} \ldots & \quad m_1 \alpha_{ab} + m_1 \alpha_{ad} - m_2 \beta_{ab} - (m_3 - m_5) \beta_{ad} = \frac{A}{L} + \theta_{ab} \\
\text{ABC} \ldots & \quad m_2 \alpha_{ba} + m_2 \alpha_{bc} - m_1 \beta_{ba} - m_1 \beta_{bc} = 2 \frac{A}{L} - \theta_{ba} - \theta_{bc} \\
\text{ADE} \ldots & \quad m_3 \alpha_{de} + (m_3 - m_4) \alpha_{de} - m_4 \beta_{de} - m_4 \beta_{bc} = \frac{A}{L} + \theta_{de} \\
\text{GDE} \ldots & \quad m_3 \alpha_{de} + m_5 \alpha_{de} - m_4 \beta_{de} = \frac{A}{L} + \theta_{de} \\
\text{DEF} \ldots & \quad m_3 \beta_{ed} + m_4 \beta_{ed} + m_4 \beta_{ef} - m_3 \beta_{ef} = 2 \frac{A}{L} - \theta_{ed} - \theta_{ef} \\
\text{So.} \ldots & \quad \frac{m_1 + m_2}{10} + \frac{m_3 + m_4}{14} = W_{ed} 
\end{align*}
\]

(159)

**Three-bay Trusses.**—Three-bay trusses (Fig. 127) require, in general, the setting out of nine equations of equilibrium to calculate six statically-indeterminate

A.S.—H
bending moments, one sway deflection, and two vertical deflections $\Delta_1$ and $\Delta_2$. These are shown in (160). It should be noted that $\Delta_1$, $\Delta_2$, and $\lambda$ can be eliminated, leaving six equations for calculating six unknown bending moments.

\[
\begin{align*}
\text{EAB} & : m_1 \alpha_{ab} + m_1 \alpha_{ae} - m_1 \beta_{ae} - m_2 \beta_{ab} = \frac{\Delta_1}{L_1} - \frac{\lambda}{h} \\
\text{ABC} & : m_2 \alpha_{ba} + m_3 \alpha_{bc} - m_1 \beta_{ba} - m_4 \beta_{bc} = \frac{\Delta_1}{L_1} + \frac{\Delta_1-\Delta_2}{L_2} \\
\text{ABF} & : m_2 \alpha_{ba} + (m_2-m_3) \alpha_{bf} - (m_2-m_3) \beta_{bf} - m_1 \beta_{ba} = \frac{\Delta_1}{L_1} - \frac{\lambda}{h} \\
\text{BCD} & : m_5 \alpha_{cd} - m_4 \alpha_{cb} + m_5 \beta_{cd} - m_6 \beta_{bc} = \frac{\Delta_2}{L_3} - \frac{\Delta_1-\Delta_2}{L_2} \\
\text{GCD} & : m_5 \alpha_{cd} + (m_4+m_3) \alpha_{cg} - (m_4+m_3) \beta_{cg} - m_6 \beta_{cd} = \frac{\Delta_2}{L_3} + \frac{\lambda}{h} \\
\text{CDK} & : m_6 \alpha_{dc} + m_6 \alpha_{dk} - m_5 \beta_{dc} - m_6 \beta_{dk} = \frac{\Delta_2}{L_3} + \frac{\lambda}{h} \\
\text{S.c.BF} & : \frac{m_1+m_2}{L_1} + \frac{m_2+m_3}{L_2} = \frac{W_1}{2} \\
\text{S.c.CG} & : \frac{m_5+m_6}{L_3} - \frac{m_3+m_4}{L_2} = \frac{W_2}{2} \\
\text{S.c.AD} & : \frac{m_1+m_2}{h} + \frac{m_2+m_3}{h} = \frac{m_4+m_5}{h} + \frac{m_6}{h} 
\end{align*}
\]

(160)

For a three-bay Vierendeel truss with symmetrical loading (Fig. 128), equations (160) reduce to (161). If the members have equal moments of inertia, equations (161) reduce to (162), from which (163) is derived.

\[
\begin{align*}
\text{EAB} & : m_1 \alpha_{ab} + m_1 \alpha_{ae} - m_1 \beta_{ae} - m_2 \beta_{ab} = \frac{A}{L} \\
\text{ABC} & : m_2 \alpha_{ba} + m_3 \alpha_{bc} + m_3 \beta_{bc} - m_1 \beta_{ba} = \frac{A}{L} \\
\text{ABF} & : m_2 \alpha_{ba} + (m_2-m_3) \alpha_{bf} - (m_2-m_3) \beta_{bf} - m_1 \beta_{ba} = \frac{A}{L} \\
\text{S.c.} & : \frac{m_1+m_2}{L} = \frac{W}{2} 
\end{align*}
\]

(161)
EXAMPLE 46.—For the three-bay Vierendeel truss shown in Fig. 129 and from equations (162) we derive (162a), from which \( m_1 = 58,774 \text{ ft.-lb.}, \)

\( m_2 = 61,222 \text{ ft.-lb.}, \)

\( m_3 = 12,244 \text{ ft.-lb.}, \) and \( \frac{EIA}{L} = 200,830. \) By assuming the

\[
\begin{align*}
EAB... \ & m_1 \frac{L}{3L} + m_1 \frac{h}{3L} - m_1 \frac{h}{6L} - m_2 \frac{L}{6L} = \frac{\Delta}{L} E \\
ABC... \ & m_2 \frac{L}{3L} + m_3 \frac{L}{6L} + m_3 \frac{L}{6L} - m_1 \frac{L}{6L} = \frac{\Delta}{L} E \\
ABF... \ & m_2 \frac{L}{3L} + (m_2 - m_1) \frac{h}{3L} - (m_2 - m_3) \frac{h}{6L} - m_1 \frac{L}{6L} = \frac{\Delta}{L} E \\
S.C. ... \ & m_1 + m_2 = \frac{W}{2} \\
EAB... \ & \frac{12}{3} \cdot m_1 + \frac{9}{3} \cdot m_1 - \frac{9}{6} \cdot m_1 - \frac{12}{6} \cdot m_2 = \frac{\Delta}{L} EI \\
ABC... \ & \frac{12}{3} \cdot m_2 + \frac{12}{3} \cdot m_3 - \frac{12}{6} \cdot m_2 - \frac{12}{6} \cdot m_1 = \frac{\Delta}{L} EI \\
ABF... \ & \frac{12}{3} \cdot m_2 + (m_2 - m_3) \frac{9}{3} - (m_2 - m_1) \frac{9}{6} - \frac{12}{6} \cdot m_1 = \frac{\Delta}{L} EI \\
S.C. ... \ & m_1 + m_2 = \frac{1}{2} \times 20,000 \times 12'0
\end{align*}
\]
\[ \begin{align*} 
    m_1 &= \frac{3Wl^2}{2} + \frac{3L + 2h}{18l^2 + 12hl + h^2} \\
    m_2 &= \frac{Wl}{2} \cdot \frac{(3L + h)^2}{18l^2 + 12hl + h^2} \\
    m_3 &= \frac{Whl}{2} \cdot \frac{3L + h}{18l^2 + 12hl + h^2} 
\end{align*} \] \hspace{1cm} (163)

cross-sectional dimensions of all members to be 2 in. by 12 in. and

\[ E = 4,000,000 \text{ lb. per square inch.} \]

\[ \Delta = \frac{200,830 \times 12^4}{4,000,000 \times 926} = 0.1124 \text{ in.} \]

**Multiple-bay Vierendeel Trusses.**—Multiple-bay Vierendeel trusses are analysed by the method used for three-bay trusses. In four-bay trusses with a uniformly-distributed load placed symmetrically on the top and bottom chords (Fig. 130) the angular gaps at each joint are influenced by the translations \( \Delta_1 \) and \( \Delta_2 \) of joints B and C and also by angular deformations \( \theta \) due to the external system of loading between the joints. Due to geometrical symmetry of the truss and the load there is no sway, and the equations of equilibrium are as in (164), from which (165) are obtained.

\[ \begin{align*} 
    FA \ldots & m_1 \frac{L}{3} + m_1 \frac{h}{3} + m_r \frac{h}{6} - m_2 \frac{L}{6} = \frac{Wl^3}{24} + \frac{\Delta l E I}{L} \\
    AB \ldots & m_2 \frac{L}{3} + \left[m_2 + m_3\right] \frac{h}{3} - \left[m_2 + m_3\right] \frac{h}{6} - m_1 \frac{L}{6} = \frac{\Delta l E I}{L} - \frac{Wl^3}{24} \\
    GBC \ldots & m_3 \frac{L}{3} + \left[m_3 + m_4\right] \frac{h}{3} - \left[m_3 + m_4\right] \frac{h}{6} - m_1 \frac{L}{6} = \frac{\Delta 2 - \Delta 1 E I}{L} + \frac{Wl^3}{24} \\
    BCD \ldots & m_4 \frac{L}{3} - m_3 \frac{L}{6} = \frac{\Delta 2 - \Delta 1 E I}{L} - \frac{Wl^3}{24} \\
    S_{cb}B6 \ldots & \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = \frac{Wl}{L} \\
    S_{cb}CH \ldots & \frac{m_3 + m_4}{L} = \frac{1}{2} \frac{Wl}{L} \hspace{1cm} (164) 
\end{align*} \]

For a four-bay symmetrical truss with a symmetrical system of loading (Fig. 131), equations (164) require only slight alteration as in (166), from which (167) are obtained.
\[ m_1 = \omega L^3 \frac{33L + 16h}{36L^2 + 18hL + h^2} \]
\[ m_2 = \omega L^2 \frac{22L^2 + 22hL + 3h^2}{36L^2 + 18hL + h^2} \]
\[ m_3 = \omega L^1 \frac{33L + 16h}{6L + h} \]
\[ m_4 = \omega L^1 \frac{33L + 16h}{6L + h} - m_3 \]

\[ \begin{align*}
FAB... & \quad m_1 \frac{L_1}{3I_1} + m_i \frac{h}{3I_3} - m_i \frac{h}{3I_3} - m_2 \frac{L_1}{6I_1} = \frac{A1}{E} \\
ABG... & \quad m_2 \frac{L_1}{3I_1} + (m_2 + m_3) \frac{h}{3I_4} - (m_2 + m_3) \frac{h}{6I_4} - \frac{L_1}{6I_1} \frac{A1}{E} \\
GBC... & \quad m_3 \frac{L_1}{3I_1} + (m_2 + m_3) \frac{h}{3I_4} - (m_2 + m_3) \frac{h}{6I_4} - m_4 \frac{L_2}{6I_2} = \frac{A2-A1}{L_2} \frac{E}{L_2} \\
BCD... & \quad m_4 \frac{L_2}{3I_2} - m_3 \frac{L_2}{6I_2} = \frac{A2-A1}{L_2} \frac{E}{L_2} \\
SC.B6... & \quad \frac{m_1 + m_2}{L_1} - \frac{m_3 + m_4}{L_2} = \frac{i}{2} W_1 \\
SC.CH... & \quad \frac{m_3 + m_4}{L_2} = \frac{i}{4} W_2
\end{align*} \]

\[ \begin{align*}
m_i &= \frac{3L^2}{2(36L^2 + 18hL + h^2)} \left[ 3(2I_1 + hL_i + (3L + 2h) W_2) \right] \\
m_2 &= \frac{1}{L_1} \cdot (2W_1 + W_2) - m_i \\
m_3 &= \frac{3L + h}{h} \cdot (m_i - m_2) \\
m_4 &= \frac{i}{4} W_2 L - m_3
\end{align*} \]

Example 47.—For the truss shown in Fig. 132, from equations (164) we get (167a), from which \( m_1 = 62,000 \) ft.-lb., \( m_2 = 58,000 \) ft.-lb., \( m_3 = 20,220 \) ft.-lb., and \( m_4 = 39,780 \) ft.-lb.
The relation between the external bending moments and the internal forces in the trusses may be obtained by cutting the structure at any point and setting out a static equation of equilibrium (168):

$$m_0 = t_v \cdot h + m_u + m_i \quad \ldots \quad \ldots \quad \ldots \quad (168)$$

in which $m_0$ is the bending moment due to external forces at any point, $t_v$ is direct force in the truss, and $m_u$ and $m_i$ are bending moments in the upper and lower chord respectively at the imaginary cut. For the truss shown in Figs. 132 and 133 external bending moments at B and C are 240,000 ft.-lb. and 360,000 ft.-lb. respectively (see Example 47), and from equation (167):

For the section at C, $31,160 \times 9 + 39,780 \times 2 = 360,000$ ft.-lb.

For the section at B, $13,780 \times 9 + 57,980 \times 2 = 240,000$ ft.-lb., or

$$31,160 \times 9 - 20,220 \times 2 = 240,000 \text{ ft.-lb.}$$
Trusses Loaded Unsymmetrically.—For this case it is generally necessary to derive equations of equilibrium for all joints in the top or bottom chord. The number of equations, however, may be halved if for the actual system of loading for the purpose of the analysis are substituted two equivalent systems of loading, one symmetrical and the other unsymmetrical. The final result is obtained by combining the results of the two equivalent cases.

For a four-bay truss (Fig. 134) with a concentrated load at B, the two equivalent systems of loading are shown in Figs. 135 and 136. For symmetrical loading (Fig. 135) the equations of equilibrium are as in (169).

For unsymmetrical loading the deflected shape of the truss and angular gaps

\[ m_1 \alpha_{ab} + m_1 \alpha_{af} - m_1 \beta_{af} - m_2 \beta_{ab} = \frac{AP}{L} \]
\[ m_2 \alpha_{ba} + m_3 \alpha_{bc} - m_1 \beta_{ba} + m_3 \beta_{bc} = \frac{A2 - AP}{L} \]
\[ m_2 \alpha_{ba} + (m_2 - m_3) \alpha_{bg} - (m_2 - m_3) \beta_{bg} = \frac{A1}{L} \]
\[ m_3 \alpha_{cb} + m_3 \beta_{cb} = \frac{A2 - A1}{L} \]
\[ \frac{m_1 + m_2}{L} = \frac{1}{2} W \]

\[ \text{FAB...} m_1 + \frac{9}{3} m_1 - \frac{9}{3} m_1 - \frac{12}{6} m_2 = \frac{A1}{L} EI \]
\[ \text{ABC...} m_2 + \frac{12}{3} m_2 + \frac{12}{6} m_3 - \frac{12}{6} m_1 = \frac{A1}{L} EI - \frac{A2 - A1}{L} EI \]
\[ \text{ABG...} \frac{12}{3} m_2 + \frac{9}{3} (m_2 - m_3) - \frac{9}{6} (m_2 - m_3) - \frac{12}{6} m_2 = \frac{A1}{L} EI \]
\[ \text{BC...} \frac{12}{3} m_3 + \frac{12}{6} m_3 = \frac{A2 - A1}{L} EI \]
\[ S.c. \ldots m_1 + m_2 = \frac{1}{2} \times 10,000 \times 12^0 \]
are shown in Fig. 136, and the equations of equilibrium are as in (170), in which $\Delta$ is the deflection at B and D, and $\lambda$ is the horizontal sway of the truss.

\[
\begin{align*}
FAB... & \quad m_1 \alpha_{ab} + m_1 \alpha_{af} - m_1 \beta_{af} - m_2 \beta_{ab} = \frac{\Delta}{L} - \frac{\lambda}{h} \\
ABC... & \quad m_2 \alpha_{ba} + m_3 \alpha_{bc} - m_1 \beta_{ba} - m_3 \beta_{bc} = \frac{2\Delta}{L} \\
GBC... & \quad m_3 \alpha_{bc} + (m_3 - m_2) \alpha_{bg} - (m_3 - m_2) \beta_{bg} - m_4 \beta_{bc} = \frac{\Delta}{L} + \frac{\lambda}{h} \\
BCH... & \quad m_4 \alpha_{cb} + 2m_4 \alpha_{ch} - 2m_4 \beta_{ch} - m_3 \beta_{cb} = \frac{\Delta}{L} + \frac{\lambda}{h} \\
S \cdot B6... & \quad \frac{m_1 + m_2}{L} + \frac{m_3 + m_4}{L} = \frac{1}{A} W \\
S \cdot AC... & \quad \frac{m_1}{h} - \frac{m_3 - m_2}{h} - \frac{m_4}{h} = 0
\end{align*}
\]

\( \text{(170)} \)

**Example 48.**—As a numerical example, consider a four-bay truss (Fig. 137) with a concentrated load $W = 10,000$ lb. at B. For a symmetrical load of 5000 lb. at B and D, from equations (169) we derive (169a), from which

$m_1 = 29,663$ ft.-lb., $m_2 = 30,336$ ft.-lb., and $m_3 = 3370$ ft.-lb.

\[
\begin{align*}
FAB... & \quad \frac{12}{3} m_1 + \frac{9}{3} m_2 - \frac{9}{6} m_1 - \frac{12}{6} m_2 = \frac{2A}{L} E I - \frac{\Delta}{h} E I \\
ABC... & \quad \frac{12}{3} m_2 + \frac{12}{3} m_3 - \frac{12}{6} m_1 - \frac{12}{6} m_4 = \frac{2\Delta}{L} E I \\
GBC... & \quad \frac{12}{3} m_3 + \frac{9}{3} (m_3 - m_2) - \frac{9}{6} (m_3 - m_2) - \frac{12}{6} m_4 = \frac{\Delta}{L} E I + \frac{\lambda}{h} E I \\
BCH... & \quad \frac{12}{3} m_4 + \frac{2\times 9}{3} m_4 - \frac{2\times 9}{6} m_4 - \frac{12}{6} m_3 = \frac{A}{L} E I + \frac{\lambda}{h} E I \\
S \cdot B6... & \quad m_1 + m_2 + m_3 + m_4 = \frac{I}{10000 \times 2} \times 12' 0 \\
S \cdot AC... & \quad m_1 + m_2 - m_3 - m_4 = 0
\end{align*}
\]

\( \text{(170a)} \)
For an unsymmetrical load of 5000 lb. at B and D, from equations (170) we derive
(170a), from which \( m_1 = 13,210 \) ft.-lb., \( m_2 = 16,789 \) ft.-lb.,
\( m_3 = 17,891 \) ft.-lb., and \( m_4 = 12,108 \) ft.-lb.
By adding the results of these two cases,
\[
\begin{align*}
m_1 &= 21,437 \text{ ft.-lb.}, \\
m_2 &= 23,562 \text{ ft.-lb.}, \\
m_3 &= 10,630 \text{ ft.-lb.}, \\
m_4 &= 4369 \text{ ft.-lb.}, \\
m_5 &= 77,395 \text{ ft.-lb.}, \\
m_6 &= 72,605 \text{ ft.-lb.}, \\
m_7 &= 6773 \text{ ft.-lb.}, \\
m_8 &= 8226 \text{ ft.-lb.}
\end{align*}
\]
The final bending moments are shown in Fig. 138.

