LIMIT-STATE DESIGN
OF PRESTRESSED CONCRETE

Volume 1
The Design of the Section
Consultant Editor

F. H. TURNER,
LIMIT-STATE DESIGN
of
PRESTRESSED CONCRETE

Volume 1
The Design of the Section

61820
Y. GUYON

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PREFACE

This book is dedicated to the memory of Eugene Freyssinet; as with its predecessor, the author hopes that it will not be found unworthy of his memory.

It is based on a series of lectures given at the Centre des Hautes Études du Béton Armé et du Béton Précontrainte (Centre for Advanced Studies of Reinforced and Prestressed Concrete). The author is grateful to M. Fougèa and M. Lebelle, at whose request the lectures were given, for this opportunity for close contact with the coming generation of engineers; it is hoped that some of the more difficult aspects of the subject have been made clearer as a result of this contact, and of the discussions and experimental work which arose from the lectures.

The author’s collaboration in the work of various committees, particularly the joint FIP–CEB and the ASP Committees, has made it necessary to modify or develop many aspects of the lectures. Here again, the exchanges have been extremely useful, and thanks are due to those colleagues who took part in the discussions.

These were mainly concerned with extending the scope of prestressing to include applications in which some degree of tensile strain is permitted, as has occurred in recent years.

The reason for this is obvious; reinforced concrete designers naturally sought to obtain the benefit, to a greater or lesser degree, of the advantages which they recognised in prestressing. There were no regulations to prevent it, prestressed reinforced concrete being simply reinforced concrete subjected to composite bending.

It was nevertheless necessary to determine the limits within which this type of construction could be applied, by combining the experience obtained with both prestressed and reinforced concrete.

One result of this work was the division of concrete structures into various classes, a step advocated by the author for some time. This system
of classification should make it possible to devise a general method of design.

The only real criterion for structural safety is, in fact, that the probability of reaching one of the limit states causing failure should be so small as to make the risk acceptable. If it is assumed that the laws controlling the limit state which is being considered (that is, the stress distribution at the particular section where it is reached or about to be reached) are known, the corresponding design method should give a single equation, stating that this state is effectively reached, when the most unfavourable assumptions corresponding to the accepted probabilities for the loading and the strength are made.

However, most codes do not yet recognise in their design regulations the changes occurring in the stress distribution as the limiting state is approached.

For example, although there is no disagreement on the distribution of stress at the ultimate load in a beam subjected to bending, the codes impose, in general, a double check of the compression zone; that is, in addition to a check on the factor of safety at failure, an elastic check is imposed. For the latter, it is implicitly assumed both that the compressive strength reaches the lowest value that it can possess as a function of the accepted probability, and that the stress distribution nevertheless remains elastic. Obviously, such a hypothesis cannot be justified. We have nevertheless concluded that it is essential to retain the use of the elastic theory. It cannot be ignored. Indeed, this theory enables a good assessment to be made of the most probable conditions in a structure, apart from the case of certain complex statically-indeterminate systems (which are not dealt with in this book).

This is not a contradiction in terms, because design methods which are based on limit states relate to states which it is desired to render improbable, with a sufficient margin of safety.

It is not the elastic theories themselves which are debatable, but the idea that failure depends only on the stresses reached, whereas, for the majority of critical cases, it depends only on limiting values of the deformations.

Moreover, one of the aims of these lectures was to assist engineers to conform to the codes currently in force. For this, a study of elastic behaviour was necessary, as well as a study of the design methods which are deduced from it, even if reservations must be placed on some or all of the formulae derived.

These reservations have been made where necessary; but at the same
time a large part of the book has been devoted to the experimental study of limit states, and to the corresponding design methods.

The phenomena which precede and accompany these limit states bring about changes of behaviour of varying degrees of importance. For those which precede failure due to bending under maximum load, the change is very pronounced, to the extent where there is in effect no difference in the behaviour of prestressed concrete and reinforced concrete at the limit state, but only a difference in the manner of collapse, depending on the percentage of reinforcement which is present. It is this phenomenon in particular which justifies the generality of the design laws for these two materials.

There are differences, however, in the definitions of limit states in the fibres subjected to tension, depending on whether or not tensile failure is accepted (that is, whether cracking is permitted), and depending also, when this is not accepted, upon the factor of safety which is required by reason of the use to which the structure is to be put.

It is the acceptance of cracking as a normal phenomenon which differentiates Class II from Classes III and IV, Class IV being reinforced concrete.

The differentiation between Class I and Class II lies in the definition of the limit state of strain of concrete in tension; Class I excludes any elongation and Class II allows it up to a certain percentage of the elongation which normally causes cracking.

For this state of limited elongation, the author had previously proposed (Prestressed Concrete, Volume 1) a law of distribution of tensile stress based on a plastic deformation of the tensile zone.

The Soviet Code has adopted a very similar limit state, and experience has shown that the behaviour of structures so designed has been satisfactory.

Plasticity in tension may be more apparent than real; it is probably due to elastic phenomena caused by microcracking, and may also be reversible, but its true nature has little practical importance for those structures designed on the basis of this limit state.

Of course, the margin given by this pseudo-plasticity is available only if shrinkage cracks are prevented during construction; prestressing allows this condition to be achieved.

The difference between Class III and Class IV lies in the magnitude of the loading at which cracking is assumed to commence. Beyond this level, the limit state (distribution and width of the cracks) is the same for the two classes.
This division into various classes should logically be extended to the design criteria for shear; but prestressing gives conditions which are so favourable to shear resistance that it is not generally necessary to define a safety factor with regard to web cracking, by the introduction or otherwise of transverse prestress by means of tensioned stirrups, and it is sufficient to design the web thickness on the basis of tensile resistance.

Apart from certain particular cases, such as that related to structures subjected to repeated loads, this solution is satisfactory in practice. However, in the author's opinion it has been unnecessarily complicated by the introduction of rather arbitrary safety criteria, which differ from one country to another. As a result, there is confusion when the various codes are compared.

The same confusion is apparent in the regulations concerning compressive failure of webs and in the design of stirrups.

An attempt has been made to clarify these aspects in one of the chapters of this book.

Independently of this evolution, the range of application of prestressing is being enlarged by the use of other materials and by the development of new types of structures.

In connection with the use of new materials, both lightweight concretes and very high strength concretes must be specially noted.

The former give rise to interesting designs, by the reduction of self-weight. This can have considerable advantages, especially in earthquake zones. They create some problems in connection with the losses of prestress, because of their lower elastic moduli, but the practical applications, which are already numerous, show that these problems can be solved.

High strength concrete (higher than 1 000 kg/cm²) will undoubtedly provide the answers to prefabricated industrial structures in the near future.

Among the new types of structures, prestressed concrete reactor pressure vessels must be mentioned; they give rise to different problems, such as the size of the prestressing tendons, the deformation of concrete subjected to high temperature gradients, the relaxation of steel, and so on.

It was not possible in the lectures to devote to all of these aspects the treatment which they deserve. More emphasis has been placed on explaining the nature of the problems, rather than on their resolution; for those who might wish to pursue them further, a bibliography is given. Certain aspects of design which have been dealt with in the author's previous books have not been repeated in this work. This applies in particular to the subject of fire resistance, the end blocks of beams, and the anchorage forces in
members prestressed by pretensioned reinforcement. It also applies to the detailed description of tests on various beams. References can be made to the earlier works for information on these subjects.

Nevertheless, on the important subject of end loads, the author has endeavoured, without attempting a new theoretical treatment, to give simplified methods for the design of the binding and reinforcement in these areas.

The lectures also included papers on statically-indeterminate systems. They are not included in the present volume; the author’s previous book (*Prestressed Concrete*, Volume II) deals with this aspect.

Finally, the author wishes to thank all those who by their criticisms or queries have enabled him to refine the treatment of parts of this book: to the engineers of STUP, his colleagues of CHEBAP and ASTEF; the engineers who attended the lectures delivered in various countries; and especially his friend M. Spasky who undertook the difficult task of giving, in parallel with the author’s lectures, the course on Practical Applications.
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SYMBOLS

At the start of the author’s lectures covering this course, he used the same symbols as in his previous work.

Other documents were later published: Recommendations of the ASP, Circular of the 12th August 1965 of the Public Works and Transport Ministry, Recommendations of the FIP–CEB Committee.

Different symbols are used in these documents. The author concluded that it was preferable to retain the majority of his own symbols, so as not to confuse the reader who wished to refer to one of his earlier books.

The main symbols used in Chapters I to XI are summarised in Table I below.

General symbols only are quoted. The symbols referring to particular applications are defined in the text as and when they occur.

The equivalent FIP–CEB symbols concerning safety are given in Table II.

Table I. Symbols for Chapters I to XI

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_a)</td>
<td>Cross-sectional area of complementary reinforcement</td>
<td>(C)</td>
<td>Ratio of strain at cracking to elastic strain</td>
</tr>
<tr>
<td>(A_c)</td>
<td>Cross-sectional area of resultant cable</td>
<td>(D)</td>
<td>Concrete density</td>
</tr>
<tr>
<td>(B')</td>
<td>Cross-sectional area of an imaginary strip consisting of that part of the bottom zone whose centroid is coincident with that of the reinforcement</td>
<td>(E)</td>
<td>Elastic moduli: (i) Concrete. (E_b) in general, (E_t, E_d, E_\infty) moduli corresponding to instantaneous, long-term, and total deformations (total at (t_\infty)) (E_{b0}) or (E_0): modulus at origin (for small values of (\sigma))</td>
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<tr>
<td>Symbols</td>
<td>Description</td>
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<tr>
<td>E</td>
<td>Generally, a factor</td>
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<tr>
<td></td>
<td>(i) Shape coefficient occurring in the determination of concrete tensile stresses due to shrinkage (Ch. XI)</td>
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<td></td>
<td>(ii) Ratio compressive strength tensile strength (see Volume 2)</td>
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<tr>
<td>K</td>
<td>Span</td>
<td></td>
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<tr>
<td>M</td>
<td>Moment, generally Where not otherwise stated, indicates a moment due to external forces</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$M_p, M_a, M_s$: moments due to loads $p$ (self-weight or permanent loads), $q$, $s$ (live loads)</td>
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<tr>
<td></td>
<td>$M_1$: moment at state 1 (basically, minimum loads)</td>
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<td></td>
<td>$M_2$: moment at state 2 (basically, maximum loads)</td>
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<tr>
<td></td>
<td>$M_R$: elastic moment of resistance</td>
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<tr>
<td></td>
<td>$M_r$: moment of resistance at failure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ΔM</td>
<td>Variation of moment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>General external normal or axial load</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td>General loading (see Table II)</td>
<td></td>
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</tr>
<tr>
<td>R</td>
<td>R: upper stress limit</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>R': lower stress limit</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$R_1, R'_1$: for state I (basically minimum loads)</td>
<td></td>
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<tr>
<td></td>
<td>$R_2, R'_2$: for state II (basically maximum loads)</td>
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E—(contd.)

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<table>
<thead>
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<tbody>
<tr>
<td>$E_M$ or $E_{ol}$: modulus at origin for instantaneous deformations</td>
<td></td>
</tr>
<tr>
<td>$E'<em>b$ or $E'$: modulus of green concrete, used for the determination of shrinkage stresses (ii) Steel. $E_s$ in general $E</em>{so}$: modulus at origin (for low stresses)</td>
<td></td>
</tr>
<tr>
<td>$E_{oT}, E_T$: tangent modulus at stress $T$</td>
<td></td>
</tr>
</tbody>
</table>

F

(i) Prestressing force $F$: permanent force, generally $F_p$: force at end of cable $F_s, F_{ol}$: corresponding values under initial tension $F_r$: force in cable at failure $F_f$: force reached at the limit state of failure in compression or cracking $F_{fI}, F_{fII}$: $F_f$ for Class I, for Class II (ii) $F_s$: force in complementary reinforcement $F_{se}$: force in complementary reinforcement at the elastic limit stress $F_C$: compressive force in reinforcement due to shrinkage

H Static moment at centroid (see Volume 2)

I General moment of inertia (second moment of area) $I_{v1}, I_{v2}$: section moduli $I_{1}, I_{2}$ (if applicable)
<table>
<thead>
<tr>
<th>Symbols</th>
<th>xix</th>
</tr>
</thead>
<tbody>
<tr>
<td>R—(cont’d.)</td>
<td>T: Generally, cable stresses</td>
</tr>
<tr>
<td>As a result of this convention, for a simply supported beam, for decreasing loads:</td>
<td>T: permanent state, unless otherwise stated</td>
</tr>
<tr>
<td>$R_1$ applies to the bottom fibre</td>
<td>Equally: stress at failure $(T = \lambda T_p)$</td>
</tr>
<tr>
<td>$R'_1$ applies to the top fibre</td>
<td>$T_s$: stress at end of cable</td>
</tr>
<tr>
<td>$R_2$ applies to the top fibre</td>
<td>$T_i$: initial stress</td>
</tr>
<tr>
<td>$R'_2$ applies to the bottom fibre</td>
<td>$T_{ei}$: stress at end under initial conditions</td>
</tr>
<tr>
<td>$\Delta R$: Variation of stress at an extreme fibre:</td>
<td>$T_f$: stress at failure</td>
</tr>
<tr>
<td>$(R_1 - R'_2$ for bottom fibre)</td>
<td>$T_G$: characteristic stress (ASP)</td>
</tr>
<tr>
<td>$(R_2 - R'_1$ for top fibre)</td>
<td>(practically, limit at 0.1% proof stress)</td>
</tr>
<tr>
<td>$R_b$, $R'_b$: Generally:</td>
<td>$\Delta T$: (i) Stress losses</td>
</tr>
<tr>
<td>(when there is no risk of misunderstanding, the suffix b is omitted)</td>
<td>$\Delta T_s$: due to shrinkage</td>
</tr>
<tr>
<td>$R_b$: compressive strength</td>
<td>$\Delta T_d$: due to long-term deformation</td>
</tr>
<tr>
<td>$R'_b$: tensile strength</td>
<td>$\Delta T_c$: due to differences in times of tensioning</td>
</tr>
<tr>
<td>$R_{b,cyl}$, $R_{b,cube}$: cylindrical, cubic strength, at day $j$</td>
<td>$\Delta T_p$: due to relaxation</td>
</tr>
<tr>
<td>$(R_{b,cyl,28}$, $R_{b,cube,28}$, at 28 days)</td>
<td>(ii) Stress variations</td>
</tr>
<tr>
<td>$R_{adh}^*$: Bond strength between reinforcement and concrete (Chapter IX)</td>
<td>$\Delta T_I$: increase in stress due to loss of compression (Class I)</td>
</tr>
<tr>
<td>$R^*$: Apparent tensile strength</td>
<td>$\Delta T_{II}$: increase in stress due to loss of compression at the limit state of cracking (Class II)</td>
</tr>
<tr>
<td>$R_{bt}$, $R^*$: See Table II</td>
<td></td>
</tr>
<tr>
<td>$S$: Generally: cross-sectional area of a concrete section</td>
<td>$V$: Generally, shear force under external loads (see Volume 2)</td>
</tr>
<tr>
<td>More particularly:</td>
<td>$W$, $W^*$: Specific cracking resistance moduli:</td>
</tr>
<tr>
<td>$S_0$, $S_1$, $S_2$: of an approximate section, a section with holes, an homogeneous section</td>
<td>$W = \frac{1}{v} R^* = k \frac{1}{v} R'$</td>
</tr>
<tr>
<td>$S_0$, $S_p$, $S_t$: partial areas of compressed concrete at various modes of failure (see Chapters IX and X)</td>
<td>$W^* = \frac{1}{v'} R^* = k \frac{1}{v'} R'$</td>
</tr>
<tr>
<td>$a$: Shear span (Volume 2)</td>
<td></td>
</tr>
<tr>
<td>$b$: Breadth of a section different suffixes: $b$, $b_1$...in accordance with the text)</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$b'$</td>
<td>In general: web thickness</td>
</tr>
<tr>
<td>$d$</td>
<td>(i) Bar diameter</td>
</tr>
<tr>
<td>$(d, d')$</td>
<td>(ii) Distance of cable from nearest face</td>
</tr>
<tr>
<td>$e$</td>
<td>Eccentricity of prestressing force (positive upwards) $e$, in general, is the cable eccentricity. When it is necessary to distinguish: $e_a$ cable eccentricity $e_1$ eccentricity of centre $e_2$ of compression at state 1 state 2... $e'$ generally, absolute value of $e$ (Chapter VIII)</td>
</tr>
<tr>
<td>$e_a$</td>
<td>Eccentricity of complementary steel</td>
</tr>
<tr>
<td>$f$</td>
<td>(i) Coefficient of friction (ii) Versed sine of beam deformation (Chapter V, Section 8) (iii) Versed sine of parabola, of an arc</td>
</tr>
<tr>
<td>$g$</td>
<td>Co-ordinates of a centroid (different origins and suffixes, according to the text)</td>
</tr>
<tr>
<td>$h$</td>
<td>Depth of a beam, thickness of a wall $h_1$: effective depth relative to the cable $h_a$: effective depth relative to the complementary reinforcement</td>
</tr>
<tr>
<td>$k$</td>
<td>Generally, a coefficient: (i) $k = \frac{R_2}{R_1}$ (ratio of allowable stresses at top and bottom fibres, Chapter VIII, Section 10) (ii) $k = \frac{R'}{R}$ (ratio of apparent and elastic tensile strengths) $k_a$ (Correction factors for the bottom fibre) $k_s$ (Stress (Chapter XI))</td>
</tr>
<tr>
<td>$l$</td>
<td>General span $l_{crit} = $ critical span</td>
</tr>
<tr>
<td>$m$</td>
<td>Modular ratio $\frac{E_a}{E_b}$ $m_t, m_d$ relative to instantaneous, long-term deformations</td>
</tr>
<tr>
<td>$m'$</td>
<td>$= \frac{E_a}{E'_b}$ modular ratio for a green concrete (calculation of shrinkage stresses, Chapter XI)</td>
</tr>
<tr>
<td>$p$</td>
<td>Self-weight (or permanent initial load) per unit length</td>
</tr>
<tr>
<td>$q$</td>
<td>Permanent loads other than self-weight, per unit length</td>
</tr>
<tr>
<td>$r$</td>
<td>(i) Radius of gyration $r_2/r_1 \frac{r_2}{v}$ bottom and top extremities of central core (ii) Occasionally, radius of curvature</td>
</tr>
<tr>
<td>$s$</td>
<td>(i) Live load per unit length (ii) Occasionally, special meaning (see text)</td>
</tr>
<tr>
<td>Symbols</td>
<td>xx1</td>
</tr>
<tr>
<td>---------</td>
<td>-----</td>
</tr>
<tr>
<td>$t$</td>
<td>In general, time</td>
</tr>
<tr>
<td>$v, v'$</td>
<td>Distances of centroid from top and bottom fibres</td>
</tr>
<tr>
<td>$w$</td>
<td>Width of a crack</td>
</tr>
<tr>
<td>$w_1$</td>
<td>Maximum strain of reinforcement in its surrounding concrete before rupture of that concrete (Chapter IX, Section 6, Fig. 10)</td>
</tr>
<tr>
<td>$x$</td>
<td>(i) Abscissa of a section (ii) Depth of concrete in compression at time of failure</td>
</tr>
<tr>
<td>$x'$</td>
<td>Depth of concrete in tension at time of cracking</td>
</tr>
<tr>
<td>$y$</td>
<td>(i) Ordinate relative to the centroid (ii) Depth over which it is assumed that compression is uniform at time of failure</td>
</tr>
<tr>
<td>$y'$</td>
<td>Depth over which it is assumed that tension is uniform at time of cracking (Class II)</td>
</tr>
<tr>
<td>$z$</td>
<td>Elastic lever arm (I/H) $z_e$: elastic lever arm relative to cables $z_a$: elastic lever arm relative to complementary reinforcement</td>
</tr>
<tr>
<td>$z_r$</td>
<td>Lever arm at failure</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>(i) Angle of cable to the horizontal (ii) Curvature of a cable, for calculation of friction losses</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>(iii) Proportion of live load at which loss of compression is reached (Chapter XI)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>(i) Angle of principal compressive stresses (see Volume 2) (ii) Parasitic curvature of cable per unit length (friction)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>(i) Coefficient of expansion for concrete (ii) Safety factor (see Table II)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>(i) Mean square deviation (ii) Distance (exceptionally)</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Generally, strain (i) Concrete in compression $\varepsilon, \varepsilon_b$ strain $\varepsilon_i, \varepsilon_d$ instantaneous, long-term strain $\varepsilon_r$ strain at failure (suffixes are omitted when there is no risk of confusion) (ii) Concrete in tension $\varepsilon'_b$ strain $\varepsilon'_b$ strain at cracking (suffixes are omitted when there is no risk of confusion) (iii) Steel $\varepsilon'_a$ strain (at stress $T$) $\varepsilon'_oa$ strain (at stress $T_o$) $\varepsilon'_r$, cable breaking strain $\varepsilon_a$, compressive strain in complementary reinforcement due to shrinkage</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Strain due to shrinkage $\eta, \eta_{\infty}$ at time $t, at t_{\infty}$</td>
</tr>
</tbody>
</table>
| \( \theta \) | (i) Temperature
(ii) Angle |
| \( \lambda \) | Generally, coefficient
(i) Fraction of long-term deformation which remains after steel relaxation (Chapter II, Section 19)
(ii) Coefficient of cable stress for the calculation of the moment at failure \( (\lambda = T/T_c) \)
(iii) Loss due to friction per unit length
(iv) \( \lambda = A_a/S \), percentage of non-tensioned reinforcement (Chapter XI) |
| \( \mu \) | Coefficient \( \mu = \sigma_b/R' \) (Chapter IX, Section 7) |
| \( \zeta \) | Abscissa |
| \( \bar{\omega} \) | In general, mechanical percentage
\[
\bar{\omega} = \frac{A_c}{bh_1} \times \frac{T^{*}_c}{R^{*}} \% \text{ for cables}
\]
\[
\bar{\omega}_a = \frac{A_a}{bh_a} \times \frac{\sigma'_{e}}{R^{*}} \% \text{ for reinforcement (see Table II)}
\]
| \( \rho \) | In general, efficiency of a section \( (\rho = r^2/wv') \) |
| \( \sigma, \sigma' \) | Stresses (\( \sigma \) top fibre, \( \sigma' \) bottom fibre)
(i) Concrete
\( \sigma_{o1}, \sigma'_{o1} \) prestress
\( \sigma_{o1}, \sigma'_{o1} \) prestress under initial tension |
| \( \sigma, \sigma' \) | (contd.)
\( \sigma_p, \sigma'_p \) self-weight or permanent loading stress
\( \sigma_s, \sigma'_s \) live load stress
\( \sigma_1, \sigma'_1 \) at state 1 (minimum loads)
\( \sigma_2, \sigma'_2 \) at state 2 (maximum loads)
\( \sigma_y \) stress at ordinate \( y \)
\( \sigma_b \) stress at centroid
\( \sigma_c \) stress at cable level
\( \sigma_x, \sigma_y \) stresses at faces perpendicular to \( x, y \) (see Volume 2)
(ii) Steels
\( \sigma'_{a} \) complementary reinforcement tensile stress
\( \sigma'_{e} \) elastic limit of reinforcement |
| \( \tau \) | In general, shear stress (see Volume 2) |
| \( \varphi \) | Coefficient
(i) Coefficient of long-term deformation
\[
\left( \varphi = \frac{\text{long-term deformation}}{\text{instantaneous deformation}} \right)
\]
(ii) Angle
(iii) Coefficient in Chapter IX \( (\varphi = \sigma'_a/mR', \text{ see text}) \) |
| \( \chi \) | (i) Dispersion coefficient (see Table II)
(ii) Bonding coefficient \( (\chi = R'_{eb}/R'_b) \) |
| \( \Gamma \) | Overall safety factor |
### TABLE II. Safety

#### A. Quantities occurring when considering:

<table>
<thead>
<tr>
<th>Mean value</th>
<th>( R_m )</th>
<th>( Q_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean square deviation of test results</td>
<td>( \delta )</td>
<td>( \delta )</td>
</tr>
<tr>
<td>Range</td>
<td>( \chi )</td>
<td>( \chi )</td>
</tr>
<tr>
<td>Characteristic value*</td>
<td>( R_k = R_m(1 - \chi \delta) )</td>
<td>( Q_k = Q_m(1 \pm \chi \delta) )</td>
</tr>
<tr>
<td>Safety factor</td>
<td>( \gamma_m )</td>
<td>( \gamma_e )</td>
</tr>
<tr>
<td>Value for design</td>
<td>( R^* = \frac{R_k}{\gamma_m} )</td>
<td>( Q^* = \gamma_e Q_k )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>((\gamma_e &gt;) or &lt; 1)</td>
</tr>
</tbody>
</table>

* The plus or minus sign is taken, according to which gives the most adverse condition.

### B. Comparison of symbols with the FIP–CEB symbols

#### (i) SAFETY FACTORS

<table>
<thead>
<tr>
<th>Symbols</th>
<th>FIP–CEB symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete compressive strength</td>
<td>( \gamma_b )</td>
</tr>
<tr>
<td>Concrete tensile strength</td>
<td>( \gamma'_b )</td>
</tr>
<tr>
<td>Reinforcement strength</td>
<td>( \gamma_a )</td>
</tr>
<tr>
<td>Strength of prestressing steel (considered as a material);</td>
<td>( \gamma_oa )</td>
</tr>
</tbody>
</table>

\( \gamma_oa \) is taken as being equal to \( \gamma_a \) and \( \gamma_{ap} \) is taken as being equal to \( \gamma_a \).
### Symbols

<table>
<thead>
<tr>
<th>Symbols</th>
<th>FIP-CEB symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b) EXTERNAL LOADING</td>
<td></td>
</tr>
<tr>
<td>Self-weight (and permanent loads, generally)</td>
<td>$\gamma_p$</td>
</tr>
<tr>
<td>Live loads</td>
<td>$\gamma_s$</td>
</tr>
<tr>
<td>Shrinkage, creep, temperature (not explicitly considered)</td>
<td>$\gamma_s$</td>
</tr>
<tr>
<td>Additional allowances for dynamic effects, etc. (not explicitly considered)</td>
<td>$\gamma_c$ the factor $\gamma_c$ (FIP-CEB) combines with $\gamma_s$, the overall factor becoming $\gamma_s\gamma_c$</td>
</tr>
</tbody>
</table>

(c) PRESTRESSING FORCES

<table>
<thead>
<tr>
<th>Symbols</th>
<th>FIP-CEB symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permanent</td>
<td>$\gamma_p$</td>
</tr>
<tr>
<td>Initial</td>
<td>$\gamma_{ot}$ (not explicitly considered)</td>
</tr>
</tbody>
</table>

*Note.* The above table only quotes the symbols. The values given to the factors vary according to the test or design methods which are used. For example, although the FIP-CEB Committee only uses the symbol $\gamma_s$ in general, for loading allowances, the value given to $\gamma_s$ can vary, depending on the type of loading; similarly for the prestress (permanent or initial).

### (ii) DESIGNATION OF VALUES

<table>
<thead>
<tr>
<th>Symbols</th>
<th>FIP-CEB symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) STRENGTH</td>
<td></td>
</tr>
<tr>
<td>Compressive strength of concrete</td>
<td>$R_b$ $R_{bk}$</td>
</tr>
<tr>
<td>Value for design</td>
<td>$R_b^* = \frac{R_{bk}}{\gamma_b}$</td>
</tr>
<tr>
<td>Concrete tensile strength</td>
<td>$R_b'$ $R_{bk}'$</td>
</tr>
<tr>
<td>Value for design</td>
<td>$R_b'^* = \frac{R_{bk}'}{\gamma_b'}$</td>
</tr>
<tr>
<td>Symbols</td>
<td>FIP-CEB symbols</td>
</tr>
<tr>
<td>---------</td>
<td>---------------</td>
</tr>
<tr>
<td>Strength of reinforcement (elastic limit)</td>
<td>( \sigma'<em>e ) ( \sigma'</em>{ek} )</td>
</tr>
<tr>
<td>Value for design</td>
<td>( \sigma'<em>e = \frac{\sigma'</em>{ek}}{\gamma_a} )</td>
</tr>
<tr>
<td>Breaking strength of prestressing steels (for calculation of moment at failure)</td>
<td>( T_r ) ( T_{rk} )</td>
</tr>
<tr>
<td></td>
<td>( T_r = \frac{T_{rk}}{\gamma_{oa}} )</td>
</tr>
<tr>
<td></td>
<td>(in practice, ( \gamma_{oa} = \gamma_a ) is used)</td>
</tr>
<tr>
<td>Moment of resistance at failure</td>
<td>The design value ( M^* ), is obtained by using the strength design values in formulae. Similarly for other values defining the breaking strength of a section (( N^*r, V^*r \ldots ))</td>
</tr>
<tr>
<td>(b) LOADING</td>
<td>( M_p ) ( M_s )</td>
</tr>
<tr>
<td></td>
<td>( M^*_p = \gamma_p M_p )</td>
</tr>
<tr>
<td></td>
<td>( M^*_s = \gamma_s M_s )</td>
</tr>
<tr>
<td></td>
<td>( N^*_p = \gamma_p N_p )</td>
</tr>
<tr>
<td></td>
<td>( N^*_s = \gamma_s N_s )</td>
</tr>
<tr>
<td></td>
<td>( V^*_p = \gamma_p V_p )</td>
</tr>
<tr>
<td></td>
<td>( V^*_s = \gamma_s V_s )</td>
</tr>
<tr>
<td>(c) PRESTRESSING FORCES</td>
<td></td>
</tr>
<tr>
<td>Permanent force</td>
<td>( F )</td>
</tr>
<tr>
<td>Value for design</td>
<td>( F^* = \gamma_s F )</td>
</tr>
<tr>
<td>Initial force</td>
<td>( F_i )</td>
</tr>
<tr>
<td>Value for design</td>
<td>( F^*<em>i = \gamma</em>{oi} F_i )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter I

GENERAL

In its most widely used form, the strength of prestressed concrete (like that of reinforced concrete) depends upon the strengths of the concrete and the steel. There is, however, a basic difference between the two.

Reinforced concrete is a combination of concrete and steel, in which each component fulfils a definite function; the concrete resists compression and the steel resists tension. The tensile weakness of the concrete is offset by suitable steel reinforcement.

Fully prestressed concrete is not a combination in this sense; it is essentially concrete (that is, a material which, when used in its natural state, is highly resistant to compression, but unpredictably weak in tension) which has been subjected to a ‘mechanical treatment’ thus enabling it to resist both tension and compression.

This mechanical treatment consists of loading the material in advance in order to induce in it stresses of a favourable nature; compressive stresses are created in areas which will eventually be subjected to tension.

The idea of improving the behaviour of a material which has a low resistance to one type of stress, by subjecting it to previously applied stresses in the opposite direction, is used in other contexts.

Examples of this technique include:

- Guy-ropes for radio masts, which are pre-tensioned so that they can subsequently resist compressive forces; the masts are subjected to compression because of the pre-tensioning of the guys.
- The spokes of a bicycle wheel, which are tensioned to enable them to resist subsequent compression; the tensioning loads are applied to the rim of the wheel which is thus compressed.
- Barrel staves which are subjected to ring compression by means of hoops, which are themselves in tension.
- The air in tyres, which is compressed inside envelopes with controlled deformation under tension.
By virtue of this previous ‘stressing’ (that is, the application of suitable forces) it is possible to create conditions which permit the use of a material which in all other respects is basically the most suitable for the application.