**Fig. 138.**

**Fig. 139.**

Example 49.—The five-bay truss shown in Fig. 139 may be analysed from
equations (171). The elastic constants are calculated by the method of summation
(Fig. 140 and Table XII).

\[
\begin{align*}
g a b \ldots m_1 \cdot \alpha_{a b} + m_1 \cdot \alpha_{c l} - m_1 \cdot \beta_{c l} - m_2 \cdot \beta_{b} &= \frac{\Delta l}{E} \\
a b h \ldots m_2 \cdot \alpha_{b} + (m_2 + m_3) \cdot \alpha_{c 2} - m_1 \cdot \beta_{b} - (m_2 + m_3) \cdot \beta_{c 2} &= \frac{\Delta l}{E} \\
h b c \ldots m_3 \cdot \alpha_{b} + (m_3 + m_4) \cdot \alpha_{c 2} - (m_2 + m_3) \cdot \beta_{c 2} - m_4 \cdot \beta_{b} &= \frac{\Delta 2 - \Delta 1}{E} \\
b c d \ldots m_4 \cdot \alpha_{b} + m_5 \cdot \alpha_{b} - m_3 \cdot \beta_{b} + m_5 \cdot \beta_{b} &= \frac{\Delta 2 - \Delta 1}{E} \\
b c k \ldots m_4 \cdot \alpha_{b} + (m_4 + m_5) \cdot \alpha_{c 2} - (m_4 + m_5) \cdot \beta_{c 2} - m_5 \cdot \beta_{b} &= \frac{\Delta 2 - \Delta 1}{E} \\
5 c b h \ldots \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} &= \frac{1}{2} \cdot W_i \\
5 c b k \ldots \frac{m_3 + m_4}{L} &= \frac{1}{2} \cdot W_i
\end{align*}
\]
\[ E_\varepsilon = 58.92 \times 1.5 = 88.38 \text{ ft.}; \quad x = \frac{272.07}{58.92} = 4.62 \text{ ft.}; \quad \varepsilon = 7.38 \text{ ft.}; \]

\[ E\alpha_b = 88.38 \times \frac{7.38}{12} = 54.35; \quad E\beta_b = 88.38 - 54.35 = 34.03. \]

The elastic constants for the columns are—External columns: \( I = 0.282 \text{ ft.}^4 \);

\[ E_\varepsilon = \frac{7}{2 \times 0.282} = 12.4; \quad E\alpha_{e_1} = \frac{8}{3} \times 12.4 = 8.27; \quad E\beta_{e_1} = 4.13. \]

Intermediate columns: \( I = 0.0834 \text{ ft.}^4 \); \[ E_\varepsilon = \frac{7}{2 \times 0.0834} = 42; \]

\[ E\alpha_{e_1} = \frac{8}{3} \times 42 = 28; \quad E\beta_{e_1} = 14. \]

By substituting these values in (I71), we derive (I71a), from which

\[ m_1 = 96.367 \text{ ft.-lb.}, \quad m_2 = 83.632 \text{ ft.-lb.}, \quad m_3 = 53.658 \text{ ft.-lb.}, \]

\[ m_4 = 66.341 \text{ ft.-lb.}, \quad \text{and} \quad m_5 = 9068 \text{ ft.-lb.}. \]

\[ \begin{align*}
G A B & \ldots 54.35 m_1 + 8.27 m_2 - 4.13 m_1 - 34.03 m_2 &= \frac{\Delta_1}{E}
\\
A B H & \ldots 54.35 m_2 + 28.0 \left(m_2 + m_3 \right) - 14.0 \left(m_2 + m_3 \right) - 34.03 m_3 &= \frac{\Delta_2}{E}
\\
H B C & \ldots 54.35 m_3 + 28.0 \left(m_2 + m_3 \right) - 14.0 \left(m_2 + m_3 \right) - 34.03 m_4 &= \frac{\Delta_3 - \Delta_1}{L}
\\
B C D & \ldots 54.35 m_4 + 54.35 m_5 - 34.03 m_3 - 34.03 m_5 &= \frac{\Delta_2 - \Delta_1}{L}
\\
B C K & \ldots 54.35 m_4 + 28.0 \left(m_4 - m_5 \right) - 14.0 \left(m_4 - m_5 \right) - 34.03 m_5 &= \frac{\Delta_3 - \Delta_2}{L}
\\
S c B H & \ldots m_1 + m_2 - m_3 - m_4 = \frac{1}{2} \times 10,000 \times 12
\\
S c C K & \ldots m_3 + m_4 = \frac{1}{2} \times 20,000 \times 12
\end{align*} \]
Bending moments, shearing forces, and axial forces are shown in Fig. 141. The deflections $\Delta_1$ and $\Delta_2$, if secondary stresses are neglected, are

$$\Delta_1 = \frac{38,262.6}{E} \times 12 = 0.648 \text{ in. and } \Delta_2 = \frac{69,239.64}{E} \times 12 = 1.47 \text{ in.},$$

where $E = 4,000,000 \text{ lb. per square inch.}$

**Influence of Direct Stresses.**—The influence of direct stresses on bending moments is usually small and is generally neglected. These, however, if required, may be taken into account when the equations of equilibrium are derived. For the truss in Fig. 139 the extension or shortening of members due to axial forces is shown in Fig. 142 and is denoted by $\lambda$. The equations of equilibrium are as shown in (172). The values of $\lambda_1$, $\lambda_2$, and $\lambda_3$ are given in Table XIII.

$$\lambda_1 = \lambda' + \lambda'' + \frac{1}{2}\lambda''' = \frac{1,478,070}{E}, \quad \lambda_2 = \lambda'' + \frac{1}{2}\lambda''' = \frac{1,178,490}{E},$$

and

$$\lambda_3 = \frac{1}{2}\lambda''' = \frac{452,230}{E}.$$
TABLE XIII.

<table>
<thead>
<tr>
<th>Pt.</th>
<th>AB</th>
<th>BC</th>
<th>CD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_{1000}$</td>
<td>$P_{1000}$</td>
<td>$P_{1000}$</td>
</tr>
<tr>
<td>b x d</td>
<td>$P_{E_A}$</td>
<td>$P_{E_A}$</td>
<td>$P_{E_A}$</td>
</tr>
<tr>
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<td>135.075</td>
<td>27.17</td>
<td>0.0245</td>
</tr>
<tr>
<td>2</td>
<td>100.785</td>
<td>27.17</td>
<td>0.0175</td>
</tr>
<tr>
<td>3</td>
<td>085.935</td>
<td>27.17</td>
<td>0.0175</td>
</tr>
<tr>
<td>4</td>
<td>076.070</td>
<td>27.17</td>
<td>0.0150</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>225/25</td>
<td>$-455/4$</td>
<td>$-455/4$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\lambda_1$</td>
<td>450.5</td>
<td>$\lambda_2$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
6AB... & m_1 + \alpha_1 + m_2, \alpha_1 = -m_1 + \beta_1 + m_2, \beta_1 = \frac{A_1}{L} - \frac{2\lambda_1}{h} \\
A\, BH... & m_2 + \alpha_2 + (m_2 + m_3) \alpha_3 = -m_2 + \beta_2 + (m_2 + m_3) \beta_2 = -\frac{A_2 + A_1}{L} - \frac{2\lambda_2}{h} \\
HBC... & m_3 + \alpha_2 + (m_2 + m_3) \alpha_3 = -m_3 + \beta_3 + (m_2 + m_3) \beta_2 = \frac{A_2 + A_1}{L} \\
BCD... & m_4 + \alpha_3 + m_5, \alpha_3 = -m_4 + \beta_3 + m_5, \beta_3 = \frac{A_2 + A_1}{L} - \frac{2\lambda_3}{h} \\
S.C.BH... & \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = \frac{1}{2} W_1 \\
S.C.CK... & \frac{m_3 + m_4}{L} = \frac{1}{2} \cdot W_2
\end{align*}
\] (172)

\[
\begin{align*}
6AB... & 54.35m_1 + 827m_1 - 413m_1 - 34.03m_2 = \frac{A_1}{L} - \frac{2 \times 1.4760.070}{80} \\
A\, BH... & 54.35m_2 + 28.0(m_2 + m_3) - 34.03m_1 - 14.0(m_2 + m_3) = \frac{A_1}{L} - \frac{2 \times 1.7784.90}{80} \\
HBC... & 54.35m_3 + 28.0(m_2 + m_3) - 14.0(m_2 + m_3) - 34.03m_4 = \frac{A_2 + A_1}{L} - \frac{2 \times 1.7784.90}{80} \\
BCD... & 54.35m_4 + 54.35m_5 - 34.03m_3 - 34.03m_5 = \frac{A_2 + A_1}{L} - \frac{2 \times 4.52250}{80} \\
S.C.BH... & m_1 + m_2 - m_3 - m_4 = 60,000 \\
S.C.CK... & m_3 + m_4 = 120,000
\end{align*}
\] (172a)

Substituting these values in (172), we derive (172a), from which

\[
\begin{align*}
m_1 &= 96,058 \text{ ft.-lb., } m_2 = 83,941 \text{ ft.-lb., } m_3 = 54,608 \text{ ft.-lb., } m_4 = 65,391 \text{ ft.-lb., and } m_5 = 10,043 \text{ ft.-lb.}
\end{align*}
\]

and, assuming that $E = 4,000,000$ lb. per square inch, $A_1 = 0.797$ in., and $A_2 = 1.442$ in.

Example 50.—Consider the truss shown in Fig. 143, which is supported elastically at K and N on a frame with a circular member. The bending moments on and deflections of the truss without the frame are as calculated for the truss in Example 49. By applying unit forces acting upwardly at K and N, and calculating the bending moments on and deflections of the truss due to them,
and by assuming that the frame is removed from (171) we derive (172b), from which \( m_1 = 3.2823 \text{ ft.-lb.}, m_2 = 2.7177 \text{ ft.-lb.}, m_3 = 1.6536 \text{ ft.-lb.}, \\

\[ m_4 = 4.3464 \text{ ft.-lb.}, m_5 = 0.5942 \text{ ft.-lb.}, \text{ and } \delta_{ku} = \frac{4I46.29}{E}, \]

in which \( \delta_{ku} \) is the deflection of the truss at K due to unit forces acting at K and N.

To calculate the bending moment at R and the deflection of the frame at K, unit forces acting downwards are applied to the frame at points K and N. The

**TABLE XIV.**

<table>
<thead>
<tr>
<th>Pt.</th>
<th>y ft</th>
<th>( m_x )</th>
<th>( m_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.85</td>
<td>2.10</td>
<td>389</td>
</tr>
<tr>
<td>2</td>
<td>5.08</td>
<td>6.55</td>
<td>3330</td>
</tr>
<tr>
<td>3</td>
<td>8.80</td>
<td>11.30</td>
<td>8810</td>
</tr>
<tr>
<td>4</td>
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<td>16.40</td>
<td>16040</td>
</tr>
<tr>
<td>5</td>
<td>11.20</td>
<td>21.75</td>
<td>24350</td>
</tr>
<tr>
<td>6</td>
<td>11.90</td>
<td>24.00</td>
<td>28600</td>
</tr>
<tr>
<td>Σ</td>
<td>—</td>
<td>8210</td>
<td>81539</td>
</tr>
</tbody>
</table>
elastic constants for this frame are the same as those calculated for Fig. 81, and the load functions are as in Table XIV. \( ds = 5.52 \text{ ft.} \)

\[ EI\theta_0 = 82.1 \times 5.52 = 453.29, \text{ and } EI\Delta_0 = 815.39 \times 5.52 \times 2 = 9003.86. \]

By substituting these values in (100),

\[
m_r \left( 22.1 + 11.05 + \frac{16}{3} + \frac{262.5}{16} + \frac{262.5}{16} + \frac{2510}{2 \times 16^2} \right) = 453.29 + \frac{9003.86}{2 \times 16},
\]

from which \( m_r = 9.6414 \text{ ft.-lb.} \); also \( \delta_{kr} = \frac{1300}{EI} \), in which \( \delta_{kr} \) is the deflection of the frame at point K due to unit forces applied at K and N. Assuming that the cross section of the frame is 21 in. by 12 in., then \( \delta_{kr} = \frac{1950}{E} \). The statically-indeterminate forces \( P \) acting at K and N can now be calculated from

\[ P(\delta_{kr} + \delta_{kr}) = \Delta_{kw},\]

in which \( \delta_{kw} \) is the deflection of the truss at K due to the applied load, assuming that the frame is removed; this has been calculated for Fig. 132 in Example 49. Substituting values in the foregoing, \( P \left( \frac{4146.290}{E} + \frac{1950}{E} \right) = \frac{69,239.640}{E} \), from which \( P = 11,360 \text{ lb.} \)

The bending moments, forces, and deflections can be calculated in terms of \( P \) as follows. The bending moments on the frame are

\[ m_r = 9.6414 \times 11,360 = 109,500 \text{ ft.-lb.} \quad m_{rp} = 80,960 \text{ ft.-lb.} \]

Bending moments on the truss are obtained by combining the bending moments due to the applied load when the frame is removed, and the bending moments on the truss due to \( P = 11,360 \text{ lb.} \) acting upwardly at K and N. The final bending moments, shearing forces, and axial forces in the truss and frame are shown in Fig. 144. The final deflection of the truss and frame at K is

\[ \Delta_{kw} = \Delta_{kr} = \frac{69,239,600 - 4146.29 \times 11,360}{E} = \frac{22,137,746}{E}. \]

Assuming that \( E = 4,000,000 \text{ lb. per square inch} \), then \( \Delta_k = 0.4614 \text{ in.} \).
EXAMPLE 51.—An open-web beam is sometimes made by cutting a rolled steel joist, displacing both parts, and welding all the joints, as shown in Fig. 145(a).

The distribution of internal stresses in these beams is complicated due to the high ratio of depth to length, particularly of the columns. In the following example an open-web beam is analysed as a Vierendeel truss, but, for the reason stated, some experimental data would be necessary to check the validity of the analysis for practical application.

For an open-web beam made of an 18 in. by 8 in. British standard beam section loaded as shown in Fig. 145(a), the equations of equilibrium are first set out in a general form (173). Fig. 145(b) shows the deflected shape of the truss

\[
\begin{align*}
\text{FAB} & : m_1 \alpha_{ab} + m_2 \alpha_{bf} - m_1 \beta_{af} - m_2 \beta_{ab} = \frac{\Delta y}{L} \\
\text{ABC} & : m_2 \alpha_{bc} + m_3 \alpha_{bc} + m_4 \beta_{bc} - m_2 \beta_{ba} = \frac{\Delta y - \Delta \delta}{L} \\
\text{ABG} & : m_2 \alpha_{ba} + (m_2 + m_3) \beta_{bg} - (m_2 + m_3) \beta_{bg} - m_2 \beta_{ba} = \frac{\Delta y}{L} \\
\text{BCD} & : m_4 \alpha_{cb} - m_5 \alpha_{cb} + m_6 \beta_{cd} - m_3 \beta_{cb} = \frac{\Delta y - \Delta \delta}{L} - \frac{\Delta \delta}{L} \\
\text{BCH} & : m_4 \alpha_{cb} + (m_4 + m_5) \alpha_{ch} - (m_4 + m_5) \beta_{ch} - m_5 \beta_{cb} = \frac{\Delta y - \Delta \delta}{L} \\
\text{CDE} & : m_6 \alpha_{bc} - m_7 \alpha_{de} + m_7 \beta_{de} - m_6 \beta_{cd} = \frac{\Delta y - \Delta \delta}{L} - \frac{\Delta \delta}{L} \\
\text{CDK} & : m_6 \alpha_{de} + (m_7 + m_8) \alpha_{dk} - (m_7 + m_8) \beta_{dk} - m_8 \beta_{dc} = \frac{\Delta y - \Delta \delta}{L} \\
\text{DE} & : m_8 \alpha_{dk} - m_7 \beta_{dk} = \frac{\Delta y - \Delta \delta}{L} \\
\text{Sc.BG} & : m_1 + m_2 - m_3 - m_4 = \frac{1}{2} W_1 L \\
\text{Sc.CH} & : m_3 + m_4 - m_5 - m_6 = \frac{1}{2} W_2 L \\
\text{Sc.DK} & : m_5 + m_6 - m_7 - m_8 = \frac{1}{2} W_3 L \\
\text{Sc.EL} & : m_7 + m_8 = \frac{1}{2} W_4 L
\end{align*}
\]
with all angular gaps marked, and the equations of equilibrium are as shown in (173a). For the truss shown in Fig. 145(a) the elastic constants are calculated by the method of summations. These are: For beams $E \alpha = 195, E \beta = 107$; for columns $E \alpha = 48.7, E \beta = 34$. By substituting these values equations (173) become (173a), from which

$$m_1 = 27,150 \text{ ft.-lb.}, m_2 = 25,350 \text{ ft.-lb.}, m_3 = 18,380 \text{ ft.-lb.},$$

$$m_4 = 19,120 \text{ ft.-lb.}, m_5 = 9890 \text{ ft.-lb.}, m_6 = 12,610 \text{ ft.-lb.},$$

$$m_7 = 3360 \text{ ft.-lb.}, \text{ and } m_8 = 4140 \text{ ft.-lb.}$$

$\text{FAB...} \begin{align*}
1950 m_1 + 48.7 m_2 - 340 m_3 - 1070 m_2 &= \frac{\Delta_1}{L} E \\
1950 m_2 - 1950 m_3 + 1070 m_4 - 1070 m_2 &= \frac{\Delta_2 - \Delta_1}{L} E \\
1950 m_4 + 1950 m_2 + 870 m_2 + m_3 - 340 (m_2 + m_3) - 1070 m_1 &= \frac{\Delta_1}{L} E \\
1950 m_4 - 1950 m_5 - 1070 m_5 - 1070 m_3 &= \frac{\Delta_2 - \Delta_1}{L} E \\
1950 m_4 + 340 m_5 + m_3 - 1070 m_3 &= \frac{\Delta_2 - \Delta_1}{L} E \\
1950 m_6 - 1950 m_7 + 1070 m_6 - 1070 m_3 &= \frac{\Delta_3 - \Delta_2}{L} E \\
1950 m_6 + 48.7 (m_6 + m_7) - 340 (m_6 + m_7) - 1070 m_5 &= \frac{\Delta_3 - \Delta_2}{L} E \\
1950 m_7 - 1070 m_7 &= \frac{\Delta_4}{L} E \\
\text{S.c.BG...} m_1 + m_2 - m_3 - m_4 &= \frac{1}{2} 	imes 10,000 \times 3.0 \\
\text{S.c.CH...} m_3 + m_4 - m_5 - m_6 &= \frac{1}{2} 	imes 10,000 \times 3.9 \\
\text{S.c.BK...} m_5 + m_6 - m_7 - m_8 &= \frac{1}{2} 	imes 10,000 \times 3.0 \\
\text{S.c.EL...} m_7 + m_8 &= \frac{1}{4} \times 10,000 \times 8.0
\end{align*}$

Fig. 146 shows the distribution of the bending moments and the shearing and axial forces. Assuming that $E = 30,000,000 \text{ lb. per square inch, the deflections are}$

$$\Delta_1 = 0.0227 \text{ in.}, \Delta_2 = 0.0409 \text{ in.}, \Delta_3 = 0.0545 \text{ in.}, \text{ and } \Delta_4 = 0.0583 \text{ in.}$$

**Vierendeel Trusses with Unsymmetrical or Inclined Chords.**—If the upper and lower chords are not symmetrical about a longitudinal axis (when
corresponding chord members are inclined or have different moments of inertia),
the bending moments and shearing forces on the upper and lower members are
not equal, and the points of contraflexure in the columns are no longer at mid-
height. It is therefore necessary to have equations of equilibrium for joints at
both the upper and lower chord. This may be avoided, and the number of
equations considerably reduced, if it is assumed that the bending moments and
shearing forces on each corresponding member are in the ratio of their respective
stiffnesses. For the truss shown in Fig. 147 the bending moments on the lower
chord can therefore be calculated from the relations

\[ m'_1 = k_1 m_1, \quad m'_2 = k_1 m_2, \quad m'_3 = k_2 m_3, \quad m'_4 = k_2 m_4. \quad \text{(174a)} \]

in which \[ k_1 = \left( \frac{L_2}{L'_1} \right)^2 \frac{I_1'}{I_1} \text{ and } k_2 = \left( \frac{L_2}{L'_2} \right)^2 \frac{I_2'}{I_2}. \quad \text{(174b)} \]

For trusses with inclined members the horizontal translations \( \lambda \) of the joints
(Fig. 147) due to vertical deflections \( \Delta \) can also be taken into account when

writing the equations of equilibrium. The values of \( \lambda \) do not represent separate
unknowns, but can be expressed in terms of \( \Delta \) as in (174c):

\[ \lambda_1 = \frac{h_2 - h_1}{L_2'} (\Delta_2 - \Delta_1); \quad \lambda_0 = \lambda_1 + \frac{h_1 - h_0}{L_1'} \Delta_1. \quad \text{(174c)} \]

The equations of equilibrium can now be written in the usual way as in (174).
EXAMPLE 52.—Consider the four-bay truss with inclined top chord in Fig. 148. First calculate the coefficients \( k \) and the horizontal translations \( \lambda \) of joints A and B:
\[
k_1 = \left(\frac{12.65}{12.0}\right)^2 \times \frac{0.1628}{0.2812} = 0.6434 ; \quad k_2 = \left(\frac{12.17}{12.0}\right)^2 \times \frac{0.2812}{0.6666} = 0.4338 ;
\]
\[
\lambda_1 = \frac{12 - 10}{12.0} (A_2 - A_1) = 0.1667 (A_2 - A_1);
\]
\[
\lambda_0 = 0.1667 (A_2 - A_1) + \frac{10 - 6}{12.0} A_1 = 0.1667 A_1 + 0.1667 A_2;
\]
\[
\cos \phi_0 = \frac{12.0}{12.65} = 0.9486 ; \quad \cos \phi_1 = \frac{12.0}{12.17} = 0.9860.
\]
Substituting these values in equations (174) we obtain (174a), from which
\[
m_1 = 119,910 \text{ ft.-lb.} ; \quad m_2 = 74,420 \text{ ft.-lb.} ; \quad m_3 = 22,430 \text{ ft.-lb.} ;
\]
\[
m_4 = 105,590 \text{ ft.-lb.} ;
\]
\[
m_1' = 77,150 \text{ ft.-lb.} ; \quad m_2' = 47,880 \text{ ft.-lb.} ; \quad m_3' = 9730 \text{ ft.-lb.} ;
\]
and
\[
m_4' = 45,800 \text{ ft.-lb.}
\]
Assuming that \( E = 4,000,000 \text{ lb. per square inch} \), then \( A_1 = 0.2572 \text{ in.} \) \( A_2 = 0.4027 \text{ in.} \), \( \lambda_1 = 0.024 \text{ in.} \), and \( \lambda_0 = 0.1068 \text{ in.} \).

Viaduct Trestles.—Trestles for viaducts (Fig. 149) can be analysed in a similar way to Vierendeel trusses. Denoting by \( A_1, A_2, \) and \( A_3 \) the horizontal translations of joints B, D and F, the equations of equilibrium are as in (176).
VIERENDEEL TRUSSES

**Fig. 149.**

\[ ABD \ldots m_1 \alpha_{ba} + m_1 \alpha_{bd} - m_1 \beta_{ba} - m_2 \beta_{bd} = \frac{\Delta_1 - \Delta_2}{h_1} \]

\[ CDF \ldots m_3 \alpha_{df} + (m_2 + m_3) \alpha_{dc} = (m_2 + m_3) \beta_{dc} - m_4 \beta_{df} = \frac{\Delta_2 - \Delta_3}{h_2} \]

\[ BDF \ldots m_2 \alpha_{db} - m_1 \beta_{db} - m_3 \alpha_{df} + m_4 \beta_{df} = \frac{\Delta_1 - \Delta_2}{h_1} - \frac{\Delta_2 - \Delta_3}{h_2} \]

\[ EFK \ldots m_5 \alpha_{fx} + (m_4 + m_5) \alpha_{fe} - (m_4 + m_5) \beta_{fe} = \frac{\Delta_3}{h_3} \]

\[ DFK \ldots m_4 \alpha_{fd} - m_3 \beta_{fd} - m_5 \alpha_{fx} = \frac{\Delta_2 - \Delta_3}{h_2} - \frac{\Delta_3}{h_3} \]

\[ S_c AB \ldots \frac{m_1 + m_2}{h_1} = \frac{1}{2} W_1 \]

\[ S_c CD \ldots \frac{m_3 + m_4}{h_2} = \frac{m_1 + m_2}{h_1} = \frac{1}{2} W_2 \]

\[ S_c BF \ldots \frac{m_5}{h_3} = \frac{m_3 + m_4}{h_2} = \frac{1}{2} W_3 \]

**Fig. 150.**

\[ ABD \ldots \frac{12}{3x1} m_1 + \frac{16}{3x0.5} m_1 - \frac{12}{6x1} m_1 - \frac{16}{6x0.5} m_2 = \frac{\Delta_1 - \Delta_2}{h_1} \]

\[ CDF \ldots \frac{17.34}{3x4} (m_2 + m_3) + \frac{18}{3x1} m_3 - \frac{17.34}{6x4} (m_2 + m_3) = \frac{\Delta_2 - \Delta_3}{h_2} \]

\[ BDF \ldots \frac{16}{3x0.5} m_2 - \frac{16}{6x0.5} m_1 - \frac{18}{3x1} m_3 + \frac{18}{6x1} m_4 = \frac{\Delta_1 - \Delta_2}{h_1} - \frac{\Delta_2 - \Delta_3}{h_2} \]

\[ EFK \ldots \frac{23.34}{3x6} (m_4 + m_5) + \frac{20}{3x4} m_5 - \frac{23.34}{6x8} (m_4 + m_5) = \frac{\Delta_3}{h_3} \]

\[ DFK \ldots \frac{18}{3x1} m_4 - \frac{18}{6x1} m_3 - \frac{20}{6x4} m_5 = \frac{\Delta_2 - \Delta_3}{h_2} - \frac{\Delta_3}{h_3} \]

\[ S_c AB \ldots m_1 + m_2 = \frac{1}{2} \times 2000 \times 16 \]

\[ S_c CD \ldots 16 (m_3 + m_4) = \frac{1}{2} \times 4000 \times 16 \times 18 \]

\[ S_c BF \ldots 18 m_5 - 20 (m_3 + m_4) = \frac{1}{2} \times 6000 \times 16 \times 20 \]
EXAMPLE 53.—For the trestle shown in Fig. 150 and from equations (176) we derive (176a), from which

\[ m_1 = 9500 \text{ ft.-lb.}, \quad m_2 = 6500 \text{ ft.-lb.}, \quad m_3 = 12,570 \text{ ft.-lb.}, \]

\[ m_4 = 41,360 \text{ ft.-lb.}, \quad \text{and} \quad m_5 = 119,700 \text{ ft.-lb.} \]

**Vierendeel Trusses with Solid Portions.**

Occasionally a Vierendeel truss has solid central or end portions (Figs. 151 and 153). The distribution of internal stresses in these parts is usually two-dimensional, and therefore beyond the scope of this book. However, if the ratio of depth to length of these parts is small, they can be treated as elastic beams and the truss designed in the usual way. In the following examples two types of trusses with solid portions are analysed.

![Diagram of a Vierendeel truss with solid portions](image)

**Fig. 151.**

**Truss with Solid Ends.**—The deflected shape of the truss and all angular gaps are shown in Fig. 151(b), and the equations of equilibrium are set out in (177).
VIERENDEEL TRUSSES

\[ \begin{align*}
ABC & : m_1 \alpha_{ba} + m_3 \alpha_{bc} + m_2 \beta_{dc} = \frac{\Delta_1}{L_1} - \frac{\Delta_2 - \Delta_1}{L_2} \\
BCD & : m_2 \alpha_{cb} - m_3 \alpha_{cd} + m_4 \beta_{cd} + m_1 \beta_{cb} = \frac{\Delta_2 - \Delta_1}{L_2} - \Delta_3 - \Delta_2 \\
BCG & : m_2 \alpha_{cb} + (m_2 + m_3) \alpha_{cg} - (m_2 + m_3) \beta_{cg} + m_1 \beta_{cb} = \frac{\Delta_2 - \Delta_1}{L_2} \\
CDE & : m_4 \alpha_{dc} - m_5 \alpha_{de} + m_6 \beta_{de} - m_3 \beta_{dc} = \frac{\Delta_2 - \Delta_1}{L_2} - \frac{\Delta_4 - \Delta_3}{L_4} \\
CDH & : m_4 \alpha_{dc} + (m_4 + m_5) \alpha_{dh} - (m_4 + m_5) \beta_{dh} - m_3 \beta_{dc} = \frac{\Delta_2 - \Delta_1}{L_3}
\end{align*} \]

\[ DE \ldots m_6 \alpha_{ed} - m_5 \beta_{ed} = \frac{\Delta_2 - \Delta_1}{L_4} \]

\[ \text{s.c.BF} \ldots \frac{m_1}{L_1} - \frac{m_2 - m_3}{L_2} = \frac{1}{2} W_1 \]

\[ \text{s.c.CG} \ldots \frac{m_1 - m_2}{L_2} - \frac{m_2 + m_4}{L_3} = \frac{1}{2} W_2 \]

\[ \text{s.c.DH} \ldots \frac{m_3 + m_4}{L_3} - \frac{m_3 + m_6}{L_4} = \frac{1}{2} W_3 \]

\[ \text{s.c.EK} \ldots \frac{m_3 + m_6}{L_4} = \frac{1}{4} W_4 \]

\[ \text{Example 54.} \]—For the truss shown in Fig. 152(a) equations (177) become (177a), from which \( m_1 = 315,000 \text{ ft.-lb.}, m_2 = 240,000 \text{ ft.-lb.}, m_3 = -7630 \text{ ft.-lb.}, m_4 = 52,630 \text{ ft.-lb.}, m_5 = -1120 \text{ ft.-lb.}, \) and \( m_6 = 16,120 \text{ ft.-lb.} \). Fig. 152(b) shows the distribution of the bending moments and the shearing and axial forces in the truss.

**Vierendeel Truss with Solid Central Portion** (Fig. 153).—To reduce the number of simultaneous equations it is assumed that the bending moments on each chord are in proportion to the ratio of their respective stiffnesses, as shown by equations 175a and 175b. The horizontal translations of the joints
due to the curvature of the lower chord are also taken into account as in equations (175c). The deflected shape of the truss and all angular gaps are shown in Fig. 153(b), and the equations of equilibrium are given in (178).

\[
\begin{align*}
\text{FAB... } & m_1 \alpha_{ab} + m_0 \alpha_{af} - m_0 \beta_{af} - m_0 \beta_{ab} = \frac{AL_1}{L_1} - \frac{\lambda_1}{h_1} \\
\text{ABC... } & m_2 \alpha_{bc} - m_3 \alpha_{bc} + m_6 \alpha_{bc} - m_1 \beta_{ba} = \frac{AL_1}{L_2} - \frac{\Delta t - \Delta l}{L_2} \\
\text{ABG... } & m_2 \alpha_{ba} + (m_2 + m_3) \alpha_{bg} - (m_4 + m_5) \beta_{bg} - m_0 \beta_{ba} = \frac{AL_1}{L_2} - \frac{\lambda_2}{h_2} \\
\text{BCD... } & m_6 \alpha_{cb} - m_5 \alpha_{cd} + m_6 \alpha_{cd} - m_3 \beta_{cb} = \frac{\Delta t - \Delta l}{L_2} - \frac{\Delta t - \Delta l}{L_2} \\
\text{BCG... } & m_4 \alpha_{cb} + (m_4 + m_5) \alpha_{cb} - (m_4 + m_5) \beta_{cb} - m_3 \beta_{cb} = \frac{\Delta t - \Delta l}{L_2} - \frac{\lambda_3}{h_3} \\
\text{CDE... } & m_6 \alpha_{de} - m_5 \beta_{de} \left( \frac{m_6 + m_5}{L_3} \right) \varepsilon_{de} = \frac{\Delta t - \Delta l}{L_3} - \theta_{de} \\
\text{SC.BG... } & \frac{m_3 + m_4}{L_2} - \frac{m_3 + m_4}{L_3} - \frac{m_5 + m_6}{L_3} \cos \phi_1 - \frac{m_5 + m_6}{L_2} \cos \phi_2 = W_1 \\
\text{SC.CH... } & \frac{m_3 + m_4}{L_2} + \frac{m_3 + m_4}{L_3} + \frac{m_5 + m_6}{L_3} \cos \phi_2 - \frac{m_5 + m_6}{L_2} \cos \phi_1 = W_2 \\
\text{SC.DK... } & \frac{m_5 + m_6}{L_3} + \frac{m_5 + m_6}{L_3} \cos \phi_1 = W_3 + W_{dk} \\
\end{align*}
\]

(178)

EXAMPLE 55.—For the truss shown in Fig. 154 the coefficients \( k \) are calculated from equations (175(b)) and the horizontal translations \( \lambda \) from equations (175(c)). The elastic constant \( \varepsilon_{de} \), and load function \( \theta_{de} \) for the portion DE, are as follows:

\[
\begin{align*}
\lambda_1 &= \left( \frac{12}{13.4} \right)^2 = 0.802; \\
\lambda_2 &= \left( \frac{12}{12.68} \right)^2 = 0.8957; \\
\lambda_3 &= \left( \frac{12}{12.25} \right)^2 = 0.9596; \\
\lambda'' &= \frac{3}{12} (\Delta_3 - \Delta_2) = 0.204 (\Delta_3 - \Delta_2); \\
\end{align*}
\]
\[ \lambda'' = \frac{9.5 - 5.45}{12}(A_2 - A_1) = 0.337(A_2 - A_1); \quad \lambda' = \frac{15.4 - 9.5}{12}A_1 = 0.492A_1. \]

\[ \lambda_2 = \lambda'' = 0.204(A_3 - A_2); \quad \lambda_2 = \lambda'' = -0.337A_1 + 0.133A_2 + 0.204A_3; \]

\[ \lambda_4 = \lambda' = 0.155A_1 + 0.133A_2 + 0.204A_3. \]

\[ E\theta_{de} = 515; \quad E\varepsilon_{de} = 14.7. \]

\[ \cos \phi_1 = \frac{12}{13.4} = 0.8955; \quad \cos \phi_2 = \frac{12}{12.68} = 0.9464; \quad \cos \phi_3 = \frac{12}{12.25} = 0.9796. \]

By substituting these values in (178) we obtain (178a), from which

\[ m_1 = 107,020 \text{ ft.-lb.}, \quad m_2 = 118,560 \text{ ft.-lb.}, \quad m_3 = 48,050 \text{ ft.-lb.}, \]

\[ m_4 = 118,400 \text{ ft.-lb.}, \quad m_5 = 66,420 \text{ ft.-lb.}, \quad m_6 = 27,290 \text{ ft.-lb.}. \]

Also

\[ m'_1 = 85,830 \text{ ft.-lb.}, \quad m'_2 = 119,150 \text{ ft.-lb.}, \quad m'_3 = 43,040 \text{ ft.-lb.}, \]

\[ m'_4 = 106,050 \text{ ft.-lb.}, \quad m'_5 = 63,740 \text{ ft.-lb.}, \quad m'_6 = 26,190 \text{ ft.-lb.}. \]

**Fig. 155** shows the distribution of bending moments and shearing and axial forces in the truss. The deflections calculated from equations (178a) by assuming

\[ \text{FAB...} \quad \frac{12}{31}m_1 + \frac{15.4}{31}m_r - \frac{0.802}{61}m_r - \frac{12}{61}m_2 = \frac{A_1}{L}E = \frac{1}{15.4}(0.155A_1 + 0.133A_2 + 0.204A_3)E \]

\[ \text{ABC...} \quad \frac{12}{31}m_2 - \frac{12}{61}m_3 + \frac{12}{61}m_4 - \frac{12}{61}m_r = \frac{A_1}{L}E = -\frac{A_2}{L}E \]

\[ \text{ABG...} \quad \frac{12}{31}m_2 + \frac{25}{61}(m_r + m_3) - \frac{9.5}{61}(0.802m_r + 0.8957m_3) - \frac{12}{61}m_r = \frac{A_1}{L}E - \frac{1}{7.5}(-0.337A_1 + 0.133A_2 + 0.204A_3)E \]

\[ \text{BCD...} \quad \frac{12}{31}m_6 - \frac{12}{61}m_r + \frac{12}{61}m_6 - \frac{12}{61}m_7 = \frac{A_2}{L}E - \frac{A_3}{L}E \]

\[ \text{BC...} \quad \frac{12}{31}m_4 + \frac{5.45}{61}(m_r + m_5) - \frac{5.45}{61}(0.8957m_4 + 0.9796m_5) - \frac{12}{61}m_r = \frac{A_1}{L}E - \frac{0.204}{5.45}(-A_3 + A_2)E \]

\[ \text{CDE...} \quad \frac{12}{31}m_6 - \frac{12}{61}m_5 + 14.7(m_6 + 0.9796m_r) = \frac{A_3}{L}E - 515 \]

\[ \text{SC...} \quad \frac{m_1 + m_2}{12} - \frac{m_3 + m_4}{12} + \frac{0.802 + 0.902}{12.68}(m_4 + m_5) - \frac{0.8957 + 0.8957}{12.68}(m_5 + m_6) = 10,000 \]

\[ \text{S.C.CH...} \quad \frac{m_3 + m_4}{12} + \frac{0.8957 + 0.8957}{12.68}(m_4 + m_5) - \frac{0.9796 + 0.8957}{12.25}(m_5 + m_6) = 10,000 \]

\[ \text{S.C.H...} \quad \frac{m_5 + m_6}{12} + \frac{0.8957 + 0.8957}{12.25}(m_5 + m_6) = 10,000 + 5,000 \]
that $E = 4,000,000$ lb. per square inch are $A_1 = 0.45$ in., $A_2 = 1.91$ in., and $A_3 = 2.54$ in.

If the supports F and M are hinged, equations (178) are only the first stage in the solution; a unit horizontal force acting along the line of the hinges FM must be applied and the bending moments and the horizontal translation $\lambda^h$ of the hinges due to this force must be calculated. The horizontal force $H$ at F and M is calculated from $H = \frac{2\lambda^h}{\lambda^h}$. The final bending moments are calculated by adding the bending moments of the first stage to those of the second stage multiplied by $H$.

**Multiple-story Vierendeel Trusses.**

Multiple-story Vierendeel trusses are occasionally used in buildings. These trusses consist of three or more beams tied by posts with all the joints rigid.

For symmetrically-loaded symmetrical trusses the deflection diagram and angular gaps are as shown in Fig. 156, and the equations of equilibrium are in (179).
EXAMPLE 56.—For the truss shown in Fig. 157 with concentrated loads at the junctions on the lowest chord and with uniformly-distributed loads on all horizontal members, \( w = 2000 \) lb. per foot, and \( V_o = \frac{1}{4} \times 2000 \times 12 = 6000 \) lb.

\[
m_o = \frac{1}{12} \times 2000 \times 12^2 = 24,000 \text{ ft.-lb.}, \quad EI\theta_o = \frac{5}{192} \times 2000 \times 12^3 = 90,000,
\]

\[
\begin{align*}
\text{EAB} & \quad m_1\alpha_{ab} + m_1\alpha_{ae} - m_1\beta_{ae} - m_2\beta_{ab} = \frac{A_1}{L} + \theta \\
\text{ABC} & \quad m_2\alpha_{ba} - m_3\alpha_{bc} - m_1\beta_{ba} + m_4\beta_{bc} = \frac{A_2 - A_1}{L} - \frac{A_1}{L} - 2\theta \\
\text{ABF} & \quad m_2\alpha_{ba} + (m_2 + m_3) \alpha_{bf} - (m_2 + m_3) \beta_{bf} - m_1\beta_{be} = \frac{A_3 - A_2}{L} - \theta \\
\text{BCD} & \quad m_4\alpha_{cb} + m_5\alpha_{cd} + m_5\beta_{cd} - m_3\beta_{cb} = \frac{A_4 - A_3}{L} - 2\theta \\
\text{BCG} & \quad m_4\alpha_{cb} + (m_4 - m_5) \alpha_{cf} - (m_4 - m_5) \beta_{cg} - m_3\beta_{cb} = -\frac{A_5 - A_4}{L} - \theta \\
\text{Sc BF} & \quad \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = W_b \\
\text{Sc CG} & \quad \frac{m_3 + m_4}{L} = W_c
\end{align*}
\]

\[\text{Fig. 157.}\]
and equations (179) become (179a), from which, assuming \( \frac{I_b}{I_o} = 4 \),

\[
\begin{align*}
  m_1 &= 186,500 \text{ ft.-lb.,} \\
  m_2 &= 191,500 \text{ ft.-lb.,} \\
  m_3 &= 38,400 \text{ ft.-lb.,} \\
  m_4 &= 165,600 \text{ ft.-lb.,} \text{ and} \\
  m_5 &= 75,300 \text{ ft.-lb.}
\end{align*}
\]

It will be noted that the intermediate chords are subjected to bending moments and shearing forces but not to axial forces. This is due to the presence of equal but opposite shearing forces on the posts at their junctions, so that no direct force is transmitted to the intermediate chords. Also, if all the posts were acting as ties the bending moments at B and C would be 378,000 ft.-lb. and 597,000 ft.-lb. respectively, compared with 191,500 ft.-lb. and 165,600 ft.-lb.