The pre-loading of concrete—that is, the prestressing (which is the name given to it by Freyssinet)—is achieved by imposing suitably distributed compressive forces upon the concrete. These forces can be applied in various ways. The most widely used method is to pre-tension the reinforcement, to which the concrete is subsequently bonded, and is thus put into compression. The mechanics of this method are considered later.

It should be said immediately that this idea was not novel at the time of Freyssinet’s first successful applications, which are considered to constitute the ‘Birth’ of prestressed concrete technology. Previous attempts, however, had been unsuccessful. For instance, in 1902, Rabut tried to prestress the arcades along the Rue de Rome, using threaded bars and nuts, but without appreciable success; and in 1907, Koenen designed reinforced concrete beams, in which the reinforcement was pre-tensioned to 6 kg/mm². At the first loading the beams acted as prestressed beams, but the prestress disappeared completely after a short time.

Freyssinet showed that the long-term deformations of concrete, which had been the subject of his famous experiments at Plougastel, would cause all the pre-compression to disappear if the reinforcement were subjected to only a small pre-tensioning stress. He demonstrated that permanent prestress could be sustained only by creating much higher initial stresses in the steel, and by raising these tensile stresses to the order of 80 to 100 kg/mm².

Freyssinet immediately obtained the desired results. In this way, the steel no longer acted merely as a substitute for concrete, but became a force—or, more exactly, the medium through which the necessary external forces could be applied.

These concepts, which seem commonplace today, were revolutionary at the time. We owe them to Freyssinet.

1. Difference between reinforced and prestressed concrete
Consider a beam (Fig. 1), of rectangular cross-section, supported at both ends; two loads, each equal to P, are applied at a distance \( a \) from each support.

The central portion AB is subjected to a constant bending moment \( M = Pa \). In the following, the portion AB is designed first as a reinforced concrete beam and then as a prestressed concrete beam.
In the reinforced concrete design, reinforcement is provided in the lower part of the beam, where cracks occur because of the low tensile strength of the concrete; the steel is subjected to axial tension (in a direction normal to the cracks); in order to balance this tension, the concrete above the cracks (that is, throughout a certain depth $x$) is compressed.

![Fig. 1.](image)

In the plane of the crack, the moment $M$ is resisted by the moment due to the internal forces. This is represented on the usual stress diagram (Fig. 2); the diagram illustrates the separate roles played by the steel and by the concrete, and it shows that the function of the steel is to act as a substitute for the concrete, when the strength of the concrete itself is exceeded.

![Fig. 2.](image)

In the case of prestressed concrete, the mode of behaviour is different. The concrete is first compressed, so that it will not crack. Hence, there exists under the moment $M$ a pattern of stresses corresponding to the resultant of the artificially induced stresses and those due to the moment $M$. Since the concrete does not crack, it is no longer necessary to determine the depth $x$ of the zone in compression (that is, the position of the neutral axis), and the concrete can be considered as an homogeneous material.
The stress diagram due to the applied loading only is shown in Fig. 3, and the procedure by which cracking can be avoided is discussed in the following.

It is apparent that a uniform compression could be applied. Assuming that concrete has no tensile strength, the prestress must then be equal to \((6M/bh^2)\)† if no resultant tensile stresses are allowed. The stress diagrams for this case are shown in Fig. 4.

![Stress Diagram](image)

**Fig. 3.**

When the loading has been applied, the resultant stress at the top of the beam is \(12M/bh^2\). Since this stress must not exceed the permissible compressive stress \(R\), then for maximum economy of material \(12M/bh^2 = R\). This equation determines the dimensions of the beam; if \(h\) is known, then \(b = 12M/Rh^2\) and if \(b\) is known, \(h = (12M/Rb)^\dagger\). However, this

![Stress Diagram](image)

**Fig. 4.**

† Compressive stresses are considered as positive; tensile stresses as negative.
solution is not very economical. It requires about twice as much concrete as a design in reinforced concrete, in which $bh^2 = 6M/R$ approximately. The solution is uneconomical because insufficient advantage is taken of the properties of the concrete in its unloaded state. Since $12M/bh^2 = R$, the stress $6M/bh^2$ in the concrete before loading is only equal to $R/2$; that is, to half the allowable stress. The solution is also inadequate because, in addition to compressing the zone which will eventually be subjected to tensile stresses, the zone which stays in compression is also unnecessarily pre-compressed.

\[
\frac{6M}{bh^2} + \frac{6M}{bh^2} = \frac{6M}{bh^2}
\]

\text{Prestress (unloaded state) + added loading = resultant (final state)}

\text{FIG. 5.}

The correct solution is shown in Fig. 5. Instead of a uniform pre-compression, a pre-compression giving a triangular stress diagram is created, the stress varying from zero at the top to $6M/bh^2$ at the bottom of the beam.

When the beam is loaded, the stresses cancel out at the bottom, and the resultant stress at the top is $6M/bh^2$. Since this stress must equal $R$, $bh^2 = 6M/R$. For the same depth $h$, therefore, the area of concrete is half that which was previously obtained. Thus, full use is made of the material, since the stress distribution in both the unloaded and the loaded states varies from the minimum allowable (in this case zero), to the maximum ($R$).

Figure 5 makes it clear that a single stress condition can no longer be considered, as with conventional materials,† but that two extreme condi-

† At least where statically-determinate structures subjected to unidirectional loading are concerned.
tions exist, each with a maximum and a minimum stress limit (in this example R and 0 in both cases). Under these conditions, the stresses must remain within the allowable range and, if possible, they must be equal to their limit values in order to make the most efficient use of the material.

2. Point of application of pre-compression
For the example shown in Fig. 4, the force to be exerted is equal to \(6M/bh^2 \times bh\); since \(R = 12M/bh^2\), the value of this force is equal to \(F = R/2 \times bh\) and it must be applied at the centroid of the section.

For the example shown in Fig. 5, where \(6M/bh^2 = R\), the force \(F\) is again equal to \(R \times bh \times \frac{1}{2}\) (triangular diagram), and it must be applied at two-thirds of the depth of the beam. Therefore, the use of a prestressing force \(F\) which is offset from the centroid makes it possible to resist twice the bending moment (\(M = R(bh^2/6)\) instead of \(M = R(bh^2/12)\), with the same quantity of concrete. In other words, if the force is applied at the wrong point, as in Fig. 4, twice the quantity of concrete and twice the prestressing force are required in order to resist the bending moment of Fig. 5, and the design is twice as expensive.

3. Application of compressive forces
The reaction to the prestressing forces must be provided by equipment which is outside the concrete. In some cases, steel need not be used for prestressing. Two fixed abutments are used, for example, to apply the force \(F\) in Fig. 6.

![Fig. 6.](image-url)

In general, however, it is difficult to provide abutments. Two end blocks taking the reactions from compression jacks can be tied together as shown diagrammatically in Fig. 7; the ties between the blocks are subjected to a tensile force \(-F\), which balances the compressive force in the concrete.

Once the force is applied, wedges are introduced between the blocks A and B and the ends of the beam in order to anchor A and B. The jacks are then removed.
Usually, the prestressing steel is located in sheaths previously positioned inside the concrete. The sheaths protect the steel, which acts as the load-equalising agent between external devices, which in turn compress the concrete, and to which the steel is anchored after it has been tensioned. Anchoring is equivalent to wedging in Fig. 7 and, in the same way, it permits the jacks to be removed afterwards (Fig. 8).

A number of systems are available in practice for carrying out these operations. Some of them are described in Chapter III. The previous example shows that the function of the steel is quite different to its function in reinforced concrete; it becomes the means of supplying the reaction to the prestressing force, and consequently it is the medium through which this force is exerted.

4. Centre of compression
The prestressing force is defined by its magnitude and by its line of action at each section. The line of action is defined by its eccentricity with respect to the centroid of the section. Let the eccentricity be \( e \), positive when the force \( F \) is applied above the centroid.
In the case of Fig. 5, the line of action of $F$ is distant $h/3$ from the bottom of the beam. Therefore $e = h/3 - h/2 = -(h/6)$. The force therefore acts at the lower limit of the middle third.

In the case of a section of any shape, the prestress diagram is triangular (as in Fig. 5) if the line of action of the force is through the lower limit of the central core or kern. Under the applied load, the resultant of the internal forces acts at a distance $h/3$ from the top (triangular diagram with zero stress at the base). The eccentricity of this resultant is $2h/3 - h/2 = +(h/6)$; that is, equal to the eccentricity of the upper boundary of the central kern.

Therefore, the eccentricity of compression passes, under the influence of the load, from the lower boundary to the upper boundary of the central kern. This is true for any section when the shape of the stress diagrams is triangular in both the unloaded and the loaded conditions, as in the case of Fig. 5.

The product of the force and its displacement from the centroid under the influence of loading must be equal to the moment $M$ which is due to the load. This can be seen immediately from Fig. 5, since $F = R \times bh/2$, and the displacement is equal to $h/6 - [-(h/6)] = h/3$. Therefore, the moment is:

$$R \times \frac{bh}{2} \times \frac{h}{3} = R \times \frac{bh^2}{6}$$

Hence

$$M = F \times \frac{h}{3}$$

This leads to the important concept of the centre of compression.

The centre of compression is the point of action of the resultant of the compressive forces, that is, of $F$, in a section. In the case of pure prestress, where the only force acting on the beam is the prestressing force, the centre of compression at a section coincides with the point of intersection of the prestressing steel with the section.

The centre of compression is subsequently displaced under the influence of the moment, and the displacement is equal to $M/F$. In order to keep the stresses within their permissible limits, the centre of compression must remain within the boundaries of a certain 'limiting core', under any moment to which the section may be subjected. The limiting core coincides with the central core or kern when the lower limiting stress is zero.
5. Variation of prestressing force

It has been assumed so far that the force F remains constant when loading is applied. This is only approximately true. In practice, F increases slightly under loading. This increase may be easily calculated in the case of the beam shown in Fig. 1 subjected, as in Fig. 5, to a prestress which is applied by means of linear steel members at the lower third of the section.

Under load, the steel undergoes the same deformation as the adjacent concrete. The compression in the concrete below the centroid is reduced when the beam is loaded; this reduction in compressive stress is accompanied by an elongation of the concrete relative to its condition in the prestressed but unloaded state.

![Diagram](image)

**Fig. 9.**

If $\Delta \sigma_b$ is the variation of stress in the concrete adjacent to the steel, the relative strain is $\Delta \sigma_b/E_b$, in which $E_b$ is the modulus of elasticity of the concrete. The steel is subjected to the same strain. If $E_a$ is the modulus of elasticity of steel, the increase in the steel stress is $E_a \times \Delta \sigma_b/E_b$, or $m \Delta \sigma_b$ in which $m$ is the modular ratio.

Let the two triangular stress diagrams be as shown in Fig. 9, the maximum stress in the concrete being 120 kg/cm². Since the steel is located at the lower third of the section, the stress in the concrete at this level changes from 80 kg/cm² in the unloaded condition to 40 kg/cm² when the loading is applied. Therefore $\Delta \sigma_b = 40$ kg/cm².

For the long-term behaviour of a typical concrete, $E_b$ is of the order of 150 000 kg/cm², as seen in Chapter II. $E_a = 2 000 000$ kg/cm²; therefore $m$ is of the order of $2 000 000 / 150 000 = 14$.

The variation of stress in the steel is, therefore:

$$14 \times 40 = 560 \text{ kg/cm}^2$$

The usual working stress in prestressing steel is of the order of 90 kg/cm². The increase of stress in the steel is therefore about 6% in the example considered, under sustained loading conditions.
In the case of instantaneous loading, $E_b$ is about 450,000 kg/cm$^2$ and $m$ is about 5. The increase in the steel stress is then only 2 kg/mm$^2$, or about 2%.

The variations of stress in the steel are therefore relatively low, at least in the case where the lower permissible stress in the concrete is approximately zero, and where tensile stresses are not permissible. Such a condition of prestress is called total prestress. In this case, variations in the prestressing force are often neglected for simplification.

A prestressed section may therefore be considered as a section under combined bending and compression, in which the normal force $F$ is constant but with a variable eccentricity, the displacement of the centre of pressure under the effect of a moment $M$ being equal to $M/F$.

The design calculations consist of finding the prestressing force and its position, such that the centre of compression remains within a certain area termed the limit kern, under all conditions of loading. The increase in stress in the prestressing steel can of course be taken into consideration if so desired.

In cases where the prestressing is not total, but where limited prestress or prestressed reinforced concrete are involved, the increase in stress in the prestressing steel can no longer be ignored as is shown later.

6. Losses of prestress

The increase in stress considered in the foregoing is added to the 'permanent stress'; that is, to the stress in the cable in the prestressed but unloaded condition.

In the example considered in Fig. 7, however, this permanent stress is not equal to the 'initial stress'. The prestressing steel undergoes a loss of stress in the interval between the application of the initial prestress and the time when working conditions are established, because of the amount of concrete compression which occurs in that period.$\dagger$

As shown in Chapter II, the compressive strain of concrete depends upon the length of time it is under load, and it reaches about three times its initial value after a prolonged period.

In other words, if $\varepsilon_i$ is the instantaneous strain due to the compressive stress, and if the compressive stress is constant, the final compressive strain is about $3\varepsilon_i$. This increase in deformation is called the long-term deformation; it is of the order of $3\varepsilon_i - \varepsilon_i = 2\varepsilon_i$. More generally, it is denoted by $\varepsilon_d$.

$\dagger$ To the losses mentioned in this section, must be added other losses described in Chapter IV.
The long-term deformation is accompanied by a corresponding stress reduction in the steel and consequently by a reduction in the ‘permanent’ prestress.

Let $\sigma_f$ be the final stress in the concrete adjacent to the steel, when stable conditions are attained; the loss of stress in the steel after its initial tensioning is $E_a \varepsilon_d$. If the initial stress is $T_0$, the final stress is $T_0E_a \varepsilon_d$.

Assuming that $\varepsilon_d = 2\varepsilon_i$, and if $E_i$ is the ‘instantaneous’ modulus of elasticity of the concrete, the loss of stress is

$$2E_a \varepsilon_i = 2E_a \frac{\sigma_b}{E_i}$$

If, as in the previous example, $E_i = 450\,000\,\text{kg/cm}^2$ and $E_a = 2\,000\,000\,\text{kg/cm}^2$, then the loss of stress in the steel is $2 \times 2\,000\,000/450\,000 \sigma_b = 9\sigma_b$. If $\sigma_b = 100\,\text{kg/cm}^2$, which is a typical value for the permanent stress local to the steel, the loss of stress is $9 \times 100 = 900\,\text{kg/cm}^2$.

This loss is independent of the initial stress $T_0$; the smaller the value of $T_0$, the greater a fraction of $T_0$ is the loss, and the greater is the percentage of additional steel required to compensate for this.

This explains why, with steel tensioned only to $10\,\text{kg/mm}^2$, a significant and lasting prestress cannot be maintained, and why a successful design can be obtained only by using high-tensile steels.

7. Continuing for the present with the assumption of total prestress, the preceding concepts are now clarified by means of a simple example which leads to further important conclusions.

Example: Design of a prestressed beam. Compensation for self-weight
Consider a beam of $12\,\text{m}$ span, with a rectangular cross-section, $1\,\text{m}$ wide, subjected to two point loads of $6\,\text{tonnes}$ at distances of $\frac{1}{3}$ and $\frac{2}{3}$ of the span from one end.

Limiting concrete stresses:† $0$ and $1\,200\,\text{t/m}^2$.

It is required to calculate the depth of the beam and to design the prestressing steel at the mid-span section.

(a) Self-weight of beam neglected. (This applies to a beam which is supposed to be lying flat on a horizontal surface and subjected to the given loads.) So as to make the best use of material, the stress diagrams must be as shown in Fig. 10. Consequently: $M/(bh^2/6) = 1\,200\,\text{t/m}^2$.

But $M = 6t \times 4 = 24\,\text{tm}$ and $b = 1\,\text{m}$. Therefore $l \times h^2/6 = 24/1\,200$, or $h^2 = 144/1\,200 = 0.12$, and $h = 0.346\,\text{m}$.

† In general, it is better to express the stresses in $\text{t/m}^2$ rather than in $\text{kg/cm}^2$; $1\,200\,\text{t/m}^2 = 120\,\text{kg/cm}^2$. 
The prestressing force can be calculated by equating it to the resultant of the forces in the concrete using the unloaded triangular stress diagram, or: \( F = bhR/2 \), \( R \) being the permissible compressive stress.

Therefore \( F = 1 \times 0.346/2 \times 1200 = 207 \) t.

The force can also be found from the fact that the displacement of the centre of compression between the unloaded and loaded conditions is \( M/F \). The displacement in this case, is equal to \( h/3 \), and the solution is again \( F = 3M/h = (3 \times 24)/0.346 = 207 \) t. The prestressing steel must be located at the lower third of the section, on the line of action of the resultant forces in the unloaded condition.

(b) The self-weight of the beam is taken into account (as a result of the upward deflection of the beam in a vertical plane).

The area of the section is 0.346 m\(^2\), and the density of concrete is 2.4 t/m\(^3\), so that the weight per unit length is: 0.346 \( \times \) 2.4 = 0.83 t/m.

The corresponding moment at mid-span is: 0.83 \( \times \) \( 12^2 \)/8 = 15 t.m.

The previous solution is no longer applicable, since this self-weight moment produces stresses which are equal to: 1200 \( \times \) 15/24 = 750 t/m\(^2\). Combining this with the previous solution, the stress diagrams are as shown in Fig. 11.

Figure 11 shows that the stresses exceed their permissible limits.

It appears, therefore, that the eccentricity of this prestressing force must be reduced and its value increased so that under the condition of self-weight plus moment \( M \), the stresses will remain within acceptable limits.

In reinforced concrete design, a reasonable size of concrete section is first assumed. The moments due to self-weight and external loading are
determined, and the stresses are checked. If they are unacceptable, the section is modified until a satisfactory solution is reached.

In the case of prestressed concrete, with the present example, the solution is easier.

If prestress is applied to the beam in its actual condition (that is, with its self-weight acting, and the beam freely supported), the initial condition is not that of pure prestress, as represented by the triangular diagram on the left-hand side of Fig. 11.

\[ \text{Prestress} \quad \text{Self weight} \quad \text{Unloaded} \quad \text{Added loads} \quad \text{Loaded} \]

\[ +750 \quad +1200 \quad +1950 \text{ t/m}^3 \]

\[ -750 \quad -450 \quad -750 \text{ t/m}^3 \]

\[ 1200 \quad +750 \quad +750 \]

\[ \text{Fig. 11.} \]

Indeed, prestress cannot be applied without some transverse loading acting on the beam, because the beam deflects when it is prestressed (Fig. 12) since the stress is greater at the bottom than at the top. The initial condition is therefore not one of pure prestress, but a condition corresponding to the superposition of prestress and self-weight, these two effects acting simultaneously. It is therefore the third diagram in Fig. 11 which represents the initial condition.

\[ \text{Fig. 12.} \]

If this diagram can be made identical with the first diagram in Fig. 10, the same conditions as for Fig. 10 will apply.

8. In other words, consider that the prestress diagram is indeterminate. The successive stress diagrams of Fig. 13 are determined, the unknown prestress diagram being denoted by X. The diagram in Fig. 14 is taken as the unknown diagram X (pure prestress).
The prestressing force does not change, since the resultant is
\[
\frac{1950 - 750}{2} \times bh = \frac{1200}{2} bh
\]
as before.
It could not in any case be otherwise, since the unloaded diagram is the
same as that of Fig. 11. Therefore \( F = 207 \text{ t.} \)

\[
\begin{array}{c}
\text{X} \\
-750
\end{array}
= \frac{750}{0} \quad \frac{1200 - 1200}{0}
= \frac{1200}{0}
\]

\text{Self weight} \quad \text{Unloaded} \quad \text{Added loading} \quad \text{Loaded}

\text{FIG. 13.}

The eccentricity, however, is different. Let it be denoted by \( e \). The
prestressing moment is \( Fe \), and the normal force is \( F \).
The edge stresses are \( F/bh \pm 6Fe/bh^2 \).
To obtain the diagram of Fig. 14, it is necessary to have:

\[
\begin{align*}
\frac{F}{bh} + \frac{6Fe}{bh^2} &= -750 \\
\frac{F}{bh} - \frac{6Fe}{bh^2} &= +1950
\end{align*}
\]

\[
\begin{array}{c}
750 \text{ t/m}^2 \\
+1950 \text{ t/m}^2
\end{array}
\]

\text{FIG. 14.}
Hence, the difference is:

\[
\frac{12Fe}{bh^2} = -2700
\]

Since:

\[
F = 207 \text{ t} \quad \text{and} \quad bh^2 = \frac{144}{1200}
\]

\[
e = -\frac{2700}{12 \times 207} \times \frac{144}{1200} = -\frac{2700}{207} \times \frac{12}{1200} = -\frac{27}{207} = -0.13
\]

That is, \(e = -0.13\) m.

But, in the solution shown in Fig. 11, the eccentricity is equal to:

\[
e = -\frac{0.346}{6} = -0.058 \text{ m}
\]

It is therefore sufficient to move the cable† towards the bottom by an amount equal to: \(13 - 5.8 = 7.2\) cm, in order to compensate for the self-weight of the beam.

It should be noted that \(0.072 \times 207 = 15\) tm, which is the moment \(M_p\) due to self-weight. By comparison with the weightless beam, the cable is subjected to a displacement equal to \(-(M_p/F)\).

9. This result follows, without calculation, from what has been previously said. Since the normal force \(F\) is constant, the resultant is displaced by an amount \(M/F\) because of the action of a moment \(M\).

For this resultant to act at \(-(h/6)\) in the unloaded state (third diagram of Fig. 13), it must act at \(-(h/6) - M_p/F\) in the condition of pure prestress (Fig. 15).

Therefore, the self-weight of the prestressed beam under consideration can be allowed for simply by displacing the cable.

There are limits to this method of compensating for self-weight, however, since the displacement (in this case \(-(M_p/F)\)) must not take the cable outside the boundaries of the section.

As long as these conditions are met, self-weight has little effect on the stresses in prestressed concrete, in either the concrete or the steel. Its effect on the concrete is small, since it is necessary to have \(bh^2/6 = M_s/R\)

† The words 'cable' or 'tendon' are used to describe the prestressing steel where 'resultant' cable is understood—the resultant cable is a fictitious cable, applying a force equal to the resultant of the separate prestressing forces, with the same eccentricity as the resultant. The resultant cable generally represents a number of separate cables.
(Fig. 13), in which $M_s$ is the moment due to additional external loading (over and above that due to self-weight). It has little effect on the steel, since the prestressing force is the same (as shown above), in the case of a beam which is assumed weightless; in any case, it is necessary to have $M_s/F = h/3$ (or more generally $M_s/F$ equals the depth of the limit core). $F$ therefore depends on $M_s$ alone, and not on $M_p$.

In the case of a rectangular section, this makes it immediately possible to determine the limits within which the self-weight may be compensated. If the cable is to remain within the boundaries of the section, $M_p/F < h/3$. Since $Fh/3 = M_s$, $M_p < M_s$. If the loading is uniform, $M_p$ and $M_s$ are proportional to the unit self-weight $p$ and the unit additional load $s$.

![Diagram](image)

**Fig. 15.**

It is possible, therefore, to compensate for the self-weight of a beam of a rectangular cross section, provided that this self-weight is less than the additional externally applied loading.†

When the self-weight is greater than the added loading, then only a proportion of the self-weight can be compensated, but the advantages of doing so are nevertheless still significant.

10. Generalisation

The foregoing is related to a more general property, as follows.

Consider an entity $X$, characterised by a constant portion $X_0$ and a variable portion $\Delta X$. The extreme cases are $X_0$ and $(X_0 + \Delta X)$.

Prestressing allows these extreme conditions to be changed by the introduction of a term $P_0$. The new extreme conditions become: $P_0 + X_0$ and $P_0 + X_0 + \Delta X$. If $P_0$ is given the value $-X_0$, the constant portion is compensated and the resulting extreme conditions become 0 and $\Delta X$. If

† In practice the displacement $M_p/F$ can rarely be equated to $h/3$, since it is necessary to locate the tendon at a suitable distance from the lower face (see Chapter VIII).
$P_0$ is given the value $-(X_0 + \Delta X/2)$, the extreme conditions become $-(\Delta X/2)$ and $+(\Delta X/2)$.

From then on if the material, after prestressing (that is, as a consequence of the properties given to it by the term $P_0$), is able to resist both the negative and the positive variations, it need only resist $\Delta X/2$, whereas non-prestressed materials must resist $X_0 + \Delta X$.

The base value or zero value of the entity is changed, and the magnitude of the variations is cut by half.

\[ X_0 + \frac{\Delta X}{2} \]

**Fig. 16.**

It is not always possible to give $P_0$ the optimum value, but it is always possible to create a more favourable condition in which the magnitude of the entity $X$ which has to be resisted is reduced. This is because the potential which is present in the material before loading is exploited (potential which is not exploited in other methods).

In the cases considered above, the entity $X$ is bending; the constant portion is the moment $M_0$; the variable portion $\Delta X$ is the moment $M_a$.

With regard to these entities, prestressing determines two parameters:

the force $F$, that is to say a compression;
the eccentricity $e$, that is to say a moment $M_0 = Fe$.

Giving $M_0$ the value $-(M_p + M_s/2)$, that is $-(X_0 + \Delta X/2)$, the base value of the moments is put at $M_p + M_s/2$. In other words, under the action of the load $p + s/2$ (mean load), the beam is no longer subjected to any moment; it is in uniform compression under the action of the force $F$. The variation becomes $\pm (M_s/2)$, and the value of $F$ must be such as to make the tensile stresses created by $M_s/2$ equal to zero. Also, the maximum stress resulting from the compression $F$ and the moment $M_s/2$ must not exceed the permissible value $R$. 

This is expressed in the case of a rectangular section by:

\[ F_e = -\left( M_p + \frac{M_z}{2} \right) \] (displacement of the origin);

\[ \frac{F}{bh} = \frac{6}{bh^2} \times \frac{M_z}{2} \] (cancellation of tensile stresses);

\[ \frac{F}{bh} + \left( \frac{6}{bh^2} \times \frac{M_z}{2} \right) = R \] (resistance to the maximum compressive stresses)

whence:

\[ F = \frac{3M_z}{h}; \quad \frac{2F}{bh} = R; \quad e = -\frac{h}{6} - \frac{M_p}{F} \]

11. Cost comparison

Compare the design in prestressed concrete for the example shown in Fig. 10 with a design in reinforced concrete. If the same maximum stress \( R = 120 \) kg/cm\(^2\) is assumed, with a stress of 1 440 kg/cm\(^2\) for steel, \( m = 15 \), and an effective depth \( d = 0.92 \) h, then \( Q/R = 0.191 \) and the resisting moment is \( M_R = 0.191 bh^2 \) R or, for \( R = 1 \) 200 t/m\(^2\), \( M_R = 229 bh^2 \) tm.

The weight per unit length is \( bh \times 2.4 \) (the density of concrete being 2.4 t/m\(^3\)); hence \( M_p = bh \times (2.4l^2/8) = bh \times 2.4 \times 144/8 = 43.2 \) bh tonne metres, \( M_z = 24 \) tm.

Therefore 229 \( bh^2 = 24 + 43.2 bh \), and since \( b = 1 \) m, \( h \) is obtained from the equation 229 \( h^2 - 43.2 h - 24 = 0 \).

Hence, \( h = 0.43 \) m instead of 0.346 m in the case of prestressed concrete.

The ratio is 0.346/0.43 = 0.8.

The self-weight is therefore 0.43 \times 2.4 = 1.03 t/m, whence \( M_p = 18.5 \) tm.

The total moment is 18.5 + 24 = 42.5 t/m.

If \( h' \) is the effective depth, assumed equal to 0.9h, the moment arm is 0.815\( h' \) = 0.75h = 0.322 m. In the case of reinforced concrete, the tensile force to be balanced by the reinforcement is

\[ F = \frac{42.5}{0.322} = 132 \text{ t} \]

Since the stress in the steel is 14.4 kg/mm\(^2\), the cross-sectional area of reinforcement required is 132 000/14.4 = 9 200 mm\(^2\).
For prestressed concrete, the force required is 207 tonnes. The stress used being 90 kg/mm², the required cable area is 207 000/90 = 2 400 mm². The weight of steel per unit length is therefore:

73·6 kg/m for reinforced concrete;
19·2 kg/m for prestressed concrete.

However, the steel used in prestressed concrete is much more expensive than the steel used in reinforced concrete. Assuming a ratio of 3 to 1 in the costs, the ratio of expense is (3 × 2 400)/9 200 = 0·78, which is of the same order as the concrete ratio.

Allowing for shuttering, which is about the same in both cases, the saving may be of the order of 10 to 15%.

However, this comparison applies only to the central zone of the beam. For a comparison of total costs, other factors must be taken into account: for example, the relative ease with which the reinforcement can be varied in reinforced concrete, and the effect of anchor costs for prestressed concrete, especially for beams of small dimensions; on the other hand, the number of stirrups is less, in the case of prestressed concrete.

It should be noted that the increase in quality which is achieved with prestressing is not accompanied by an increase in price, as might have been expected. On the contrary, prestressing is generally cheaper.

12. Cable profiles
Only the section of the beam at mid-span has so far been considered. Similar reasoning and calculations can be applied at any section, but with this difference. At mid-span (or more generally at the most highly stressed section), the aim is to obtain the maximum utilisation of materials for the two conditions of loading. This is possible for the beam in Fig. 10 (it is seen later that it is not always so). Under these conditions, it is possible to determine the three unknowns $h$, $F$ and $e$.

If $M_s$ decreases at sections remote from mid-span, and if $h$ and $F$ remain constant, the strength is excessive.

Of the three parameters $h$, $F$ and $e$, $e$ alone remains unknown. It is no longer determined by an equality, but by a series of inequalities expressing the condition that the stresses must remain within fixed limits. In other words, at any section $x$, $e$ must lie between two limit values $e_1(x)$ and $e_2(x)$, and consequently the centre of compression for pure prestress must lie between two limiting points $E_1(x)$ and $E_2(x)$. The loci of these points $E_1$ and $E_2$, as the section is displaced along the beam, are the two limits
of the cable profile. Any profile contained within the two limit profiles satisfies the strength criteria.

The concept of two limit profiles is immediately understood from a consideration of the centres of compression.

At any section \((x)\), let \(A\) and \(A'\) be the upper and lower boundaries of the limit kern; if \(M_x(x)\) and \(M_{x+}(x) + M_{x}(x)\) are the maximum moments to which the section is subjected, the lower and upper limiting points are obtained by striking the vectors \(-(M_p/F)\) and \(-(M_p + M_s/F)\) from \(A'\) and \(A\) respectively. Indeed, if the point of passage \(E_0\) of the cable were below \(E_1\), the centre of compression due to self-weight (which is derived from \(E_0\) by the displacement \((M_p/F)\) would be below \(A'\). If \(E_0\) were above \(E_1\), the centre of pressure in the loaded condition (displaced by an amount \((M_p + M_s/F)\) from the origin) would be above \(A\).

![Diagram](https://via.placeholder.com/150)

**Fig. 17.**

Generally, if \(M_1\) and \(M_2\) are the algebraic minimum and maximum moments, the two limiting cable profiles are obtained by striking the vectors \(-(M_1/F)\) and \(-(M_2/F)\) from the lines \(A\) and \(A'\), which are the loci of the boundaries of the limit kern.