**Continuous Vierendeel Trusses.**

Continuous Symmetrical Trusses.—Continuous symmetrical trusses, symmetrically loaded, may be analysed in a similar way to the trusses discussed previously. It is important that the deflected shape of the truss be clearly visualised and drawn to an exaggerated scale. This assists in setting out equations of equilibrium and in avoiding errors. The main difficulty in analysing continuous trusses lies usually in solving a large number of simultaneous equations. Although the deflections and sway entering into the equations may be easily eliminated, the number of simultaneous equations often remains large.

Some complicated symmetrical trusses may be more easily analysed by the general method described on page 137.

Examples of the analysis of continuous trusses are given to illustrate the procedure.

**Two-span Symmetrical Trusses.**—Fig. 158(b) shows the deflected shape of a truss with angular gaps marked at each joint of the top chord. There is no horizontal sway, due to the geometrical symmetry of the truss and symmetry in loading. The equations of equilibrium are set out in the usual way as in (180).
VIERENDEEL TRUSSES

\[
\begin{align*}
FAB & \ldots m_1 \alpha_{ab} + m_1 \alpha_{af} - m_1 \beta_{af} - m_2 \beta_{ab} = \frac{A_1}{L_1} \\
ABC & \ldots m_3 \alpha_{bc} - m_2 \alpha_{ba} - m_4 \beta_{bc} + m_1 \beta_{ba} = \frac{A_2 - A_1}{L_2} - \frac{A_1}{L_1} \\
ABG & \ldots n_2 \alpha_{ba} + (m_2 + m_3) \alpha_{bg} - (m_2 + m_3) \beta_{bg} - m_1 \beta_{ba} = \frac{A_1}{L_1} \\
BCD & \ldots m_4 \alpha_{cb} + m_5 \alpha_{cd} - m_6 \beta_{cd} - m_3 \beta_{cb} = -\frac{A_2 - A_3}{L_3} \\
KCD & \ldots m_5 \alpha_{cd} + (m_3 - m_6) \alpha_{ck} - (m_3 - m_6) \beta_{ck} - m_6 \beta_{dc} = \frac{A_2 - A_3}{L_3} \\
CDE & \ldots m_7 \alpha_{de} - m_6 \alpha_{dc} - m_8 \beta_{de} + m_3 \beta_{dc} = \frac{A_3}{L_4} - \frac{A_2 - A_3}{L_3} \\
CDL & \ldots m_6 \alpha_{dc} + (m_6 + m_7) \alpha_{de} - (m_6 + m_7) \beta_{de} - m_5 \beta_{dc} = \frac{A_2 - A_3}{L_3} \\
DE & \ldots m_8 \alpha_{ec} - m_7 \beta_{ed} = \frac{A_3}{L_4} \\
S.c.BG & \ldots \frac{m_1}{L_1} + \frac{m_2}{L_2} - \frac{m_3}{L_3} + \frac{m_5}{L_4} = \frac{1}{2} W_1 \\
S.c.CK & \ldots \frac{m_3}{L_2} + \frac{m_4}{L_3} + \frac{m_3}{L_4} = \frac{1}{2} W_2 \\
S.c.DL & \ldots \frac{m_7}{L_3} + \frac{m_8}{L_4} - \frac{m_5}{L_5} = \frac{1}{2} W_3
\end{align*}
\]

\[
\begin{align*}
FAB & \ldots \frac{12}{3} m_1 + \frac{9}{3} m_1 - \frac{9}{6} m_1 - \frac{12}{6} m_2 = \frac{A_1}{L} E I \\
ABC & \ldots \frac{12}{3} m_3 - \frac{12}{3} m_2 + \frac{12}{6} m_4 + \frac{12}{6} m_1 = \frac{A_2 - A_1}{L} E I - \frac{A_1}{L} E I \\
ABG & \ldots \frac{12}{3} m_2 + \frac{9}{3} (m_2 + m_3) - \frac{9}{6} (m_2 + m_3) - \frac{12}{6} m_1 = \frac{A_1}{L} E I \\
BCD & \ldots \frac{12}{3} m_4 + \frac{12}{3} m_5 - \frac{12}{6} m_6 - \frac{12}{6} m_3 = \frac{A_2 - A_3}{L} E I + \frac{A_2 - A_3}{L} E I \\
KCD & \ldots \frac{12}{3} m_5 + \frac{9}{3} (m_5 - m_6) - \frac{9}{6} (m_5 - m_6) - \frac{12}{6} m_6 = \frac{A_2 - A_3}{L} E I \\
CDE & \ldots \frac{12}{3} m_7 - \frac{12}{3} m_6 - \frac{12}{6} m_8 + \frac{12}{6} m_5 = \frac{A_3}{L} E I - \frac{A_2 - A_3}{L} E I \\
CDL & \ldots \frac{12}{3} m_6 + \frac{9}{3} (m_6 + m_7) - \frac{9}{6} (m_6 + m_7) - \frac{12}{6} m_5 = \frac{A_2 - A_3}{L} E I \\
DE & \ldots \frac{12}{3} m_8 - \frac{12}{6} m_7 = \frac{A_3}{L} E I \\
S.c.BG & \ldots \frac{m_1}{L_1} + \frac{m_2}{L_2} - \frac{m_3}{L_3} - \frac{m_4}{L_4} = \frac{1}{2} \times 10,000 \times 12 \\
S.c.CK & \ldots \frac{m_3}{L_2} + \frac{m_4}{L_3} + \frac{m_3}{L_4} = \frac{1}{2} \times 20,000 \times 12 \\
S.c.DL & \ldots \frac{m_7}{L_3} + \frac{m_8}{L_4} - \frac{m_5}{L_5} = \frac{1}{2} \times 10,000 \times 12
\end{align*}
\]

**Example 57.**—For the two-span truss shown in Fig. 159 and from equations (180) we have (180a), from which \(m_1 = 58,412\) ft.-lb.; \(m_2 = 55,015\) ft.-lb.; \(m_3 = 16,985\) ft.-lb.; \(m_4 = 36,443\) ft.-lb.; \(m_5 = 42,284\) ft.-lb.; \(m_6 = 24,286\) ft.-lb.; \(m_7 = 53,552\) ft.-lb.; and \(m_8 = 73,018\) ft.-lb.

Assuming that the cross-sectional dimension of all the members is 21 in. by 12 in., and \(E\) is 4,000,000 lb. per square inch, then \(A_1 = 0.118\) in., \(A_2 = 0.174\) in.
and $A_3 = 0.102$ in. Fig. 159 shows bending moments, shearing forces, and axial forces.

**Three-span Symmetrical Truss.**—The deflected shape of the truss and angular gaps are shown in Fig. 160(b), and the equations of equilibrium are as in (181).

**Example 58.**—For the three-span truss shown in Fig. 161 and from equations (181) we have (181a),

\[
\begin{align*}
    m_1 &= 19,927 \text{ ft.-lb.} ; \\
    m_2 &= 22,472 \text{ ft.-lb.} ; \\
    m_3 &= 12,728 \text{ ft.-lb.} ; \\
    m_4 &= 4,921 \text{ ft.-lb.} ; \\
    m_5 &= 16,819 \text{ ft.-lb.} ; \\
    m_6 &= 30,830 \text{ ft.-lb.} ; \\
    m_7 &= 65,138 \text{ ft.-lb.} ; \\
    m_8 &= 54,861 \text{ ft.-lb.} ; \\
    m_9 &= 20,567 \text{ ft.-lb.} ;
\end{align*}
\]

and $m_{10} = 39,432$ ft.-lb.

Fig. 153 shows the distribution of bending moments, shearing forces, and axial forces. The deflections, if required, can now be calculated from the equations of equilibrium. Assuming that the cross-sectional dimensions of all members are

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**Analysis of Structures**

![Diagram of a three-span symmetrical truss](image)

**Fig. 160.**
21 in. by 12 in., and \( E \) is 4,000,000 lb. per square inch, then \( \Delta_1 = 0.036 \) in., \( \Delta_2 = 0.021 \) in., \( \Delta_3 = 0.013 \) in., and \( \Delta_4 = 0.179 \) in.

It is interesting to note, comparing Examples 57 and 58 with Example 47, that in simply-supported trusses the effect of continuity on the bending moments and forces is not great. Although this must not be assumed to be general, it is in sharp contrast to the effect of continuity in continuous beams; the reason is that, in simply-supported Vierendeel trusses, continuity already exists between the chords and the columns. The continuity with other trusses modifies the bending moments and forces to a limited extent only. It should also be noted that the upper members, directly over the intermediate supports, may in certain cases be subject to an axial compressive force, in addition to the usual bending moments and shearing forces, while the corresponding lower members may still be in direct tension as in the case of single-span trusses. Continuity, therefore, will not always result in a change of sign of direct forces when compared with single-span trusses. The reason is that the bending moments and forces required to close the angular and linear gaps formed at the intermediate

\[
\begin{align*}
GAB & \quad m_1 \alpha_{ab} + m_1 \alpha_{ag} - m_1 \beta_{ag} - m_2 \beta_{ab} = \frac{\Delta_1}{L_1} \\
ABC & \quad m_2 \alpha_{ba} + m_3 \alpha_{bc} - m_2 \beta_{ba} - m_3 \beta_{bc} = \frac{\Delta_1 - \Delta_2}{L_2} \\
ABH & \quad m_2 \alpha_{ba} + (m_2 - m_3) \alpha_{bh} - (m_2 - m_3) \beta_{bh} - m_1 \beta_{ba} = \frac{\Delta_1}{L_1} \\
BCD & \quad m_4 \alpha_{cb} - m_5 \alpha_{cd} + m_6 \beta_{cd} - m_3 \beta_{cb} = \frac{\Delta_1 - \Delta_2}{L_2} - \frac{\Delta_3}{L_3} \\
BCK & \quad m_4 \alpha_{cb} + (m_4 + m_5) \alpha_{ck} - (m_4 + m_5) \beta_{ck} - m_3 \beta_{cb} = \frac{\Delta_1 - \Delta_2}{L} \\
CDE & \quad m_5 \alpha_{dc} + m_7 \alpha_{de} - m_6 \beta_{de} - m_5 \beta_{dc} = \frac{\Delta_2}{L_3} + \frac{\Delta_3}{L_4} \\
LDE & \quad m_7 \alpha_{de} + (m_7 - m_8) \alpha_{dl} - (m_7 - m_8) \beta_{de} - m_6 \beta_{dc} = \frac{\Delta_2}{L_3} \\
DEF & \quad m_8 \alpha_{ef} - m_8 \alpha_{ed} + m_7 \beta_{ed} - m_9 \beta_{ef} = \frac{\Delta_3}{L_5} - \frac{\Delta_3}{L_4} \\
NEF & \quad m_9 \alpha_{ef} + (m_8 + m_9) \alpha_{en} - (m_8 + m_9) \beta_{en} - m_{10} \beta_{ef} = \frac{\Delta_3}{L_4} - \frac{\Delta_3}{L_5} \\
EFG & \quad m_{10} \alpha_{fe} - m_9 \beta_{ef} = \frac{\Delta_3}{L_5} \\
S.BH & \quad \frac{m_1 + m_2}{L_1} + \frac{m_3 + m_4}{L_2} = \frac{1}{2} W_1 \\
S.CK & \quad \frac{m_2 + m_6}{L_3} - \frac{m_5 + m_4}{L_2} = \frac{1}{2} W_2 \\
S.EN & \quad \frac{m_7 + m_8}{L_4} - \frac{m_7 + m_9}{L_5} = \frac{1}{2} W_3 \\
S.FM & \quad \frac{m_8 + m_9}{L_5} = \frac{1}{4} W_4
\end{align*}
\]

\[\text{FIG. 161.}\]
ANALYSIS OF STRUCTURES

\[
\begin{align*}
\text{GAB} & \quad \frac{12}{3} m_1 + \frac{9}{3} m_3 - \frac{9}{6} m_2 - \frac{12}{6} m_2 = \frac{d_1}{L} \text{EI} \\
\text{ABC} & \quad \frac{12}{3} m_2 + \frac{12}{3} m_3 - \frac{12}{6} m_1 - \frac{12}{6} m_4 = \frac{d_1-d_2}{L} \text{EI} \\
\text{ABH} & \quad \frac{12}{3} m_2 + \frac{9}{3} (m_2-m_3) - \frac{9}{6} (m_2-m_3) - \frac{12}{6} m_1 = \frac{d_1}{L} \text{EI} \\
\text{BCD} & \quad \frac{12}{3} m_4 - \frac{12}{3} m_3 + \frac{12}{6} m_6 - \frac{12}{6} m_3 = \frac{d_1-d_2}{L} \text{EI} - \frac{d_2}{L} \text{EI} \\
\text{BCK} & \quad \frac{12}{3} m_4 + \frac{9}{3} (m_4+m_5) - \frac{9}{6} (m_4+m_5) - \frac{12}{6} m_5 = \frac{d_1-d_2}{L} \text{EI} \\
\text{CDE} & \quad \frac{12}{3} m_6 + \frac{12}{3} m_7 - \frac{12}{6} m_8 - \frac{12}{6} m_5 = \frac{d_2}{L} \text{EI} + \frac{d_3}{L} \text{EI} \\
\text{LDE} & \quad \frac{12}{3} m_3 + \frac{9}{3} (m_3-m_6) - \frac{9}{6} (m_3-m_6) - \frac{12}{6} m_8 = \frac{d_3}{L} \text{EI} \\
\text{DEF} & \quad \frac{12}{3} m_9 - \frac{12}{6} m_8 + \frac{12}{6} m_7 - \frac{12}{6} m_{10} = \frac{d_4-d_3}{L} \text{EI} - \frac{d_3}{L} \text{EI} \\
\text{NEF} & \quad \frac{12}{3} m_9 + \frac{9}{3} (m_9+m_8) - \frac{9}{6} (m_9+m_8) - \frac{12}{6} m_{10} = \frac{d_4-d_3}{L} \text{EI} \\
\text{EF} & \quad \frac{12}{3} m_{10} - \frac{12}{6} m_9 = \frac{d_4-d_3}{L} \text{EI} \\
\text{S.c.BH} & \quad m_1 + m_2 + m_3 + m_4 = \frac{1}{2} \times 10,000 \times 12 \\
\text{S.c.CK} & \quad m_2 + m_6 - m_3 - m_6 = \frac{1}{2} \times 5,000 \times 12 \\
\text{S.c.EN} & \quad m_7 + m_9 - m_8 - m_{10} = \frac{1}{2} \times 10,000 \times 12 \\
\text{S.c.FM} & \quad m_9 + m_{10} = \frac{1}{4} \times 20,000 \times 12 \\
\end{align*}
\]

FIG. 162.

supports, should each span be treated as simply-supported, are not always of such magnitude as to exceed the initial stresses in the trusses.

Symmetrical Vierendeel Trusses Loaded Unsymmetrically.—If a continuous symmetrical truss is loaded unsymmetrically (Fig. 163) values of bending moments and shearing and axial forces may be obtained comparatively quickly, and the number of equations reduced if, for the purpose of the analysis, the actual load is replaced by two equivalent systems of loading, one symmetrical and the other unsymmetrical.

Example 59.—For the truss shown in Fig. 163(a), first set out the equations
of equilibrium in a general form for both symmetrical and unsymmetrical loading. For symmetrical loading, from Fig. 163(b), the equations of equilibrium are as in (182). By substituting the numerical values in (182), equations (182a) are derived, from which \( m_1 = 6370 \text{ ft.-lb.}, m_2 = 5240 \text{ ft.-lb.}, m_3 = 5570 \text{ ft.-lb.}, m_4 = 6040 \text{ ft.-lb.}, m_5 = 2960 \text{ ft.-lb.}, m_6 = 8650 \text{ ft.-lb.}, m_7 = 22,470 \text{ ft.-lb.}, \) and \( m_8 = 25,920 \text{ ft.-lb.} \).

For unsymmetrical loading, from Fig. 164, the equations of equilibrium are

\[
\begin{align*}
\text{FAB} & : m_1 \cdot \alpha_{ba} + m_2 \cdot \alpha_{af} - m_1 \cdot \beta_{af} - m_2 \cdot \beta_{ab} = \frac{\Delta_1}{L_1} \\
\text{ABC} & : m_2 \cdot \alpha_{ba} - m_3 \cdot \alpha_{bc} + m_4 \cdot \beta_{bc} - m_1 \cdot \beta_{ba} = \frac{\Delta_2 - \Delta_1}{L_2} + \frac{\Delta_1}{L_1} \\
\text{ABG} & : m_3 \cdot \alpha_{ba} + (m_3 + m_4) \cdot \alpha_{bg} - (m_2 + m_3) \cdot \beta_{bg} - m_1 \cdot \beta_{ba} = \frac{\Delta_1}{L_1} \\
\text{BCD} & : m_4 \cdot \alpha_{cb} - m_5 \cdot \alpha_{cd} + m_6 \cdot \beta_{cd} - m_1 \cdot \beta_{cb} = \frac{\Delta_2 - \Delta_1}{L_2} - \frac{\Delta_3 - \Delta_2}{L_3} \\
\text{KCD} & : m_5 \cdot \alpha_{cd} + (m_5 + m_6) \cdot \alpha_{ck} - (m_4 + m_5) \cdot \beta_{ck} - m_1 \cdot \beta_{cd} = \frac{\Delta_3 - \Delta_2}{L_3} \\
\text{CDE} & : m_7 \cdot \alpha_{de} + m_6 \cdot \alpha_{dc} - m_5 \cdot \beta_{de} - m_1 \cdot \beta_{cd} = \frac{\Delta_3}{L_4} + \frac{\Delta_3 - \Delta_2}{L_3} \\
\text{CDL} & : m_6 \cdot \alpha_{dc} + (m_5 + m_6) \cdot \alpha_{dl} - (m_4 + m_7) \cdot \beta_{dl} - m_1 \cdot \beta_{dc} = \frac{\Delta_3 - \Delta_2}{L_3} \\
\text{DE} & : m_8 \cdot \alpha_{ed} - m_7 \cdot \beta_{ed} = \frac{\Delta_3}{L_4} \\
\text{Sc.BG} & : \frac{m_1 + m_2}{L_1} - \frac{m_3 + m_4}{L_2} = 0 \\
\text{Sc.CK} & : \frac{m_3 + m_4}{L_2} + \frac{m_5 + m_6}{L_3} = 0 \\
\text{Sc.DL} & : \frac{m_7 + m_8}{L_3} - \frac{m_5 + m_6}{L_4} = \frac{W}{2}
\end{align*}
\]
shown in (183), and, by substituting the numerical values in (183), we have (183a), from which

\[ m_1 = 8230 \text{ ft.-lb.}, m_2 = 6770 \text{ ft.-lb.}, m_3 = 7240 \text{ ft.-lb.}, m_4 = 7760 \text{ ft.-lb.}, m_5 = 4170 \text{ ft.-lb.}, m_6 = 10,830 \text{ ft.-lb.}, m_7 = 25,500 \text{ ft.-lb.}, \]

and \( m_8 = 19,500 \text{ ft.-lb.} \).

The final result is obtained by adding the results of these two equivalent cases. Fig. 165(c) shows the distribution of bending moments and shearing and axial forces. The effect is seen by comparing the bending moments and forces in this example with those for a single-span truss in Example 48 (Fig. 138).

\[
\begin{align*}
\text{FAB} & \quad m_1 \alpha_{a} + m_2 \alpha_{b} + m_1 \beta_{a} - m_1 \beta_{b} - m_2 \beta_{a} = \frac{A_{l_1}}{L_1} + \frac{\lambda}{h} \\
\text{ABC} & \quad m_2 \alpha_{a} - m_3 \alpha_{b} + m_1 \beta_{a} + m_1 \beta_{b} = \frac{A_{l_1}}{L_1} - \frac{A_{2}-A_{1}}{L_2} \\
\text{ABG} & \quad m_2 \alpha_{a} + (m_2 + m_3) \alpha_{b} - (m_2 + m_3) \beta_{b} - m_1 \beta_{b} = \frac{A_{l_1}}{L_1} + \frac{\lambda}{h} \\
\text{BCD} & \quad m_4 \alpha_{b} - m_5 \alpha_{c} + m_6 \beta_{b} + m_3 \beta_{b} = \frac{A_{2}-A_{1}}{L_2} - \frac{A_{3}-A_{2}}{L_3} \\
\text{BCH} & \quad m_6 \alpha_{c} + (m_4 + m_5) \alpha_{i} - (m_4 + m_5) \beta_{c} - m_5 \beta_{c} = \frac{A_{3}}{L_3} + \frac{A_{3}-A_{2}}{L_2} \\
\text{CDE} & \quad m_7 \beta_{d} + (m_7 - m_6) \beta_{d} - (m_7 - m_6) \beta_{d} = \frac{A_{4}}{L_4} - \frac{A_{3}}{L_3} - \frac{\lambda}{h} \\
\text{DEL} & \quad m_8 \beta_{e} + 2 m_8 \beta_{e} = \frac{A_{3}}{L_3} - \frac{\lambda}{h} \\
\text{s. c. B} & \quad \frac{m_1 + m_2}{L_1} - \frac{m_3 + m_4}{L_2} = 0 \\
\text{s. c. C} & \quad \frac{m_3 + m_4}{L_2} - \frac{m_5 + m_6}{L_3} = 0 \\
\text{s. c. D} & \quad \frac{m_5 + m_6}{L_3} + \frac{m_7 + m_8}{L_4} = \frac{W}{2} \\
\text{s. c. A} & \quad \frac{m_1 + m_2}{h} + \frac{m_3 + m_4}{h} - \frac{m_5 + m_6}{h} - \frac{m_7 + m_8}{h} = 0
\end{align*}
\]
Continuous Unsymmetrical Trusses.—Continuous unsymmetrical trusses may be analysed by the method discussed in the previous paragraphs, but the number of statically-indeterminate bending moments and simultaneous equations may become too large for practical application. However, the amount of work may be reduced if the following procedure is adopted.

First calculate the bending moments on all spans assuming that there is no continuity, each span being treated as if simply-supported (Fig. 166). The sway (\( \lambda \)) of each span and the angular deformations (\( \theta \)) must also be calculated from the equations of equilibrium. Fig. 166 shows diagrammatically the deflected shape of each span, the linear gaps formed due to sway, and the angular gaps at intermediate supports. Next, as with curved beams, apply the unit bending moments to each span (Fig. 166), calculate all bending moments, and, finally, calculate the angular deformations \( \alpha \beta \). Now apply unit horizontal force to each truss and calculate the bending moments, angular deformations \( \gamma \phi \), and

A.S.—K
linear deformations $\Delta h$ for each span. Finally, calculate the bending moments and forces required to close the gaps at each intermediate support. Each intermediate support will provide, generally, two conditions of equilibrium, one for angular gap and one for horizontal translation.

For a two-span truss (Fig. 166) the equations of equilibrium at the support $F$ are in (184). The final bending moments on each span are obtained by adding the bending moments in the simply-supported case to those due to the applied unit moment multiplied by $M_b$, and the bending moments due to the unit horizontal force multiplied by $H_b$.

**Example 60.**—For the unsymmetrically three-span Vierendeel truss, unsymmetrically loaded, shown in Fig. 166, first divide it into three simply-supported trusses and calculate the bending moments, angular deformations, and sway in each truss due to external forces, unit bending moments, and unit horizontal forces.

For the truss AF (Fig. 167) we have

(a) The bending moments due to external loading calculated in Example 49, from which $E\theta_1 = 1,093,400$. 

---

**Fig. 165.**

**Fig. 166.**

\[
\begin{align*}
\alpha_1 M_1 + \alpha_2 M_2 + \beta_2 M_2 + H_1 y_1 + (H_1 - H_2) y_2 &= \theta_1 + \theta_2 \\
\alpha_2 M_2 + \alpha_3 M_2 + \beta_3 M_1 + H_2 y_3 + (H_1 - H_2) y_2 &= \theta_3 + \theta_4 \\
H_1 \lambda_1 + (H_1 - H_2) \lambda_2 &= \Delta_1 + \Delta_2 \\
H_2 \lambda_2 + (H_1 - H_2) \lambda_2 &= \Delta_2 + \Delta_3
\end{align*}
\]

(184)
FIG. 167.

\[ \text{VIERENDEEL TRUSSES} \]

\[
\begin{align*}
GAB & : 5.435m_1 + 0.27m_1 - 4.13m_1 - 34.03m_2 = \frac{A_1}{L} E \\
A BC & : 5.435m_2 - 5.435m_3 + 34.03m_4 - 34.03m_5 = \frac{A_2}{L} E - \frac{A_3}{L} E \\
A BH & : 5.435m_2 + 28.0(m_2 + m_3) - 14.0(m_2 + m_3) - 34.03m_1 = \frac{A_1}{L} E \\
B CD & : 5.435m_5 - 5.435m_5 + 34.03m_6 - 34.03m_3 = \frac{A_2}{L} E - \frac{A_3}{L} E \\
B CJ & : 5.435m_4 + 28.0(m_4 + m_5) - 14.0(m_4 + m_5) - 34.03m_3 = \frac{A_2}{L} E - \frac{A_3}{L} E \\
C DE & : 5.435m_4 - 5.435m_4 + 34.03m_8 - 34.03m_6 = \frac{A_3}{L} E - \frac{A_4}{L} E \\
C DK & : 5.435m_8 + 28.0(m_6 + m_7) - 14.0(m_6 + m_7) - 34.03m_5 = \frac{A_3}{L} E - \frac{A_4}{L} E \\
DEF & : 5.435m_8 - 5.435m_9 + 34.03m_8 - 34.03m_7 = \frac{A_4}{L} E - \frac{A_5}{L} E \\
L E F & : 5.435m_9 + 28.0(m_9-m_8) - 14.0(m_9-m_8) + 34.03m_8 = \frac{A_4}{L} E \\
\text{Sc. BH} & : m_1 + m_2 - \frac{m_3 + m_4}{L} = 0 \\
\text{Sc. CJ} & : \frac{m_3 + m_4}{L} - \frac{m_5 + m_6}{L} = 0 \\
\text{Sc. DK} & : \frac{m_5 + m_6}{L} - \frac{m_7 + m_8}{L} = 0 \\
\text{Sc. EL} & : \frac{m_7 + m_8}{L} - \frac{m_9}{L} = 0 \\
\text{Sc. AF} & : \frac{m_1}{h} + \frac{m_2 + m_3}{h} + \frac{m_4 + m_5}{h} + \frac{m_6 + m_7}{h} - \frac{m_9}{h} - \frac{l - m_{10}}{h} = 0 \\
\text{A'B} & : 5.435m_1 + 0.27m_1 - 4.13m_1 - 34.03m_2 = \frac{A_1}{L} E + \frac{A_2}{L} E \\
A BC & : 5.435m_2 - 5.435m_3 + 34.03m_4 - 34.03m_5 = \frac{A_1}{L} E + \frac{A_2}{L} E \\
A BB' & : 5.435m_3 - 34.03m_1 + 28.0(m_2 + m_3) - 14.0(m_2 + m_3) = \frac{A_1}{L} E + \frac{A_2}{L} E \\
B CD & : 5.435m_4 - 5.435m_5 + 34.03m_6 - 34.03m_3 = \frac{A_2}{L} E - \frac{A_3}{L} E \\
B CC' & : 5.435m_6 - 34.03m_3 + 28.0(m_4 + m_5) - 14.0(m_4 + m_5) = \frac{A_3}{L} E - \frac{A_4}{L} E \\
\text{Sc. BB'} & : \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = 0 \\
\text{Sc. CC'} & : \frac{m_3 + m_4}{L} - \frac{2m_5}{L} = 0 \\
\text{Sc. AF} & : \frac{m_1}{h} + \frac{m_2 + m_3}{h} + \frac{m_4 + m_5}{h} = \frac{1}{h} 
\end{align*}
\]
(b) Bending moments due to applied unit bending moment at F calculated from equations of equilibrium (185), from which \( m_1 = 0.1123; \ m_2 = 0.0877; \ m_3 = 0.1008; \ m_4 = 0.0992; \ m_5 = 0.0994; \ m_6 = 0.1006; \ m_7 = 0.0904; \ m_8 = 0.1096; \ m_9 = 0.0398; \ m_{10} = 0.2398; \ E\alpha_1 = 23.7338. \) Fig. 176 shows the distribution of bending moments.

![Diagram](Image)

Fig. 168.

(c) Bending moments due to unit horizontal force applied at F (Fig. 168) are calculated from equations of equilibrium (186), from which \( m_1 = 0.4142; \ m_2 = 0.3858; \ m_3 = 0.5172; \ m_4 = 0.2828; \ m_5 = 0.4; \ E\lambda_1 = 257.9808; \ E\gamma_1 = 11.7748. \) Fig. 176(c) shows the distribution of the bending moments.

For the truss FK, we have:

(a) Bending moments due to external loading as calculated in Example 48, from which (Fig. 169) \( E\Delta_2 = 902.458; \ E\theta_2 = 359.686; \ E\theta_3 = 163.584. \)

![Diagram](Image)

Fig. 169.

(b) Bending moments due to the applied unit bending moment at K are calculated from equations of equilibrium (187), from which \( m_1 = 0.1327; \ m_2 = 0.1173; \ m_3 = 0.077; \ m_4 = 0.173; \ m_5 = 0.0573; \ m_6 = 0.1928; \ m_7 = 0.5045; \ m_8 = 0.7546; \ E\alpha_2 = 28.5259; \ E\beta_2 = 0.7077. \)

(c) Bending moments due to unit horizontal force (Fig. 170) applied at K are calculated from the equations (188), from which \( m_1 = 0.6054; \ m_2 = 0.5196; \ m_3 = 0.405; \ m_4 = 0.72; \ E\lambda_2 = 70.5; \ E\gamma_2 = 2.4708. \)
\[ \begin{align*}
F G \ldots & \quad \frac{12}{5} m_i + \frac{9}{5} m_i - \frac{9}{6} m_i - \frac{12}{6} m_2 = \frac{A_1}{L} EI \\
G H \ldots & \quad \frac{12}{3} m_2 - \frac{12}{3} m_3 - \frac{12}{6} m_1 + \frac{12}{6} m_4 = \frac{A_1}{L} EI - \frac{A_2 - A_1}{L} EI \\
G G' \ldots & \quad \frac{12}{3} m_2 + \frac{9}{3} (m_2 + m_3) - \frac{9}{6} (m_2 + m_3) - \frac{12}{6} m_1 = \frac{A_1}{L} EI \\
G H J \ldots & \quad \frac{12}{3} m_5 + \frac{9}{3} (m_5 + m_8) - \frac{9}{6} (m_5 + m_8) - \frac{12}{6} m_5 = \frac{A_1 - A_1}{L} EI \\
G H H' \ldots & \quad \frac{12}{3} m_5 + \frac{9}{3} (m_5 + m_8) - \frac{9}{6} (m_5 + m_8) - \frac{12}{6} m_5 = \frac{A_1 - A_1}{L} EI \\
H J K \ldots & \quad \frac{12}{3} m_7 + \frac{9}{3} (m_7 - m_6) + \frac{9}{6} (m_7 - m_6) - \frac{12}{6} m_7 = \frac{A_1 - A_2}{L} EI \\
H J J' \ldots & \quad \frac{12}{3} m_7 - \frac{9}{3} (m_7 - m_6) + \frac{9}{6} (m_7 - m_6) - \frac{12}{6} m_7 = \frac{A_1 - A_2}{L} EI \\
S c G \ldots & \quad m_1 + m_2 - m_3 + m_4 = 0 \\
S c H \ldots & \quad m_3 + m_4 - m_5 + m_6 = 0 \\
S c J \ldots & \quad m_5 + m_6 - m_7 = 0 \\
S c F K \ldots & \quad \frac{m_1 + m_3}{h} + \frac{m_2 + m_5}{h} + \frac{m_4 + m_8}{h} - \frac{m_7 - m_6}{h} - \frac{1 - m_8}{h} = 0 \\
F G \ldots & \quad \frac{12}{3} m_1 + \frac{3}{3} m_i - \frac{3}{6} m_i - \frac{12}{6} m_2 = \frac{A_1}{L} EI + \frac{1}{2} EI \\
G H \ldots & \quad \frac{12}{3} m_2 - \frac{12}{3} m_3 - \frac{12}{6} m_1 + \frac{12}{6} m_4 = \frac{A_1}{L} EI \\
G G' \ldots & \quad \frac{12}{3} m_2 + \frac{9}{3} (m_2 + m_3) - \frac{9}{6} (m_2 + m_3) - \frac{12}{6} m_1 = \frac{A_1}{L} EI + \frac{1}{2} EI \\
G H J \ldots & \quad \frac{12}{3} m_5 + \frac{9}{3} (m_5 + m_8) - \frac{9}{6} (m_5 + m_8) - \frac{12}{6} m_5 = \frac{A_1}{L} EI - \frac{1}{2} EI \\
S c G \ldots & \quad \frac{m_1 + m_3}{h} - \frac{m_3 + m_4}{h} = 0 \\
S c F K \ldots & \quad \frac{4 m_1}{h} + \frac{4 (m_2 + m_3)}{h} + \frac{4 m_4}{h} = 1
\end{align*}\]
\[
\begin{align*}
K'KL & \quad \frac{12}{3} m_1 + \frac{9}{3} m_4 - \frac{9}{6} m_1 - \frac{12}{6} m_2 = \frac{4}{L} EI - \frac{3}{L} EI \\
KLM & \quad \frac{12}{3} m_2 - \frac{9}{3} m_3 - \frac{12}{6} m_1 + \frac{9}{6} m_4 = \frac{4}{L} EI - \frac{4}{L} EI \\
KLL' & \quad \frac{12}{3} m_2 + \frac{9}{6} (m_2 + m_3) - \frac{9}{6} (m_2 + m_3) - \frac{12}{6} m_1 = \frac{4}{L} EI - \frac{3}{L} EI \\
LMN & \quad \frac{12}{3} m_4 + \frac{9}{6} m_5 - \frac{12}{6} m_3 - \frac{12}{6} m_6 = \frac{4}{L} EI + \frac{4}{L} EI \\
G'GN' & \quad \frac{12}{3} m_5 + \frac{9}{6} (m_5 + m_6) - \frac{9}{6} (m_5 + m_6) - \frac{12}{6} m_4 = \frac{4}{L} EI + \frac{3}{L} EI \\
MNN' & \quad \frac{12}{3} m_6 + \frac{9}{6} m_4 - \frac{9}{6} m_6 - \frac{12}{6} m_3 = \frac{4}{L} EI + \frac{3}{L} EI \\
S_{c.KL} & \quad \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = \frac{1}{2} \times 5000 \\
S_{c.NM} & \quad \frac{m_3 + m_4}{L} + \frac{m_5 + m_6}{L} = \frac{1}{2} \times 10,000 \\
S_{c.KN'} & \quad m_1 + (m_2 + m_3) = (m_5 + m_4) + m_6
\end{align*}
\]
For the truss KL:
(a) Bending moments due to external loading (Fig. 171) are calculated by equation (189), from which \( m_1 = 20.0406 \); \( m_2 = 19.9594 \); \( m_3 = 0.609 \); \( m_4 = 10.609 \); \( m_5 = 25.4519 \); \( m_6 = 25.7661 \); \( E\Delta_3 = 181,000 \); \( EI\theta_4 = 127,200 \).

(b) Bending moments due to unit bending moment (Fig. 172) at N are calculated from equations (190), from which \( m_1 = 0.181 \); \( m_2 = 0.1523 \); \( m_3 = 0.1435 \); \( m_4 = 0.1898 \); \( m_5 = 0.0793 \); \( m_6 = 0.4126 \); \( E\alpha_3 = 8.8904 \).

\[
\begin{align*}
&k'KL \ldots \frac{12}{3} m_1 + \frac{9}{3} m_1 - \frac{9}{6} m_1 - \frac{12}{6} m_2 = \frac{A_1}{I} EI \\
&kLM \ldots \frac{12}{3} m_2 - \frac{12}{3} m_2 - \frac{9}{6} m_1 + \frac{12}{6} m_4 = \frac{A_1}{I} EI - \frac{A_1}{L} EI \\
&kLL' \ldots \frac{12}{3} m_2 + \frac{2}{3}(m_5 + m_4) - \frac{9}{6}(m_2 + m_3) - \frac{5}{6} m_1 = \frac{A_1}{I} EI \\
&LMN \ldots \frac{12}{3} m_3 + \frac{12}{6} m_5 - \frac{12}{6} m_3 + \frac{12}{6} m_6 = \frac{A_1}{L} EI + \frac{A_2}{I} EI \\
&M'MN \ldots \frac{12}{3} m_5 + \frac{2}{3}(m_5 + m_4) - \frac{9}{6}(m_5 + m_4) + \frac{12}{6} m_6 = \frac{A_2}{I} EI \\
&SeLL' \ldots \frac{m_1 + m_2}{L} - \frac{m_3 + m_4}{L} = 0 \\
&SeMN \ldots \frac{m_3 + m_4}{L} - \frac{m_5 + m_6}{L} = 0 \\
&SeKN \ldots \frac{m_1 + m_3}{n} - \frac{m_5 + m_6}{n} - \frac{l - m_6}{n} = 0
\end{align*}
\]

(c) Bending moments due to unit horizontal force (Fig. 173) applied at K are calculated from equations (191), from which \( m_1 = 0.6 \); \( m_2 = 0.9 \); \( m_3 = 0.75 \); \( E\lambda_3 = 38.4 \); \( E\gamma_3 = 1.0668 \).

\[
\begin{align*}
&k'KL \ldots \frac{12}{3} m_1 + \frac{3}{4} m_1 - \frac{9}{6} m_1 - \frac{12}{6} m_2 = \frac{A_1}{I} EI + \frac{3}{4} EI \\
&kLM \ldots \frac{12}{3} m_2 - \frac{12}{3} m_2 - \frac{9}{6} m_1 + \frac{12}{6} m_3 = \frac{A_1}{I} EI + \frac{2A_2}{L} EI \\
&kLL' \ldots \frac{12}{3} m_2 + \frac{2}{3}(m_2 + m_3) - \frac{9}{6}(m_2 + m_3) - \frac{9}{6} m_1 = \frac{A_1}{L} EI + \frac{1}{4} EI \\
&SeLL' \ldots \frac{m_1 + m_2}{L} = 0 \\
&SeKN \ldots \frac{m_1 + m_3}{n} = \frac{1}{4}
\end{align*}
\]
Statically-indeterminate bending moments $M_1$ and $M_2$ (Fig. 174) and forces $H_1$ and $H_2$, required to close the angular and linear gaps formed at F and K, are now calculated from equations (184) as in equations (192) (see Fig. 175), from

Fig. 174.