If the loads and the section are uniform, the lines \(A\) and \(A'\) are both horizontal and the functions \(-(M_1/F)\) and \(-(M_2/F)\) are parabolic.

In the case of a simply supported beam, the moments are zero at the supports and consequently the profile limits at the points of support are at \(B\) and \(B'\), the points of intersection between the horizontal lines \(A\) and \(A'\) and the vertical lines drawn through the supports. The two limit profiles are given by two parabolas with chords \(B'B\), \(BB\) (Fig. 17). If it is required to obtain full use of the material at mid-span, the point of passage of the cable at that section is the point \(E_0(m)\) previously determined, and the two limiting parabolas \(B'E_0(m)B'\) and \(BE_0(m)B\) are obtained. Any cable profile within the boundaries of these two parabolas satisfies the strength criteria.
It is seen, therefore, that it is more or less necessary to adopt a parabolic cable profile shape, subject, of course, to tolerances with regard to the ideal parabolic shape; or, more generally, a catenary load distribution is required.

13. This conclusion leads to the consideration of prestressing from a different viewpoint. Consider Fig. 18, assuming that the cross section is rectangular and the lower limit stress is zero (limit kern coincident with central kern).

The tensioning of a parabolic cable inside a curved duct located within the beam brings the cable into contact with the top generatrix of the duct, since, if it was not restrained by the duct, it would take the shape of its chord CC. It remains parabolic only because it is subjected to downward forces from the duct equal to \( q \) per unit length and therefore also from the concrete.

![Fig. 18.](image)

If \( f \) is the sag or dip of the cable, \( f = ql^2/8F \). The dip of the parabola can lie between \( A'E_0(m) \) and \( AE_0(m) \) (Fig. 17). Therefore \( A'E_0(m) = M_p/F = pl^2/8F \) and \( AE_0(m) = (M_p + M_s)/F = (p + s) l^2/8F \).

The forces \( q_1 \) for the lower profile are therefore such that \( q_1 l^2/(8pl^2/8F) = F \), whence \( q_1 = p \).

Similarly, for the upper profile, \( q_2 = p + s \).

Conversely, the cable exerts forces \(-q\) on the concrete, that is to say \(-p\) for the lower limit profile and \(-(p + s)\) for the upper limit profile. In addition it exerts a compressive force \( F \) in the direction of the chord CC (B'B' or BB).

For the lower limit profile, therefore, this is equivalent to the resultant forces being zero in the unloaded condition \((p - q_1 = 0)\). The beam is then subjected to a compressive force \( F \) acting along B'B', with a constant eccentricity equal to \(-(h/6)\).

The effect of this eccentric compression is the same as that of a central compression \( F \) and a constant moment \(-(Fh)/6 = -(M_s(m))/2, M_s(m)\) being the moment due to the additional external loading at mid-span.

The external loads add the effect of a moment \(+M_s(x)\) to this unloaded condition.
For the upper limit profile, it is equivalent to the resultant forces being equal to zero in the loaded condition \((p + s - q_2 = 0)\). The beam is then subjected to a compressive force \(F\), acting along BB, with constant eccentricity \(+ (h/6)\). The effect of this eccentric compression is equivalent to that of a central compression \(F\) and of a constant moment \(Fh/6 = + [M_s(m)/2]\).

The unloaded state is obtained from this loaded state by adding the effect of the load \(- s(p + s - s = p)\); that is, by adding the effect of the moment \(- M_s(x)\).

In both cases the centre section is subjected to the compressive force \(F\) and to the moments \(- (M_s/2)\) when unloaded, and \(+ (M_s/2)\) when loaded.

![Fig. 19.](image)

If the chosen profile is the mean profile between the two limit profiles, whose chord is the mean line GG of the beam, \(q = p + s/2\). The beam is in simple compression under the load \(p + s/2\), and subjected to the resultant loads \(- (s/2)\) when unloaded and \(+ (s/2)\) when loaded. The same concept of displacement of the origin of the moments (or of the loads) as in the case of the mid-section is applicable.

This study of the increased forces which are exerted when tensioning the cable, or of the thrust of the cable, in the unloaded condition, is very useful in certain cases, and it can enable complicated calculations to be dispensed with.

14. Shear strength

It can be shown that prestressing leads to useful savings when designing for shear strength; it becomes possible to reduce the web thicknesses of T- or I-beams as well as the number of stirrups required.

The first saving arises from the thrust of the cable in the unloaded condition (examined in the preceding paragraph) which considerably reduces the resultant loads acting on the beam. If the cable profile is taken as the mean profile, in the cases examined in the previous paragraph, the resultant load is \(\pm (s/2)\). Consequently, the maximum shearing force is \((s/2)(l/2)\), instead of \((p + s)(l/2)\). The ratio is \(s/2(p + s)\), and, if \(p = s\), its value is \(\frac{1}{4}\).
A second saving accrues as a consequence of the compressive forces, which bend the lines of compression towards the horizontal. Since shear failure tends to occur in the direction of these lines, and since the stirrups are designed to act as ties between the right-hand and left-hand sides of the possible failure line (Fig. 20), it is obvious that the greater the length of the horizontal component $a$ of the possible line of failure, the smaller is the number of stirrups required per unit length.

![Fig. 20.](image)

The proportional reduction can be of the order $\frac{1}{2}$ to $\frac{1}{3}$. In total the shear stresses can often be reduced by as much as $9/10$ by comparison with reinforced concrete.

The possibility of using thinner webs is a further contribution to lighter structures.

15. **Provision of favourable initial conditions**

In many cases, prestressing enables the strength under working conditions to be greatly improved, because of the application of forces opposing those which are applied by the external loading.

The case of a vertical cylindrical tank or reservoir containing a liquid is one example.

![Fig. 21.](image)

Deformations due to liquid pressure + Deformations due to prestress above = Resultant deformation under load

The diameter of the tank increases because of tensile strain when liquid pressure is applied. If the barrel of the cylinder is fixed to the base, deformations such as those represented in Fig. 21(a) occur, and large vertical moments are present local to the base of the tank.
When a tank of this type is prestressed, circumferential compressive stresses are induced in the barrel in the unloaded condition, and the tank assumes the shape which is shown in Fig. 21(b). The deformations are opposite in direction to those created by the internal pressure. Consequently, when the tank is filled, the resultant deformations are zero or, at least, they are considerably reduced. The moments are still considerable, but they occur in the unloaded state, when leak-tightness is not so critical.

This is a particular application of the case of displacement of the origin previously discussed. Other examples may be found in the study of self-supporting cylindrical shells, folded plate roofs, etc.

16. Prestressing with bonded steel

This is an alternative prestressing technique to that described in the foregoing. It consists in first pre-tensioning the steel against independent abutments using 'long-line' prestressing beds, and then casting the concrete to the required shape around the steel. The steel wires are in contact with

![Diagram of prestressing with bonded steel](image)

**FIG. 22.** Bonded-wire prestressing bed.

the concrete (no longer in sheaths as in Fig. 8); the wires are tensioned using two terminal blocks, one of which is movable by means of jacks. The shutters are positioned and the concrete is placed. When the concrete has hardened sufficiently, the wires are cut in the spaces between the shutters, at the ends nearest to the terminal blocks. The wires tend to contract into the cast members, but are prevented from doing so by the bond between the concrete and steel. The pre-tensioning force is thus transmitted to the member.

This method is used especially for mass production in factories. The terminal blocks are generally about 100 m apart, and the reactions to the prestressing forces are provided either by anchor blocks or by the factory floor itself.
Heating is usually employed to accelerate the setting and hardening of the concrete, and it is possible to complete one or even several cycles each day.

Losses of prestress are naturally greater than in the case of post-tensioned steel, since the final shortening of the concrete is equal to the total shortening and not to the long-term shortening alone. Moreover, the prestressing steel is generally straight, which eliminates the possibility of compensating for the weight of the member; though it has been possible in some cases to curve the wires along the length of the members by means of screw mechanisms attached to the shuttering supports.

17. Prestressing by jacking
It has been seen that it is possible to prestress without using steel, if abutments are available, by compressing the member with jacks supported from abutments located at each end of the member.

These methods are used particularly for prestressing airport runways, motorways, and for longitudinal prestressing of pipes; they have also been used in the construction of lock-gate sills and dams.

In the early days of prestressing, it was a condition that all tensile stresses in the concrete should be avoided, so as to eliminate all risk of cracking.

There are cases where this is the only way of achieving a complete guarantee. Such cases include tanks or reservoirs where leak-tightness is essential, submerged structures, structures in corrosive atmospheres, and generally all structures where cracking would be detrimental. It is also the case with structures subjected to high-frequency fluctuating loads, where the closing and opening of cracks could cause fatigue of the reinforcement; it could even be necessary, in such a case, not merely to avoid tensile stresses, but to impose a residual compressive stress at the lower stress limit.

However, there are many structures in existence where the possibility of cracking is allowed (especially if cracking occurs only under unusual conditions of loading), when the complete elimination of tensile stresses would result in unjustifiable costs.

It may happen that, because of the nature of a particular structure, a design using total prestress is more costly than a design using reinforced concrete.

This does not contradict the previous statements on relative costs. For
example, this can be the case with short beams, because of the high proportionate cost of the anchors; with members which can be loaded in two opposite directions (such as electric transmission line supports), where the prestress must be axial, thereby reducing the moment arm; it is especially the case with some statically-indeterminate structures, particularly buildings, where the advantages of compensation for permanent loads, discussed in the context of statically-determinate beams, are obtained; lastly it can be the case in countries where high-tensile prestressing steel is scarce.

On the other hand, a design using partial prestress (that is, where some tensile stresses are permitted under maximum loading conditions) can usually be produced for the same price as, or at a lower price than, a design in traditional reinforced concrete; the prestressed design has equal safety, and offers certain definite advantages.

This does not imply that there is a rivalry between reinforced and prestressed concrete and that the two are irreconcilable. On the contrary, prestressing offers a valuable contribution to reinforced concrete, and reinforced concrete manufacturers have readily appreciated the benefits that can be obtained from it.

Time has proved the soundness of structures constructed in this manner, as well as the higher quality which can be achieved.

Such structures are not in general subjected to tensile stresses under permanent loading; and cracking as a result of shrinkage can be avoided if the prestress is applied judiciously. If cracking should occur under maximum loading conditions, it disappears when the loading is removed; protection of the steel is thus much better assured than in the case of reinforced concrete.

Depending on the level of prestress, the advantages of prestressed concrete from the point of view of integrity of the material can be maintained for a greater or lesser amount of overloading.

The acceptable level of prestress can be determined from the probable frequency of overloading in relation to its intensity, from the risk of cracking in relation to the possible tensile stress, and from the consequences of this eventual temporary cracking. With respect to the risks and the consequences, structures can be divided into three classes, according to the FIP–CEB Committee (International Prestressing Federation–European Concrete Committee):

† These three classes are denoted in the following chapters by: total prestress (Class I); partial prestress (Class II); prestressed reinforced concrete (Class III).
Class I: where no cracking is permitted.
Class II: where a crack, although undesirable, is not unduly important.
Class III: where cracking is permitted, the role of the prestress being to contribute the advantages which are mentioned above.

Different design and checking rules apply to each of these classes.
In Class I structures, tensile stresses are not permissible. This is the case of total prestress.
In Class II structures, tensile stresses are limited, and they are calculated as though the structure were elastic and homogeneous. Their value is restricted so that the probability of cracking remains sufficiently low. The ASP (Scientific Prestressing Association) adopts the prism measurement of tensile strength for this limit (see Chapter IX). In addition, it requires a minimum quantity of ordinary reinforcement in the tensile zones.
The FIP–CEB Committee limits the strains and not the stresses (Chapter IX).
In Class III structures (cracking considered as an inherent risk during normal functioning of the structure), the design and the checks are the same as for reinforced concrete in compound bending. Ordinary reinforcement (not tensioned), is added to satisfy the strength criteria. Again, rules are given such that the opening of the cracks, when they occur, does not exceed certain limits. This imposes limitations on the diameter and spacing of the reinforcement.
Examples of the three classes are given in the chapters dealing with the design of prestressed structures. It should be noted at this stage that there is no difference between the calculations for Class I and Class II when they are based on the homogeneous section, as advocated by the ASP; there is a difference only in the requirement for minimum reinforcement in the case of Class II.
Class III is the true solution for industrial buildings. It is studied in Chapter XI.

19. Centres of compression and curves of compression for Classes II and III
The centre of compression is defined in Section 4 of this chapter. The curve of compression is the locus of the centre of compression as the section is moved along the beam. In the case of statically-determinate beams, the curve of compression for the prestress alone is the cable profile itself.
In the case of total prestressing (Class I) if cable over-tensioning is neglected (Section 5), the curve of compression under any condition of
loading is obtained from the cable by striking from it the ordinate M/F at each section. This curve of compression must stay within the limit zone, which is the zone described by the limit of the core as the section is traversed along the beam.

At this stage, it is important to note that with Classes II and III the increase in prestress due to applied loading may be high, and also that the ordinary reinforcement (which is here termed complementary reinforcement) exerts complementary forces in the areas which are subjected to tension under the loads.

The position of the centre of compression can easily be determined by taking moments with respect to the centroid. Let $e_0$ and $e_a$ be the eccentricities of the cable and complementary reinforcement, and $F$ the ‘permanent’ Prestressing force. Let $\Delta F$ be the excess tension in the cable in the loaded condition and $F_a$ the force in the complementary reinforcement in that condition.

In pure Prestressing, the moment is $Fe$; in the loaded condition it becomes $(F + \Delta F)e_0 + F_ae_a + M$.

The eccentricity of the centre of compression becomes

$$\frac{(F + \Delta F)e_0 + F_ae_a + M}{F + \Delta F + F_a}$$

This eccentricity must lie within the limit zone.

If, as often happens, $e_0$ and $e_a$ have approximately the same value, the eccentricity under load is $e_0 + M/(F + \Delta F + F_a)$ the curve of compression is therefore obtained from the cable profile by displacements equal to $M/(F + \Delta F + F_a)$ at each section.

This can be seen in a different manner by imagining that the prestress is reduced to the previous force $F$ (that is, to the permanent prestress), and that the forces $\Delta F$ and $F_a$ are the result of the compound bending produced by the moment $M$ and the force $F$, taken as constant and having the same eccentricity as the cable at each section.

Two curves of compression can thus be defined; one for outside forces, called for short the external curve of compression.† The ordinate of the curve relative to the cable profile is equal to $M/F$ at any section. The second curve is the internal curve of compression, with ordinates as given above, deduced from compound bending analysis.

The external curve of compression can lie outside the boundaries of the material, and the forces $\Delta F$ and $F_a$ restore the internal centre of

† $F$ is considered here as an external force.
compression within the limit zone (at a distance of \( M/(F + \Delta F + F_a) \) from the cable of \( e_0 = e_a \)).

This manner of dealing with the subject is of interest for two reasons:

1. For the classes of prestress under consideration (Class II and especially Class III), it is possible in general to assume the prestressing force, without having to satisfy such stringent conditions as for Class I.

The approximate limit position of the internal centre of compression is often known; if \( F \) is known, the external curve of compression is also known. The internal compressive force can therefore be obtained by equating moments about the cable (Fig. 23). If \( Z \) and \( z \) are the distances of the cable from the external and internal curves of compression respectively, then \( F + \Delta F + F_a = F \times Z/z \), whence \( \Delta F \) and \( F_a \) are obtained and therefore the complementary reinforcement can be determined.

![Diagram](image)

**FIG. 23.**

2. It shows that the external force \( F \) can always be replaced by two internal forces: a compressive force \( (F + \Delta F + F_a) \) and a tensile force \(-(\Delta F + F_a)\). In some cases, this enables simple geometrical solutions to be obtained.

The same arguments apply to Class I prestressing, but the external and internal curves of compression are very close to each other, and the approximation that \( F \) remains constant under loading assumes that they are coincident.
Chapter II

MATERIALS†

I. CONCRETE

1. The concrete used in prestressed concrete work is similar to that used in reinforced concrete work, but it is usually subjected to higher stresses. Therefore, an increase in the quality of the concrete generally leads to more economical results. It is particularly advantageous for the zones which are ultimately in tension to be initially very strong in compression. It is also advantageous to have high elastic moduli at the time of prestress, so as to reduce deformations and consequent losses of stress in the prestressing steel.

The advantages of high strength are not so pronounced in the case of reinforced concrete, where it offers little benefit in the tensile zones.

The concrete strength (for 20-cm cubes at 28 days) required in Class I prestressing (total prestress) is of the order of 380 to 450 kg/cm². For partial prestressing or reinforced prestressed concrete (Classes II and III) a lower strength is usually acceptable, of the order of 330 to 380 kg/cm² at 28 days.

Relationships between the strength of cubes and cylinders \(\left(\frac{R_{\text{cyl.}}}{R_{\text{cube}}}\right)\) (cylinders 15 cm in diameter and 30 cm high, 20-cm cubes with capped ends).

It is generally agreed that the value of the ratio \(\left(\frac{R_{\text{cyl.}}}{R_{\text{cube}}}\right)\) is approximately constant and of the order of 0·83.

It increases with concrete strength (see Dreux, Connaissance du Béton, edited by the UTI). The value of 0·83 applies to concrete with cylinder strengths of 200 to 300 kg/cm². The ratio approaches 0·90 for concretes of 450 kg/cm² cylinder strength. Concrete of this strength is usual in prestressing applications.

† Formulae relating certain phenomena to the principal properties upon which they depend are given in this chapter. These are empirical rules, giving average values, and departures from them can be quite considerable; however, they are useful for obtaining orders of magnitude in the absence of more precise data.
As a general principle, both strengths (cylinder and cube) are indicated in the following pages so as to avoid confusion. The strengths are measured on standard test pieces unless otherwise stated (20-cm cubes, 15 × 30-cm cylinders).

It can be assumed (Dreux) that the increase in strength with time can be obtained from the following formula, \( R_j \) being the strength at day \( j \):

\[
\frac{R_j}{R_7} = 2.93 - \frac{1.77}{(\log j)^\frac{1}{2}}
\]

Therefore the strength at day 28 is \( R_{28} = 1.45 \ R_7 \) and the strength at day 90 is \( R_{90} = 1.65 \ R_7 \).

On average, \( R_{90} = 1.13 \ R_{28} \).

This enables the older rules, which have for reference the cube strength at 90 days, to be related to more recent rules which take the cube strength at 28 days as the reference.†

For example, the outdated rule which specifies the limit stress as 0.28 of the 90-day strength becomes 0.28 × 1.13 = 0.32 of the 28-day cube strength; or, when the ratio \( R_{cyl.}/R_{cube} = 0.83 \) applies, it becomes 0.32/0.83 = 0.4 of the 28-day cylinder strength.

From here on, those properties which bear most directly on prestressing are discussed: in particular the unit deformation properties, since they affect the losses of stress in the prestressing steel, and consequently the effectiveness of the prestress.

2. Shrinkage

Shrinkage is completed only after a considerable time, usually after two or three years when dealing with usual thicknesses. The rate of shrinkage, however, is relatively high at first; it can be assumed that for normal thicknesses a quarter of the total shrinkage occurs within the first 7 days, a third after 14 days, half after a month and three-quarters after 6 months.

If \( \eta_\infty \) is the total shrinkage, a reasonable value of the shrinkage \( \eta_t \) after time \( t \) is given by the following formula:

\[
\frac{\eta_t}{\eta_\infty} = \frac{(1.5 + t)t}{1 + 4t + t^2}
\]

where \( t \) is expressed in months (Brazilian Standards formula).

† Translators’ note: this relates to a feature which was largely limited to French practice only.
The total shrinkage $\eta_\infty$ depends upon many factors. It varies inversely as the percentage relative humidity (it is low at a relative humidity of 90% and very high in a very dry climate); it varies inversely with thickness (the smaller thickness when dealing with rectangular cross-sections). It increases with the amount of water which is present and with the ratio $E/C$, $E$ and $C$ being the weight of water and the weight of cement respectively per cubic metre of concrete; in this last it varies approximately as $E(1 + 3E/C)$.†

Because of this influence of the water quantity which is present, shrinkage increases when fine cement is used, when impurities are present (such as clay in badly washed aggregates) and with finer aggregates; it is reduced when a higher degree of compaction is obtained.

Shrinkage also increases with temperature. It can be deduced‡ from tests carried out by ROSS and ENGLAND (Magazine of Concrete Research, March 1962) that the total shrinkage increases at temperatures above $20^\circ$C by $0.05 \times 10^{-4}$ per $^\circ$C.

The losses of prestress in the prestressing steel are affected only by the shrinkage that occurs after tensioning.††

A reasonable value to assume is $\eta = 2/10000$ from the time of tensioning, under average conditions and with a relative humidity of 80%.

The rate of shrinkage, which would be proportional to $\eta_\infty$, if formula (1) were strictly true, depends also on the rate of evaporation of the water. Consequently, final equilibrium is reached more quickly on the exposed surfaces; this creates internal stresses which place the outside faces in tension. These stresses can be of considerable magnitude in thick-walled concrete structures.

Independently of the differing rates of shrinkage within the mass of a section, equilibrium is reached more quickly in thin sections than in

† The CEB gives empirical rules as a function of these various factors. They may be combined in the following approximate formula:

$$\eta_\infty = \frac{10}{10000} \frac{(1 - p) 0.5 + 0.3e}{0.4 + e} \frac{E}{500} \left(1 + \frac{3E}{C}\right)$$

in which $p = \text{relative humidity (\%)}$, $e = \text{least thickness (metres)}$, $E$ and $C = \text{weight of water and weight of cement (kg/m}^3\text{)}$. This formula is on the optimistic side, but indicates clearly the effect of the various factors.

‡ The tests give results after 60 days; the final value is extrapolated from the empirical law:

$$\eta_\infty = \eta_t \times \frac{1 + 4t + t^2}{(1.5 + t)t}$$

†† As a result, the whole of the shrinkage must be allowed when estimating the losses in the case of members with pre-tensioned steel.
thick sections. Hence internal stresses are set up within structures of varying thicknesses (thin webs with thick flanges and bracing, thin slabs between thick beams, and so on).

If this prestress is applied sufficiently early, it is possible to offset the effect of internal stresses by the imposition of strains which are greater than those caused by the shrinkage. However, it is not always possible to do so; in particular, it is not possible when the prestress is uni-directional and when shrinkage occurs uniformly at right angles to it.

In any case, unrestrained contraction should be permitted in order to reduce shrinkage effects as much as possible. There are several ways of achieving this: moulds may be mounted on rollers, sliding faces, or rubber pads when supported on rigid foundations.

Stiffness may be reduced by means of temporary hinges or links; construction joints may be left open and later grouted or concreted. Restraint from the shuttering must also be avoided (for example, shuttering of the web of an I-beam, which resists the vertical contraction of the web), and it must be arranged so that it can be dismantled as soon as possible after the concrete is placed. Equally, excessive differences in the ages of successive lifts must be avoided.

Curing, by means of water sprays during the setting and hardening period, or of wet straw laid over the concrete, reduces the initial shrinkage rate; internal stresses and the effects of age differences can thus be reduced. Similar results can be obtained by spraying protective coatings over the green concrete.

When concrete is allowed to harden in air and is then immersed in water, it swells; also when the face is in contact with water (as in reservoirs or pipes). It does not, however, revert to its original dimensions; part of the shrinkage is irreversible† (tests by DUTRON).

3. Coefficients of thermal expansion
The coefficient of thermal expansion of concrete depends on the source and nature of the stone used in the aggregate,‡ and can range from $0.6 \times 10^{-5}$ to $1.2 \times 10^{-5}$ per °C.

Because of its low thermal conductivity, concrete does not immediately respond to changes in the ambient temperature; simultaneously, the

† See Retrait des Ciments, Mortiers et Bétons by DUTRON (Brussels, 1934).
‡ Orders of magnitude from tests by ENDELL on various mortars made with different aggregates at normal temperatures:
- Rhine gravel \(1 \times 10^{-5}\) per °C
- Limestone aggregates \(0.9 \times 10^{-5}\) per °C
- Basalt \(0.6 \times 10^{-5}\) per °C
- Slag \(0.4 \times 10^{-5}\) per °C
hygrometric state changes with temperature, and dimensional changes may therefore occur which are of opposite sense to those created by temperature.

The effect of this complex behaviour depends on the rate of temperature change; for small changes, dimensional changes can be appreciably less than those usually allowed.

4. Deformations under loading
(a) Instantaneous deformation
The stress–strain diagram for a concrete specimen subjected to rapid loading (of the order of a minute) has the shape shown in Fig. 1.

The curve is for all practical purposes coincident with the tangent through the origin for stresses less than 1/10 of the ultimate stress R. It then curves gradually away as the stress increases.

![Stress-strain diagram for concrete.](image)

If $\sigma$ is the stress and $\varepsilon$ the strain, the modulus of elasticity is defined by the ratio $\sigma/\varepsilon$. It is given by the slope of the line OA (Fig. 1). This modulus is denoted by $E_{\text{m}}$, the suffix $i$ indicating that it is applicable to instantaneous, or immediate, loading; it is called the modulus of instantaneous deformation, or more briefly the instantaneous modulus.

The modulus has the value $E_{\text{oi}}$ (slope of the tangent at the origin) for stress values up to about $R_{\text{cube}}/10$. It then decreases as the stress increases. In order to establish orders of magnitude, it could be assumed that for $\sigma = 0.3 \ R_{\text{cube}}$, $E_i = 0.9 \ E_{\text{oi}}$ and for $\sigma = 0.5 \ R_{\text{cube}}$, $E_i = 0.75$ to $0.8 \ E_{\text{oi}}$. 
If the stress–strain diagram is extended to the point of failure, it has the shape shown in Fig. 2; the stress remains practically constant after reaching the ultimate value† while the strain continues to increase. Failure occurs when the strain reaches its limiting value \( \varepsilon_r \). This limit has no fixed value, as is often thought. Nevertheless, a mean value of 3-5/1000 can be used; in general actual values do not vary from this mean value by more than \( \pm 30\% \).

![Stress–strain diagram to failure.](image)

When the test piece is unloaded, the slope of the return path AA' (Fig. 1) is approximately equal to \( E_{ol} \); there is therefore a permanent instantaneous deformation approximately equal to

\[
\sigma \left( \frac{1}{E_t} - \frac{1}{E_{ol}} \right) = \varepsilon_i \left( 1 - \frac{E_t}{E_{ol}} \right)
\]

This formula is only approximate and the permanent deformation is slightly less than that obtained by the formula because the return path is not truly linear; the permanent deformations OA' obtained in practice are 1/10 to 1/6 of the deformations under loaded conditions, depending on the magnitude of the stress reached under load.

Under repeated loading (Fig. 3), the stress–strain diagrams still retain the general shape shown in Fig. 1, but they regress progressively until they approach the tangent whose slope is \( E_{ol} \); the permanent deformations are cumulative, but they get smaller from one cycle to the next, and the

† In practice, the stress diminishes after passing through the maximum value R (Fig. 2).
permanent deformation tends towards a limit. From this stage onward, the stress–strain diagrams during loading and unloading become stable. The forward and return paths are slightly curved, and the deformation cycle has the shape shown in Fig. 4, accompanied by the production of heat.

![Stress-Deformation Diagram](image)

**Fig. 3.**

*Value of the elastic modulus $E_{oi}$ (instantaneous).* The instantaneous modulus of elasticity at the origin is approximately equal to $E_{oi} = 19,000 (R)\, kg/cm^2$, where $R$ is expressed in $kg/cm^2$.†

![Stress-Deformation Diagram](image)

**Fig. 4.**

† It would be better to use a dimensionless formula: $(E/E_1) = (R/R_1)$, $E_1$ and $R_1$ being the modulus and the strength of a reference concrete ($E_1 = 400,000 \, kg/cm^2$; $R_1 = 450 \, kg/cm^2$ cube strength).
**Tensile deformations.** For small values of tensile stress, the elastic modulus is equal to $E_o$. The stress–strain diagram in tension is therefore a continuation of that shown in Fig. 1.

Plastic deformations in tension are small as far as concrete without reinforcement is concerned; the plastic or pseudo-plastic behaviour of reinforced concrete in tension is dealt with later.

In direct tension, the tensile strength of concrete is of the order of $1/12$ to $1/15$ of the cube compressive strength; the tensile strength in bending (from bending tests on prisms) is about $60\%$ greater than the direct strength.

(b) **Long-term deformation**

If a test piece is subjected to compression under a constant load, a shortening of the test piece is first observed, and the value of this instantaneous shortening is a function of the applied stress, in accordance with the diagram of Fig. 1.

![Diagram](image)

**Fig. 5.** A, long-term deformation; B, instantaneous (immediate) deformation; C, total deformation; D, instantaneous unloading deformation; E, long-term unloading deformation; F, residual deformation. From Le Camus.

The deformation increases with time, quickly at first, then more and more slowly, until it reaches a limit value. This is reached only after a very long time, of the order of 3 to 4 years, but 90% of the deformation occurs before the end of the first 18 months.

† It can be assumed that $R' = 8 + (R/20)$ (kg/cm$^2$) approximately (R measured in 20-cm cubes).

The CEB gives $R' = 0.5 (R_{cube})^{\frac{1}{2}}$, or $R' = 0.58 (R_{cyl})^{\frac{1}{2}}$, if it is assumed that $R_{cyl} = 0.8 R_{cube}$ (see Section 1).
This type of concrete behaviour is called long-term deformation or creep. It is of considerable importance when dealing with prestressed concrete.

Figure 5 shows the variation of long-term deformation with time.

For stresses less than 0·5 R the limit value of the long-term deformation $\epsilon_d$ is closely proportional to the instantaneous deformation $\epsilon_i$. This can be expressed as $\epsilon_d = \phi \epsilon_i$, where $\phi$ is a coefficient.

The total limit deformation is $\epsilon_{\infty} = \epsilon_i + \epsilon_d = \epsilon_i(1 + \phi)$.

Usually, $\phi$ lies between 1·5 and 2. Long-term deformation is therefore 1·5 to 2 times the instantaneous deformation and the total deformation is about 2·5 to 3 times the instantaneous deformation.

In practice, $\phi$ depends upon many factors: climatic conditions, thickness of concrete, water content, time since loading. It varies inversely as the percentage relative humidity and thickness of concrete; it increases as the percentage water content increases; it diminishes rapidly with the age at the time of loading.

Since the compressive strain also decreases with age, because of the increase in the elastic modulus, the age at the time of loading is of great importance. Taking the 28-day conditions as unit conditions, the approximate ratios are given in the following table:

<table>
<thead>
<tr>
<th>Load applied after</th>
<th>3 days</th>
<th>7 days</th>
<th>28 days</th>
<th>90 days</th>
<th>1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>At the time of loading</td>
<td>1. Strength varies as</td>
<td>0·54</td>
<td>0·74</td>
<td>1</td>
<td>1·13</td>
</tr>
<tr>
<td></td>
<td>2. Elastic moduli</td>
<td>0·67</td>
<td>0·83</td>
<td>1</td>
<td>1·07</td>
</tr>
<tr>
<td></td>
<td>3. Compressive strains</td>
<td>1·5</td>
<td>1·2</td>
<td>1</td>
<td>0·93</td>
</tr>
<tr>
<td>After a very long period</td>
<td>4. $\phi$</td>
<td>1·58</td>
<td>1·4</td>
<td>1</td>
<td>0·73</td>
</tr>
<tr>
<td></td>
<td>5. Long-term deformation</td>
<td>2·39</td>
<td>1·68</td>
<td>1</td>
<td>0·68</td>
</tr>
</tbody>
</table>

The European Committee for Concrete (CEB) has formulated empirical rules for the evaluation of the influence of the various factors.†

† If $p$ is the percentage relative humidity (100% = 1), $e$ is the least thickness in metres, $E$ and $C$ are the weights of water and cement, in kg/m$^3$ of concrete, and $t_0$ is the age at time of loading, in days, these rules can be approximately summarised by the following formula:

$$\phi = (3·6 - 2·4p^2) \frac{0·5 + 0·3e}{0·4 + e} \times \frac{E}{500} \left(1 + \frac{3E}{C}\right) (1·72 - \log_{10} (t_0))$$

These rules, like those relating to shrinkage, appear to give rather low values of $\phi$; the values given for $\phi$ in the text (1·5 to 2) seem reasonable values to use in design. The formula shows clearly the way in which the separate conditions affect the overall results.
Elastic moduli $E_i$, $E_d$, $E_\infty$ can be defined, corresponding respectively to the instantaneous deformation, long-term deformation and total deformation:

$$
\varepsilon_i = \frac{\sigma}{E_i}, \quad \varepsilon_d = \frac{\sigma}{E_d}, \quad \varepsilon_\infty = \frac{\sigma}{E_\infty}
$$

Since $\varepsilon_\infty = \varepsilon_i + \varepsilon_d$:

$$
\frac{1}{E_\infty} = \frac{1}{E_i} + \frac{1}{E_d}
$$

If $\phi = 2$:

$$
\varepsilon_d = 2\varepsilon_i \quad E_d = \frac{E_i}{2} \quad \text{and} \quad E_\infty = \frac{E_i}{3}
$$

If $\phi = 1.5$:

$$
E_d = \frac{E_i}{1.5} \quad \text{and} \quad E_\infty = \frac{E_i}{2.5}
$$

For a concrete with a cube strength of 450 kg/cm$^2$, $E_i$ is about 400 000 kg/cm$^2$.

For $\phi = 2$: $E_d = 200 000$ kg/cm$^2$ and $E_\infty = 133 000$ kg/cm$^2$.

For $\phi = 1.5$: $E_d = 265 000$ kg/cm$^2$ and $E_\infty = 160 000$ kg/cm$^2$.

If more exact data are not available, $E_d = 230 000$ kg/cm$^2$ and $E_\infty = 150 000$ kg/cm$^2$ are typical values to use.

(c) Concrete strength at time of prestress

From the table given above, it is clearly best not to apply the prestress too early, so as to avoid excessive long-term deformations. In practice, the prestress can be applied when the strength is two-thirds of the 28-day strength. The elastic modulus is then about 80% of the 28-day modulus. When a concrete with a 28-day cube strength of 380 kg/cm$^2$ is required (corresponding to about 430 kg/cm$^2$ at 90 days), the prestress may be applied, under normal conditions, when the cube strength has reached 250 to 270 kg/cm$^2$. At this stage, the prestress should be only partially applied, so that the higher deformations due to the low modulus and to the high value of $\phi$ are only partially present.

In this way, satisfactory results are obtained if the maximum compressive stress does not exceed 45 to 50% of the cube strength, bearing in mind that at this time the stress in the cables is 20 to 25% greater than the final effective stress, after losses have occurred (see Chapter IV), and
also that the permanent loading is generally only partially applied; this means that the compressive stress at the bottom edge of the beam is still higher than it will eventually become.

It is, of course, desirable that the strength required before the prestress is applied should be reached as quickly as possible; rapid-hardening cements are therefore used. A proportion of the total prestress can be generally applied after 5 or 6 days, using suitable cements.

The final prestress is applied later, but the shuttering can be removed as soon as the partial prestress is applied, if the total permanent load is not yet fully applied; this is generally the case.

A small amount of prestress can be applied very soon after casting to offset shrinkage effects. Low compressive stresses, of the order of 5 to 10 kg/cm², are sufficient to achieve this result. Although the losses are still relatively high for this small amount of prestress, they have little effect on the overall loss under the total prestress.

**(d) Variation of long-term deformation as a function of time**

If \( \varepsilon_d \) is the final long-term deformation and \( \varepsilon_d(t) \) the deformation at age \( t \) the variation of \( \varepsilon_d(t) \) can be represented by an equation of the form

\[
\varepsilon_d(t) = \varepsilon_d[1 - \exp \left(-\frac{t}{t_1}\right)]
\]

with \( t_1 \) being a time unit.

If time is expressed in months, and if \( m \) is the number of months which have elapsed, a formula due to CAQUOT can be written in the following approximate form:

\[
\varepsilon_d(m) = \varepsilon_d[1 - 10^{-\left(0.16\right)\left(m^{-1}\right)^{1/2}}]
\]

This implies that after 16 months the long-term deformation is 90% complete.

According to the formula, the fractions of final long-term deformation as a function of time are the following:

<table>
<thead>
<tr>
<th>Time after loading</th>
<th>Hours</th>
<th>Days</th>
<th>Weeks</th>
<th>Months</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>12</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Fraction of long-term deformation achieved:

|                       |       |       |       |       |       |       |       |       |       |
|                       | 0.023 | 0.03  | 0.08  | 0.11  | 0.15  | 0.26  | 0.40  | 0.44  | 0.56  | 0.75  | 0.86  | 0.92  | 0.97  | 0.99  |
The above values are indicative only but enable reasonable forecasts to be made. The rate of long-term deformation depends in practice on many factors (as in the case of shrinkage).

The variations given by formula (2) are a little high during the early stages (between 0 and 2 months).

Other formulae have been proposed, none of which can be considered as universally valid. The true result can be represented only by the use of coefficients which embrace all the parameters (hygrometry, mix, thickness, time of loading, etc.) affecting the rates at which all the changes occur.

Figure 6 indicates the measured deformations obtained in various laboratory tests and on certain structures, and it gives a guide to the accuracy of the formulas.

If \( F(t) \) is the time function \( [F(0) = 0, F(\infty) = 1] \) representing variations in the long-term deformation, the latter can be considered in the form \( \varepsilon_d(t) = \varepsilon_i \times \varphi F(t) \).

A specimen which has suffered long-term deformation undergoes a relative increase in its length when it is unloaded. This recovery is noticeably less than the original deformation. In other words, deformation is only partially reversible. The recovery is itself again made up of an instantaneous elastic component and a long-term component, but the long-term component is a lot less than previously, and equilibrium is reached much sooner (in about 2 to 3 months).

Orders of magnitude may again be obtained by assuming that the total recovery is equal to the instantaneous deformation minus the residual deformation OA' in Fig. 1 (that is, to the instantaneous elastic deformation A); and that 60 to 70% of the recovery takes place immediately, and the remaining 40 to 30% after 2 to 3 months.

Consider a structure which is first prestressed and loaded to a stress \( \sigma \) for a time \( t \), at the end of which time some fraction \( \varepsilon_d(t) \) of the total long-term deformation has occurred. It is then unloaded. When the structure is again loaded, it only undergoes the portion \( \varepsilon_d - \varepsilon_{\infty_\sigma}(t) \) of the long-term deformation; that is, the balance of the total long-term deformation.

(e) Application of load in several stages. Principle of superposition

Consider a structure to which partial loads \( P_0, P_1, P_2, P_3 \), etc. are applied successively, giving rise to stresses \( \sigma_0, \sigma_1, \sigma_2, \sigma_3 \), etc. respectively, at times \( t_0, t_1, t_2, t_3 \), etc. The order of magnitude of the deformation at any time \( t \) under the cumulative stress \( \sigma = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 \), etc. is obtained by assuming that it is equal to the sum of the deformations which would have occurred separately with each of the stresses \( \sigma_0, \sigma_1, \sigma_2, \sigma_3, \)
Fig. 6. Long-term concrete deformations.
etc., due account being taken of the age of the concrete at the time of application of each load, and of the time which has elapsed since the application of the load.

In other words, if \( \varepsilon_{01}, \varepsilon_{11}, \varepsilon_{21}, \) etc. are the respective instantaneous deformations at the stresses (the value of the elastic modulus at the moment of application of each successive loading being taken into account), \( \varphi_0, \varphi_1, \varphi_2, \) etc. are the coefficients \( \varphi \) corresponding to these times, and \( F(t) \) is the time function previously defined, then the total deformation at time \( t \) (where time \( t_0 \) is taken as the origin) is approximately equal to

\[
\varepsilon(t) = \varepsilon_{01}[1 + \varphi_0 F(t)] + \varepsilon_{11}[1 + \varphi_1 F(t - t_1)] + \varepsilon_{21}[1 + \varphi_2 F(t - t_2)], \text{ etc.}
\]

The same formula applies if some of the loads are negative (partial unloading), the terms corresponding to the deformation then being negative, and the function \( F(t) \) being replaced by the recovery function \( F'(t) \) (Fig. 5). For example, if load 2 is negative:

\[
\varepsilon(t) = \varepsilon_{01}[1 + \varphi_0 F(t)] + \varepsilon_{11}[1 + \varphi_1 F(t - t_1)] - \varepsilon_{21}[1 + \varphi'_2 F'(t - t_2)] + \varepsilon_{31}[1 + \varphi_3 F(t - t_3)], \text{ etc.}
\]

After a considerable time, and assuming that \( -\varepsilon_{21}(1 + \varphi'_2) = -(\sigma_2/E_{01}) \):

\[
\varepsilon = \varepsilon_{01}(1 + \varphi_0) + \varepsilon_{11}(1 + \varphi_1) - \varepsilon_{21} + \varepsilon_{31}(1 + \varphi_3), \text{ etc.}
\]


5. Influence of temperature on deformations
The discussion in Sections 3 and 4 relates to normal temperature conditions.

The effect of temperature must be taken into account when considering structures subjected to relatively high temperatures of the order of 100°C or more. This effect is quite considerable. Tests carried out by Ross and ENGLAND show that the total strain (shortening), including elastic strain, creep and shrinkage but excluding thermal expansion, can be more than doubled when the temperature changes from 20 to 80°C.

ROSS and ENGLAND tested cylinders 11.5 cm in diameter \( \times \) 30 cm high, using concrete with a water/cement ratio of 0.45, a 14-day cube strength of 385 kg/cm² (4-in cubes) and an elastic modulus of 350 000 kg/cm². Some of the samples were tested as made; others were coated with an
Fig. 7. Long-term deformations of test pieces at a stress of 70 kg/cm² at different temperatures (G. L. England and A. D. Ross, *Magazine of Concrete Research*, March 1962).
impermeable material to reduce evaporation, in order to simulate conditions inside a thick concrete mass. Both the tests and the results are described and summarised in the *Magazine of Concrete Research*, March 1962, to which reference should be made; the long-term deformation diagrams at a stress of 70 kg/cm² are reproduced in Fig. 7.

The magnitude as well as the laws governing deformations have a considerable effect on the stresses in thick walls when they are subjected to a steep temperature gradient (see Section 7). In particular this applies to nuclear reactor pressure vessels.

![Graph showing shrinkage of unsealed specimens at different temperatures](image)

**Fig. 8.** Shrinkage of unsealed specimens, maintained at different temperatures, as a function of time (G. L. England and A. D. Ross, *Magazine of Concrete Research*, March 1962).

### 6. Reduction with time of the stress in concrete when it is compressed at constant length (concrete relaxation). Problem of prestress from jacks supported off fixed abutments

From the preceding sections, the total deformation at age $t$, at a constant stress $\sigma$ varies according to the law

$$\varepsilon = \frac{\sigma}{E_t} [1 + \varphi F(t)]$$

(3)

To obtain an appreciation of the orders of magnitude which are involved, it can be assumed that if $\sigma$ is not maintained constant the above formula can be used to obtain a relation between the strain, stress and time variables ($\varepsilon$, $\sigma$, $t$).

Take the expression for $F(t)$ as:

$$F(t) = 1 + 2[1 - 10^{(m/16)^{1/2}}] \quad (\varphi = 2)$$

With the above assumption, the relation between the three variables is represented by a certain surface (Fig. 9).

The intersection of this surface with the plane $t = 0$ represents the instantaneous deformation as a function of the stress (Fig. 1); the intersection of the surface with the horizontal plane $\sigma$ represents the total
deformation at the stress $\sigma$ as a function of time (Fig. 5). This is called the creep of the concrete at constant stress.

If the length of the loaded member is kept constant (that is, if the deformation $\varepsilon$ is fixed), the stress decreases with time. The variation is represented by the intersection of the surface and the vertical plane $\varepsilon$. The reduction in stress as a function of time is called the relaxation of the concrete at constant length.

![Diagram](image)

**Fig. 9.**

As an example of this condition, consider the case of a member which is prestressed by means of jacks supported by a fixed abutment, the fixed deformation $\varepsilon$ being equal to the travel imparted to the jacks before locking. If $\sigma_i$ is the initial value of the stress ($\sigma_i = E_0\varepsilon$), equation (3) becomes

$$\sigma = \frac{\sigma_i}{1 + 2(1 - 10^{-(m/16)1/2})}$$

(4)

After a long period, $\sigma$ approaches $\sigma_i/3$, or more generally $\sigma_i/(1 + \phi)$. Consequently, with $\phi = 2$ it would be necessary to exert an initial stress of $3\sigma$ in order to achieve an equilibrium stress of $\sigma$. This is not possible in practice, because the abutments cannot be sized for such an overload.

Formula (4) assumes, however, that the jacking time is extremely short, and that the length is maintained constant by locking the jacks.

If the jacking force is kept constant for a certain time $t_1$, the material undergoes initially a deformation $\varepsilon$ at constant stress; the deformation
under this load increases rapidly, in accordance with formula (3). If the stress can be given a value \( \sigma_1 \), such that at the end of time \( t_1 \) the deformation \( \varepsilon_1 \) is equal to the total deformation corresponding to the required permanent stress \( \sigma \), it is sufficient at time \( t_1 \) to reduce the stress to the required value \( \sigma \), without further change of length. The stress \( \sigma \) then becomes the equilibrium stress.

But the total deformation at a stress \( \sigma \) (for the assumed value of \( \varphi = 2 \)) is equal to \( 3\sigma/E_i \) (three times the value of the instantaneous deformation). It is therefore necessary to have:

\[
\varepsilon_1 = \frac{\sigma_1}{E_i} \left[ 1 + 2\left[ 1 - 10^{-\left(m_1/16\right)^{1/2}} \right] \right] = \frac{3\sigma}{E_i}
\]

or

\[
\sigma_1 = 3 \frac{\sigma}{1 + 2\left[ 1 - 10^{-\left(m_1/16\right)^{1/2}} \right]}
\]

in which \( m_1 \) is the number of months during which the stress \( \sigma_1 \) is maintained.

The stress to be applied is therefore three times the stress to which \( \sigma \) would have dropped at the end of time \( t \). The ratio \( \sigma_1/\sigma \) is three times as great as the quantity for which the variations are shown in Fig. 10.*

![Fig. 10. Reduction in stress as a function of time at constant length.](image)

For example, with the assumed value of \( \varphi \), if the stress \( \sigma_1 \) is maintained for \( 1\frac{1}{2} \) months, Fig. 10 shows that \( \sigma \) drops to \( 0.5\sigma \). It is therefore necessary to have \( \sigma_1 = 3 \times 0.5\sigma = 1.5\sigma \). This explains a rule which is often applied: in order to obtain a permanent stress \( \sigma \), it is necessary to exert \( 1\frac{1}{2} \) times

* Generally \( (\sigma_1/\sigma) = [1 + \varphi/1 + \varphi F(t_1)] \).
the load for 1½ months. Nevertheless, it is still a difficult rule to apply because of the excessive size of abutments required.

It is therefore necessary either to maintain stresses which are not too much greater than $\sigma$ for a longer time, or to re-apply periodical compression at a slightly higher stress than the required stress, until equilibrium is reached. The residual compression in the concrete is measured by pumping up the jacks and noting the pressure at which movement begins (correcting for friction if necessary). The jacks are re-activated when the residual compression has dropped below a given value.

If, for example, compression is again applied at $1.2\sigma$ when the residual stress has dropped to $\sigma$, this is approximately equivalent to the permanent application for a time $t_1$ of a compression $1.1\sigma$ (mean of the two limits). It is found that $t_1$ is of the order of 1 year.

In most cases, the aim is to counteract the effects of temperature variations (airport runways, roads prestressed between two fixed abutments). At the minimum temperature, the compressive stress should not be less than a certain minimum value $\sigma_{\text{min}}$ to withstand the external bending forces. Under these conditions, if $\Delta\theta$ is the temperature departure from the minimum, $L$ the length between abutments (fixed) and $\gamma$ the coefficient of expansion, each temperature rise increases the stress by an amount $\Delta\sigma$. For a concrete which is entirely elastic, with an elastic modulus of $E_0$, then $\Delta\sigma = E_0\gamma \Delta\theta$, since if the concrete were free to expand its increase in length would be $\Delta L = L\gamma \Delta\theta$. The abutment maintains it at the length $L$, which is equivalent to shortening it by the amount $\Delta L$,† and therefore $\Delta\sigma = E_0(\Delta L/L)$.

The initial values of $\Delta\sigma$ are low because temperature variations occur slowly.

Each temperature cycle cold-works the concrete, imparting to it a permanent plastic deformation; the length $L_0$ at zero stress, therefore, decreases with each cycle. Consequently, the stress at the minimum temperature is gradually lowered. In order to bring it back to its required permanent value, jacking must be used to compensate for the loss of length due to cold-working. This must be done until the plastic deformations which occur during cycling through the maximum temperature, and therefore at maximum stress, are absorbed. When this state is reached, the concrete behaves elastically. The maximum stress at maximum temperature is $\sigma_{\text{min}} + E_0\gamma \Delta\theta_{\text{max}}$, and this determines the abutment strength required.

† The effect of temperature on the elastic modulus is neglected.
The total jack travel \( x \) can be estimated as follows. Let \( \theta_1 \) be the minimum temperature, \( \theta_2 \) the maximum temperature, \( \theta_b \) the temperature at time of concreting; let \( \sigma_1 \) be the required stress at minimum temperature (previously called \( \sigma_{\text{min}} \)).

(i) First assume that \( \sigma_1 = 0 \) and that the temperature is \( \theta_1 \) at the time of concreting. Assume there is zero shrinkage.

Once the concrete has reached its definite elastic state, the stress varies between 0 and \( \sigma_2 = E_0 \gamma(\theta_2 - \theta_1) \). The jacks are required to compensate for the plastic deformations corresponding to \( \sigma_2 \).

Therefore:

\[
x = \frac{\sigma_2}{E_0} \cdot L \times \varphi
\]

or

\[
x = L \cdot \gamma \varphi (\theta_2 - \theta_1) \quad (a)
\]

(ii) Concreting takes place at a temperature \( \theta_b \), however. The previous conditions apply if the jack adjustment is applied when the concrete reaches the minimum temperature \( \theta_1 \). The jacks must at that time be in contact with the concrete, so they that are given the additional travel \( L \gamma (\theta_b - \theta_1) \) to compensate for the reduction in length due to cooling.

With these hypotheses, the jack travel is:

\[
x = L \gamma [\theta_b - \theta_1 + \varphi (\theta_2 - \theta_1)]
\]

If \( \theta_b = (\theta_1 + \theta_2)/2 \) (mean temperature), \( \theta_b - \theta_1 = (\theta_2 - \theta_1)/2 \) and

\[
x = L \gamma (\frac{1}{2} + \varphi)(\theta_2 - \theta_1) \quad (b)
\]

(iii) The stress \( \sigma_1 \) is not zero, however. The total reduction in length corresponding to the required stress \( \sigma_1 \) must therefore be added to the jack travel; this amounts to

\[
(1 + \varphi) \frac{\sigma_1}{E_0} \cdot L
\]

The total travel is:

\[
x = L \left[ \gamma (\frac{1}{2} + \varphi)(\theta_2 - \theta_1) + (1 + \varphi) \frac{\sigma_1}{E_0} \right] \quad (c)
\]
The travel is generally spread among several lines of jacks, positioned in equidistant gaps parallel to the abutments. If \( l \) is the distance between gaps, the concrete is divided into \( N \) slabs, such that \( N = L/l \).

(iv) The friction of the concrete slab on the sub-surface is not allowed for. Because of friction, the stress is less at the centre of each slab. If \( f \) is the coefficient of friction, the reduction in stress at the centre is equal to \( f(l/2)(hD/h) = f(l/2)D \), \( h \) being the slab thickness and \( D \) the concrete density.

A stress of \( \sigma_1 + f(l/2)D \) must therefore be applied in the gaps between slabs in order to obtain the stress \( \sigma_1 \) at the centre of the slabs; the mean stress is therefore \( \sigma_1' = \sigma_1 + f(l/4)D \). Therefore this value \( \sigma_1' \) must be substituted for \( \sigma_1 \) in formula (c).

Finally, the shrinkage \( \eta_\infty \) must be compensated, and the term \( L \times \eta_\infty \) must be added to the jack travel.

Therefore, finally,

\[
x = L \left[ \gamma(\frac{1}{2} + \varphi)(\theta_2 - \theta_1) + (1 + \varphi) \frac{\sigma_1 + f(l/4)D}{E_0} + \eta_\infty \right]
\]

The lengths of travel obtained from this formula are quite considerable.

More optimistic assessments of abutment strength and jack travel are often made. In the opinion of the author, this is a mistake.

It is possible, nevertheless, that some relaxation can be allowed when the time interval between extreme temperatures is long: values for \( E \) slightly lower than \( E_0 \) (slope at origin of stress–strain diagram) can perhaps also be used, with Fig. 4 as a basis, the mean modulus being the chord of the deformation cycle, and by considering the effects of the accompanying hygrometric variations. These alleviations can only slightly affect the stresses. The results obtained on the Fontenay-Tresigny prestressed concrete highway (R. Peltier, Annales des Ponts et Chausées, paper on the rheology of highly prestressed concrete obtained from Fontenay-Tresigny tests) illustrate these deformation cycles (Fig. 11). The increase in stress is approximately 4 kg/cm\(^2\) per degree.

A more important reduction in jack travel can result from the value of \( \varphi \) in formula (4). The value to use is the mean value, which depends on the times of the successive applications of the jacks; that is, on the age of the concrete at the times of jacking. At first, this jacking is carried out on a relatively green concrete, and finally on a mature concrete.

Consider the application of formula (6), assuming that \( \varphi = 2 \) after 28 days and \( \varphi = 1 \) after one year, and using a mean value of \( \varphi \) equal to 1.5.
FIG. 11. Evolution of stresses with time for the Fontenay-Tresigny prestressed concrete highway. (Reproduced by permission of R. Peltier.)
Assume:
\[ \gamma = 0.8 \times 10^{-5} \quad \theta_2 - \theta_1 = 40^\circ \]
\[ \sigma_1 = 5 \text{ kg/cm}^2 \quad f = 0.4 \]
\[ l = 150 \text{ m} \quad D = 2.4 \text{ t/m}^3 \]
\[ \eta_\infty = 3 \times 10^{-4} \quad E_0 = 400 000 \text{ kg/cm}^2 \]

Then:
\[ f \frac{1}{4} D = 0.4 \times 37.5 \times 2.4 = 36 \text{ t/m}^2 \ (3.6 \text{ kg/cm}^2) \]
\[ \sigma_1 + f \frac{1}{4} D = 5 + 3.6 = 8.6 \text{ kg/cm}^2 \]

and
\[ \frac{\sigma_1 + f(l/4)D}{E_0} = \frac{8.6}{400 000} = 0.21 \times 10^{-4} \]

\[ x = L[0.8 \times 10^{-5} \times 40 \times 2 + 2.5 \times 0.21 \times 10^{-4} + 3 \times 10^{-4}] \]
\[ = [6.4 + 0.5 + 3] \times 10^{-4} = 10L \times 10^{-4}, \text{ or } 1 \text{ mm per metre} \]

The jack travel in each of the gaps \((l = 150 \text{ m})\) is about 15 cm.

The maximum stress, which determines the strength of the abutments, is
\[ \sigma_2 = 400 000 \times 0.8 \times 10^{-5} \times 40 + 8.6 = 136.6 \text{ kg/cm}^2 \]

One method of reducing the forces on the abutments is to use ‘elastic abutments’. This type of abutment was used by FREYSSINET for the runway of Maison-Blanche. The abutments consist of cables in tension located under the runway. The cables are free to slide and they run in ducts located inside a layer of concrete (Fig. 12).

---

**Fig. 12.** Diagrammatic arrangement of an elastic abutment (Maison-Blanche).
The method of assessing the forces on the abutments is briefly as follows. Let $S$ be the transverse cross-sectional area of the runway, $E_0$ the modulus of elasticity, $F$ the force exerted by the abutment for a temperature rise of $\Delta \theta$, $x$ the displacement of the section of runway joined to the abutment, $S'$ the cross-sectional area of the abutment, $L'$ its length, $E'$ its modulus of elasticity.

Then:

$$x = L\gamma \Delta \theta - \frac{FL}{E_0S}, \quad \text{also } x = \frac{FL'}{E'S'}$$

Therefore

$$F = \frac{E_0S\gamma \Delta \theta}{1 + (L'/L)(E_0/E')(S/S')}$$

The reduction coefficient is therefore:

$$\frac{1}{1 + (L'/L)(E_0/E')(S/S')}$$

This can be very small when $E'$ and $S'$ are small compared to $E_0$ and $S$.

7. Stresses in a thick-walled structure subjected to a temperature gradient

Consider the effect of a temperature difference between the inside and outside faces of a vessel, the contents being at a high temperature (as in a nuclear reactor). The temperature cycles are low during the life of the structure (and they are in any case damped by the large thermal inertia of the thick walls). Cold-working of the concrete is neglected.

Only general principles are dealt with below, and reference should be made to more detailed publications for a more thorough appreciation of the subject (in particular, reference should be made to Reinforced Concrete Under Thermal Gradients, by England and Ross—Magazine of Concrete Research, March 1962; this article contains a bibliography).

Generally, concrete envelopes of this type are cylindrical in shape. A uniform temperature would not cause stresses, except near the ends. In the barrel portion, thermal stresses are caused only by the temperature gradient.

Let $h$ be the wall thickness, $\gamma$ the coefficient of expansion, $\theta_1$ the temperature of the inside face, $\theta_2$ that of the outside face ($\theta_1 > \theta_2$). It is assumed that the temperature gradient is linear.

It can be considered that the temperature state is the superposition of a uniform temperature $\theta_m$ and of a linear temperature gradient.
In relation to \( \theta_m \), taken as the origin, the inside face is at a temperature \( \theta_2 - \theta_m = \Delta \theta_1 \) and the outside face is at a temperature \( \theta_2 - \theta_m = -\Delta \theta_2 \). \( \Delta \theta \) is the temperature difference between the two faces (\( \Delta \theta = \Delta \theta_1 + \Delta \theta_2 \)).

Consider a section along a generatrix. In addition to the uniform expansion \( \gamma \theta_m \), each radial section is subjected to a strain which varies linearly, the strains on the faces being \( +\gamma \Delta \theta_1 \) and \( -\gamma \Delta \theta_2 \). These strains tend to 'unroll' the cylinder.

The continuity of the cylinder introduces equal but opposite deformations; consequently, the inside is in compression and the outside in 'de-compression'. (It is assumed that prestress is such that the resultant stress on the outside face remains compressive.)

The resultant stress must be zero, since conditions are such that there cannot be a normal force.

In any transverse section the stress is zero at a point N (neutral point) which is at the temperature \( \theta_m \).

![Fig. 13.](image)

The distances \( x_1 \) and \( x_2 \) from this point to the inner and outer surfaces are such that

\[
\frac{x_1}{\Delta \theta_1} = \frac{x_2}{\Delta \theta_2} = \frac{h}{\Delta \theta}
\]

The position of the neutral point is not initially known. It must be determined such that the stress resultant is zero.

The neutral point varies with time, because of the variations of the elastic modulus with temperature.

Assume that the position is known, and let \( \xi \) be the distance of any point from point N (Fig. 13), considered positive towards the inner face. The point is at temperature

\[
\theta = \theta_m + \xi \frac{\Delta \theta_1}{x_1} = \theta_m + \xi \frac{\Delta \theta}{h}
\]
The strain (with respect to the origin strain $\gamma \theta_m$) is

$$\varepsilon = \gamma (\theta - \theta_m) = \gamma \xi \frac{\Delta \theta}{h}$$

When $\varepsilon$ is known, curves such as those of England and Ross (Fig. 7) enable the corresponding stress to be obtained, taking into consideration the time $t$ since prestressing, the time $t'$ at which internal pressure is applied and the time $t''$ at which temperature conditions are established.

For simplification, it is assumed that prestress is uniform.† When the vessel is pressurised, the shell is de-compressed.

Let $\sigma_o$ be the prestress, $-\sigma_p$ the pressure stress (assumed uniform) and $\sigma'$ the resultant stress ($\sigma' = \sigma_o - \sigma_p$) which would exist in the absence of a temperature gradient.

Let $\sigma$ be the unknown stress due to the gradient; $\sigma$ is variable along the transverse section (positive in the region $x_1$, negative in the region $x_2$).

The resultant stress at each point is $\sigma' + \sigma$.

The curves given in Fig. 7 enable the unit deformation during the time $t - t''$ to be determined under an applied stress of $\sigma' + \sigma$. Let $C_{\theta,t-t''}$ be the coefficient obtained from Fig. 7 for the temperature $\theta$ considered. The unit deformation is $C_{\theta,t-t''} (\sigma' + \sigma)$.

It is also possible, taking into consideration the time intervals since prestress and pressure have been applied, to determine the deformation which would have occurred in this same time interval at the stress $\sigma'$ and at normal temperature. Because of the principle of superposition, the resultant deformation is the sum of the deformations in this time interval due to prestress (applied at time 0) and those due to pressure (applied at time $t'$). This deformation is obtained from Fig. 5; for the deformation due to prestress, Fig. 5 is simply a diagram (at normal ambient temperature) of the coefficients taken from Fig. 7 at that temperature ($C_{20}$, for example, if the ambient temperature is $20^\circ$); for the deformations due to $\sigma_p$ (internal pressure), variations are shown in Fig. 5 (case of de-compression).

Let $\varepsilon'$ then equal the unit deformation in the time interval $t - t'$, at normal temperature, under the stress $\sigma'$.

To this deformation must be added the deformation due to the temperature $\theta_m$ (without gradient) at the point N, or $C_{\theta,t-t'} (\sigma')$.

The deformation due to the gradient is equal to the difference between

† In practice the stresses $\sigma_o$ and $\sigma_p$ are not uniform because of the thickness of the wall of the cylinder.
the actual deformation and that which would have occurred at uniform temperature $\theta_m$. Therefore, eliminating the suffix $t - t'$ for simplification:

$$C_\theta \cdot (\sigma' + \sigma) - (\varepsilon' + C_{\theta m}\sigma') = \varepsilon = \gamma \zeta \frac{\Delta \theta}{h}$$

The stress $\sigma$ can thus be obtained when the neutral point is known (this determines $\zeta$).

Let the position of point N vary until the absolute value of the resultant of excess compressions is equal to the resultant of de-compressions, or

$$\sum_{0}^{x_1} \sigma \Delta \zeta = \sum_{0}^{-x_2} \sigma \Delta \zeta$$

in absolute values.