Fig. 175.

which $M_1 = 28.5694$; $M_2 = 2.1484$; $H_1 = 1.5394$; $H_2 = 5.6284$. The final bending moments [Fig. 176(d)] are obtained by adding the bending moments in a simply-supported case to those due to applied forces multiplied by $M_1$ and $M_2$ plus the bending moments due to unit horizontal forces multiplied by $H_1$, $(H_1 + H_2)$, and $H_2$ respectively.

$$
\begin{align*}
F & \quad 23.77M_1 + 26.52M_2 - 0.71M_2 - 11.77H_1 - 2.47(H_1 + H_2) = 1093.400 + 357.680 \\
K & \quad 28.53M_1 + 8.89M_2 - 0.71M_1 - 1.07H_2 - 2.47(H_1 + H_2) = 163.580 - 12.7200 \\
Se.F & \quad 257.98H_1 + 70.50(H_1 + H_2) = 902.460 \\
Se.K & \quad 38.40H_2 + 70.50(H_1 + H_2) = 902.460 - 181.000
\end{align*}
$$

(192)
CHAPTER V

INFLUENCE LINES

The deformation method may conveniently be used in calculating the ordinates for influence lines for beams and frames. Influence lines are defined as curves giving the values of any function, such as a bending moment, shearing force, reaction, deflection, etc., at a particular section of a member for any position of the load. Influence lines depend only on the elastic properties and shape of a structure, and are independent of the load. Elastic constants are used to determine the ordinates of influence lines, based on the well-known observation that the deflection diagram resulting from unit displacement imposed at any point in the direction of a stress component, such as the bending moment or shearing force, is identical with the influence line for the stress component, in accordance with Clerk Maxwell’s Reciprocal Theorem.

Beams with Fixed Ends.

Reaction $R_b$.—To find the influence line for the reaction $R_b$ of a fixed-ended beam of any shape, two conditions must be considered, namely, the beam subjected to a unit load $W = 1$ (Fig. 177a) and the beam with the support B displaced by $y_b = 1$ (Fig. 177b). By the Reciprocal Theorem the work of the forces in the first condition on the corresponding displacements of the second condition is zero, hence equation (193), in which $y = \frac{x}{L} + m_a \Delta_{za} - m_a \Delta_{xa}$. $\Delta_{za}$ and $\Delta_{xb}$ are the deflections of any point due to unit bending moments at A and E, and in a general case are calculated from $\Delta_x = \int\int \frac{m}{EI} dx^2$.

$$R_b y_b - W y = 0; \quad \text{or} \quad R_b = \frac{W y}{y_b} = y.$$  \hspace{1cm} (193)

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For prismatic beams \( EI\Delta_x = \int \frac{x}{L} dx^2 = \frac{L^2}{6} \left( \frac{x}{L} \right) \left[ I - \left( \frac{x}{L} \right)^2 \right] \) \hspace{1cm} (194)

and the bending moments due to unit displacement \( y_b = x \) (see Appendix) are

\[ m_a = m_b = \frac{6EIy_b}{L^2} \]

Substituting these values in equation (193),

\[ R_b = y = \left( \frac{x}{L} \right) + \frac{6EI}{L^2} \left[ \frac{L^2}{6EI} \left( \frac{x}{L} \right) \left[ I - \left( \frac{x}{L} \right)^2 \right] - \frac{L^2}{6EI} \left( \frac{z}{L} \right) \left[ I - \left( \frac{z}{L} \right)^2 \right] \right] \]

from which

\[ R_b = \left( \frac{x}{L} \right)^2 \left[ 3 - 2 \left( \frac{x}{L} \right) \right] \hspace{1cm} \ldots \hspace{1cm} (195) \]

Fig. 177c shows the ordinates of the influence line for \( R_b \) and the shearing force at any point on the beam.

**Bending Moment at Support A.**—In the first condition it is assumed that the beam is subjected to a load \( W = x \) (Fig. 178a), and in the second condition

![Diagram of a beam with labels](image)

unit bending moment is applied at \( A \) (Fig. 178b). The work of the forces in the first condition on the corresponding displacements in the second condition is

\[ m_a \theta_a = W y \]; or \( m_a = \frac{y}{\theta_a} \hspace{1cm} \ldots \hspace{1cm} (196) \]

in which \( y = m_a' \Delta_{xa} - m_b' \Delta_{xb} \).

For non-prismatic beams, \( \theta \) and \( \Delta \) are calculated from equations (4).

For prismatic beams \( EI\theta_a = \frac{L}{3} - \frac{x}{2} \frac{L}{6} = \frac{L}{4} \), and

\[ y = \frac{L^2}{6EI} \left( \frac{z}{L} \right) \left[ I - \left( \frac{z}{L} \right)^2 \right] - \frac{1}{2} \frac{L^2}{6EI} \left( \frac{x}{L} \right) \left[ I - \left( \frac{x}{L} \right)^2 \right] = \frac{L^2}{4EI} \left( \frac{x}{L} \right) \left[ \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right) + 1 \right] \]

Substituting these values in (196),

\[ m_a = \frac{y}{\theta_a} = L \left( \frac{x}{L} \right) \left[ \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right) + 1 \right] \hspace{1cm} \ldots \hspace{1cm} (197) \]

Fig. 178c shows the ordinates of the influence line for \( m_a \) calculated from (197).
The position of $W$ for maximum bending moment is found from $\frac{dm}{dx} = 0$, and is when $x = \frac{L}{3}$; the maximum value of $\frac{m}{L}$ is then $\frac{W}{27}$.

Fixed-end bending moments calculated from equation (197) are

For uniformly-distributed load: $m_a = \int_0^L wL \left( \frac{x}{L} \right) \left[ \left( \frac{x}{L} \right)^2 - 2\left( \frac{x}{L} \right) + 1 \right] dx = \frac{wL^3}{12}$.

For concentrated load: $m_a = WL \left( \frac{a}{L} \right) \left[ \left( \frac{a}{L} \right)^2 - 2\left( \frac{a}{L} \right) + 1 \right] = \frac{Wab^2}{L^2}$.

**Bending Moment on the Span.**—The influence line for any point C (Fig. 179) is calculated from

$$m_c = m_a - \frac{b}{L} - \frac{a}{L}$$

**Fig. 179.**

For prismatic beams,

For $x < a$:

$$m_c = b \left( \frac{x}{L} \right) - \frac{b}{L} \left( \frac{x}{L} \right) \left[ \left( \frac{x}{L} \right)^2 - 2\left( \frac{x}{L} \right) + 1 \right] L - a \left( \frac{z}{L} \right) \left[ \left( \frac{z}{L} \right)^2 + 2\left( \frac{z}{L} \right) + 1 \right] L$$

$$= \left( \frac{x}{L} \right)^2 \left[ b \left( 2 - \frac{x}{L} \right) - a \left( 1 - \frac{x}{L} \right) \right]$$

$$= \left( \frac{x}{L} \right)^2 \left[ b \left( 2 - \frac{x}{L} \right) - a \left( 1 - \frac{x}{L} \right) \right]$$

For $z < b$:

$$m_c = \left( \frac{z}{L} \right)^2 \left[ a \left( 2 - \frac{z}{L} \right) - b \left( 1 - \frac{z}{L} \right) \right]$$

For a concentrated load $W$, from equation (199a) or (199b),

$$m_c = \left( \frac{a}{L} \right)^2 \left[ b \left( 2 - \frac{a}{L} \right) - a \left( 1 - \frac{a}{L} \right) \right] = \frac{Wa^2b^2}{L^3}.$$
Two-span Continuous Beam:

BENDING MOMENT AT SUPPORT B.—In the first condition a unit load \( W \) is assumed to act on the span BC (Fig. 180a) and in the second condition the beam is assumed to be cut at B and a unit bending moment applied (Fig. 180b). The sum of the angles \( \theta_1 \) and \( \theta_2 \) represents the angular deformation in the second condition, corresponding to the bending moment \( m_b \) acting in the first condition. Therefore

\[
m_b(\theta_1 + \theta_2) = Wy; \quad \text{or} \quad m_b = \frac{y}{\theta_1 + \theta_2}. \quad \quad (200)
\]

For non-prismatic beams, \( \theta_1, \theta_2, \) and \( y \) are calculated from equations (4).

For prismatic beams, \( \theta_1 + \theta_2 = \frac{L_1}{3I_1} + \frac{L_2}{3I_2}; \quad \theta_2 = \frac{L_2^2}{6I_2} \left( \frac{x_2}{L_2} \right) \left[ I - \left( \frac{x_2}{L_2} \right)^2 \right] \).

Substituting these values in (200), we have (201a), in which \( n = \frac{I_2}{I_1} \)

\[
m_b = \frac{1}{2} \cdot \frac{L_2^2}{nL_1 + L_2} \left( \frac{x_1}{L_1} \right) \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right]. \quad \quad (201a)
\]

For \( I_1 = I_2 \):

\[
m_b = \frac{1}{2} \cdot \frac{L_2^2}{L_1 + L_2} \left( \frac{x_1}{L_2} \right) \left[ I - \left( \frac{x_1}{L_2} \right)^2 \right].
\]

For \( L_1 = L_2 \):

\[
m_b = \frac{L}{4} \left( \frac{x_1}{L} \right) \left[ I - \left( \frac{x_1}{L} \right)^2 \right].
\]

The ordinates of the influence line for \( m_b \) are shown in Fig. 180c.

For a uniformly-distributed load on span BC, from (200),

\[
m_b = \int_0^L w \frac{L}{4} \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right] dx = \frac{wL^2}{16}.
\]

Similarly, for beam AB,

\[
y_1 = \frac{L_1^2}{6I_1} \left( \frac{x_1}{L_1} \right) \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right], \quad \text{and} \quad m_b = \frac{1}{2} \cdot \frac{nL_1^2}{nL_1 + L_2} \left( \frac{x_1}{L_1} \right) \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right]. \quad (201b)
\]
REACTION $R_a$ (Fig. 181a).—The influence line for $R_a$ can be calculated from (202):

$$R_a y_a = W y, \text{ or } R_a = y \quad \ldots \ldots \ldots \quad (202)$$

in which $y_1 = \left(\frac{x_1}{L_1}\right) - m_b A_{ba}$ for span AB, and $y_2 = m_b A_{be}$ for span BC.

![Diagram](image)

**Fig. 181.**

For prismatic beams $m_b$ is calculated from $m_b = \frac{\left(L_1 + L_2\right)}{3L_1 + 3L_2}$ or

$$m_b = \frac{3nEI_y y}{L_1(nL_1 + L_2)}$$

in which $n = \frac{I}{I_1}$. For span AB, therefore,

$$R_a = y_1 = \left(\frac{x_1}{L_1}\right) - \frac{3nEI_1}{L_1(nL_1 + L_2)} \cdot \frac{L_1^2}{6EI_1}\left(\frac{x_1}{L_1}\right)\left[1 - \left(\frac{x_1}{L_1}\right)^2\right]$$

$$= I - \left(\frac{x_1}{L_1}\right) - \frac{nL_1}{2(nL_1 + L_2)}\left[1 - \left(\frac{x_1}{L_1}\right)^3\right] \quad \ldots \ldots \ldots \quad (203a)$$

If $L_1 = L_2$ and $I_1 = I_2$:

$$R_a = I - \left(\frac{x}{L}\right) - \frac{x}{4}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] \quad \ldots \ldots \ldots \quad (203b)$$

For uniformly-distributed load on span AB, $R_a$ calculated from (203a) is

$$R_a = \int_0^L w\left[I - \left(\frac{x}{L}\right) - \frac{x}{4}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]\right]dx = \frac{1}{16}wL.$$

For span BC: $R_a = y_2 = \frac{3nEI_1}{L_1(nL_1 + L_2)} \cdot \frac{L_2^2}{6EI_2}\left(\frac{x_2}{L_2}\right)\left[1 - \left(\frac{x_2}{L_2}\right)^2\right]$$

$$= \frac{L_2^2}{2L_1(nL_1 + L_2)}\left[\frac{x_2}{L_2} - \frac{x_2}{L_2}\right] \quad \ldots \ldots \ldots \quad (203c)$$

If $L_1 = L_2$ and $I_1 = I_2$:

$$R_a = \frac{x}{4}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^3\right] \quad \ldots \ldots \ldots \quad (203d)$$

**Fig. 181b** shows the ordinates of the influence line for $R_a$ calculated from (203a) and (203c). For a uniformly-distributed load on span BC, $R_a$, calculated from (203a), is

$$R_a = \int_0^L w\frac{x}{4}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^3\right]dx = \frac{1}{16}wL.$$
REACTION $R_b$ (Fig. 182).—In a similar way the influence line for $R_b$ can be found from $R_by_b = Wy$. For the span AB, $y_1 = \left(\frac{x_1}{L_1}\right) + m_b A_{ba}$, in which $m_b$ is calculated from $m_b = \left(\frac{L_1}{3I_1} + \frac{L_2}{3I_2}\right) = \frac{y_b}{L_1} + \frac{y_b}{L_2}$, or $m_b = \frac{3I_2}{nL_1 + L_2} \cdot \frac{L_1 + L_2}{L_1L_2} \cdot y_b$.

Finally,

$$R_b = y_1 = \left(\frac{x_1}{L_1}\right) + \frac{3I_2}{nL_1 + L_2} \cdot \frac{L_1 + L_2}{L_1L_2} \cdot \frac{L_1^2}{6I_1} \left[ I - \left(\frac{x_1}{L_1}\right)^2 \right]$$

$$= \left(\frac{x_1}{L_1}\right) + \frac{1}{2} \cdot \frac{L_1}{L_2} \left[ I - \left(\frac{x_1}{L_1}\right)^2 \right] \quad \ldots \quad (204a)$$

If $I_1 = I_2$: $R_b = \left(\frac{x_1}{L_1}\right) + \frac{1}{2} \cdot \frac{L_1}{L_2} \left[ I - \left(\frac{x_1}{L_1}\right)^2 \right] \quad \ldots \quad (204b)$

If $L_1 = L_2$: $R_b = \left(\frac{x_1}{L}\right) + \frac{1}{2} \left[ I - \left(\frac{x_1}{L}\right)^2 \right] \quad \ldots \quad (204c)$

For a uniformly-distributed load on span AB, $R_b$, calculated from (204c), is

$$R_b = \int_0^L w \left( \left(\frac{x}{L}\right) + \frac{1}{2} \left[ I - \left(\frac{x}{L}\right)^2 \right] \right) dx = \frac{1}{6} wL.$$

For span BC: $R_b = \left(\frac{x_2}{L_2}\right) + \frac{1}{2} \cdot \frac{nL_2}{L_1 + L_2} \cdot \frac{L_1 + L_2}{L_1L_2} \left[ I - \left(\frac{x_2}{L_2}\right)^2 \right] \quad \ldots \quad (205)$

*Fig. 182(b)* shows the ordinates of the influence line for $R_b$ calculated from (204) and (205).

**BENDING MOMENT ON SPAN AB** (Fig. 183).—The influence line for the bending moment on span AB is calculated from (206a).

For span AB: $x_1 < a$: $m_e = m_e - m_b \cdot \frac{a}{L} \quad \ldots \quad (206a)$

For prismatic beams, $m_b$ is calculated from (206a).

$$m_b \left(\frac{L_1}{3I_1} + \frac{L_2}{3I_2}\right) = \frac{Wab}{6L_1L_1} (L_1 + a); \quad m_b = \frac{1}{2} \cdot \frac{nL_1^2}{nL_1 + L_2} \left[ I - \left(\frac{x_1}{L_1}\right)^2 \right];$$
\[ m_e = b \frac{x_1}{L_1} - a \frac{x}{L_2} \frac{nL_1^2}{2} \left( \frac{x_1}{L_1} \right) \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right] \]

\[ = \left( \frac{x_1}{L_1} \right) \left\{ b - a \frac{nL_1^2}{2} \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right] \right\} \]

\[ \text{(206b)} \]

If \( L_1 = L_2 \) and \( I_1 = I_2 \), then:

\[ m_e = \left( \frac{x}{L} \right) \left\{ b - a \frac{x}{4L} \left[ I - \left( \frac{x}{L} \right)^2 \right] \right\} \]

\[ \text{(206c)} \]

The bending moments, calculated from (206b), are:

For a uniformly-distributed load on span AB:

\[ m_e = \int_0^L \frac{w}{L} \left( \frac{x_1}{L_1} \right) \left( \frac{L}{2} - \frac{L}{8} \left[ I - \left( \frac{x}{L} \right)^2 \right] \right) dx = \frac{3wL^2}{32}. \]

For the span AB, \( x_1 \leq b \), therefore

\[ m_e = m_e^0 - a \frac{x}{L} m_e = a \frac{x_1}{L_1} - a \frac{x}{L_2} \frac{nL_1^2}{2} \left( \frac{x_1}{L_1} \right) \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right] \]

\[ = a - a \left( \frac{x_1}{L_1} \right) \left\{ I + \frac{nL_1^2}{2(L_1 + L_2)} \left[ I - \left( \frac{x_1}{L_1} \right)^2 \right] \right\} \]

\[ \text{(206d)} \]

If \( L_1 = L_2 \) and \( I_1 = I_2 \), then:

\[ m_e = a - a \left( \frac{x}{L} \right) \left\{ I + \frac{1}{4L} \left[ I - \left( \frac{x}{L} \right)^2 \right] \right\} \]

\[ \text{(206e)} \]

For a concentrated load \( W \) at \( a = \frac{L}{2} \), \( m_e = \frac{13}{4}WL. \)

For span (BC) (Fig. 183),

\[ m_e = m_e a \frac{I_1}{L_1} = a \frac{L_2^2}{2(L_1 + L_2)} \left( \frac{x_1}{L_2} \right)^2 \left[ I - \left( \frac{x_2}{L_2} \right)^2 \right] \]

\[ \text{(207a)} \]

If \( L_1 = L_2 \) and \( I_1 = I_2 \), then:

\[ m_e = a \left( \frac{x_2}{L} \right) \left[ I - \left( \frac{x_2}{L} \right)^2 \right] \]

\[ \text{(207b)} \]
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For a uniformly-distributed load on span BC, from (207b),

\[ m_e = \int_0^L wL^2 a \left( \frac{x^2}{L} \right) \left[ 1 - \left( \frac{x^2}{L} \right)^2 \right] dx = \frac{wL^4}{32} \]

The ordinates of the influence line for the bending moment at \( a = \frac{L}{2} \), calculated from (206c) and (207b), are shown in Fig. 183(c).

Three-span Continuous Beams.

The ordinates of the influence lines for continuous beams of any number of spans are calculated in a similar way.

**REACTION** \( R_a \) (Fig. 184).—The influence line for \( R_a \) is calculated from

\[ R_ay_a = Wy, \text{ or } R_a = y \quad \ldots \quad . \quad (208a) \]

![Fig. 184.](image)

In a general case, for non-prismatic beams: For span AB, \( y_1 = \left( \frac{x_1}{L_1} \right) - m_bA_{ba} \); For span BC, \( y_2 = m_bA_{bc} - m_cA_{cb} \); For span CD, \( y_3 = m_cA_{cd} \).