If the vessel is spherical, the variation in the width of the transverse section between two radii must be considered. If $r$ is the internal radius,

![Diagram](image)

**Fig. 14.**

the width varies as $r + x_1 - \zeta$ (Fig. 14). The position of the neutral point must therefore be determined by trial and error in order to obtain

$$\sum_{0}^{x_1} (r + x_1 - \zeta) \sigma \Delta \zeta = \sum_{0}^{-x_2} (r + x_1 - \zeta) \sigma \Delta \zeta$$

In practice, the problem is even more complex because of:

(a) shrinkage, which is itself affected by temperature (Fig. 8);
(b) different deformation conditions at the faces and inside the wall (conditions for the latter are intermediate between those of Figs. 7 and 8);
(c) the slow rate of heating, so that time zero $t'$ is not the same at all points;
(d) the non-uniformity of the stresses $\sigma_o$ and $\sigma_p$.

ROSS and ENGLAND have shown how problems of this nature can be resolved [taking shrinkage, non-uniformity of stresses ($\sigma_o$ and $\sigma_p$), conditions of evaporation, and so on, into account], by dividing the wall into elements (in this case concentric) and the time into intervals.

The condition which is obtained at the end of each time interval is considered as the initial condition for the next interval. It is not possible to deal with the detail of the method in this book, but it can be found in the ROSS and ENGLAND publication previously referred to.

In the case of partial prestressing, the method is still applicable; those elements which, according to the calculations, appear to be cracked (that is, for which the tensile resistance is exceeded) are ignored, and the calculations are repeated for the remaining elements. The reinforcement is considered separately as a further concentric element. Tests made by the previously mentioned authors using gauges in reinforced concrete, as well as the confirmations obtained by their method (see Magazine of Concrete Research, March 1962), can also be applied in this case.

Calculations made using these modifications give very reasonable results.

As previously noted these calculations are valid only so long as the conditions assumed are consistent with those actually obtaining; that is, as long as the reactor concrete is relatively new. After a few years the concrete becomes stabilised and the return cycles will be obtained purely from elastic conditions, though the favourable conditions acquired during the earlier stage will be regained with each drop in temperature and pressure.

8. Procedures for improving the resistance of concrete and its setting time
To improve the resistance of concrete, it is necessary to reduce the ratio E/C (water/cement ratio). This may be achieved by vibration, pre-vibration, vibration under compression, or curing. These processes were mostly used by FREYSSINET (posts at Forclum, 1933; pipes for l'Oued Fodda, 1940; and posts at Brommat Marèges, 1942).

Vacuum de-watering operates in the same manner.

Heating improves the rate of hardening of concrete. It is used particularly for members where it is desirable to strike the moulds as quickly as possible: this applies especially with factory-made units. The heat is
provided by circulating steam or hot water around the moulds (pipes for l'Oued Fodda).

It must also be borne in mind that there is a critical temperature which must not be exceeded, otherwise the composition of the concrete may be altered.

The better the compaction the higher this temperature limit is. For vibrated and compressed concrete, the temperature of the autoclave can be up to 100°C and hardening may occur after about an hour.

However, these conditions are optimal; in general the temperature employed should not exceed 80°C. This question is considered in more detail in Chapter II, Volume 2.

*Very high strength concrete*

Concrete units with very high strengths (greater than 100 kg/cm²), for particular applications such as triangulated frames and machine foundations, can be obtained by autoclave methods using temperatures of up to 200°C and a pressure of about 10 kg/cm² (see the report of the Commission des Bétons à Haute Résistance, 5th Congress of the FIP, Paris, 1966).

9. **Expanding concrete. Self-stressing**

Many investigations have been made into the possibility of obtaining a degree of prestress by restraining, by means of cages of high tensile reinforcing steel, the expansion of ‘expanding concrete’ which is obtained by introducing into the cement a certain proportion of high alumina cement of suitable composition.

In this way, LOSSIER obtained positive results, but his methods, which he has termed ‘self-stressing’, have not been widely applied.

More recent Russian research (V. V. MIKHAILOV, LITVER and POPOV) has been applied in industrial processes such as the manufacture of pipes using expanding mortar, employing either a centrifugal casting process or a cement gun in conjunction with a rotating mould.

The cement comprises three main constituents: finely ground Portland cement (66%), high alumina cement of suitable composition (20%) and gypsum (14%).

According to the Russian authors, it is possible to obtain a prestress of the order of 60 kg/cm², for which the strength of the mortar must be between 800 and 1 000 kg/cm².

These values of prestress and strength are obtained using a particular type of curing treatment, which consists of storage in air within a tube containing a coating of paraffin (18 to 24 hours), followed by immersion
first in boiling water (6 hours) then in cold water (5 to 6 days), and lastly with open-air storage for 21 to 28 days. The self-stressing is produced during the period of immersion in the cold water; its magnitude depends on the percentage of the high tensile steel.

The expansion is caused by the formation of sulpho-aluminate crystals with a high sulphate content; it is important to prevent too rapid a growth of these crystals for fear of disrupting the concrete. The different phases of the curing sequence and their durations have been systematically investigated in order to obtain the optimal amount of self-stress and strength.

On this subject it is not possible to improve on the original sources, and in particular a paper by V. V. MIKHAILOV at the Fourth International Symposium on the Chemistry of Cement (Washington 1960): Stressing Cement and the Mechanism of Self-stressing Regulation.

This paper contains a bibliography.

10. Lightweight concrete

The use of lightweight concrete for prestressed structures has been widespread since 1955 in the United States and Russia, and also in Australia and Canada; more recently it has been applied and developed in many other countries.

Lightweight concrete is made by using artificial lightweight aggregates (such as expanded slate or clay sintered in a rotary kiln) with a specific gravity of about 0.5. The whole of the aggregate, including the fine aggregate, may be lightweight but for prestressed concrete it is more usual for only the coarse aggregate to be lightweight, the fines being natural sand of normal density.

By comparison with concrete made solely with lightweight aggregate, the use of natural sand increases the density of the mix but lessens the scatter of the strengths and the elastic moduli and improves the mean values of these properties.

The density of concrete made with aggregates of this type is between 1.5 and 1.9 t/m³.

The properties of lightweight concrete depend to a great extent on the quality and properties of the artificial aggregate, the control of which during manufacture is of fundamental importance.

The values of the properties given in the following are extracted for the most part from a paper presented by BEN GERWICK at the Fifth Congress of the FIP, Paris, 1966, and from various tests carried out in different laboratories.
The compressive cube strength of the lightweight concrete ranged between 280 and 400 kg/cm².† These tests relate to concretes with quantities of cement similar to those used for ordinary concrete.

According to Vironnaud (Proceedings of the ITBTP, June 1965) it would be difficult to improve the strength beyond 400 kg/cm² by adding more cement, since the strength of the concrete is limited by the strength of the aggregate. Nevertheless, it may be possible to improve the strength of the concrete to some extent; Ben Gerwick indicates that strengths of about 550 kg/cm² can be obtained with certain methods of compaction.

The tensile strength depends on the environment in which the concrete is placed. It is about the same as that of ordinary concrete when the lightweight concrete is placed in a humid atmosphere or in water, and is naturally less (75%) in a dry atmosphere; this reduction is possibly caused by differential shrinkage between the matrix and the aggregate. This reduction of tensile strength, and consequently of shear strength (Chapter II, Volume 2), must be taken into account in the design of the end blocks.

The instantaneous elastic modulus is about 140 000 to 170 000 kg/cm², which is significantly less than the values for ordinary concrete.

The ratio between the instantaneous deformation and the long-term deformation (the coefficient φ defined in Section 3) is slightly larger than for ordinary concrete (see Ben Gerwick’s paper, Table 2E); the long-term deformation takes place more quickly (80% of the delayed deformation occurs within 2 months compared with 60% during the same period with ordinary concrete).

The shrinkage is greater (by 20 to 40%) than with ordinary concrete.

The differences between the properties of lightweight and ordinary concretes become less as the strengths of the concretes increase. On the whole, the properties of lightweight concrete are improved by steam curing at atmospheric pressure (see Ben Gerwick’s report for suitable times and temperatures).

It is clear from the foregoing that the losses of prestress due to shrinkage and creep are greater than for ordinary concrete (see Chapter IV) and it is worthwhile to reduce such losses by using lightweight concretes of sufficiently high strengths and to subject them to steam curing. In addition it is clearly desirable to apply the highest practicable initial stresses to the prestressing steel.

† 500 kg/cm² according to Ben Gerwick’s report; but the strengths were measured using cylinders and converted to equivalent cube strengths by multiplying by 1.25, a factor which is probably too high.
Clearly the main advantage of lightweight concrete lies in the reduction of self-weight, as demonstrated by its extensive use for floors, roofs, shell roofs, footbridges, and so on in the USA and Russia.

On the other hand, its unit cost is higher (lightweight aggregate costs about twice as much as ordinary aggregate).

However, in making such comparisons it is necessary to bear in mind the indirect advantages of lightweight concretes. There are economies for example in steel (10% to 18%); foundations (20% to 25%); and transport. By and large, judging from a number of studies made in the USA and Russia, the economic comparison is in general favourable to lightweight concrete.

Other advantages should also be borne in mind; these include greater resistance to damage by fire, good thermal and acoustic qualities, higher resistance to damage by frost and also the reduction of weight is advantageous in zones subject to earth tremors.

The relative weights to be attached to each of these considerations must be assessed in each separate case.

Ben Gerwick's report contains tables presenting the various possible uses for lightweight structural concrete together with the reasons for their adoption in these cases.

This report also contains a bibliography.

II. STEELS

The types of steel used in prestressed concrete are: high tensile steels (the active reinforcement), which are the agency by means of which the prestressing force is applied (these steels are used under tension); non-tensioned steels, which can be mild steel or cold-worked steel which may also be deformed (providing passive reinforcement, stirrups and hooks).

11. Non-tensioned steels
(a) Mild steels

These are the same as those used for ordinary reinforced concrete. The stress–strain diagram (Fig. 15) comprises a linear section OP, a horizontal section PP' and a curved phase P'R, leading to rupture.

The slope of the line OP gives the elastic modulus of the steel. This modulus is about 21 000 kg/mm² (with a variation of 10%). The elastic
limit is the stress corresponding to the point P; it is about 20 to 22 kg/mm² for the steels normally used.

Beyond P', the steel undergoes a phase of strain hardening, its strength increases and it is possible to ensure that at the time of rupture of the prestressed member the mild steel close to the tensile face has already entered this strain-hardening phase.

The order of magnitude of the ratio of the compressive strain in the compressed concrete and the tensile strain at the tensile face is about 4 to 10; since the limiting compression in the concrete is about 3·5/1 000, the corresponding tensile strain in the steel can be up to 3·5/100. Hence the stress causing rupture of the steel (the peak value R in the diagram of Fig. 15) is never reached, as the strain corresponding to this value is about 20%.

(b) Cold-worked steels
Steels used are usually steels with high bond characteristics (deformed or indented bars) which are obtained by cold rolling.

The stress–strain diagrams show no true yield point and an elastic limit or proof stress of 0·2% is arbitrarily taken instead; this point is obtained in the diagram by the intersection of a straight line with a slope equal to 21 000 kg/mm² and the stress–strain curve itself (Fig. 16 shows a typical stress–strain curve for this type of steel). The 0·2% proof stress can vary between 40 and 80 kg/mm². In the case of ‘partial prestress’, it is unnecessary for the 0·2% proof stress to be very high because the strength is not fully utilised. Good bonding properties reduce the opening of cracks. With reinforced prestressed concrete, a 0·2% proof stress of 50 kg/mm² is a useful maximum to use as a starting point. Medium strength steels
are not essential with total prestress and ordinary mild steel (structural quality) suffices. In this case also, the use of a steel with improved bonding properties increases the factor of safety against cracking.

12. High tensile steels
These steels are used for applying the prestressing forces. They can take the form of wires, bars or strands. The RILEM terminology is used (Réunion Internationale des Essais de Matériaux).
A bar is an elemental piece of material which can only be provided straight.

![Stress-strain diagram for cold-worked medium hard steel](image)

**Fig. 16.** Stress-strain diagram for cold-worked medium hard steel (typical diagram).

A wire is an elemental piece of material which is, or can be, provided in rolls. In practice, wire is manufactured in diameters of up to 12 mm.
A strand is an assembly of helically wound wires, of equal pitch, wound in the same direction over a central straight wire.
The steels contain 0.7 to 0.9% of carbon, and in general about 0.6% of manganese and about 0.1% of silicon.
13. Wires. Heat and mechanical treatments
The various types of wire are first obtained by hot drawing and their properties are determined by their subsequent heat and mechanical treatments.

(a) Heat treatments†
These may consist of heating, quenching and also tempering.

The grain structure of the steel depends upon the temperature and duration of the treatment, and upon the speed of quenching where this is concerned.

The main treatments are:
quenching, followed by tempering;
patenting (isothermal quenching);
mar-tempering, followed by tempering;
artificial ageing by heating;
stress relieving.

Quenching. Rapid cooling to ambient temperature increases the hardness and brittleness of the material (martensitic state). Quenching is followed by tempering at 400 to 500°C, of sufficient duration to relieve locked-in stresses and to restore ductility.

Patenting. This process is similar to quenching but it takes place at a temperature which is higher than the martensitic temperature, and it is of sufficient duration (hence the term isothermal) to achieve a stable state in the whole of the mass at that temperature.

Mar-tempering. This is isothermal quenching in stages, with a final stage at a temperature higher than the martensitic temperature, with uniform temperature in the mass, followed by cooling in the martensitic range, and tempering.

Artificial ageing. This consists of heating at a temperature less than 300°C. The elastic limit is raised and cold-rolling properties are retained.

Stress relieving. This consists of heating for a short period at about 400°C. The elastic limit, and the stress and strain at failure, are raised. Cold-rolling properties are retained.

† The terminology and definitions are those of the RILEM, to which further detail has been added (cf. in particular, ASP, First Study Session, 1956).
(b) Mechanical treatments
These include either a calibration, in which the wire is drawn to a slightly reduced diameter, principally in order to obtain a more uniform diameter, or processes which substantially reduce the diameter of the wire, modifying the mechanical properties at the same time: increasing the strength, the elastic limit and the hardness, with a consequent reduction in the ductility and the rupture strain.

These processes include:

(i) Cold working.

(ii) Wire-drawing through a series of dies, with large reductions in the diameter of the wire, in one or in several passes.

The wire may also undergo a cold-pulling process. This consists of stretching the wire in order to give it a permanent set. This increases the 'stiffness' of the stress–strain diagram, as is shown later, and it can improve the relaxation properties of the steel. The wire may also be deformed or indented during the rolling operation in order to increase its bonding efficiency.

(iii) Heat treatment can be applied either before or after mechanical treatment, or it can be integrated between the various phases of mechanical treatment; for example, isothermal quenching between progressive wire-drawing operations restores ductility to the wire. The combination of various mechanical and heat treatments, as well as their sequence, gives a wide range of properties.

It is the task of the designer to specify the properties which he requires: the shape of the stress–strain diagram and the values attached to it (strength and rupture strain); relaxation, brittleness, resistance to corrosion, surface finish and hardness.

14. Stress–strain diagrams for high strength wires
Stress–strain diagrams for high strength wires do not show a zone of yielding. They include:

(a) A sensibly linear portion OP corresponding to the 'elastic state'. The stress \( P \) is sometimes referred to as the limit of proportionality. In practice, except for steel previously subjected to 'pulling', the straight-line interpretation of phase OP is only an approximation and the characteristic OP is not strictly valid. The slope \( T/e \) of the diagram at the origin is by definition the elastic modulus of the steel. The mean value is again 21 000 kg/mm\(^2\) \(\pm 10\%\), the variations being possibly incurred by errors in assessing the slope.
(b) A curved portion PAR; the maximum stress R is the rupture stress; the corresponding strain is the rupture strain. These two properties are extremely important, as well as the value of the 0.2% proof stress (the intersection of the curve with a line parallel to the tangent at the origin, passing through the value 0.2% on the horizontal axis).

When a test specimen is tensioned to a value $T_A$ (point A on the diagram) and then unloaded, the return path is very nearly along a line AA' approximately parallel to the tangent OP at the origin. There is a permanent set or residual extension, OA', in the unloaded test specimen.

![Fig. 17. Stress–strain diagram for a high tensile prestressing steel.](image)

When the test specimen is again loaded, the return path is sensibly along the same straight line A'A and when the specimen is loaded to rupture, the stress–strain relation is approximately along the path A'AR. Therefore, that part of the diagram which is beyond the stress reached in the first test coincides very nearly with the portion that would be followed during the continued loading of the initial specimen.

Although not strictly true, because long-term deformations affect the phenomenon, this approximation is considered to be sufficiently exact. It illustrates the effects of cold working by ‘pulling’, when new properties are imparted to a steel: the new limit of proportionality is A, the stress–strain diagram is ‘stiffened’ for stresses less than the pull stress, but the rupture strain is reduced by an amount equal to OA'.
The shape of the diagram (that is, its curvature between O and R) is a characteristic of the steel which it is essential to know.

A diagram is said to be 'soft' when the slope decreases rapidly from O to R; it is said to be 'stiff' when the slope remains closer to the slope at the origin.

Figure 19 shows two extreme diagrams: one is very soft and the slope continuously decreases from its value at the origin; the other is very stiff, and the slope is equal to the slope at the origin \( E_a \) until the stresses are close to the rupture stress. The latter diagram can be approximated to two
straight lines, the line OE and the horizontal threshold ER, with a transition region in the neighbourhood of E. In practice, stress–strain diagrams lie between these extreme shapes. Typical diagrams are shown in Fig. 18.†

There is no absolute criterion defining the most suitable shape of stress–strain diagrams. Nevertheless, a diagram which is too soft implies excessive tensile elongations during the application of prestress; the steel is also more prone to large long-term deformations (relaxation). Also, a steel which is characterised by a soft diagram may be more liable to cracking, as will be shown later.

A stiff diagram is in principle more favourable, but excessive stiffness can imply brittleness, even more so if the rupture strain is low.

\[ \epsilon = \frac{\Delta l}{l} \]

**Fig. 19.**

Most of the codes define ‘conventional elastic limits’ which determine the shape of the diagrams.

The 0.2% elastic limit (0.2% proof stress) is the most frequently used. It is the stress \( T_{0.2} \) (Fig. 17) at which the permanent set \( \varepsilon_{0.2} \) is equal to 0.2%. In addition, a limit of 0.1% is defined in some countries. Since the points \( \varepsilon_{0.1} \) and \( \varepsilon_{0.2} \) are on straight lines with slopes equal to \( E_a \) (21 000 kg/mm²), and since the slope at the origin is equal to \( E_a \), the value of the proof stress and the values of the stress and strain at rupture (point R) define the shape of the diagram with sufficient accuracy.

† Stress–strain diagrams for bars and strands are also shown.
The term 'proof stress' is largely conventional. Its value can be exceeded without risk during tensioning, and it is difficult to assign to it a value which it can truthfully be said must not be exceeded.

Nevertheless, the Scientific Prestressing Association specifies that the 0.1% proof stress should be less than 0.95 of the rupture strength; this specification is intended to discourage the use of steel with diagrams which are too stiff. There is no specification to limit the use of steels with diagrams which are too soft, however, and it is probable that their use is governed anyway by economic considerations.

A specification proposed by the Chambre Syndicale des Constructeurs en Béton Armé et Béton Précontrainte de France appears reasonable in this respect: it stipulates that the stress–strain diagram should not encroach upon a line OBC, defined by its slope OB (15 000 kg/mm²) and a horizontal line BC (80% of rupture strength) (Fig. 20).

Provided that brittleness and reproducibility criteria are respected, there is no specified limit to the strength at rupture; usual values are of the order of 140 to 180 kg/mm² (see Fig. 18).

The rupture strain is the most important characteristic of the diagram. To comply with the French ASP Code, it is measured between two datums 500 mm apart. Under these conditions, the strains vary between 3 and 5% for 5 to 8 mm diameter wires. The elongation excluding the contracted zone is about a third of the total elongation measured including contraction.

15. Bars
The bars which are used in prestressed concrete work are between 14 and 32 mm in diameter; 50 mm diameter bars have been considered, but have not so far been used in practice.
In the same manner as for wires, the stress–strain diagrams for bars are continuous, without a yield zone. Rupture strengths are of the order of 90 to 115 kg/mm². The 0.2% proof stress is of the order of 75 to 80% of the strength at rupture (see Fig. 18 for diagrams applicable to bars).

16. Strands
Strands are made up from wires which are helically wound over a central king wire.

The simple 7-wire strand (1 + 6) is made up from a king wire which is slightly larger in diameter (4 to 6%) than the peripheral wires. This leaves a slight circumferential gap between the peripheral wires, thereby avoiding the possibility of binding of the wires, and ensures that the king wire is in contact with the six outer wires.

The pitch of the helix is twelve to sixteen times the diameter of the strand.

The wire diameter is usually small (2 to 4 mm) and the strength of the wire is high (180 to 200 kg/mm²). Larger diameter wires can be used, however, and strands have been made with 7 mm diameter wire. The wire has the properties given in Sections 10 and 11.

The elastic modulus of the strand is less than that of the separate wires because of the effect of helical winding and because of various deformations which occur during tensioning (such as crushing, bending and straightening). The modulus varies between 16 000 and 19 000 kg/mm² and varies over too wide a range to be used as a criterion. The modulus during de-tensioning, that is, during removal of the load, is usually specified; it does not vary so much and lies between about 19 000 and 20 000 kg/mm².

Multilayer strands are also used (that is, not only one layer of six wires about a core wire, but several concentric layers); the pitch of the layers is constant, and they are wound in one direction so that the wires in the outside layer are in contact over their whole length with the corresponding wires in the layer beneath.

The methods employed to anchor strands must ensure that slipping of the central king wire cannot occur.

Die-formed strands
This treatment tightens the wires against each other by deforming them in a transverse direction. The strand diameter is reduced and the outside surface is more regular. The strand is more compact and it approximates to a bar, the transverse sections of the wires being approximately in the
shape of contiguous hexagons. This method of treatment can be applied to 7-wire strands and to multilayer strands. It greatly reduces the risk of slipping of the central wire.

17. Relaxation of steels at constant length
The ways in which relaxation is measured are not dealt with in this book, and only the physical behaviour is explained.

If a wire (or bar, or strand) is tensioned between two fixed ends, the tension decreases with time. It is said that relaxation of the wire has occurred. The loss of tension is greater the more the reference point A on the stress–strain diagram (Fig. 17) for the initial tension is distant from the tangent OP at the origin. In other words, the greater the permanent set OA’, the greater is the loss of tension through relaxation.

![Diagram of stress-strain relationship](image)

Fig. 21.

Relaxation can be explained by assuming two limit stress–strain diagrams, I and II. Diagram I is obtained by a standard test and is the instantaneous diagram. Diagram II corresponds to a test of infinite duration, where plastic deformations have fully occurred. Diagram II is the equilibrium diagram. If the stress is kept constant, the strain increases with time, and the reference point A moves along the horizontal line AA (Fig. 21). This is called creep. If the length (and hence the strain) is kept constant, the tensile stress diminishes with time and the reference point A moves along the vertical line AA’. This is called relaxation. The stress approaches the equilibrium stress fairly rapidly. The amount of relaxation
is the distance between curves I and II, measured along the vertical line corresponding to the appropriate strain. The greater this distance of point A from the tangent at the origin, the greater the relaxation; alternatively, the higher the instantaneous plastic strains in relation to the elastic strains $A_E$, the greater the relaxation.

If $E_T$ is the slope of the tangent at A, and $E_o$ the slope at the origin, the final loss of stress after a very long time varies approximately as $\log(E_o/E_T)$. This is merely an indication, but it illustrates clearly the manner of the variation.

Relaxation is closely related to the slope of the stress–strain diagram. Since relaxation is accompanied by a loss of prestress, and therefore requires higher initial stresses or a greater quantity of steel, stiff diagrams are preferable from this aspect to soft diagrams.

Stiff diagrams must not be detrimental to ductility, of course, as happens with certain methods of heat treatment. In about 1950, steels of this type were the cause of accidents because of corrosion of the steel; reservations about the use of stiff diagrams have persisted because of these accidents. This reticence is not entirely justified, because a stiff diagram does not necessarily imply brittleness. Mechanical treatments such as ‘pulling’ and thermo-mechanical treatments such as stabilisation (Section 18) enable the diagram to be stiffened without risk.

Although it is necessary to pay particular attention to corrosion resistance properties when choosing a steel (Section 22), it is not necessary to discard a steel out of hand because of the shape of the diagram.

The rate of relaxation of steel is initially much greater than that of concrete.

It is difficult to formulate laws for its behaviour because its rate is considerably affected by the shape of the stress–strain diagram and by the value of the initial stress.

In the absence of more precise experimental data, reasonable orders of magnitude are obtained by assuming that the loss of tension $\Delta T(h)$ after $h$ hours is related to the final loss of tension $\Delta T_\infty$ in accordance with the expression

$$\Delta T(h) = \Delta T_\infty[1 - 10^{-4(h/10000)^{1/2}}]$$

This expression provides the following values of total percentage relaxation with time:

<table>
<thead>
<tr>
<th>After</th>
<th>1 hour</th>
<th>100 hours</th>
<th>1000 hours</th>
<th>10 000 hours</th>
<th>100 000 hours</th>
<th>(1½ months)</th>
<th>(14 months)</th>
<th>(10 years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.21</td>
<td>0.51</td>
<td>0.72</td>
<td>0.9</td>
<td>0.98</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Curves:**

1. CHAGNEAU-DAWANCE $\Phi$ 5 Wire drawn tensioned 0-7 $T_r$
2. $\Phi$ 5 percentage tensioned (0-7 $T_r$)
3. $\Phi$ 5 drawn tensioned to 0-7 $T_r$ (tested by DE STRYCKER)
4. Law $I = 10^{-\frac{h}{10,000}}$ (arbitrary scale)
5. DUMAS - Steel tensioned to 155 kg/mm² Pulled to 155 kg/mm²
6. $\Phi$ 2-5 tensioned to 170 kg/mm² (0-75 $T_r$)

**Materials:**

- SOMERSET $\Phi$ 5 mm stabilized ($T_i = 0-8 T_r$)
- Strand dia 12-7 mm tensioned to 0-7 $T_r$
- Strand dia 12-7 mm tensioned to 0-58 $T_r$

**Fig. 22.**
72% of the total relaxation will have therefore occurred after 1½ months, whereas it takes 5 months for the same percentage of the concrete long-term deformation.

It should be noted that the rate of relaxation is at first very high. Consequently, experimental results which are obtained in different laboratories can only be compared if the first reading is taken at a well defined time, since the amount of relaxation occurring in the first few minutes is a high proportion of the total.

It must also be noted that during prestressing the cables are not anchored instantaneously; the tensioned cable stays for a time at a constant stress and this slightly reduces the amount of relaxation between curves I and II.

The final relaxation loss depends on the type of steel and the shape of the stress-strain diagram (that is, on the slope and the curvature at the point representing the initial stress).

Relaxation diagrams for different steels and different initial stresses are shown in Fig. 22.

In most specifications, the steel supplier is required to provide guaranteed values of relaxation losses corresponding to values of the initial stress. It is therefore possible to obtain quite definite information once the type of steel is known.

In the initial design stages, when a choice of steel has not been made, it is usually safe to assume a relaxation loss of 8 to 10% of the initial stress up to stress values equal to 80% of the rupture strength.†

It has already been mentioned that relaxation times can be reduced by special treatment ('pulling', stabilisation, etc.).

It will be shown (Section 19) that, where relaxation and long-term concrete deformation occur simultaneously, the resultant loss of stress is not equal to the sum of the two separate losses.

**Creep**

It can be necessary in certain cases to take into account the creep characteristic of the steel, $\varepsilon_T$; that is, its strain at constant load.

If a stable stress-strain diagram, as shown in Fig. 21, diagram II, is assumed, which is reached after a considerable period of time, then

† For other values of initial stress, approximate orders of magnitude can be obtained by assuming that if the percentage loss is taken as one (that is, as the unit) when $T_l = 0.8T_r$, it becomes (as a percentage of $T_l$) about 0.7 for $T_l = 0.7T_r$ and 1.3 for $T_l = 0.9T_r$. This applies only to wires without previous 'pulling' treatment. For such wires, the percentage losses for the same initial stresses are 0.9, 1 and 1.07. These are only indicative; the true values depend above all on the shape of the stress-strain diagram.
\[ \varepsilon_{f\infty} = \Delta T_{\infty}/E_T \] approximately, where \( \varepsilon_{f\infty} \) is the creep strain, \( \Delta T_{\infty} \) is the final relaxation from the initial stress \( T \) and \( E_T \) is the slope of the diagram at the stress \( T \). (The slope should really be taken on curve II at a point between \( A_r \) and \( A_f \), Fig. 23.)

It is not certain that \( \varepsilon_f(t)/\varepsilon_{f\infty} = \Delta T(t)/\Delta T_{\infty} \) is truly valid.

If it is assumed to be approximately true, however, then the strain \( AA' \) can be obtained. This is the strain resulting from a stress which is maintained constant for a time \( t \). The relaxation \( A'A' \), can then be obtained.

This is a little smaller than the relaxation \( AA_r \), which would have been obtained if anchoring had been instantaneous.

Consider the calculation for a wire 5 mm in diameter. The wire has a strength of 160 kg/mm\(^2\) and is stressed to 112 kg/mm\(^2\). Relaxation is 8.8 kg/mm\(^2\); the slope \( E_T \) (measured on curve II approximately drawn) is 4 000 kg/mm\(^2\). The total creep strain, if the stress were maintained indefinitely, would be 8.8/4 000 = 0.22/100.

If the stress is maintained for 15 min, then

\[ \frac{\varepsilon_f(\frac{1}{4} \text{ hour})}{0.22/100} = \frac{\Delta T(\frac{1}{4} \text{ hour})}{\Delta T_{\infty}} \]

approximately.
If the relaxation law previously given is assumed:

\[
\left( \frac{\Delta T}{\Delta T_\infty} = 1 - 10^{-4(9/100000)1/2} \right)
\]

then at the end of 15 min, \( \Delta T/\Delta T_\infty = 0.15 \).

At this time, the creep strain is

\[
\frac{0.22}{100} \times 0.15 = \frac{0.033}{100}
\]

By marking off this strain as AA' on the graph shown in Fig. 23, it is found that A'A', = 0.85 AA'. Maintaining the stress constant therefore reduces the relaxation by 15%.

18. Temperature effects. ‘Stabilisation’ of wires and strands

De Strycker has shown (Berlin Congress, 1958) that relaxation losses increase very rapidly with temperature, even at the temperatures which are encountered on sites in tropical countries.

The subject of high temperature relaxation losses has become of the greatest importance with the advent of concrete reactor pressure vessels.

Tests conducted by the Group Research Laboratory of GKN (Report no. 647 by T. Cahill, April 1962) on 5 mm diameter wires with a strength of 160 kg/mm² give the following results at 1000 hours:

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Initial tension (kg/mm²)</th>
<th>Loss of tension (kg/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>124</td>
<td>9</td>
</tr>
<tr>
<td>100</td>
<td>107</td>
<td>20</td>
</tr>
<tr>
<td>200</td>
<td>88</td>
<td>26</td>
</tr>
</tbody>
</table>

The higher losses at the higher temperatures would obviously have been greater still had the initial tensions been the same as at 20°C.

In another report (The Long-term Relaxation Behaviour of Prestressing Wires and Strands, by G. D. Branch and T. Cahill, May 1963), experimental and theoretical comparisons are made of tensile losses in a seven-5-mm wire strand, stressed to 75% of its breaking strength. The losses after 1000 hours were in the ratios:

at 20°C 40°C 60°C 100°C
1 1.9 3.3 8
After 30 years, and for the same temperatures, these ratios would become:

\[
\begin{array}{cccc}
20^\circ C & 40^\circ C & 60^\circ C & 100^\circ C \\
1 & 1.9 & 2.7 & 6.2 \\
\end{array}
\]

It is doubtful, however, that these extrapolations are entirely valid. According to PAPPSDORF and SCHWIER (Stahl und Eisen, July 1958), the final loss would be the same and only the rate of relaxation would change, curve II of Fig. 21 being reached earlier. This statement does not seem justifiable in view of the 1000-hour tests on 5 mm wire mentioned above. The best course is perhaps to temper the 30 year extrapolation.

In any event, relaxation of the steel is a troublesome feature to cope with. It can be avoided by previously treating the wires or strands. The treatment is called ‘stabilisation’ by the Somerset Wire Company, who, so far as the author knows, were the first to apply it industrially.

The treatment consists of tensioning the wires or strands at a suitable temperature to a stress which is higher than their working stress. According to the Somerset Wire Company reports, this reduces relaxation to normal temperature values, and even lower, as a result of the pretensioning. More detailed information can be obtained from the reports.

![Graph](image)

**Fig. 24.** Relaxation of high strength steels at various temperatures—tests by the Somerset Wire Company (5 mm diameter wires, 160 kg/mm² ultimate strength).
Experimental work in the laboratories of Liege University, referred to by Dumas (7th Day of the ASP, 1964), gives comparable results with wires which had been treated in a similar manner.

19. Resultant effect of steel relaxation and long-term concrete deformation
For relaxation at constant length, it can be considered that the reduction in stress from an initial stress $T_i$ is such that it compensates at each moment the creep elongation which would occur if the stress were kept constant.

It is therefore necessary to have

$$\frac{T_i - T(t)}{E_a} = \varepsilon_f(t)$$  \hspace{1cm} (a)

all the time, where $E_a$ is the elastic modulus corresponding to the loss of stress $T_i - T(t)$, read from curve $(t)$ on Fig. 23.

If the wire is integral with a concrete beam which shortens by an amount $\varepsilon_d(t)$ during the same period:

(i) The stress in the wire is reduced by an amount $E_{ao}\varepsilon_d(t)$; in accordance with Fig. 17, the modulus of elasticity $E_{ao}$ is equal to the slope at the origin of the stress–strain diagram for the steel.

(ii) The excess length which it is necessary to compensate through loss of stress becomes $\varepsilon_f(t) - \varepsilon_d(t)$. It is therefore necessary to have:

$$\frac{T_i - E_{ao}\varepsilon_d(t) - T(t)}{E_a} = \varepsilon_f(t) - \varepsilon_d(t)$$  \hspace{1cm} (b)

Therefore if $E_a$ (the modulus for the relaxation loss of stress) had the value $E_{ao}$ (the modulus for the long-term deformation loss of stress), $\varepsilon_d(t)$ could be eliminated in expression (b) and the loss through long-term deformation would therefore not appear.

If, on the other hand, as assumed in Fig. 23, the modulus for the relaxation loss of stress (cancelling creep) is $E_T$, the slope of the stress–strain diagram at tension $T$ (which can be taken as being approximately equal to $E_{Tl}$), then

$$T(t) = T_i - E_T \varepsilon_f(t) - (E_{ao} - E_T)\varepsilon_d(t)$$  \hspace{1cm} (c)

If it is considered that relaxation of the steel is complete when there still remains a fraction $\lambda \varepsilon_f \infty$ of concrete long-term deformation, a loss of stress equal to $E_{ao} \lambda \varepsilon_d \infty$ results. It is then possible to write:

$$T \infty = T_i - E_T \varepsilon_f \infty - [(E_{ao} - E_T)(1 - \lambda) + E_{ao} \lambda] \varepsilon_d \infty$$  \hspace{1cm} (d)
Whatever assumption is made for the modulus $E_a$ used in the determination of the loss of stress through relaxation, the term $E_{Tr} \varepsilon_{f \infty}$ is defined in Section 17 as the relaxation loss at constant length; let it be called $\Delta T_{\rho}$.

The loss would be $E_{ao} \varepsilon_{d \infty}$ if only long-term concrete deformation occurred: call it $\Delta T_d$.

The term corresponding to this long-term deformation in eqn. (d) can be written:

$$E_{ao} \varepsilon_{d \infty} \left[ 1 - \frac{E_{Tr}}{E_{ao}} (1 - \lambda) \right]$$

The same argument applies to shrinkage deformation. Assuming that the fraction $\lambda_r$ of shrinkage which remains after complete relaxation of the steel is equal to the value $\lambda$ of the long-term deformation, then, calling $\Delta T_r$ the shrinkage loss and $\Delta T$ the total loss:

$$\Delta T = \Delta T_{\rho} + \left[ 1 - \frac{E_{Tr}}{E_{ao}} (1 - \lambda) \right] (\Delta T_d + \Delta T_r) \quad (e)$$

This equation indicates that the resultant loss is less than the sum of the losses.

It is customary in practice to summate the losses, and it therefore seems that the total loss is overestimated.

This cannot be definitely said, however, because $\Delta T_d$ is underestimated and this may average out the final result. The deformation $\varepsilon_{d \infty}$ is indeed calculated, on the basis that the compressive stress is constantly equal to its final value after the losses have occurred.

Yet it appears obvious that it should depend on the history of the concrete right from the time of loading.

It is difficult to be precise on this aspect; the following approximate reasoning can be followed, however.

The initial compression in the concrete is $\sigma_{bi}$ and the final compression is

$$\sigma_b = \sigma_{bi} \times \frac{T_i - \Delta T}{T_i} = \sigma_{bi} - \frac{\sigma_{bi}}{T_i} \Delta T$$

It can be assumed, from the principle of superposition (Section 5), that the resultant long-term reduction in length is equal to the sum of the long-term reduction in length corresponding to $\sigma_{bi}$, or $\sigma_{bi}/E_b \times \phi$; and the elastic compression (see Section 4) $\sigma_{bi}(\Delta T/T_i)$, corresponding to the strain $-(\sigma_{bi}/E_b)(\Delta T/T_i)$. 
The total long-term reduction in length is therefore
\[
\frac{\sigma_{bi}}{E_b} \left( \phi - \frac{\Delta T}{T_i} \right) = \phi \frac{\sigma_{bi}}{E_b} \left( 1 - \frac{1}{\phi} \frac{\Delta T}{T_i} \right)
\]
or
\[
\sigma_{bi} = \sigma_b \frac{T_i}{T_i - \Delta T} = \sigma_b \frac{T + \Delta T}{T} = \sigma_b \left( 1 + \frac{\Delta T}{T} \right)
\]

Let $\Delta T_d$ be the true loss of stress due to long-term deformation, and let $\Delta T_d$ be the loss of stress which is usually considered, calculated on the basis of the final compressive stress. Then:
\[
\Delta T_d = \phi \frac{E_{ao}}{E_f} \sigma_{bi} \left( 1 - \frac{1}{\phi} \frac{\Delta T}{T_i} \right) = \phi \frac{E_{ao}}{E_b} \sigma_b \left( 1 + \frac{\Delta T}{T} \right) \left( 1 - \frac{1}{\phi} \frac{\Delta T}{T_i} \right)
\]

But
\[
\phi \frac{E_{ao}}{E_b} \sigma_b = \Delta T_d
\]

Therefore
\[
\Delta T_d = \Delta T_d \left( 1 + \frac{\Delta T}{T} \right) \left( 1 - \frac{1}{\phi} \frac{\Delta T}{T_i} \right)
\]

Since $\Delta T/T$ is small and if $\Delta T/T_i$ is taken as equal to $\Delta T/T$, then:
\[
\Delta T_d = \Delta T_d \left[ 1 + \left( 1 - \frac{1}{\phi} \right) \frac{\Delta T}{T} \right]
\]

approximately.

Finally, then:
\[
\Delta T = \Delta T_\rho + \left[ 1 - \frac{E_{Ti}}{E_{ao}} (1 - \lambda) \right] \left[ \Delta T_d \left( 1 + \frac{\phi - 1}{\phi} \frac{\Delta T}{T} \right) + \Delta T_r \right] \tag{f}
\]
or, with $\phi = 2$,
\[
\Delta T = \Delta T_\rho + \left[ 1 - \frac{E_{Ti}}{E_{ao}} (1 - \lambda) \right] \left[ \Delta T_d \left( 1 + \frac{1}{2} \frac{\Delta T}{T} \right) + \Delta T_r \right]
\]

Although the above reasoning is based entirely on assumptions, it does illustrate the complexity of the problem; there are, on the one hand, reduction factors which depend to a very high degree on the steel stress-strain diagram ($E_{Ti}/E_{ao}$), on the properties of steel relaxation and on the
concrete creep ($\lambda$), and, on the other hand, multiplying factors of
$[1 + \frac{1}{4}(AT/T)]$.

It would be very useful if systematic tests were conducted with steels
of different characteristics.

Some tests of great interest are due to Chagneau and Dawance, on
the effect of simultaneous steel relaxation and concrete creep.

In these tests, prisms were compressed by means of steel in tension,
and the steel relaxation, the prism deformations and the stress in the
steel inside the prisms were measured separately.

Figure 25 shows stress–strain diagrams for the steels, diagrams for the
relaxation of the wires alone at various stresses, diagrams of the stresses in
the wires within the prisms and the deformation diagrams for the prisms.

The tests were carried out on specimens one year old, compressed by
means of 5 mm wire-drawn steel with relatively ‘rounded’ stress–strain
diagrams, stressed to 70% of the rupture stress, and also on specimens one
month old, compressed by means of wires treated by patenting, stressed
to various values. The tests lasted 13 and 10 years respectively.

In order to draw conclusions, the three deformations (steel, concrete
and prism) must be taken at the same age on the graphs. It is necessary to
take the results at 2 000 days ($5\frac{1}{2}$ years) for the first series and at 1 000 days
(2 years 9 months) for the second series, because the measurements of the
slow concrete deformations were not taken over as long a period as the
other measurements.

The results can be obtained from Test Report No. 223 of the Laboratoires
du Bâtiment et des Travaux Publics (March 1964).

They can be interpreted in the following manner. Consider, for example,
prism No. 2, compressed after 1 year to a stress of 101 kg/cm$^2$ by tensioning
the wires to a stress of 101 kg/mm$^2$.

After 2 000 days the compressive stress in the concrete was 81 kg/cm$^2$.
The loss of stress in the steel in the prism was 19·5 kg/mm$^2$.

The relaxation, measured on the separate test bed, from the same initial
stress of 101 kg/mm$^2$, was 12·5 kg/mm$^2$. The strain (compressive) in the
prism was $510 \times 10^{-6}$.

The corresponding loss, with $E_u = 20 000$ kg/mm$^2$, is 10·2 kg/mm$^2$.

The sum of the losses—shrinkage loss plus true long-term deformation
loss—is therefore $12·5 + 10·2 = 22·7$ kg/mm$^2$. The effective loss is only
19·5 kg/mm$^2$. The coefficient of reduction is $19·5/22·7 = 0·86$.

However, the long-term deformation is normally calculated (see above)
on the basis that the compressive stress is constant at 81 kg/cm$^2$. The
N.B. The long-term deformation curves for concrete initially compressed to 120 and 10 kgf/cm² are practically coincident.
Fig. 25. Loss of stress under the simultaneous effect of steel relaxation and long-term deformation (tests by Dawance and Chagneau).
### Table A

**Losses through relaxation and simultaneous long-term deformation**

<table>
<thead>
<tr>
<th>No.</th>
<th>(1) Wire stress (kg/mm²)</th>
<th>(2) Concrete compressive stress Initial (kg/cm²)</th>
<th>(3) Concrete compressive stress Final (kg/cm²)</th>
<th>(4) Prism compressive strain (10^-6)</th>
<th>(5) Relaxation loss (kg/mm²)</th>
<th>(6) Long-term deformation loss (kg/mm²)</th>
<th>(7) Sum of the losses (kg/mm²)</th>
<th>(8) Effective loss (kg/mm²)</th>
<th>(9) Coefficient of reduction</th>
<th>(10) Conventional reduction in length (kg/mm²)</th>
<th>(11) Sum of the losses (kg/mm²)</th>
<th>(12) Coefficient of reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99</td>
<td>80</td>
<td>65</td>
<td>390</td>
<td>12.5</td>
<td>7.8</td>
<td>20.3</td>
<td>17.5</td>
<td>0.86</td>
<td>7</td>
<td>19.5</td>
<td>0.90</td>
</tr>
<tr>
<td>2</td>
<td>101</td>
<td>101</td>
<td>81</td>
<td>510</td>
<td>12.5</td>
<td>10.2</td>
<td>22.7</td>
<td>19.5</td>
<td>0.86</td>
<td>9.2</td>
<td>21.7</td>
<td>0.90</td>
</tr>
<tr>
<td>3</td>
<td>103</td>
<td>134</td>
<td>106</td>
<td>570</td>
<td>12.5</td>
<td>12.4</td>
<td>24.9</td>
<td>21</td>
<td>0.84</td>
<td>10</td>
<td>22.5</td>
<td>0.90</td>
</tr>
<tr>
<td>4</td>
<td>99</td>
<td>169</td>
<td>131</td>
<td>620</td>
<td>12.5</td>
<td>12.4</td>
<td>24.9</td>
<td>20</td>
<td>0.80</td>
<td>10.8</td>
<td>23.3</td>
<td>0.86</td>
</tr>
<tr>
<td>5</td>
<td>121</td>
<td>93</td>
<td>78</td>
<td>720</td>
<td>6.2</td>
<td>14.4</td>
<td>20.6</td>
<td>18.2</td>
<td>0.88</td>
<td>13.1</td>
<td>19.3</td>
<td>0.94</td>
</tr>
<tr>
<td>6</td>
<td>121</td>
<td>117</td>
<td>97</td>
<td>770</td>
<td>6.2</td>
<td>15.4</td>
<td>21.6</td>
<td>20</td>
<td>0.92</td>
<td>14</td>
<td>20.2</td>
<td>0.99</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>149</td>
<td>125</td>
<td>980</td>
<td>6.2</td>
<td>19.6</td>
<td>25.8</td>
<td>19.6</td>
<td>0.76</td>
<td>17.8</td>
<td>24.0</td>
<td>0.81</td>
</tr>
<tr>
<td>8</td>
<td>132</td>
<td>213</td>
<td>167</td>
<td>1300</td>
<td>10.2</td>
<td>26</td>
<td>36.2</td>
<td>28.2</td>
<td>0.78</td>
<td>22.8</td>
<td>33.0</td>
<td>0.85</td>
</tr>
</tbody>
</table>

\[
(\text{Col. 11} = \text{Col. 6} \times \frac{2\sigma_p}{\sigma_{br} + \sigma_p}; \quad \text{Col. 12} = \frac{\text{Col. 8}}{\text{Col. 11}})
\]
calculated long-term deformation loss would have been, therefore, according to the reasoning followed above:

\[ 10.2 \times \left( 1 - \frac{1}{2} \frac{\Delta T}{T} \right), \quad \text{or with } \Delta T = 20 \quad \text{and} \quad T = 100 \]

\[ 10.2 \times 0.9 = 9.2 \text{ kg/mm}^2 \]

The loss as normally calculated would have been:

\[ 12.5 + 9.2 = 21.7 \text{ kg/mm}^2 \]

instead of only 19.5 kg/mm². The coefficient of reduction is about 0.9.

The same calculations have been conducted for the other prisms by evaluating the true total losses (\(\Delta T_d + \Delta T_p\)) and the calculated total losses (\(\Delta T_d + \Delta T_p\)), and they have been compared with the effective losses.

It is also possible to calculate the conventional long-term deformation loss (that is, based on the hypothesis that the compression is at any time equal to its final value) by applying the coefficient \(\sigma_b/\sigma_{b\text{mean}}\) to the true long-term deformation loss, where \(\sigma_b\) is the final compressive stress and \(\sigma_{b\text{mean}}\) is the mean value \((\sigma_{bI} + \sigma_f)/2\) between the initial and the final stress.

Table A gives a summary of the results.

The conclusion to be drawn from this summary is that simple addition of the losses through relaxation and reduction in length leads to an overestimate of 10 to 25% (col. 9) compared with the actual losses; the greater the compressive strain, the greater is this overestimation.

In calculating long-term deformation losses in the conventional manner, values are closer to the actual values, but insufficiently so. The error can be of the order of 20% (prism 7).

It would be appropriate to apply a reduction factor of 0.7 to 0.8 to these long-term deformation losses.

20. Fatigue failure of steels

Particular attention should be given to the variations of the tension in the steel when structures are subjected to high-frequency alternating loads. Tests by XERCAVINS (Amsterdam Congress, 1955) indicate that failure occurs after 1 000 000 cycles for a stress amplitude variation of 15 kg/mm², independently of the mean stress. Figure 26 summarises these test results.

Such variations can occur only in the case of partial prestress (Class II and Class III). For Class II (limited prestress), where the stresses in the steel are calculated on the basis of an homogeneous section, it is necessary to limit the concrete stress variations to 1500/m kg/cm², \(m\) being the
modular ratio. Since rapid phenomena are concerned and since the concrete behaves elastically anyway because of the resorption of plastic phenomena due to load cycling, the values of \(m\) are small, and concrete stress variations of 150 kg/cm\(^2\) local to the cables could be acceptable (+120, −30). Limited prestress (Class II) is not to be advised for such applications; prestressed reinforced concrete is quite unsuitable.

21. Strength tests for steels
It is relatively simple to eliminate the use of a steel whose quality is known to be unsuitable. It is much more difficult to detect local defects in a steel whose quality is otherwise acceptable. If such defects do exist, tests do not provide any guarantee, since there is every chance that the tests will have been conducted on samples which do not contain the defects.

![Diagram](image)

**Fig. 26.** Fatigue tests (Xercavins).

Stated thus, this statement seems obvious. It is nevertheless the drawback with all strength tests. The only course which can be adopted, in these circumstances, and this concerns the metallurgists and not the users of the steel, is to exercise extreme care in the formulation of treatments. Should there be any risk involved in exceeding the parameter limits characterising the treatment, then the treatment should be maintained well within those limits, and with sufficient margins.
It could be (so wrote Laravoire in ‘Travaux’ of November 1952) that too much has been expected of a single heat treatment in certain cases, and this could explain certain incidents which occurred around the year 1950.

If this was the fault, it has been corrected, and it can be taken that the quality of the steels actually used is sufficiently reproducible to necessitate only sample quality control.

Should confidence in the quality not exist, strength control is simply illusory. Apparatus for continuous control does exist (electromagnetic control), which monitors the wire as it leaves the wire-drawing rolls. Its use has not been pursued in France, and it has not appeared to be necessary.

The three practical strength tests are bending, cyclic torsion and winding.

**Bending**
The wire is bent over a mandril with a diameter, defined by the ASP code, as a function of the diameter D of the wire (about 15D: 45 mm for diameters from 5 to 12 mm; 20 mm for wire diameters less than 4 mm; 30 mm for wires between 4 and 5 mm). The wire is bent alternately to the right and to the left at 90° (sufficiently slowly so as not to heat the wire) and the number of cycles N before failure occurs is noted. The minimum value of N is defined by the codes. It is generally taken that N should be equal to at least 3.

**Alternating tension**
The wire is held between two clamps 50 D apart, and twisted one turn to the left, two turns to the right, one turn to the left, at a speed of 10 seconds per turn. These four turns represent a cycle. The number of cycles n is noted. This very severe test is not very representative and is seldom used. It is mentioned for the record.

**Winding**
The wire is wound in contiguous spirals of 2.5 D diameter at a maximum speed of \( \frac{1}{2} \) rev/sec. There should not be any breakage or cracking in the wire after twelve consecutive spirals.

**22. Corrosion resistance**
Around 1950, as mentioned in the previous section, breakages of steel occurred in various structures. They were attributed to a phenomenon, new in the field of prestressed concrete, called stress corrosion. It was
magnified into a ‘disease’ of prestressed concrete at the time, and fear of it still remains.

In truth, this fear has been very greatly exaggerated. With good procedures and suitable steels the risk is practically non-existent; or, more accurately, it is in the same class as all other risks of construction.

It cannot, however, be denied that incidents did occur, and a lesson must be drawn from this fact. Some accidents were caused by the type of steel which was used. Reference to this was made in the preceding section and it can be confidently assumed that this cause of failure has now disappeared. To avoid the recurrence of similar faults, the use of new types of steel should be explored with extreme caution, and they should be used only after prolonged testing in unfavourable conditions with regard to corrosion.

Other accidents have been due to inadequate construction methods. It is not possible to list them in this book, and it is sufficient to say that failures have been blamed on corrosion as a cover for other faults, often of an extremely gross nature.

Attempts were made to place the blame on certain mechanical treatments: it was maintained that rolled steel was more prone to corrosion than drawn steel. This was based on laboratory test results on wires bent by winding and immersed in ammonium nitrate baths heated to 105°C. The steels were classified according to the time taken to fracture. This test does not seem to be truly representative.† At a private meeting held in Paris in 1954, all known cases of failure at that date were examined. Drawn steel was used in 80% of the cases and rolled steel in 20% of the cases. These percentages corresponded approximately to the percentages used in all of the countries represented, and rolling or drawing could not in any of the cases be blamed for the failures. In any case, it is not possible to discriminate on the basis of this mechanical treatment alone, since the subsequent heat treatment is far more important.

In fact, the author’s personal experience has shown that with the steels used by his company, and with a good grout protection, done sufficiently early, there is no risk of corrosion. This opinion is shared by all French engineers using the same steels and the same methods of protection. An identical opinion has been expressed by Dr Gilchrist, following an enquiry on the steels used in England.

† The time during which the wire can remain unprotected is more important, in any case, and this can depend on the mechanical treatment (surface hardening), rather than on the intrinsic quality of the steel (brittle elements).
However, it is certain beyond any doubt that unprotected tensioned steel will fail.

An experiment was made by DUMINY in Rouen in 1951 on four bare cables, 33 mm long, stressed to 90 kg/mm$^2$ in a slightly humid atmosphere. The wires successively failed; one cable failed completely after 6 months and the others at the end of 9 months.

Similar failures have occurred on bare cables inside hollow beams. It had been assumed that weather protection was sufficient; but renewal of the oxygen must be prevented and only a protective envelope around the cables can achieve this end.

The best protection is achieved by the injection of grout or mortar under pressure. This method is adopted with ducted cables. It must be done sufficiently early on; it seems that the delay should not exceed one month, unless the work is in a very dry atmosphere. The delay should be less in corrosive atmospheres and in offshore works.

It is interesting to note that incidents due to corrosion have occurred mainly with circular structures. The cause has been attributed to electrolytic phenomena related to the curvature of the steel; in other words, to the deformation gradient across a transverse section. It is appropriate to query the effectiveness of protection in certain of these works. Reservoirs and pipes bound with external wires are coated for protection; this coating needs to be perfectly applied. It is also necessary that the coating should not contain any corrosive elements, and this can proscribe the use of blast furnace cement or similar, containing sulphur in various forms, and various hardening agents (calcium chloride in particular). In such cases it is still the protection and not the prestress or the steel which is the prime cause of corrosion.

Various theories on the mechanism of corrosion have been advanced, but it does not appear to be necessary to expound them. From the above, it is possible to conclude that the risk is practically non-existent if the work is properly executed. A bibliography on this subject is given at the end of this chapter.

23. Resistance to fire
It is proposed to deal with this subject only in outline. A more detailed study can be found in the author's book 'Prestressed Concrete', Vol. 1.

The strength of a structure for only a limited time can be considered when discussing resistance to fire, since there is no structure which can withstand indefinitely the high temperatures which are reached during a fire.

Various regulations define the times for which structures must withstand
a typical fire, defined by a relation between temperature rise and time, which varies according to the location of the structure. The German Code, for example, defines constructions which are resistant to fire (duration of resistance 1\frac{1}{2} hours) and constructions which are very resistant to fire (duration of resistance 3 hours).

At high temperature, deterioration and transformation of materials occur, accompanied by a large reduction in strength, which is dependent upon the temperatures. Thermal expansions also give rise to thrusts and additional forces.

The compressive strength of concrete decreases by about one-third at 400°C and two-thirds at 800°C. The strength of steel decreases by 50\% at 400°C and by 85\% at 600°C.

The moment of resistance therefore decreases progressively and when it becomes equal to the moment to which a particular element of the structures is subjected, this member collapses.

Concrete structures (reinforced or prestressed) are helped by the low thermal conductivity of concrete, which prevents the steel being exposed to excessive temperatures, especially if the concrete is thick—provided, of course, that help arrives in time.

In this respect, a temperature of 400°C can be considered as critical; if the steel reaches this temperature, the moment of resistance decreases by about half of its value, and this causes failure of a member designed with a safety factor of 2.

The duration of the resistance to fire damage therefore depends above all else on the thickness of concrete covering the steel.

It is estimated that the time to reach a temperature of 400°C is proportional to the protective thickness, at the rate of 20 min/cm.

Plaster finishes are equivalent to additional protective thickness, and a 1 cm plaster thickness is equivalent to 2 or 2.5 cm of concrete. Plaster is only effective as long as it remains adherent.

The Department of Scientific and Industrial Research and the Fire Offices Committee of Great Britain specify a thickness of 2\frac{1}{2} in for 2 hours protection.

Structures which have been subjected to fire can usually be reinstated, provided that they have not been exposed for more than half their theoretical protection time.

A steel which has been heated to 400°C recovers 80\% of its initial strength after cooling (at 400°C the strength of the steel is only half of the cold strength). If, due to the protection, the steel only reaches 200°C, the initial strength is almost entirely recovered (92 to 100\%) on cooling.
BIBLIOGRAPHY

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2. Steel relaxation at ambient temperature

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4. Corrosion

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Chapter III

PRESTRESSING EQUIPMENT
Equipment for post-tensioning

Post-tensioning† is the application of prestress by means of steel tendons which are tensioned after the concrete has set and hardened. The methods adopted for post-tensioning are more or less the same for all prestressing systems.

The cables are usually arranged within the concrete and they must be free to slide whilst being tensioned; therefore, they must be positioned before the concrete is placed, or inserted after placing, inside longitudinal ducts (sheaths) which terminate at the end faces of the concrete.

The tendons are tensioned by means of jacks which bear on the concrete, thus placing the concrete in compression. Each tendon must be anchored after it is tensioned, so that the jacks can be removed, and each tendon must also be grouted in so as to eliminate the risk of corrosion.

Finally, with certain methods the anchorages themselves, as well as the portions of the cables which are outside the concrete, are given a protective treatment against corrosion.

In some cases, the tendons are placed outside the concrete. Corrosion protection in these cases must be 100% efficient (see Section 14).

The components and procedures described in the following pages deal mainly with the following:

cables and tendons;
safeguards for the integrity of the cable ducts;
anchors;
groutings.

The plant and methods used by FREYSSINET are first briefly described. Other systems using identical or similar principles are then examined.

† As opposed to pre-tensioning, where the prestressing steel is tensioned before the concrete is placed. Problems peculiar to this method are not described in this chapter.
A. FREYSSINET SYSTEM FOR 12-WIRE CABLES

1. The cable
The cable comprises twelve parallel, high tensile steel wires, 5, 6, 7, 8 or 12 mm in diameter. Twelve-strand cables, using strands of several diameters (12-7 mm, 15 mm and 21 mm), are also employed. The wires or strands are unwound twelve at a time, one from each of twelve cable drums which are mounted with their axes vertical. The wires or strands are drawn through a die which groups them into parallel bundles. The several steps in the fabrication of the tendon (tying, sheathing, cropping) are carried out subsequently on a bench located behind the die.

![Diagram of cable and die](image)

**FIG. 1.** Sketch of FREYSSINET 12 parallel wire cable.

The core of the bundle is often formed from a helically wound wire, 1.5 to 2.5 mm in diameter, with a helical pitch of 1 to 3 cm, depending on whether the cable is to be curved or straight. The object of the helix is to provide a spacer for the wires, to prevent the wires from becoming wedged and also to provide a channel for grouting. Ties, at intervals of about 0.7 m, keep the cable tight against the central spiral. It is sometimes desirable to remove the ties once the cable is in its sheath (or during the fabrication of the sheath, if it is fabricated around the cable) in order to avoid the possibility of jamming or interference during tensioning. This can also occur if the curvature of the sheath inside the concrete is too sharp.

The central spiral is not absolutely necessary and it is not always used.

The cable must project beyond the face of the anchorage, at each end, by about 50 cm to provide an adequate gripping length for the tensioning jacks.

2. Longitudinal cable ducts
These can take the form of tubes previously arranged within the concrete, or they can be provided by means of an arrangement of rubber-covered rods which are withdrawn one or two days after the concrete has been
placed. The cables are then threaded through the ducts which have thus been formed inside the concrete. This method may be used when the cable is straight or only slightly curved, but the rough surface of the ducts introduces high friction losses (see Chapter IV) if the curvature is pronounced; tests have shown that the coefficient of friction can be twice as high as with metal-lined ducts.

This latter type of duct is generally formed by using metallic sheaths, between 0.15 to 0.2 mm thick, which remain in the concrete. The surface of the sheaths can be smooth, but annular corrugations are preferable. They provide a higher transverse rigidity which is an important property, especially when the ducts are curved, since the formation of wrinkles which could otherwise increase friction losses are avoided during bending.

![Diagram](image)

**Sheath**

**Diagrammatic section**

*Fig. 2. Sheath with double corrugation.*

The sheath must also be flexible in the longitudinal direction so that it can easily take up a curved profile. Sheaths which are both supple longitudinally and rigid transversely are made from corrugated strip which is spirally wound in a manner similar to that in which webbing is wound to form a gaiter (Fig. 2).

The sheath must be watertight. The joints between sections are sleeved, or they are made by threading the sections one into the other (sheaths with helicoidal corrugations), or by inserting one into the other. Tightness of the joints is completed by the application of adhesive tape.

Plastic sheaths may also be used.

The cable, with its metallic or plastic sheath, is usually positioned within the shuttering, in the same manner as ordinary reinforcement. When curved, the sheaths and the cables are given a smooth continuous shape. Sudden changes in direction must be avoided, since they prevent the cable from sliding freely. It is preferable that the cables should assume their natural freely suspended shape, which is very nearly parabolic, before tying the sheaths to the stirrups or bars. The starting point of a design should
always be based on the natural form of the cable and complicated cable profiles should be used only in special circumstances.

3. Anchors
The cable projects from its sheath at each end. It passes through a female cone, which is a cylinder made of steel or reinforced concrete, with an axial tapered hole of mean slope equal to $\frac{1}{6}$ to $\frac{1}{10}$. In the case of reinforced concrete cones, the tapered hole is banded with a close spiral of small diameter, high tensile steel wire. The spiral resists the greater part (about 90%) of the radial pressures exerted through the wedging action of the male cone. The female cone is also banded on the outside with multiple layers of mild steel wire, having four to five times the cross-sectional area of the inner banding. This balances almost entirely the remainder of the radial pressures at a very low working stress. The external banding is the last protective barrier against excessive deformations and cracking of the surrounding concrete.