In a general case the values of \( A \) are calculated from equations (2), and \( m_b \) and \( m_c \) from equation (22). For prismatic beams the values of \( A \) are in (194), and the bending moments \( m_b \) and \( m_c \) are calculated from

\[ m_b \left( \frac{L_1}{3I_1} + \frac{L_2}{3I_2} \right) - m_c \frac{L_2}{6I_2} = y_a \quad \text{and} \quad m_c \left( \frac{L_2}{3I_2} + \frac{L_3}{3I_3} \right) - m_b \frac{L_2}{6I_2} = 0. \]

For three equal spans of constant cross sections, \( m_b = \frac{8y_aEI}{5L^2} \) and \( m_c = \frac{2y_aEI}{5L^2} \) [see equation (24)].

Substituting these values in equation (208a),

For span AB:

\[ R_a = \left( \frac{x_1}{L} \right) - \frac{8EI}{5L^2} \frac{L^2}{6EI} \left( \frac{x_1}{L} \right) \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] = \frac{4}{15} \left( \frac{x_1}{L} \right)^3 - \frac{19}{15} \left( \frac{x_1}{L} \right) + 1. \quad (208b) \]
For span BC:
\[
R_a = \frac{8EI}{5L^2} \left( \frac{x_2}{L} \right) \left[ I - \left( \frac{x_2}{L} \right)^2 \right] - \frac{2EI}{6EL} \left( \frac{x_2}{L} \right) \left[ I - \left( \frac{x_2}{L} \right)^2 \right] \\
= \frac{I}{2} \left( \frac{x_2}{L} \right) \left[ 5 \left( \frac{x_2}{L} \right)^2 - 12 \left( \frac{x_2}{L} \right) + 7 \right] \quad (208c)
\]

For span CD:
\[
R_a = \frac{2EI}{5L^2} \left( \frac{x_2}{L} \right) \left[ I - \left( \frac{x_2}{L} \right)^2 \right] = \frac{1}{15} \left( \frac{x_3}{L} \right) \left[ \left( \frac{x_3}{L} \right)^2 - 3 \left( \frac{x_3}{L} \right) + 2 \right] \quad (208d)
\]

The reactions \( R_a \) for a uniformly-distributed load, calculated from equations (208a), are:

On span AB: \( R_a = \int_0^L w \left[ \frac{4}{15} \left( \frac{x}{L} \right)^2 - \frac{10}{15} \left( \frac{x}{L} \right) + 1 \right] dx = \frac{1}{36} wL \);  

On span BC: \( R_a = \int_0^L \frac{1}{15} \left( \frac{x}{L} \right) \left[ 5 \left( \frac{x}{L} \right)^2 - 12 \left( \frac{x}{L} \right) + 7 \right] dx = \frac{wL}{20} \);  

On span CD: \( R_a = \int_0^L \frac{1}{15} \left( \frac{x}{L} \right) \left[ \left( \frac{x}{L} \right)^2 - 3 \left( \frac{x}{L} \right) + 2 \right] dx = \frac{1}{60} wL \).

Reactions \( R_b \) (Fig. 185a).—Similarly for \( R_b \),
\[
R_b y_b = Wy, \text{ or } R_b = y \quad (209a)
\]

For AB, \( y_1 = \left( \frac{x_1}{L_1} \right) + m_b \Delta_{bs} \). For BC, \( y_2 = \left( \frac{x_2}{L_2} \right) + m_b \Delta_{bc} - m_c \Delta_{cb} \). For CD, \( y_3 = m_c \Delta_{cd} \).

For a beam with three equal spans of a constant moment of inertia, \( m_b \) and \( m_c \) are [see (21)] \( m_b = \frac{18EI}{5L^2} \) and \( m_c = \frac{12EI}{5L^2} \). Substituting these values and the values of \( \Delta \) from (194) in (209a),

For span AB:
\[
R_b = \left( \frac{x_1}{L} \right) + \frac{18EI}{5L^2} \cdot \frac{L^2}{6EI} \left( \frac{x_1}{L} \right) \left[ I - \left( \frac{x_1}{L} \right)^2 \right] = \frac{1}{5} \left( \frac{x_1}{L} \right) \left[ 8 - 3 \left( \frac{x_1}{L} \right)^2 \right] \quad (209b)
\]

Fig. 185.
For span BC:

\[
R_b = \left(\frac{x_2}{L}\right) + \frac{18EI}{5L^2} \cdot \frac{L^2}{6EI} \left(\frac{x_2}{L}\right)^2 \left[ 1 - \left(\frac{x_2}{L}\right)^2 \right] - \frac{12EI}{5L^2} \cdot \frac{L^2}{6EI} \left(\frac{x_2}{L}\right) \left[ 1 - \left(\frac{x_2}{L}\right)^2 \right]
\]

\[
= \frac{L}{5} \left[ 5 \left(\frac{x_2}{L}\right)^3 - 9 \left(\frac{x_2}{L}\right)^2 + \left(\frac{x_2}{L}\right) + 5 \right]
\]  

(209c)

For span CD:

\[
R_b = \frac{12EI}{5L^2} \cdot \frac{L^2}{6EI} \left(\frac{x_2}{L}\right) \left[ 1 - \left(\frac{x_2}{L}\right)^2 \right] = \frac{2}{5} \left(\frac{x_2}{L}\right) \left[ 1 - \left(\frac{x_2}{L}\right)^2 \right]
\]  

(209d)

The reactions \( R_b \) for uniformly-distributed load, calculated from (209b, c, d), are,

On span AB: \( R_b = \int_0^L \frac{w}{L} \left[ \frac{x}{L} \left( \frac{x}{L} \right)^3 - 3 \left( \frac{x}{L} \right)^2 \right] dx = \frac{1}{4} \frac{3}{2} wL \);

On span BC: \( R_b = \int_0^L \frac{w}{L} \left[ \frac{x}{L} \left( \frac{x}{L} \right)^3 - 9 \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right) + 5 \right] dx = \frac{1}{2} \frac{3}{2} wL \);

On span CD: \( R_b = \int_0^L \frac{w}{L} \left[ \frac{x}{L} \right]^3 \left[ 1 - \left( \frac{x}{L} \right)^2 \right] dx = \frac{1}{10} wL \).

**Bending Moments at Supports B and C.**—The influence line for the bending moment at support B (Fig. 186a) is calculated from

\[
m_b(\theta_1 + \theta_2) = Wy; \text{ or } m_b = \frac{y}{\theta_1 + \theta_2} \]

(210a)

---

In a general case of non-prismatic beams, for span AB: \( m_b = \frac{A_{ba}}{(\theta_1 + \theta_2)} \);

For span BC: \( m_b = \frac{A_{bc} - m'_{c} \cdot A_{cb}}{\theta_1 + \theta_2} \); For span CD: \( m_b = \frac{m'_{c} \cdot A_{cd}}{\theta_1 + \theta_2} \), in which
\( \theta_1, \theta_2, \text{ and } m'_c \) are calculated as follows:

\[
m'_{L_2} \left( \frac{L_2}{3} \right) = m'_b \frac{L_2}{6} \quad \text{or} \quad m'_c = \frac{L_2}{12} \frac{L_2}{L_2 + L_3}
\]

\[
EI\theta_1 = \frac{L_1}{3}, \quad EI\theta_2 = m'_b \left( \frac{L_2}{3} \right) - m'_c \frac{L_2}{6} = \frac{L_2}{12} \frac{3L_2}{L_2 + L_3}
\]

For all spans equal, \( m'_c = \frac{1}{4}; \quad EI\theta_1 = \frac{L}{3}, \quad EI\theta_2 = \frac{L}{24} \quad \text{or} \quad EI(\theta_1 + \theta_2) = \frac{L}{8}\)

For span AB:

\[
m_b = \frac{8}{5L} \frac{L^2}{6} \left( \frac{x_1}{L} \right) \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] = \frac{4}{15} \frac{x_1}{L} \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] L \quad \ldots \quad (210b)
\]

For span BC:

\[
m_b = \frac{8}{5L} \frac{L^2}{6} \left( \frac{x_2}{L} \right) \left[ 1 - \left( \frac{x_2}{L} \right)^2 \right] - \frac{8}{5L} \frac{L^2}{6} \left( \frac{x_2}{L} \right) \left[ 1 - \left( \frac{x_2}{L} \right)^2 \right] \frac{1}{4}
\]

\[
= \frac{L}{15} \left[ 5 \left( \frac{x_2}{L} \right)^3 - 12 \left( \frac{x_2}{L} \right)^2 + 7 \left( \frac{x_2}{L} \right) \right] \quad \ldots \quad \ldots \quad (210c)
\]

For span CD:

\[
m_b = \frac{8}{5L} \frac{L^2}{6} \left( \frac{x_3}{L} \right) \left[ 1 - \left( \frac{x_3}{L} \right)^2 \right] = \frac{L}{15} \left[ \left( \frac{x_3}{L} \right)^3 - 3 \left( \frac{x_3}{L} \right)^2 + 2 \left( \frac{x_3}{L} \right) \right] \quad \ldots \quad (210d)
\]

Bending moments \( m_b \) for uniformly-distributed load, calculated from (210b, c, d), are

On span AB:

\[
m_b = \int_0^L \left[ \frac{4L}{15} \frac{x}{L} \left[ 1 - \left( \frac{x}{L} \right)^2 \right] \right] dx = \frac{wL^2}{15}.
\]

On span BC:

\[
m_b = \int_0^L \left[ \frac{L}{15} \left[ 5 \left( \frac{x}{L} \right)^3 - 12 \left( \frac{x}{L} \right)^2 + 7 \left( \frac{x}{L} \right) \right] \right] dx = \frac{wL^2}{20}.
\]

On span CD:

\[
m_b = \int_0^L \left[ \frac{L}{15} \left[ \left( \frac{x}{L} \right)^3 - 3 \left( \frac{x}{L} \right)^2 + 2 \left( \frac{x}{L} \right) \right] \right] dx = \frac{wL^2}{60}.
\]

Similarly, the influence line for \( m_c \) (in the same system of ordinates) is

For span AB:

\[
m_c = \frac{m'_b A_{ba}}{\theta_1 + \theta_2} = \frac{L}{15} \left( \frac{x_1}{L} \right) \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] \quad \ldots \quad \ldots \quad (211a)
\]

For span BC:

\[
m_c = \frac{(m'_c A_{cb} - m'_b A_{bc})}{\theta_1 + \theta_2} = \frac{L}{15} \left[ -5 \left( \frac{x_2}{L} \right)^3 + 3 \left( \frac{x_2}{L} \right)^2 + 2 \left( \frac{x_2}{L} \right) \right] \quad (211b)
\]

For span CD:

\[
m_c = \frac{m'_c A_{cd}}{\theta_1 + \theta_2} \quad \ldots \quad \ldots \quad \ldots \quad (211c)
\]

**Bending Moment on First Span (Fig. 187).—**The ordinates of the influence lines for the bending moment on the first span are calculated as follows.
For $W = 1$ on span $AB$: $x_1 \leq a$: $m_e = m^o_e - \frac{a}{L}m_b$.  \hspace{1cm} (212a)

in which $m_b$ and $m_e$ are calculated from:

$$
\left(\frac{L_1}{3I_1} + \frac{L_2}{3I_2}\right)m_b - \frac{L_2}{6I_2}m_e = \frac{Wx_1}{6L_1I_1}(L_1 + x_1); \hspace{0.5cm}
\left(\frac{L_2}{3I_2} + \frac{L_3}{3I_3}\right)m_e - \frac{L_2}{6I_2}m_b = 0; \hspace{0.5cm}
$$

from which, if $L_1 = L_2 = L_3$ and $I_1 = I_2 = I_3$, $m_b = \frac{4L}{15}\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_1}{L}\right)^2\right]$ and $m_e = \frac{L}{15}\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_1}{L}\right)^2\right]$.

Substituting these values in (212a), $m_e = \left(\frac{x_1}{L}\right)\left\{a - \frac{4a}{15}\left[1 - \left(\frac{x_1}{L}\right)^2\right]\right\}$.  \hspace{1cm} (212b)

For $W = 1$ on span $AB$, and $z_1 < b$: $m_e = m^o_e - \frac{a}{L}m_b$.  \textit{If all spans are equal and $I$ is constant},

$$
m_e = \left(\frac{z_1}{L}\right)a - \frac{4L}{15}\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_1}{L}\right)^2\right]$$

$$
= a\left\{1 - \left(\frac{x_1}{L}\right) - \frac{4}{15}\left(\frac{x_1}{L}\right)\left[1 - \left(\frac{x_1}{L}\right)^2\right]\right\}. \hspace{1cm} (212c)
$$

For a concentrated load at $x_1 = a = \frac{L}{2}$, from (212a) or (212c), $m_e = \frac{1}{8}WL$.

For load $W = 1$ on span $BC$ (Fig. 188), $m_e = m_b \frac{a}{L}$.  \textit{If all the spans are}
equal and \( I \) is constant,

\[
m_b = \frac{L}{15} \left( \frac{x_2}{L} \right) \left( 1 - \frac{x_2}{L} \right) \left( 7 - \frac{5x_2}{L} \right); \quad m_c = \frac{L}{15} \left( \frac{x_2}{L} \right) \left( 1 - \frac{x_2}{L} \right) \left( 2 + \frac{5x_2}{L} \right);
\]

\[
m_e = \frac{a}{4} \frac{L}{15} \left( \frac{x_3}{L} \right) \left( 1 - \frac{x_3}{L} \right) \left( 7 - \frac{5x_3}{L} \right) = \frac{a}{15} \frac{L}{15} \left( \frac{x_3}{L} \right) \left( 1 - \frac{x_3}{L} \right) \left( 7 - \frac{5x_3}{L} \right). \quad (212d)
\]

Similarly for \( W = T \) on span CD,

\[
m_e = \frac{a}{L} m_b = \frac{a}{L} \frac{L}{15} \left( \frac{x_3}{L} \right) \left( 1 - \left( \frac{x_3}{L} \right)^2 \right). \quad (212e)
\]

For a concentrated load \( W \) at \( x_2 = \frac{L}{2} \) and \( a = \frac{L}{2} \), from (212d), \( m_e = \frac{3}{8} WL \).

For a load \( W \) at \( x_2 = \frac{L}{2} \) and \( a = \frac{L}{2} \), from (212e), \( m_e = \frac{1}{8} WL \).

**Bending Moment on Middle Span (Fig. 189a).—**

For span AB:

\[
m_e = m_b \frac{b}{L} - m_c \frac{a}{L}. \quad (213a)
\]

![Fig. 189](image.png)

If all spans are equal and \( I \) is constant,

\[
m_e = \frac{b}{L} \frac{4L}{15} \left( \frac{x_1}{L} \right) \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] - a \frac{L}{15} \left[ \left( \frac{x_1}{L} \right)^3 - 3 \left( \frac{x_1}{L} \right)^2 + 2 \left( \frac{x_1}{L} \right) \right]
\]

\[
= \frac{1}{15} \left\{ 4b \left( \frac{x_1}{L} \right) \left[ 1 - \left( \frac{x_1}{L} \right)^2 \right] - a \left[ \left( \frac{x_1}{L} \right)^3 - 3 \left( \frac{x_1}{L} \right)^2 + 2 \left( \frac{x_1}{L} \right) \right] \right\}. \quad (213b)
\]

Similarly, for span CD,

\[
m_e = \frac{1}{15} \left\{ 4a \left( \frac{x_3}{L} \right) \left[ 1 - \left( \frac{x_3}{L} \right)^2 \right] - b \left[ \left( \frac{x_3}{L} \right)^3 - 3 \left( \frac{x_3}{L} \right)^2 + 2 \left( \frac{x_3}{L} \right) \right] \right\}. \quad (213c)
\]

From (213b), for \( x = \frac{L}{2} \) and \( a = \frac{L}{2} \), \( m_e = \frac{3}{8} WL \). For a uniformly-distributed load on span AB, and with \( a = \frac{L}{2} \), \( m_e = \frac{1}{4} wL^2 \).
For span BC and $x_2 \leq a$ (Fig. 190): 
\[ m_e = m_e^0 - \frac{b}{L} m_b - \frac{a}{L} m_c \]  \hspace{1cm} (214a) 

If all the spans are equal and $I$ is constant,
\[ m_e = b \left( \frac{x_2}{L} \right) - \frac{b}{L} \cdot \frac{L}{I_5} \left[ 5 \left( \frac{x}{L} \right)^3 - 12 \left( \frac{x}{L} \right)^2 + 7 \left( \frac{x}{L} \right) \right] - \frac{a}{L} \cdot \frac{L}{I_5} \left[ -5 \left( \frac{x}{L} \right)^3 + 3 \left( \frac{x}{L} \right)^2 + 2 \left( \frac{x}{L} \right) \right] \]
\[ = \frac{b}{I_5} \left[ -5 \left( \frac{x}{L} \right)^3 + 12 \left( \frac{x}{L} \right)^2 + 8 \left( \frac{x}{L} \right) \right] - \frac{a}{I_5} \left[ -5 \left( \frac{x}{L} \right)^3 + 3 \left( \frac{x}{L} \right)^2 + 2 \left( \frac{x}{L} \right) \right] \]  \hspace{1cm} (214b) 

Similarly for span BC and $x_2 = b$,
\[ m_e = \frac{a}{I_5} \left[ -5 \left( \frac{z}{L} \right)^3 + 12 \left( \frac{z}{L} \right)^2 + 8 \left( \frac{z}{L} \right) \right] - \frac{b}{I_5} \left[ -5 \left( \frac{z}{L} \right)^3 + 3 \left( \frac{z}{L} \right)^2 + 2 \left( \frac{z}{L} \right) \right] \]  \hspace{1cm} (214c) 

Fig. 189b shows the ordinates of the influence line for the bending moment at E ($a = 0.6L$), calculated from (213b and c) and (214b and c). For a concentrated load $W$ at $x = a$, from (214b), 
\[ m_e = \frac{2Wab}{I_5L} \left[ 4L^2 + 5aL - 5a^2 \right] \] 
For $x = a = \frac{L}{2}$, 
\[ m_e = \frac{7}{40} WL. \]
APPENDIX

In the following the load functions and elastic constants are shown for various loads and shapes of beams. These are calculated from equations (2), and only the final results are shown. It is not essential to refer to these formulae, but their use will facilitate calculation.