A few spirals, using bars of 8 to 10 mm diameter, depending on the diameter of the cones, are also arranged in the surrounding concrete, around the female cones.

The radial thickness of the female cone is sized to ensure that the pressures on the bearing face are kept within allowable limits, which are usually high for these localised pressures. Female cones are generally set in high-strength, reinforced concrete prefabricated blocks located inside the face of the shuttering or at intermediate points along the span of the member.

The cones are sometimes located outside the member and grouted, or they are supported on an incompressible material on the free face of the member.

**FIG. 3. FREYSSINET concrete anchoring cone.**
The male cone is made of concrete, reinforced longitudinally. The conical surface is provided with twelve longitudinal grooves for housing and centering the tendon wires. The dimensions and arrangement of male and female cones are shown in Table I.

<table>
<thead>
<tr>
<th></th>
<th>D (mm)</th>
<th>h (mm)</th>
<th>i (mm)</th>
<th>d₁ (mm)</th>
<th>d₂ (mm)</th>
<th>Weight (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete cones</td>
<td>12/5</td>
<td>100</td>
<td>8</td>
<td>28</td>
<td>46.8</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>12/7</td>
<td>120</td>
<td>10</td>
<td>36</td>
<td>61.5</td>
<td>4.7</td>
</tr>
<tr>
<td></td>
<td>12/8</td>
<td>135</td>
<td>11</td>
<td>44</td>
<td>70.0</td>
<td>6.3</td>
</tr>
<tr>
<td>Steel cones</td>
<td>12-strand 12-7</td>
<td>225</td>
<td>127</td>
<td>33</td>
<td>72</td>
<td>106</td>
</tr>
<tr>
<td></td>
<td>12-strand 15</td>
<td>285</td>
<td>165</td>
<td>42</td>
<td>83</td>
<td>180</td>
</tr>
</tbody>
</table>

The external profiles of 12-strand cones differ from those shown in the sketch and they are not detailed in the table; for these, D is the diameter of the bearing surface and i is the diameter of the hole at the base of the steel cone.

4. Jacks. Tensioning

The wires are fanned out inside the female cone and they project beyond the outside face of the cone by about 50 cm. The male cone is located inside the female cone with the wires lying in the grooves. The cable is then tensioned by means of a jack at each end. In some cases only one end is jacked; the cable is then anchored at the other end by its movement when the active end is tensioned. Figure 4 shows the arrangement of a jack.

The nose of the jack rests on the female cone. The wires are threaded through radial openings in the nose of the jack and are keyed to the jack cylinder. Hydraulic pressure causes the cylinder to move out, taking the wires of the prestressing tendon with it. Once the required extension (which is previously calculated from the required stress and length of the cable) is obtained and when the pressure gauge on the hydraulic pump indicates the correct pressure (thereby providing a check), the male cone is rammed home by an auxiliary piston contained within the body of the jack. The pressure is then released; the cylinder is returned by means of an internal spring and emptied. The male cone wedges itself in position and the wires are unfastened from the jack.
FIG. 4. FREYSSINET jack.
TABLE II

Jack characteristics

Jacks with automatic unlatching most commonly used are among the following types:

- Type U₁: Jack for 5-mm diameter, 12-wire cable, travel 200 mm
- Type U₃: Jack for 7-mm and 8-mm diameter, 12-wire cable, travel 200 mm
- Type U₅: Jack for 7-mm and 8-mm diameter, 12-wire cable, travel 300 mm

<table>
<thead>
<tr>
<th>Type of jack</th>
<th>U₁</th>
<th>U₃</th>
<th>U₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum effective travel</td>
<td>200 mm</td>
<td>200 mm</td>
<td>300 mm</td>
</tr>
<tr>
<td>Cross-sectional area of tensioning cylinder</td>
<td>78·2 cm²</td>
<td>157·8 cm²</td>
<td>159·8 cm²</td>
</tr>
<tr>
<td>Maximum pressure</td>
<td>448 kg/m²</td>
<td>520 kg/cm²</td>
<td>520 kg/cm²</td>
</tr>
<tr>
<td>Maximum force</td>
<td>35 000 kgf</td>
<td>82 000 kgf</td>
<td>82 000 kgf</td>
</tr>
<tr>
<td>Cross-sectional area of blocking ram</td>
<td>31·67 cm²</td>
<td>71·3 cm²</td>
<td>71·3 cm²</td>
</tr>
<tr>
<td>Total weight</td>
<td>40 kg</td>
<td>70 kg</td>
<td>80 kg</td>
</tr>
<tr>
<td>Length of jack, closed</td>
<td>60 cm</td>
<td>72 cm</td>
<td>82 cm</td>
</tr>
<tr>
<td>Length of jack, extended</td>
<td>80 cm</td>
<td>92 cm</td>
<td>112 cm</td>
</tr>
<tr>
<td>Maximum diameter</td>
<td>15·5 cm</td>
<td>21·5 cm</td>
<td>21·5 cm</td>
</tr>
</tbody>
</table>

For 12·7 mm diameter, 12-stranded cables 165-tonne jacks are used, and 240-tonne jacks are used for 15 mm diameter, 12-stranded cables. The effective travel is 300 mm and the weights are 188 and 265 kg respectively.

5. Action of the anchor

In its early days, prestressing was criticised on the grounds that it created a statically-indeterminate state, without any guarantee that the wires were equally loaded, since it relies on tensioning twelve wires simultaneously.

These criticisms were unfounded. The wires are all stressed to very nearly the same extent, due to the flatness of the stress-strain diagram in the zone concerned, and because the strain in all the wires is almost exactly the same through the action of the jacks. The risk of non-uniformity lies mainly in the radial direction, and the radial compression of the wires in the anchor must be as uniform as possible. In practice, this condition is met because of the plasticity of the anchor under the extremely high pressures which are developed.
Forces in the anchors

Each wire is in equilibrium under the action $R$ from the male cone, and the reaction $R_1$ from the female cone. Once the male cone is in equilibrium, after ramming, the reactions of the wire on the cone must be perpendicular to the axis of the cone.

Consequently the slope $\alpha$ of the generatrices of the cone, relative to the axis of the cone, must be less than the coefficient of friction $\mu$ (male cone/wire); if $\alpha$ were greater than $\mu$, the cone would be ejected. The slope of the cone must therefore be small; it is also an advantage to increase the coefficient of friction $\mu$ so as to minimise the chances of ejection, and to

![Diagram](image)

**Fig. 5.** Diagram showing the forces applied to the cones.

increase the efficiency during the initial stages of 'self-wedging'. To provide the initial grip, it is essential that the wires should not slip relative to the cone, and that the male cone/wire assembly should be driven home until the required compression is established. This occurs automatically because of the increasing pressure as the cone enters further into the female cone; the movement ceases only when equilibrium is reached. With concrete cones, the coefficient is sufficiently high; in the case of male cones made of steel, $\mu$ is increased by grooving or etching the surface.

The reaction from the female cone is inclined to the normal to the generatrix by the angle of friction $\varphi$ (female cone/wire).

The force $F$ exerted by the cable is in equilibrium with the components of the reactions $R_1$ parallel to the axis. Therefore, if $N$ is the number of wires:

$$NR_1 \sin (\alpha + \varphi) = F$$
Also:
\[ R = R_1 \cos (\alpha + \varphi) \]

Therefore:
\[ R_1 = \frac{F}{N \sin (\alpha + \varphi)} \quad \text{and} \quad R = \frac{F}{N \tan (\alpha + \varphi)} \]

The value of the sum of the reactions \( R \) is:
\[ NR = \frac{F}{\tan (\alpha + \varphi)} \approx \frac{F}{\alpha + \varphi} \]

This total force is distributed over the surface of the male cone. If \( r \) is the mean radius of the cone and \( l \) its useful length, then the surface area is \( 2\pi rl \). Therefore, the bearing pressure on the male cone is:
\[ p = \frac{F}{(\alpha + \varphi) \times 2\pi rl} \]

The diametral compressive stress in the cone is equal to \( p \).

The bearing pressure \( p \) is considerable. Usual orders of magnitude are \( \alpha = 0.125, \ \varphi = 0.15 \). For a 7 mm diameter 12-wire cable, of 461 mm\(^2\) cross-sectional area, the initial tensile stress can be as high as 135 kg/mm\(^2\) and \( F = 461 \times 135 = 62000 \) kg.

The total bearing force is:
\[ \frac{62000}{0.125 + 0.15} = 225000 \text{ kg} \]

The male cone (concrete) has a mean radius of 2.5 cm at mid-height of the effective length for a 7 mm diameter 12-wire anchorage. The effective length is 8 cm. The surface area is:
\[ 2\pi \times 2.5 \times 8 = 126 \text{ cm}^2 \]

The bearing pressure \( p \) is, therefore:
\[ p = \frac{225000}{126} = 1880 \text{ kg/cm}^2 \]

The male cone is wholly in the plastic state at this pressure. The flow within the anchor is contained by the longitudinal reinforcement and the central tube. Friction is very high and safety is assured.

Much higher bearing pressures are possible with steel cones and the length of the anchors can therefore be reduced. This is important where
very high loads are applied, because dimensions and weight increase rapidly with load.† This cannot be taken to the limit, however, because the wires are damaged if the diametral pressures exerted upon them are too great.

For a 12 mm diameter, 12-wire cable (1 360 mm$^2$) stressed to 120 kg/mm$^2$ the force to be anchored is equal to 160 tonnes. If $\alpha = 0.125$ and $\varphi = 0.125$ ($\varphi$ less than in the case of a concrete female cone), the total force distributed on the surface of the male cone is:

$$\frac{160,000}{0.25} = 640,000 \text{ kg}$$

With $r = 3$ cm and $l = 8.5$ cm,‡ the surface area is 160 cm$^2$.

The bearing pressure on the cone is:

$$p = \frac{640,000}{160} = 4,000 \text{ kg/cm}^2$$

The annular thickness between the central hole in the male cone and the base of the grooves is 1 cm. The circumferential force on the annular thickness is, therefore, for unit height:

$$pr = 4,000 \times 3 = 12,000 \text{ kg/cm}$$

The circumferential stress is:

$$\frac{12,000}{1 \times 1} = 12,000 \text{ kg/cm}^2 (120 \text{ kg/mm}^2)$$

The female cone is also subjected to high circumferential stresses. Its mean inside radius is 4.2 cm. The effective surface area is:

$$2\pi \times 4.2 \times 8.5 = 224 \text{ cm}^2$$

The bearing pressure is, therefore:

$$\frac{640,000}{224} = 2,860 \text{ kg/cm}^2$$

† Other things being equal, the weight of an anchorage increases with the anchor load to the power 3/2.

‡ These dimensions apply to a type with reduced height. The dimensions of the standard type are slightly greater.
Let $S$ be the transverse cross-sectional area of the female cone. The stress is equal to:

$$\frac{2prl}{S} = \frac{640,000}{2\pi rl} \times \frac{2rl}{S} = \frac{204,000}{S}$$

With $S = 3350 \text{ cm}^2$, the stress is 61 kg/mm.

Stresses of this magnitude require special steels, treated (female cone) or treated and case hardened (male cone).

With regard to the bearing pressures on the wires, the total wedging force of 640 tonnes is shared between the 12 wires, giving 53.5 tonnes per wire, acting along a length of wire equal to 8 cm. The force per centimetre of generatrix is therefore 6.6 tonnes, and the diametral pressure is $(6600/1.2 \times 1) = 5500 \text{ kg/cm}^2$.

When strands are used the contact no longer occurs along the generatrix of a single wire, because of the helical winding of the wires. The discontinuity reduces the contact length and the bearing pressures are increased.

![Diagram](image)

**Fig. 6.**

*Safety with regard to slip of the separate wires*

Should there be a tendency for a wire to slip in the anchor, the cone reaction is immediately inclined by an angle to the normal.

The component of the male cone reaction parallel to the wire becomes:

$$R \cos \alpha \tan \mu = R_1 \cos (\alpha + \varphi) \cos \alpha \tan \mu$$
The component of the female cone reaction parallel to the wire is $R_1 \sin \varphi$. The total reaction resisting slip is:

$$R_1 \left[ \varphi + \cos (\alpha + \varphi) \cos \alpha \tan \mu \right] = R_1 (\varphi + \mu)$$

Since $R_1 = [F/N(\alpha + \varphi)]$, the reaction is equal to:

$$\frac{F}{N} \times \frac{\varphi + \mu}{\alpha + \varphi}$$

The force to be anchored being $F/N$ per wire, the safety factor against slip is $(\varphi + \mu)/(\alpha + \varphi)$. It is greater for higher values of $\mu$, a condition previously recognised, and for lower values of $\varphi$. The female cones must therefore be smooth and they are sometimes greased. The amount by which $\varphi$ can be reduced is limited, however, by the value of the stresses on the male cones and by the lateral compression in the wires.

6. Dead-end anchors

In certain cases the cable is tensioned from one end only; the anchor at the other end then works as a ‘dead’ anchor, or ‘self’-anchor. The male cone of the dead anchor is hammered home before tension is applied at the other end, in order to initiate ‘self-anchoring’. If the radial pressure exerted by this driving-in is sufficient, the male cone/wire assembly is pulled inwards when the tension is applied at the other end, and anchorage is effected in the manner previously described.

Dead-end anchors can also be provided by means of a round bar 40 mm in diameter, over which the cable wires are looped. The cable in this case is made up of six wires (doubled) throughout the whole of its length, each with a hairpin bend at one end, so that there are twelve free wires at the active end for attaching to the jack. At the dead end, the round bar bears on the concrete through a thick steel plate, which incorporates an opening through which the wires pass (Fig. 7).

The employment of dead anchors is useful where space for accommodating jacks is restricted. Dead anchors cannot be used when the cables are very long because the friction between the cable and the duct, acting over its whole length, reduces the effectiveness of the cable towards the ‘dead’ end. (Friction losses over only half of the cable length need be considered when the cable is tensioned from both ends (see Chapter IV).

Anchors of the type shown in Fig. 7 also have the disadvantage that the wires must be bent to a tight radius to provide the loops. This can cause the wire to wrinkle, which increases the possibility of corrosion under tension. The loops must be very efficiently protected, therefore, and
grouting must be carried out very carefully so that it can, if need be, also provide an anchorage through bonding.

7. Sealing—Grouting
The wires project beyond the cones after tensioning. They are cropped and folded back, and sealed with rich mortar (cement content 600 kg/m³). The mortar must not contain any ingredient which can have an adverse effect on the steel.

![Diagram of a 'Dead' anchor by looping over a bar of circular cross-section.](image)

Fig. 7. 'Dead' anchor by looping over a bar of circular cross-section.

A grouting hole is left in the seal. The seal prevents water seeping into the anchor and it must be made with care, especially on exposed horizontal surfaces (bridges).

Grouting
Grouting is not a minor operation, as is often thought. Its quality is of prime importance with regard to corrosion protection and bonding.

Grout† must
- be sufficiently fluid at the time of pumping;
- be of good consistency, without excessive water, and it must shrink very little on setting;

† See the recommendations of the FIP-RILEM Committee on Grouting (report by Lyse at the Fourth Prestressing Congress in Rome, 1962).
have good mechanical strength, necessary for bonding, after setting and hardening;
be resistant to frost;
not contain any ingredient which could corrode the steel.

These conditions are achieved in practice by:

(i) pumping under pressure;
(ii) the composition of the mix—this depends on the length of sheath to be grouted, its profile, its material and the amount of steel within the sheath;
(iii) the addition of small quantities of expanding agents;
(iv) reducing the water–cement ratio.

Where the cable duct is long and when the steel is congested, the mix must not contain any sand (water + Portland cement + plasticiser). Where the ducts are shorter, with smooth curves, and when more space is available, very fine and pure silica sand is added.

There are many types of expanding agents and they usually contain some quantity of aluminium powder. They must be used with caution. Their purpose is not to give excessive swelling, but to compensate for shrinkage.

The water content must be low enough to obtain the necessary strength. It must be even lower when frost resistance is necessary.

It is difficult in normal conditions to make the water–cement ratio less than 0.40 for mixes without sand and less than 0.45 for mixes with sand.

Expanding agents increase frost resistance. Quick-setting additives containing calcium chloride are to be avoided; also, generally, any additive which has not been proved in practice and any cement which contains corrosive agents.

The strength should be as high as 300 kg/cm² on 28-day, 7-cm cubes (the cubes being maintained in an atmosphere with a relative humidity of 70%), or as high as 240 kg/cm² on cylinders 7 cm diameter by 14 cm high.

Two mixes which are used in France are quoted as an example:

mixture without sand: 12 kg Portland cement
5.5 litres water
0.360 kg additive

mixture with sand: 12 kg Portland cement
3 kg Fontainebleau sand
6 litres water
0.360 kg additive

(the water contents should be a little less for colder countries).
The locations of grouting holes and air vents must be carefully selected. Vents are required at the highest points and points of grout injection at the lowest points, when the sheaths are long.

The sheaths must be flushed out with water before injection and purged with compressed air before grouting commences. Grout pumping must be stopped only when a full flow of grout is discharged through the air vents. Grouting must take place as soon as possible after the cables have been tensioned (within about 1 month).

**B. PRINCIPAL PRESTRESSING SYSTEMS**

There are many prestressing systems, which combine a limited number of principles and types of equipment.

They differ in:

(i) The arrangement of the cables, which can be:
    circular, with only one layer of wires—*example*: FREYSSINET 12-wire system;
    circular, with several layers of wires—*example*: BBRV, FREYSSINET;
    made up from successive horizontal layers—*example*: MAGNEL, PHILIPPE HOLTZMANN.

(ii) The tensioning system:
    tensioning applied directly to the cables themselves;
    tensioning by means of a movable head which can be pushed or pulled.

(iii) The system of anchorage.

(iv) The transfer system, which transfers the forces exerted by the jacks to the permanent structures, enabling the jacks to be unloaded and removed.

The last two points are the most characteristic of a particular system. When the wires are pulled directly by the jacks, anchorage is usually obtained through friction, and the transfer and anchorage are realised simultaneously.

When use is made of a tensioning head, anchorage is obtained by connecting the wires to the head in various ways, and the transfer is achieved by fixing the head in the final position with wedges or other devices. The diagrams in Figs. 8 and 9 show some anchorage and transfer techniques. It is not possible within the scope of this book to deal in
detail with all the various systems. An outline description of a few of them is given below; a more complete description can be found in the author's book *Prestressed Concrete*.

8. Anchorage systems (Fig. 8)
With anchors of the first type (wires tensioned directly by the jacks, and anchoring by friction), friction is provided through the wedging action between the male and the female anchor components. The tension in the wires provides the energy which is required for establishing the pressure, by means of the taper between the male and female anchor cones, as seen in the case of the Freyssinet anchor.
Figure 8 shows the following:

1. Male cone or key with outward action; that is, displacing the wires towards the outside, against the surface of the female cone—FREYSSINET and others (BARREDO, MAGNEL).

2. Key with inward action, displacing the wires towards the inside, against a central mandril (2a), or squeezing them together (2b).

3. Key located circumferentially between the wires, so as to increase the crown diameter of the wires until the crown (or the keys) makes contact with the female anchor.

4. Anchorage of single wires (Gifford Udall, Freyssinet).

5. Anchorage by means of spacers placed between horizontal layers of wires, with a wedge forcing the assembly against a steel anchor box (see Fig. 8, Chapter V) (Philippe Holtzmann).

For head anchors:

6. Wires attached to the head by bonding.

7. Wires attached by looping (Leoba).

8. Attachment by upset nail heads (BBRV, Boussiron).

9. Attachment by crimping (Rheinhausen, GTM).

10. Attachment through friction (10b, VSL).

11. Bundles of large numbers of wires passing over heads made of concrete, which are jacked out (Leonhardt).

12. Bonded attachment of a large number of wires to a head made of concrete, moved by means of jacks (Leonhardt).

9. Transfer systems (Fig. 9)

For anchors of the first type (wires pulled directly by jacks, simultaneous transfer and anchorage operations), the following systems are shown in Fig. 9:

1. Buried cone, or external cone (Freyssinet and similar).

2. Female cone set in a steel plate.

3. Sandwich plate (MAGNEL).

4. Bearing against circumferential keys (of Fig. 8, Chapter III).

5. Wedge anchor (for system shown in Fig. 8, Chapter V).

For head anchorages:

6, 7, 8. Fixing by mortar behind the wire-carrying head (system 8 is a dead-end anchor).
9. Fixing through a threaded bar.
10, 11. Fixing the movable head (Leonhardt) either by means of wedges (or concreting), or by grouting in the jacks.

All transfer systems give rise to an elastic deformation of the anchorage system, whether it be in the anchor itself or in the separate system used for keying and fixing; this deformation is accompanied by an inward movement of the cable with a resulting loss of tension of varying magnitude, depending on the degree of the inward movement and on the length of the member which is being prestressed.

10. Use of strands
Strands can be used in place of wires in certain of the systems described above, enabling much higher forces to be applied. In the case of a 7-wire strand, where the diameter of the king wire is a little greater than the diameter of the outside wires, as described in Chapter II, anchoring of the strand is carried out in the same manner as the anchoring of a wire. Difficulties can arise, however, because of the possibility of indentations on the king wire which are caused by the outer wires; failure of the anchor can result if these occur over too short a length. Failure can also occur through the unravelling of the strand when it is tensioned. The
anchors must be sufficiently long and of reasonable slope to avoid cramping with excessive diametral pressures.

Difficulties with gripping the king wire or the inside layers of wire can be experienced in the case of strands made up from several peripheral layers of wire (19, 37, 61 wires). It can become necessary in these cases to provide a base at the ends of the cable and to use it as a head.

The difficulties of anchoring the king wire or the intermediate layers of wire are reduced with die-formed strands (Chapter II, Section 16).

11. Dead-end anchors. Loops and embedded spirals

The wires can be looped at their exit from the sheath (as seen in Section 6, the cable is then comprised of wires with return bends). The loops are spaced apart a distance \( a \), equal to about \( 7d \), where \( d \) is the wire diameter. Alternatively, the loops can be spaced out along the axis of the cable, a distance \( b \) apart, where \( b \) is about \( 15d \). Anchors using spirals can also be provided.

Dead-end anchors should occupy the minimum amount of space, but this is limited by the risk of cutting through the concrete if the radius of curvature \( r \) is too small, because of an excessive contact pressure

\[
\omega = \frac{\pi (d^2/4)T}{rd} = \frac{\pi d}{4r} T
\]

Permissible pressures \( \omega \) are nevertheless higher than the general allowable compressive stress \( R \) in the member.

If \( a \) is the distance between the extremities of the loops (Fig. 10a), and if \( e \) is the distance of the wire from the nearest free face, the maximum permissible pressure is given by the smaller of the following quantities (BA 45):

\[
\omega = R \left[ 1 + \left( 3 - \frac{2d}{a} \right) \left( 1 - \frac{d}{a} \right) \right]
\]

or

\[
\omega = R \left[ 1 + \left( 3 - \frac{d}{e} \right) \left( 1 - \frac{d}{2e} \right) \right]
\]

The values \( d/a \) and \( d/e \) are negligible by comparison with the full depth of the concrete, so that in this case the pressure can be as much as four times as high as the general level of permissible stress \( R \). The permissible pressure decreases rapidly with tighter loops and with their proximity to the free concrete face.
Fig. 10: Dead-end anchors with loops. FH = hairpin bending. FE = helical bending. Sketch (a) on the left: theoretical logarithmic spirals for various values of \( \varphi \) (see text); full line for \( \varphi = 0.6 \); dotted line for \( \varphi = 0.4 \). Sketch (b): approximations to spiral with arcs of circles. \( \rho_0 \) is the radius of curvature at the origin; 1, 2, 3, 4 are successive centres (45° arcs); Sketch (c): three centre hook (1, 2, 3).
The reduction in the permissible pressure can be offset by providing suitable helical reinforcement in the concrete, to prevent bursting of the concrete inside the loop adjacent to the wire. In this case, code BA 45 allows the thickness of $a$ and $e$ to be increased by an amount which is equal to the cross-sectional area of the reinforcement which is present over a 1-m length of wire, in cm$^2$.

Where the loops exert a radial thrust in the unloaded state, radial reinforcement is needed around the loops (mesh or helices) in order to balance the tensile forces (Figs. 10a and 10b).

In order to reduce the contact pressure $\omega$, it is advantageous to make the wire straight or only slightly curved at its exit from the sheath where the dead-end anchor starts. The wire is maintained straight for a length $l$ equal to about $30d$, so that the stress in the wire is reduced by bonding. If $R'_d$ is the bonding stress, the reduction in tensile stress is equal to

$$\frac{\pi dlR'_d}{\pi (d^2/4)} = 4\frac{l}{d} R'_d$$

When $l = 30d$ and $R'_d = 30$ kg/cm$^2$ this is $4 \times 30 \times 30 = 3600$ kg/cm$^2$, or 36 kg/mm$^2$.

The stress at the point of exit from the sheath is of the order of 100 kg/mm$^2$, the cable being tensioned from the opposite end. The tensile stress at the origin of the loops is thus reduced to between 60 and 70 kg/mm$^2$. For a concrete in which $R = 120$ kg/cm$^2$, and for which $\omega = 400$ kg/cm$^2$ is permissible, the radius $r$ at the origin of the loop can be reduced to about $14d[r \geq (\pi/4)(T/\omega)d]$.

The length $l$ is reduced, with some systems, by forming a wave into the wire before the beginning of the curve. This is equivalent to increasing $R'_d$.

Test results given in _Prestressed Concrete_, Volume 1, show that it is possible to obtain anchors which are even more compact than those obtained by following the rules given above. A few of the principles involved are mentioned below.

The radius of curvature $r$ must decrease progressively from the point of origin of the loop (or spiral) in order to maintain the pressure $\omega$ within reasonable limits, without the need for excessive strength or volume of concrete.

If $f$ is the coefficient of friction of steel on concrete, $q = \pi dR'_d$ the bonding force per unit length, $T_o$ the tension at the origin of the loop,
\( \alpha \) the slope with respect to the origin, \( s \) the distance from the origin measured along the curve and \( T \) the tension in the wire at any point, then:

\[
T = T_o \exp (-f \alpha - \frac{qs}{\pi(d^2/4)}) = T_o \exp (-f \alpha - 4 \frac{s}{d} R_d')
\]

The design condition is, then, that \( \omega = (\pi/4)(d/r)T \) at each point must be less than the maximum allowable value; that is, \((\pi/4)(d/r)T < \omega_{\text{permissible}}\).

It can be shown that a logarithmic spiral profile, with a polar equation relative to a pole \( P \) of the form \( \rho = \rho_o \exp (-f'\alpha) \), where \( f' = f + \pi R_d'/\omega \), gives rise to a pressure \( \omega \) which is uniform along the length of the spiral. \( f, R_d' \) and \( \omega \) depend on the quality of the concrete, on the reinforcement and on the type of wire. If \( \omega = 4R \) and \( R_d' = R/5 \), the term \( \pi(R_d'/\omega) \) can be as high as 0·16; generally, the value of \( f \) is between 0·4 and 0·5.

Figure 10c shows the principles for plotting such spirals.

Once the radius vector has rotated from 180° to 270°, the spiral can be continued by a circle of constant radius, or by a tangent, and the finished shape is a hook.

The spirals can be manufactured with the use of variable radius wrenches. (The elasticity of the wire must be taken into consideration since the spring-back increases the radius when the wrench is disengaged.)

The theoretical spiral shape can be approximated using arcs of circles with successive centres and of diminishing radii (Fig. 10c).

Many combinations of spirals have been perfected so as to achieve compactness in dead-end anchor design; these include hooks of variable curvature. Figure 10d shows an example. Sound concrete and grease-free oxidised wires are essential.

### 12. Use of bars

Some prestressing systems make use of bars instead of wires or strands. The bars are usually of high strength steel, but with a rupture strength which is lower than that of wires, of the order of 90 to 115 kg/mm\(^2\) compared to 140 to 180 kg/mm\(^2\) (Chapter II, Section 14).

The bar is threaded at the end and can be tensioned by jacking against a nut screwed onto the end of the rod. The load is transferred through a second nut which bears on a steel plate, which in turn bears on the concrete. This second nut is tightened while the bar is still tensioned by the jack, thereby resisting the elastic strain in the rod.

The thread at the end of the bar is a disadvantage since it reduces its strength. There are three ways of overcoming this (Fig. 12):
(i) thread of variable depth starting from zero near the bearing plate and finishing at the normal depth at the outside end of the rod (Lee-McCall);

(ii) cold rolled thread with a 50% depth reduction into the bar, half of the depth being in the bar, therefore, and the other half being taken up in the thickening due to cold rolling (Dywidag);

(iii) extremity of the bar rolled locally to a larger diameter, so that threading is possible without reducing the effective strength of the bar (Russian method).

There is a tendency to replace these systems with systems which rely on friction for locking the bars (keys or split-cones). These two systems are similar to those shown in Fig. 8, Chapter IV.

Strands (monostrands) can be used instead of bars. Anchoring is effected through friction in a female cone.

There is little difficulty with 7-wire strands, providing of course that, as in the case of multiple systems, the diameter of the core wire is a little greater than the diameter of the circumferential wires, and that the anchors
are of reasonable length. Difficulties can be experienced with 19-wire strands (two circumferential layers and the core) as previously mentioned. Satisfactory results are also obtained with drawn strands.

13. External cables
The use of internal cables imposes certain restrictions: increases in web thickness, the need for threading (in the case of precast sections), friction in the sheaths, and so on. On the other hand, the risk of corrosion is minimised.

In some structures cables have been located outside the members. In these cases, very efficient corrosion protection must be provided.

One solution is to sheath the cables in ducts which are made sufficiently strong to withstand pressure grouting, in the same way as for internal cables.

In certain instances, outside cables may be the only possible solution. This is the case, for example, with bridges having a large span, made up of central beams which are supported from high piers, in which cables form the top flange and the roadway forms the lower flange (example: Maracaibo bridge). The safety of these structures depends entirely upon the cables and their protection must be completely assured. Grouting appears to be the most satisfactory solution. In other cases, nuclear power stations for instance, outside cables are often specified in order to permit easy inspection during the life of the station. In these cases, pitch can be applied for protection. It is not as good a means of protection as grout, and, in any case, the requirement for outside cables in the author’s opinion does not seem justified.

14. Evolution of prestressing systems
In the early days of prestressed concrete, FREYSSINET was anchoring forces of the order of several hundred tonnes by means of anchors similar to those shown on sketches 6 to 12 of Fig. 8; the anchors were cast in-situ.

Then followed the ‘industrial’ period, when it was generally considered that it was more beneficial to shop-fabricate and mass-produce standard-type anchors. Requirements were at that time relatively modest, and 20 to 60 tonne tendons and anchors were adequate.

The demands made on prestressing have progressively increased, and the concentration of the prestressing steel was gradually increased to meet these demands within the restricted spaces which were available, and 200 to 300 tonnes prestressing units were then used.

Still more powerful units (500 to 1000 tonnes) are now required for nuclear applications.
In parallel, the scope of prestressing was being extended to include lesser structures, floor beams for example, which had until then been made of reinforced concrete, and this required the development of much smaller tensioning units.