Straight Prismatic Beams.
Simply-supported Beam Subjected to Unit Bending Moment (Fig. 191).

\[ EI\varepsilon = \int_A^B m \, dx = \frac{L}{2}; \quad \tilde{x} = \frac{L}{3}; \quad EI\alpha = \frac{L}{3}; \quad EI\beta = \frac{L}{6}. \]

\[ EI\phi_x = x - \frac{x^2}{2L} - \frac{L}{3}; \quad EI\Delta_x = \frac{1}{2}L\lambda \left( 2 - \frac{3x}{L} + \frac{x^2}{L^2} \right). \quad \text{For} \quad x = L \left( 1 - \frac{\sqrt{3}}{3} \right), \]

max. \( EI\Delta_x = \frac{L^3}{9\sqrt{3}}. \)

**UNIFORMLY-DISTRIBUTED LOAD OVER WHOLE SPAN (Fig. 192).** \( EI\varepsilon = \frac{1}{12}wL^3. \)

\[ EI\theta = \frac{1}{12}wL^3. \quad EI\phi_x = \frac{1}{12}wL^3 - \frac{1}{6}wx^2 \left( \frac{L}{2} - \frac{x}{3} \right). \quad EI\Delta_x = \frac{wx}{24} \left[ L^3 - x^2(2L - x) \right]. \]

For \( x = \frac{L}{2}, \) \( EI\phi_x = 0 \) and max. \( EI\Delta = \frac{1}{128}wL^4. \)

**UNIFORMLY-DISTRIBUTED LOAD OVER PART OF SPAN (Fig. 193).** \( \xi = \frac{2L^2 - a^2}{2(3L - 2a)}. \)

\[ EI\theta_a = \frac{wa^2}{24L}(2L - a)^2; \quad EI\theta_b = \frac{wa^2}{24L}(2L^2 - a^2). \]

\[ m_a = \frac{wa^3}{12L^2}(6L^2 - 8aL + 3a^2); \quad m_b = \frac{wa^3}{12L^2}(4L - 3a). \]

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For \( a = \frac{L}{2} \):
\[ EI\theta_a = \frac{3wL^3}{128} \; ; \; EI\theta_b = \frac{7wL^3}{384} \; ; \; m_a = \frac{11wL^2}{192} \; ; \; m_b = \frac{5wL^2}{192} \]  

**Uniformly-distributed Load as in Fig. 194.**

\[ EI\theta_a = \frac{w}{12} \left[ 2L(a^2 - b^2) - 2(a^2 - b^2) + \frac{1}{2L}(a^4 - b^4) \right] \]
\[ EI\theta_b = \frac{w}{12}(a^2 - b^2) \left[ L - \frac{1}{2L}(a^2 + b^2) \right] \]
\[ m_a = \frac{2}{L}(2\theta_a - \theta_b) \; ; \; m_b = \frac{2}{L}(2\theta_b - \theta_a). \]

**Uniformly-distributed Load as in Fig. 195.**

\[ EI\theta = \frac{wa^2}{12} (3L - 2a). \; \; m^F = \frac{wa^2}{6L} (3L - 2a). \]

**Concentrated Load (Fig. 196).** — \( EI\varepsilon = \frac{w}{3}m_0L \; ; \; \varepsilon = \frac{L + a}{3} \; ; \; \bar{z} = \frac{L + b}{3} \)
\[ EI\theta_a = \frac{1}{8}m_o(L + b) = \frac{Wa_b}{6L}(L + b). \quad EI\theta_b = \frac{1}{8}m_o(L + a) = \frac{Wa_b}{6L}(L + a).\]

For \( a = \frac{L}{2}, \ EI\theta = \frac{WL^2}{16}.\)

For \( x < a: \ EI\phi_x = \frac{1}{8}m_o(L + b) - \frac{m_o x^2}{2a}; \ EI\Delta_x = \frac{1}{8}m_o[(L + b)x - \frac{x^3}{a}].\)

For \( x < b: \ EI\phi_x = \frac{1}{8}m_o(L + a) - \frac{m_o x^2}{2a}; \ EI\Delta_x = \frac{1}{8}m_o[(L + a)x - \frac{x^3}{b}].\)

For \( x = a \) (or \( x = b): \ EI\Delta_x = \frac{1}{8}m_o x.\)

For \( a = \frac{L}{2}, \max. \ EI\Delta = \frac{wL^3}{48}; \ m_a = \frac{Wab}{L}; \ m_b = \frac{Wab}{L}.\)

**Two Concentrated Loads Symmetrically Disposed** (Fig. 197).

\[ EI\theta = \frac{1}{8}Wa(L - a); \ EI\Delta = \frac{Wa^2}{6}(3L - 4a). \ m_F = \frac{Wa}{L}(L - a).\]

For \( a = \frac{L}{3}; \ m_F = \frac{3}{8}WL.\)

**Crane Load** (Fig. 198).—\( m_o = We; \ V_a = V_b = \frac{m}{L}; \ m_{ca} = m_o \frac{a}{L}; \ m_b = m_o \frac{b}{L}.\)

\[ EI\theta_a = \frac{m}{6L}[a^3 + 3a^2b - 2ab^2]. \ EI\theta_b = \frac{m}{6L}[b^3 + 3ab^2 - 2a^3].\]

\[ m_a = \frac{2}{L}(2\theta_a + \theta_b) = m_o \frac{b}{L}(2 - \frac{b}{3L}). \ m_b = \frac{2}{L}(2\theta_b + \theta_a) = m_o \frac{a}{L}(2 - \frac{a}{3L}).\]

For \( a = b = \frac{L}{2}; \ EI\theta_a = EI\theta_b = m_o \frac{5L}{48}; \ m_a = m_b = m_{ca} = \frac{1}{4}m_o.\)

**Triangular Load** (Fig. 199).—\( EI\theta_a = \frac{7}{8 \times 45}pL^3. \ EI\theta_b = \frac{7}{8 \times 45}pL^3; \ x = \frac{2}{3}L.\)

\[ m_x = \frac{pxL}{6}[1 - \left(\frac{x}{L}\right)^2].\]

For \( x = \frac{L}{\sqrt{3}}, \ m_o \max. = \frac{pL^2}{9 \sqrt{3}}. \ m_a = \frac{1}{8}pL^2, \ m_b = \frac{1}{8}pL^2.\)
APPENDIX

Fig. 199.

**Triangular Load as in Fig. 200.**

\[ EI\theta = \frac{5}{6} \times \frac{1}{32} pL^3; \quad m_x = \frac{p}{12} \left[ \frac{1}{2} Lx - \frac{1}{2} x^2 \right]; \quad m_o \text{ max.} = \frac{1}{4} pL^2; \quad m_F = \frac{3}{8} pL^2. \]

\[ V_o = \frac{1}{4} pL. \]

**Translations of Joints (Fig. 201).**—For non-prismatic beams with fixed ends

Fig. 200.

\[ m_a \text{ and } m_b \text{ are calculated from} \]

\[ \alpha_{ab} m_a - \beta m_b = \frac{\Delta}{L} \text{ and } \alpha_{ba} m_b - \beta m_a = \frac{\Delta}{L}. \]

For prismatic beams, \[ \frac{L}{3EI} m^F - \frac{L}{6EI} m^P = \frac{\Delta}{L}, \]

from which \[ m^P = \frac{6EI\Delta}{L^2}, \text{ and} \]

\[ V = \frac{12EI\Delta}{L^2}. \]

**Beam with One End Hinged** (Fig. 202).—\[ \alpha_{ba} m^P = \frac{\Delta}{L} \text{ or } \frac{L}{3EI} m^P = \frac{\Delta}{L}, \]

from which \[ m^P = \frac{3EI\Delta}{L^2}; \quad V = \frac{3EI\Delta}{L^3}. \]
APPENDIX

FIG. 202.

Symmetrical Sloping Beam.—For unit end bending moment \( m_a = 1 \) (Fig. 203),

\[
EIa = \frac{3}{8}s; \quad EI\beta = \frac{3}{4}s; \quad EI\Delta^m = \frac{1}{3}ps.
\]

For unit horizontal force \( H = 1 \) (Fig. 204), \( EI\gamma = EI\delta = \frac{1}{3}ps; \quad EI\Delta^h = \frac{2}{3}p^2s \)

Fig. 203.

For uniformly-distributed load on the whole beam (Fig. 205),

\[
EI\theta_a = EI\theta_b = \frac{2}{3}wL^2s; \quad EI\Delta^o = \frac{2}{3}wL^2p.\]

Fig. 204.

For uniformly-distributed load on part of the beam (Fig. 206),

Fig. 205.

Fig. 206.
\[ EI \varepsilon = \frac{wa^2s}{6L}(3L - 2a) \]; \[ \delta = \frac{2L^2 - a^2}{2(3L - 2a)} \];
\[ EI \theta_a = \frac{wa^2s}{12L}(2L - a) \delta; \quad EI \theta_b = \frac{wa^2s}{12L}(2L^2 - a^2); \]
\[ EI \Delta^o = \frac{wa^2ps}{12L^2}(3L^2 - 2a^2). \]

For \( a = \frac{L}{2} \):
\[ EI \theta_a = \frac{1}{8} wL^2 \delta; \quad EI \theta_b = \frac{1}{16} wL^2 \delta; \quad \delta = \frac{1}{2} L; \quad EI \Delta^o = \frac{1}{3} wL^2 \delta. \]

For Concentrated Load (Fig. 207), \( EI \varepsilon = m_o \delta \); \[ \delta = \frac{1}{2}(L + a); \]

For uniformly-distributed load on inclined beam (Fig. 208),
\[ EI \theta_a = \frac{Wabs(L + b)}{3L^2}; \quad EI \theta_b = \frac{Wabs(L + a)}{3L^2}; \quad EI \Delta^o = \frac{Wap}{6L^2}(3L^2 - 4a^2). \]

For \( a = \frac{L}{2} \):
\[ EI \theta = \frac{1}{8} WL \delta; \quad EI \Delta^o = \frac{1}{8} WL \delta. \]

For uniformly-distributed load on inclined beam (Fig. 208),
\[ EI \varepsilon = \frac{1}{8} Wp \delta; \quad \delta = \frac{1}{5} L; \]
\[ EI \theta_a = \frac{1}{16} Wp \delta; \quad EI \theta_b = \frac{1}{32} Wp \delta; \quad EI \Delta^o = \frac{1}{16} Wp \delta. \]

Unsymmetrical Sloping Beams.

For bending moment \( m_a = 1 \) (Fig. 209),
\[ EI \varepsilon = \frac{1}{2L}[s_1(L + b) + s_2b]; \quad \delta = \frac{s_1a(L + 2b) + s_2b(L + 2a)}{3 s_1(L + b) + s_2b} \]
\[ EI \theta_{ab} = \frac{r_a}{2L}; \quad EI \beta = \frac{r_a}{L}; \quad EI \Delta^{ma} = \frac{p}{6L}[s_1(L + 2b) + 2s_2b]. \]
For bending moment $m_b = 1$ (Fig. 210),

$$EIe_{ba} = \frac{1}{2L}[s_1(L + a) + s_1a]$$

$$z = \frac{1}{3} \cdot \frac{s_2b(L + 2a) + s_1a(L + 2b)}{s_2(L + a) + s_1a}$$

$$EI\alpha_{ba} = e_{ba}\bar{r}; \quad EI\beta = e_{ba}\bar{z}; \quad EI\Delta_{mb} = \frac{p}{6L}[s_2(L + 2a) + 2s_1a].$$

For horizontal force $H = 1$ (Fig. 211),

$$EI\theta_a = \frac{1}{2}p(s_1 + s_2); \quad \bar{x} = \frac{2as_1 + s_2(L + 2a)}{3(s_1 + s_2)};$$

$$EI\gamma = \frac{p}{6L}[s_1(L + 2b) + 2s_2b]; \quad EI\delta = \frac{p}{6L}[s_2(L + 2a) + 2s_1a]; \quad EI\Delta^h = \frac{p}{3}s_1 + s_2.$$

For uniformly-distributed load (Fig. 212), $EI\theta = \frac{w}{12L}[3as_1 + s_2(L + 3a)].$

$$\bar{x} = \frac{4a^2s_1 + s_2(8L^2 - 11bL + 4b^2)}{2[3as_1 + s_2(L + 3a)]}$$

$$EI\theta_a = e_{ba}\bar{r}; \quad EI\theta_b = e_{ba}\bar{z}; \quad EI\Delta_0 = \frac{w}{24L}[4as_1 + s_2(5L - 4b)].$$
For concentrated load (Fig. 213),\( EI\alpha = \frac{m_o}{2bd}[abs_1 + s_2(L^2 - cL - a^2)] \).

\[
\bar{x} = \frac{1}{3}\frac{2a^2bs_1 + s_2(2L^3 - 3cL^2 + c^2L - 2a^2)}{abs_1 + s_2(L^2 - cL - a^2)}.
\]

\[
EI\Delta_b = \frac{m_o b}{6d}\left[2as_1 + \frac{s_2}{b}(L^3 - 3a^2L + 2a^2)\right].
\]

**Inclined Beam with Vertical Member.**

For bending moment \( m_a = 1 \) (Fig. 214), \( EI\varepsilon_{ab} = \frac{1}{3}(2s_1 + s_2) \); \( \bar{x} = \frac{1}{3}\frac{Ls_2}{2s_1 + s_2} \).

\[
EI\varepsilon_{ab} = \frac{1}{3}(3s_1 + s_2); \quad EI\beta = \frac{s_2}{6}; \quad EI\Delta^{ma} = \frac{\phi}{6}(3s_1 + 2s_2).
\]

For bending moment \( m_b = 1 \) (Fig. 215),

\[
EI\varepsilon_{ba} = \frac{s_2}{2}; \quad \bar{x} = \frac{L}{3}; \quad EI\alpha_{ba} = \frac{s_2}{3}; \quad EI\beta = \frac{s_2}{6}; \quad EI\Delta^{mb} = \frac{\phi s_2}{6}.
\]
For horizontal force \( H = 1 \) (Fig. 216), \( EI\epsilon = \frac{p}{2}(s_1 + s_2) \).

\[
EI\gamma = \frac{p}{6}(3s_1 + 2s_2) ; \quad EI\delta = \frac{ps_2}{6} ; \quad EI\Delta = \frac{p^2}{3}(s_1 + s_2).
\]

![Fig. 216.](image)

For uniformly-distributed load (Fig. 217), \( EI\epsilon = \frac{WL^2s_2}{12} ; \quad \bar{x} = \frac{L}{2} \).

\[
EI\theta_a = EI\theta_b = \frac{WL^2s_2}{24} ; \quad EI\Delta_a = \frac{WL^2s_2f}{24}.
\]

**Trapezoidal Beams.**

For bending moment \( m_a = 1 \) (Fig. 218),

\[
EI\epsilon = s + \frac{b}{2} ; \quad \bar{x} = \frac{2as(3L - 2a) + L^3 - 6a^2L + 4a^3}{3L(2s + b)} \]

\[
EI\alpha = \frac{\bar{x}}{L} ; \quad EI\beta = \frac{\bar{x}}{L} ; \quad EI\Delta = \frac{1}{2}(s + b).
\]

![Fig. 218.](image)

For horizontal force \( H = 1 \) (Fig. 219),

\[
EI\gamma = EI\delta = \frac{1}{2}p(s + b) ; \quad EI\Delta = p^2(\frac{1}{2}s + b).
\]

For uniformly-distributed load on whole beam (Fig. 220),

![Fig. 220.](image)
\[ EI\theta = \frac{wsa}{12} (3L - 2a) + \frac{wa^3}{24} (L^3 - 6a^2L + 4a^3). \]
\[ EI\Delta_0 = \frac{Waps}{12} (4L - 3a) + \frac{wp}{12} (L^3 - 6a^2L + 4a^3). \]

For uniformly-distributed load on horizontal member (Fig. 221),
\[ EI\theta = \frac{wabs}{4} + \frac{w}{2} \left( \frac{L^3}{12} - a^3L + \frac{4a^3}{3} \right); \quad EI\Delta_0 = \frac{wabps}{3} + wp \left( \frac{L^3}{12} - a^3L + \frac{4a^3}{3} \right). \]

For uniformly-distributed load on inclined member (Fig. 222),
\[ EI\varepsilon = \frac{wa^3}{12} [4s + 3(L - 2a)]; \quad \varepsilon = \frac{as(11L - 8a) + L^3 - 6a^2L + 4a^3}{2L[4s + 3(L - 2a)]}; \]
\[ EI\Delta_0 = \frac{1}{3} wa^2p(\frac{3s}{2} + L - 2a). \]

For concentrated loads \( W \) (Fig. 223),
\[ EI\theta = Wa \left( \frac{s}{2} + b \right); \quad EI\Delta_0 = 2Wap \left( \frac{s}{3} + b \right). \]

**Stepped Beams.**

For uniformly-distributed load (Fig. 224),
\[ EI = \frac{wL^3}{12} + \frac{1}{2} wabp; \]
\[ \bar{x} = \frac{L}{2} \left( 1 + \frac{\frac{a}{L}}{1 + \frac{b}{L} - \frac{p}{L}} \right); \quad EI\theta_a = \bar{x} \bar{\theta}_a; \quad EI\theta_b = \bar{x} \bar{\theta}_b; \quad EI\Delta_a = \frac{1}{2} wabp^2 + p\theta. \]
For bending moment $m_{ab} = 1$ (Fig. 225),

$$EI\varepsilon = \frac{L}{2} + \frac{bp}{L} + p; \quad \Delta E = \frac{1}{3} \frac{L^3 + 6abp}{L^3 + 2bp + 2Lp}; \quad EI\Delta = \rho(a + p)\left(1 - \frac{a}{2L}\right).$$

**Fig. 225.**

For bending moment $m_{ab} = 1$ (Fig. 220),

$$EI\varepsilon = \frac{L}{2} + \frac{ap}{L}; \quad \Delta E = \frac{L^3 + 6abp}{3(L^3 + 2ap)}; \quad EI\Delta = \frac{ap}{2L}(a + p).$$

For unit horizontal force (Fig. 227):

$$EI\gamma = \rho(a + p)\left(1 - \frac{a}{2L}\right); \quad EI\delta = \frac{ap}{2L}(a + p); \quad EI\Delta = \rho^2(a + \frac{3}{2}p).$$

**Circular Beams.**

For unit bending moment $m_a = 1$ (Fig. 228),

$$EI\varepsilon = \phi R; \quad EI\alpha = \frac{3}{8}\phi R; \quad EI\beta = \frac{1}{8}\phi R; \quad EI\Delta = R^2(\sin \phi - \phi \cos \phi) = cR^2.$$

For $\phi = \frac{\pi}{4}$, $EI\Delta = R^2$. 

**Fig. 228.**
APPENDIX

For horizontal force $H = 1$ (Fig. 229),

\[ EI\gamma = EI\delta = R^2(\sin \phi - \phi \cos \phi) = c_1 R^3. \]

\[ EI\Delta = R^2\left(\phi \cos^2 \phi - \frac{3}{4} \sin 2\phi + \frac{\phi}{2}\right) = c_2 R^3. \]

For $\phi = \frac{\pi}{2}$: $EI\gamma = R^3$; $EI\Delta = \frac{\pi}{4} R^3$.

For uniformly-distributed load (Fig. 230),

\[ EI\theta = \frac{1}{4} w R^3\left(\phi \sin^2 \phi + \frac{1}{2} \sin 2\phi - \frac{\phi}{2}\right) = \frac{1}{4} wc_3 R^3. \]

\[ EI\Delta_o = wR^4\left[\frac{3}{8} \sin^2 \phi + \phi \cos \phi\left(\frac{1}{4} - \sin^2 \phi\right) - \frac{1}{4} \sin 2\phi \cos \phi\right] = wc_4 R^4. \]

For $\phi = \frac{\pi}{2}$: $EI\theta = \frac{1}{4} w R^3$; $EI\Delta_o = \frac{3}{8} w R^4$.

For constants $c_1$, $c_2$, $c_3$, and $c_4$, see Table XV.

For concentrated load $W$ at crown (Fig. 231),

\[ EI\theta = m_o R \frac{\phi \sin \phi + \cos \phi - 1}{\sin \phi}. \]

\[ EI\Delta_o = m_o R \frac{1 + \phi \sin 2\phi + \cos^2 \phi - 2 \sin^2 \phi - 2 \cos \phi}{\sin \phi}. \]

For $\phi = \frac{\pi}{2}$: $EI\theta = m_o R \left(\frac{\pi}{2} - 1\right)$; $EI\Delta_o = m_o R^3$. 
Relation between Horizontal and Vertical Translation (Fig. 232).—The relation between horizontal and vertical translation is calculated from

$$(L + \Delta)^2 + (h - \lambda)^2 - (L^2 + h^2) = 0,$$

from which $\lambda = h - \sqrt{h^2 - 2L\Delta - \Delta^2}$; or, for small movements, $\Delta \frac{L}{h}$ or $= \frac{\Delta}{tga}$ (approx.). The same relation applies for a beam rotating at one end (Fig. 232(b)).
TABLE XV.—COEFFICIENTS FOR THE ELASTIC CONSTANTS AND LOAD FUNCTIONS
OF CIRCULAR BEAMS (see page 61).

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