The present-day importance of prestressing in engineering constructions is well illustrated by this two-fold development of its applications.
Chapter IV

CALCULATION OF CABLE LOSSES

I. FRICTION

1. Losses in curved and rectilinear ducts
It is assumed that the cable itself has no resistance to bending. When the cable profile is curved, the tension in the cable is not uniform because of the friction between the cable and the sheath. The effect is similar to that between a belt and its driving pulley. If \( f \) is the coefficient of friction, and if \( \theta \) is the angle between the tangent at any point \( M \) on the curve and the tangent at the origin \( A \), the tension at \( M \) is equal to \( T = T_0 \exp(-f\theta) \), \( T_0 \) being the tension at \( A \).

\[
F_0 = A_c T_0 \\
F = A_c T = F_0 e^{-f\alpha} = A_c T_0 e^{-f\alpha}
\]

**Fig. 1.**

\( \theta \) is called the angular deviation, or more simply the deviation, from the origin.

At the end of the curve at \( B \), the deviation is \( \alpha \) and the tension is equal to \( T_0 \exp(-f\alpha) \).

\( f \) is a dimensionless coefficient and \( \alpha \) is measured in radians.
Values of $f$ vary quite widely, depending upon the care which is taken during construction; the condition, nature and stiffness of the sheaths; the type of joints; and the condition of the wire (oxidised or otherwise).

In order to minimise friction, it is essential to maintain uniform curvature of the cables during construction. The best way of achieving this is to secure the cable at both extremities and to let it hang naturally. This is possible when the curve lies in a vertical plane and when it is continuous, as, for example, the cables in a simply supported beam. Other points on the curve should not be dimensionally referenced, as this could give rise to cumulative design and construction errors, resulting in undulations or waviness, leading to increased friction losses.

On a well-conducted site, it should be possible to achieve values of $f$ less than 0·25 for parallel-wire cables in ordinary cylindrical sheaths.

Soluble oils which are subsequently displaced by grouting can be used to reduce friction losses. Coefficients between 0·22 and 0·18 have been achieved in this manner in the approach spans of the Tancarville bridge.

Coefficients of friction are slightly lower when strands are used.

In some systems, with the wires arranged in layers, paraffin-coated sheets are placed between the layers.

Leonhardt, using bundles of the type described in Chapter III, claims to have obtained coefficients of friction as low as 0·03.

Coefficients approximately equal to 0·06 have been obtained in STUP experiments with paraffin-coated strands.

Such low values of the coefficients are economically attractive.

The means adopted to reduce the friction losses should not of course impair the bond between the cable and its sheath after grouting, otherwise a reduction in strength would result, leading to an increase instead of a decrease in the number of cables required.

In most cases, the value of the friction coefficient could be of secondary importance, within limits, if it were known exactly and if it were constant, since friction losses can be compensated by applying additional tension at the anchorage.\* The variation in the value of the coefficient can be particularly troublesome, because it introduces an unknown which could require an additional margin. The variation in the coefficient depends largely on the care which is taken during construction. On well-run sites, the variation should not exceed 5%. The errors are then insignificant when a sufficient number of cables is provided.

\* It is assumed that codes do not limit the excess stresses; they decrease rapidly because of cone movements and relaxation.
Losses also occur in linear (straight) cable because of unavoidable undulations in the cable during its installation and setting, and because of the supplementary lateral friction, considered in Section 7. The quality of construction is again of great importance. With careful setting, the linear loss should not exceed 0·1 to 0·15 kg/mm² per metre.

The cost of large-span beams with linear cables could become excessive if the friction losses were too high.

2. Practical formulae for the evaluation of losses and extensions
(a) In order to evaluate the friction losses in a beam at a section distant \(x\) from the anchorage, the sum of the absolute values of the deviations must be determined; if \(\alpha\) is the total estimated deviation, \(f\) is the coefficient of friction and \(\lambda\) the loss per unit length of straight cable, the friction loss in the curve, or curves, is \(T_0[1 - \exp(-fx)]\), and the loss over the length \(x\) is \(\lambda x\).

The total loss is therefore: 
\[
\Delta T = T_0[1 - \exp(-fx)] + \lambda x. \quad \alpha \text{ is measured in radians; } x \text{ is hereafter expressed in metres.}
\]

Since \(fx\) is generally small, \(\exp(-fx) \approx 1 - fx\). Consequently, the loss is approximately equal to
\[
\Delta T = T_0fx + \lambda x \tag{1}
\]

STUP use the following formula, where \(\alpha\) is in degrees and \(x\) in metres:
\[
\Delta T = 0·4(x + \alpha) \% \tag{2}
\]

This is equivalent to taking, for the curvature losses \(\Delta T_1\):
\[
\Delta T_1/T_0 = 0·004\alpha.
\]
Since the factor for expressing \(\alpha\) in degrees is 57 times the number expressing the same angle in radians:
\[
\frac{\Delta T_1}{T_0} = 0·004 \times 57\alpha_{\text{radians}}
\]

or:
\[
\Delta T = T_0 \times 0·23\alpha_{\text{radians}}
\]

Formula (2) therefore assumes that \(f = 0·23\). This assumption is justifiable in the case of parallel-wire cables. The coefficient should be slightly lower with strands.

On the other hand, a loss per unit length of 0·4\% per metre is too high in the case of an efficient, well-run site. With \(T_0 = 125\) kg/mm², it implies a loss of \(0·4 \times 125/100 = 0·05\) kg/mm².
It should be possible to reduce this loss to a half or a third of this value, and the following formula could be used:

\[ \Delta T_{friction} = \frac{T_0}{100} \times 0.4 \left( \frac{x_{metres}}{3} + \alpha^2 \right) \]  

(2b)

Coefficients of friction can be reduced with the use of soluble oils, and the formula then reads:

\[ \Delta T_{friction} = \frac{T_0}{100} \times 0.3 \left( \frac{x_{metres}}{2} + \alpha^2 \right) \]  

(2c)

Formula (2b) is the one which is used in the following text. It is applicable to all prestressing systems where non-oiled, parallel wires are used. The anchor system does not affect the magnitude, which depends only on the contact between the sheaths and the wires.

Formula (2b) assumes careful setting of the cables; if for some reason or other this cannot be guaranteed, \( \frac{x}{3} \) could be replaced by \( \frac{x}{2} \) in the formula.

(b) Since the friction forces are tangential reactions on the sheath in the opposite direction to the relative displacement, there exists a neutral point on the cable, if it is tensioned from both ends, where there is zero displacement. The position of the neutral point can be easily determined by equating the stress values at the neutral point when calculated from either end of the cable.

Let \( x_n \) and \( x'_n \) be the distances of the neutral point N from the two anchors \( A \) and \( A' \); let \( T_0 \) and \( T'_0 \) be the tensions at the anchors, \( \alpha \) and \( \alpha' \) the cable deviations between \( A \) and \( N \), and \( A' \) and \( N \), respectively, in degrees.

Assuming formula (2b) to apply, the following relation defines \( x_n \) and \( x'_n \):

\[ T_0 \left( \frac{x_n}{3} \times \alpha \right) = T'_0 \left( \frac{x'_n}{3} + \alpha' \right) \]

If \( T_0 = T'_0 \), \( x_n + 3\alpha = x'_n + 3\alpha' \).

Since \( x'_n = l - x_n \), where \( l \) is the length between anchors,

\[ x_n = \frac{l}{2} + \frac{3}{2}(\alpha' - \alpha) \]

(c) Extensions. The extensions required at transfer should be specified because the measurement of the extension is a better indication that the
required force has been applied than the reading on the pressure gauge. The strain is measured between \( x_n \) and 0. At the anchor, the tension is \( T_0 \). This tension is specified and calculated by taking into consideration the losses which occur, in order to obtain a given permanent tension within the span. At \( x_n \), it is equal to \( T_0 - 0.4 \frac{T_0}{100}((x_n/3) + \alpha^2) \).

The extension can only be calculated if the steel stress–strain diagram is available, since cables are usually tensioned beyond the limit of proportionality.

Theoretically, the strain can be calculated by numerical integration. Having drawn the graph of tensions along the cable, made up of straight lines if the simplified formulae quoted in the foregoing are assumed, the length between the anchor and the neutral point is subdivided into lengths \( \Delta x \). The mean tension \( T \) in each subdivision is equal to the ordinate at the centre of each subdivision. The strain corresponding to this tension \( T \) is read from the stress–strain diagram. The displacement at the anchor is:

\[
u = \varepsilon_1 \Delta x_1 + \varepsilon_2 \Delta x_2 + \cdots + \varepsilon_n \Delta x_n
\]

In practice, approximate formulae are sufficient.

One formula is derived by assuming that the strain is the same as that obtained by subjecting the cable to a mean tension \( T_m \) equal to \((2/3)T_n + T_0/3\); then \( u = \varepsilon_m x_n \), where \( \varepsilon_m \) is the strain at tension \( T_m \), obtained from the stress–strain diagram.

The following formula can also be used:

total strain = strain due to minimum tension \( T_n + \frac{1}{2} \) of the strain due to the difference \( T_0 - T_n \) or:

\[
u = x_n[\varepsilon_n + \frac{1}{2}(\varepsilon_0 - \varepsilon_n)]
\]

If the neutral point is at mid-span, then:

\[
x_n = \frac{l}{2} \quad \text{and} \quad T_n = T_0 - 0.4 \frac{T_0}{100} \left(\alpha^2 + \frac{1}{3} \cdot \frac{l}{2}\right)
\]

These formulae can be applied to cables which are straight in the central portion of the beam, terminating in a single curve local to the anchors. The formulae assume that the variation of tension in relation to the abscissa, measured from mid-span, can be approximated to a parabola, as opposed to two straight lines as in Fig. 2. Practical experience shows that in this particular case the approximation is sufficiently accurate.
3. Determination of coefficients $f$ and $\lambda$ in Formula (1)
These coefficients, assumed equal to 0.23 for $f$ and 0.13 kg/mm$^2$ per metre for $\lambda$,† are not known exactly during the development of a project design. They can be determined at the start of construction, however, and it is necessary to do so in order not to penalise a well-run, efficient site, or encourage an inefficient one. Two jacks are positioned at each end of a cable. Only one end of the cable is jacked; the jack at the other end is previously pumped and the cable slightly tensioned, and the feed and drain valves closed against a slight pressure. Tensioning by the active jack increases the pressure inside the passive jack. The total loss, which is the difference between the pressure in the active jack and the increase of pressure in the passive jack, includes the friction losses in addition to the internal jack losses and the anchor losses. These latter losses (jacks and anchors) are of the order of 5 to 8%; they are assumed to be known.

† More exactly 0.13 × (T$_0$/100) kg/mm$^2$ per metre (giving 0.13 kg/mm$^2$ when T$_0$ = 100 kg/mm$^2$).
They can be determined, if required, by means of a previous test on the jacks. The jacks are attached to a cable in pairs, and the cable is arranged to slide freely within a metal trough. One of the jacks is attached to a spring balance and the other is subjected to a hydraulic pressure \( P_1 \). From the differences between \( P_1 \) (active jack) and \( P_2 \) (passive jack), the loss \( (P_1 - P_2)/2 \) for each jack is determined.

If, for example, the loss in each jack and associated anchor is 7%, and if, during a friction-loss test, the pressures in the active and passive jacks are respectively \( P_1 \) and \( P_2 \), the corrected pressures to use are 0.93\( P_1 \) and 1.07\( P_2 \) respectively. Then: 

\[
0.93P_1/1.07P_2 = \exp(-\alpha x) + \lambda l.
\]

The losses for different cables having different values for \( \alpha \) and \( l \) can be determined experimentally. A certain number of equations is obtained, enabling \( f \) and \( \lambda \) to be determined by the method of least squares.†

Tests have also been done by arranging openings, in beams, which allow measurements of the displacements and strains in the steel to be made in different positions, especially near the anchors and at mid-span.

Some of the results obtained during such tests are summarised in Section 5.

4. Site checks to demonstrate that specifications relating to friction are met

For each tensioning operation, it is necessary to check that the specified extensions and pressures are met. In practice, it is not possible to avoid some discrepancy. In principle, the extension specifications must first be satisfied, since the extension is a means of measuring the stresses which the pressure gauge merely controls. This is not an absolute rule, in particular when the cable curvature is large; in this case the extension may have little significance (see Section II), and therefore the pressure gauges must be frequently calibrated.

In the case of straight beams, the principal measurement is that of the extension. One minor difficulty has to be overcome: at the start of tensioning, the cable ‘settles’ (taking up of the slack, radial displacement towards the centre of curvature, etc.) without any increase, or only a slight increase, in the pressure. Also, some parasitic friction has to be overcome. These effects change the relation between the pressure and the strain at the start of operations.

The following procedure is usually adopted: the gauge pressure is increased in stages to 100–200–300 kg/cm\(^2\) ... The corresponding displacement is noted and the strain–pressure diagram is plotted. This

† Method due to J. R. ROBINSON.
enables the zero elongation to be determined (Fig. 3). As explained in Section 7, this method can lead to errors, since the diagram is not necessarily linear at the start. Therefore the first readings must be taken with low values of pressure.

A rule which is sometimes followed, which does not give predominance to either of the two parameters (strain or pressure), is the rule of 5%. This was indicated by Dreux at the Berlin Congress (Session II, discussion). It states that the first of the two values to reach its specified value can be increased by 5% in order to enable the second of the values to reach its own specified value. Should it be necessary to exceed the 5% increment, the engineer in charge should be notified. This simple rule has the advantage that it brings to light anomalies caused by bad workmanship, such as those mentioned in Section 7.

5. Values of design coefficients
Values obtained from various sources show a considerable scatter.

At one extreme the coefficients have values which are extremely high (bordering on the ridiculous), because they result from badly-conducted tests on badly-engineered specimens. They can only be regarded as such.

At the other extreme, optimistic assertions are made, often without foundation.

It should be possible to find more realistic values in the codes adopted by various countries. Here again, abnormal scatter is encountered because the values given depend upon the bias given to them by the users of the
various systems, according to whether or not they need to be cleared from the risk of exceeding specified values at site.

Indeed, different values of the coefficients are found in the codes for identical cables, contained in identical sheaths. Since the anchorage methods do not affect the problem, these differences are irrational.

The most reliable values with which to work are those obtained from well-run and reproducible tests, conducted on a purely technical basis.

Results of tests on active and passive jacks (Section 3) are given below. Their accuracy has been checked.† Mean values are given in the first table, obtained from five sites where parallel-wire cables were used (7 tests), and from five sites where strands were used (22 tests). The scatter is approximately ±20% from the mean tabulated values.

<table>
<thead>
<tr>
<th></th>
<th>Parallel wires</th>
<th>Strands</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f$ for curve</td>
<td>$\lambda$ per m</td>
</tr>
<tr>
<td>Site 1</td>
<td>0.218</td>
<td>0.28</td>
</tr>
<tr>
<td>2</td>
<td>0.236</td>
<td>0.30</td>
</tr>
<tr>
<td>3</td>
<td>0.228</td>
<td>0.30</td>
</tr>
<tr>
<td>4</td>
<td>0.312*</td>
<td>0.40*</td>
</tr>
<tr>
<td>5</td>
<td>0.228</td>
<td>0.30</td>
</tr>
</tbody>
</table>

The coefficients are reduced by 10 to 15% if the cables are oiled.

The results given in the above table are reasonably good, and they agree with the values which should be obtained on a well-run site, with the exception of site No. 4, where the results were, in fact, not acceptable.

It is possible to achieve better results when the site is clearly aware of the precautions which are required during transport of the cables (reasonably close pitching of support points), laying of the cables (well-set supports at reasonable pitch), inspection after laying, which is essential, and fixing to the supports or stirrups in order to avoid any movement during concreting.

This carefulness is well worthwhile, more so in the case of $\lambda$ than in the case of $f$: it is in fact more difficult to check the straightness than the curvature of a cable. DREUX explains (Berlin Congress, 1958) that, thanks to a careful and thorough inspection procedure, it was possible to reduce $\lambda$ from 0.35 to 0.10 at Tancarville bridge.

† The tabulated values of $\lambda$ are expressed as percentages of the stress $T_0$; in other words, if $T_0 = 100$ kg/mm², a coefficient $\lambda = 0.1$ indicates a loss of 0.1 kg/mm² per metre.
DREUX also states that the values of \( f = 0.22 \) with non-oiled cables and \( f = 0.18 \) with oiled cables were consistently obtained (parallel wires).

CAMPENON BERNARD obtained the following values for the Saint-Jean bridge in Bordeaux; \( f = 0.20 \) and \( \lambda = 0.2 \) for parallel-wire cables, and \( f = 0.15 \) and \( \lambda = 0.15 \) for stranded cables.

All these results were obtained with the use of corrugated sheaths, similar to those shown in Fig. 2, Chapter III.

Based on the above values, it is estimated that whatever the anchor system, and with corrugated sheaths, the following range of values can be used in preliminary design:

for parallel-wire cables: \( f = 0.20 \) to \( 0.25 \), \( \lambda = 0.2 \) to \( 0.3 \);

for stranded cables: \( f = 0.16 \) to \( 0.20 \), \( \lambda = 0.15 \) to \( 0.25 \).

The values to be chosen within this range depend on the degree of control that the designer has over the site operations.

Laboratory tests on single wires are often mentioned in the literature. They consist of measuring the horizontal force which is necessary to move a wire subjected to a vertical load \( V \) on a given support (such as various types of sheaths, or sheaths treated in different ways).

These tests are of interest when making comparisons between various types of contact surfaces, but they should not be quoted, as sometimes happens, as being relevant to cables in practical applications. Indeed, with a cable, the wires are pushed towards the boundary by radial forces caused by the curvature \( (F/r) \), but only some of the wires make contact: the other wires bear against them, and the frictional behaviour can be modified by this interference (Fig. 4).
With this proviso, some coefficients which have been measured by STUP are given below (mean of 10 tests with a 5 cm length of contact).

<table>
<thead>
<tr>
<th>Type of sheath</th>
<th>Leaded</th>
<th>Zinc-coated</th>
<th>Clean, bright sheet-metal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled wire</td>
<td>0.30</td>
<td>0.21</td>
<td>0.25</td>
</tr>
<tr>
<td>Extruded wire</td>
<td>0.23</td>
<td>0.18</td>
<td>0.23</td>
</tr>
<tr>
<td>Strands</td>
<td>0.22</td>
<td>0.17</td>
<td>0.25</td>
</tr>
<tr>
<td>$V = 150 \text{ kg}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extruded wire</td>
<td>0.22</td>
<td>0.20</td>
<td>0.22</td>
</tr>
<tr>
<td>$V = 75 \text{ kg}$ (with alternate sliding)</td>
<td>0.11</td>
<td>0.09</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The above results show that of the various types tested, the best values are obtained with zinc-coated sheaths, that friction is less with extruded wires than with rolled wires, and that it is also less with strands than with wires.

Finally, it should be mentioned that friction coefficients could be reduced by a suitable treatment of the sheaths or cables.

Elskasser (Schweizerische Bauzeitung, February 1963) states that friction losses can be considerably reduced with a teflon film in the sheaths.

STUP have also tested paraffin-coated curved cables. The paraffin wax was applied hot by passing the cable through a trough before feeding it into the sheath. The following results were obtained, at a temperature of 20°C:

4 hours after coating  0.042
3 days after coating   0.065 and 0.062

As mentioned in Section 1, these results are of interest only if bonding can be restored with grouting, and, of course, if the cost of the special treatment is recovered by the savings on the steel.

6. Effect of the radius of curvature of the cable on the coefficient of friction
To the author's knowledge, there is no experimental work which enables the effects of the radius of curvature $r$ of the cable to be determined. He has himself attempted to tension cables with a similar deviation at various curvatures without observing any marked differences in the results.† Yet

† See Prestressed Concrete, Vol. 1, by the author.
there must be differences: for a symmetrical cable bent to a curve of radius \( r \) with two straight portions at the ends and with a displacement equal to \( u \) at each anchorage, the straightening of the cable from the radius \( r \) to a radius of infinite value requires a certain amount of work which is deducted from the useful work which is done.

Small cable radii are not advisable, anyway, and in general the radii are not less than 1·50 m + 700d, \( d \) being the unit diameter of the wire. If absolutely necessary, smaller radii than this can be accepted (in some cases at Esbly 1 m radii for 5 mm diameter, 12-wire cables have been used). In these cases, the work must be extremely well executed and the sheaths must be very strong (heavy tubing).

If the sheaths are too weak they crease when they are bent to the small radius; this increases the friction losses and can make tensioning impossible. This statement is not a contradiction of the above, concerning the small effect which the radius of curvature seems to have on the coefficient of friction in laboratory tests; it results from an imperfection which is purely an indirect consequence of the tight curvature.

7. Parasitic friction losses
Friction in the curved sheaths under normal conditions is a consequence of the pressure exerted by the cable (\( F/r \)) on the wall of the sheath nearest to the centre of the curve; the tangential friction reaction is then \( f(F/r) \) and the familiar formula \( F = F_0 \exp (-f\theta) \) mentioned above is deduced.

In this case, as \( F_0 \) increases during tensioning, so the force at any point in the cable increases proportionally to \( F_0 \). If, in the jack-to-jack test described in Section 3, \( F \) is the force on the passive side and \( \alpha \) is the deviation between the two ends, then: \( F = F_0 \exp (-f\alpha) \). Therefore: \( F_0 = F \exp (f\alpha) \).

If the variations of \( F \) as a function of \( F_0 \) are represented by plotting \( F \) as the abscissa and \( F_0 \) as the ordinate, a straight line graph is obtained, passing through the origin. The slope of the line is equal to \( \exp (f\alpha) \).

In some cases, however, and particularly when the cables have a small radius, as in the case of pipes of 4 to 6 m diameter, the pressure gauge in the passive jack begins to register only when \( F_0 \) has reached a threshold value \( F_{0\text{th}} \).

The graph is then a straight line. The ‘apparent’ coefficient \( f \), deduced again from the slope \( F_0/F \) (slope of line OM in Fig. 6), decreases as \( F_0 \) increases. The ‘true’ coefficient \( f \) corresponds to the slope \( (\Delta F_0/\Delta F) \) of the line \( F_0/M \).

Once \( F_0 \) has reached the maximum test value \( F_{0\text{max}} \), \( F \) has reached the
FIG. 5.

FIG. 6.
value $F_{\text{max}}$ and the pressure of the active jack is reduced for the return path, the force on the passive jack remains constant at $F_{\text{max}}$ until $F_o$ reaches a certain value $F'_o$; $F$ then decreases, but when $F_o$ is back to zero a residual force $F'_s$ remains on the passive side, of the same order of magnitude as the threshold force $F_{\text{as}}$ on the tensioning path.

This can be explained by assuming that, in addition to the tangential reaction which is proportional to the normal component $[f(F/r)]$, there exists a parasitic frictional force which is independent of pressure, and

![Diagram](image)

\[ \frac{F}{r}fds + qds \]

\[ F + dF \]

\[ Fo \]

\[ \theta \]

\[ F + q \frac{r_f}{f} = C \exp \left(-f\theta\right) \]

which is due to various imperfections (wrinkles in the sheaths, interference of cable ties with the sheaths, lateral forces due to flattening of the sheaths, and so on). Let $q$ be this parasitic frictional force per unit length.

The equilibrium equation for the forces at the start and finish of a segment $r \, d\theta$ is (Fig. 7):

\[ dF = -\left(\frac{F}{r}fds + q \, ds\right) = -(Ff + qr) \, d\theta = -f \left(F + \frac{qr}{f}\right) \, d\theta \]

Therefore:

\[ F + q \frac{r_f}{f} = C \exp \left(-f\theta\right) \]
For
\[ \theta = 0, \quad F = F_o \quad \text{hence} \quad C = F_o + q \frac{r}{f} \]

Hence the relation:
\[ F = F_o \exp (-f\theta) - q \frac{r}{f} [1 - \exp (-f\theta)] \quad (3) \]

If F and \( \theta \) are small, the following approximate equation is obtained:
\[ F = F_o (1 - f\theta) - qr\theta \]

If \( x \) is the abscissa measured along the curve \( (x = r\theta) \), then:
\[ F = F_o (1 - f\theta) - qx \quad (4) \]

At the passive jack \( (\theta = \alpha, \ x = l) \), therefore:
\[ F = F_o (1 - f\alpha) - ql \quad (5) \]

For \( F_o = 0 \), F is negative, so that the formula is not applicable in that case. F remains equal to zero up to a value of \( F_o \) such that:
\[ F_o = \frac{ql}{1 - f\alpha} \]

This is the threshold value \( F_{os} \) of Fig. 6. As \( F_o \) increases beyond this threshold, F increases linearly in relation to \( F_o \) as shown by the line \( F_{os}F_{omax} \) in Fig. 6.

On the return path the tangential forces shown in Fig. 7 are reversed. The relation between F and \( F_o \) is therefore obtained by changing the signs of f and q. Therefore:
\[ F = F_o (1 + f\alpha) + ql \quad (6) \]

However, since F cannot increase as \( F_o \) is reduced, this formula is not strictly correct. It would show, for \( F_{omax} \), a greater value of F than that obtained with formula 5. F therefore remains constant as long as \( F_o \) does not reach a value \( F'_o \) such that \( F'_o (1 + f\alpha) + ql = F_{omax} (1 - f\alpha) - ql \).

Or:
\[ F'_o = F_{omax} \frac{1 - f\alpha}{1 + f\alpha} - \frac{2ql}{1 + f\alpha} \]

and
\[ F'_o = F_{omax} (1 - 2f\alpha) - 2ql (1 - f\alpha) \]

This explains the datum \( F_{omax}F'_o \) in Fig. 6.
When $F_o$ becomes less than $F'_o$, formula 6 is applicable; and $F$ decreases linearly with respect to $F_o$. For $F_o = 0$, $F$ is not zero. The residual force is obtained by substituting $F_o = 0$ in formula 6: $F's = ql$.

Figure 8 shows results obtained on two cables during tests conducted on a 6 m diameter water main. The angle of deviation between the tangents at entry and exit was $180^\circ$.

The forces shown on Fig. 8 are nett, corrected for the jack internal losses. The cables were unusually defective because of accidental parasitic friction, caused by sheath breakages during bending; but this was almost entirely eliminated during construction. However, since large values of parasitic friction accentuate the sensitivity of the phenomenon, it is interesting to note that, on the whole, the results agree with those predicted by the previous theory. It is not strictly fair to seek a numerical check for the theory using such bad examples. The linear formulae in any case are no longer valid for such large deviations, and the irregularity of the parasitic friction losses does not allow them to be represented by a constant force per unit length. It can be noted that although the true coefficients of friction were relatively low, estimated from the slopes of the diagrams (with exponential formulae and not linear formulae, which are no longer true for such large deviations), the parasitic friction losses were considerable. The true coefficient of friction $f$ was 0.21 for cable 1 and 0.16 for cable 2. These are normal values; they are indeed even lower than usual. On the other hand, the forces $q$, which can be estimated in relation to the threshold $F_{os}$ with the use of the exponential formulae, are 0.36 tonne per metre for cable 1 and 0.33 tonne per metre for cable 2. The cross-sectional area of
the cable was 461 mm\(^2\) so that the loss is 0.8 kg/mm\(^2\) per metre for cable 1 and 0.7 kg/mm\(^2\) per metre for cable 2. These values are obviously unacceptable.

The reason for emphasising the importance of parasitic losses is that it is unavoidable that they should occur, despite all the care and precautions that can be taken during construction. The tests described above show their existence on an exaggerated scale.

There is good reason to believe that they are the major cause of the losses which occur along the length of the cable, represented by the term \(\lambda x\) of formula 1 in Section 2.

A different explanation has been proposed by Cooley (Friction in post-tensioning prestressed concrete systems—Cement and Concrete Association, October 1953). He supposes that parasitic friction is due entirely to the wobble effect. The wobbles in the sheath would cause small deviations, alternating in sign, and their sum would be equivalent to an appreciable deviation per unit length, to be added to the angular deviation.

If \(\beta\) is the parasitic deviation per unit length, then:

\[
T = T_o \exp \left[-f(\alpha + \beta x)\right] = T_o (1 - f\alpha - f\beta x)
\]

or

\[
T = T_o (1 - f\alpha - \mu x)
\]

in which \(\mu = f\beta\)

It is difficult to agree with this reasoning, since it means that the cable itself has to follow the wobble in the sheath. In fact, along the straight portions, the cable must bear only against the humps, and the pressure on the humps is virtually independent of the tension in the cable (Fig. 9).

![Fig. 9.](image)

Experience in practice indicates that parasitic friction losses are constant, and are not proportional to the tension in the cable.

If this is in fact the case, the tension at a distance \(x\) from the anchorage must stay equal to zero until the force \(F\) at the anchor reaches a threshold value of \(q\alpha/(1 - f\beta)\) (formula 4).

If the cable is tensioned from both ends, the tensions at the various points along the length \(l/2\) between the anchor and the neutral point are
out of phase with respect to the tension at the anchor, and tension is registered at the neutral point only when $F$ reaches the threshold:

$$F_s = \frac{ql/2}{1 - f\alpha}$$

From this point onwards the parasitic friction losses are overcome, and the increases in tension at every point become proportional to the increase in $F$; consequently the measured increases in the strain are also proportional to the increases in $F$. The strain diagram is therefore a straight line when $F > F_s$.

Between 0 and $F_s$, it is easily shown that the diagram is a parabola with its apex at the origin.

![Stresses vs Elongations Diagram](image)

**Fig. 10.** Displacement of the origin under the influence of parasitic friction.

As a result, when the diagram is drawn on site from readings at 100, 200, 300 kg/cm$^2$, as described above (Fig. 3), it must intersect the $F$-axis at $F_1$, above O when extended. This agrees with practice, in general, and it illustrates that the effects of parasitic friction are generally more important than the effects of taking up the slack in the cable.

If the diagram enables $F_s$ to be determined, the origin could be found; this, as explained in the case of Fig. 3, is unknown. This could be done by drawing from point $F_s$ on the diagram a line $F_sA$ with twice the slope of the line $F_sF$ (Fig. 10).
But $F_2$ is the point of contact between the linear portion of the diagram and the initial parabolic portion. The parabolic portion can be plotted by taking small increments of $F$ during the initial stages of tensioning.

As long as parasitic friction is small, the possible errors are small. If parasitic friction is high, a significant error can arise by considering as the origin of strain the intersection $B$ of the straight line diagram with the strain axis. By measuring strain from this point $B$, and having to meet the total specified strain, the value obtained for the final force $F$ is too low. This was noticed by DREUX with some of the cables in the access spans of the Tancarville bridge, and it appears that this was due to the above reason. The particular cables had probably not been laid as carefully as the others. The correction which DREUX recommended amounted to establishing the line $F_2A$, thus fixing the origin from which the strains are measured. This correction is sufficient in principle when parasitic friction losses are too high.

8. Curves and reverse curves

Because of the progress which has been made in reducing the values of the friction coefficients, it is possible, when necessary, to arrange the cables in a beam so that their paths follow several smooth bends. A safe assessment of all possible losses should be made, and the greatest care is necessary in laying and setting the cables.

![Diagram showing 20° deviations between each anchor and the neutral point (intermediate support).](image)

The angle of flexure should never exceed 30° and the limit should be set at 20° if possible. The radii should be sufficiently large, as mentioned in Section 6.

The beam shown in Fig. 11 has three 20° deviations between each anchor and the neutral point (intermediate support). Therefore, the total deviation is 60°. With $f = 0.22$ and a loss per metre of 0.13 kg/mm², the friction loss with $\alpha = 60°$ and $l = 40$ m is:

$$\Delta T = 0.4 \frac{T_o}{100} \left(60 + \frac{40}{3}\right) = \frac{29}{100} T_o$$
This represents a high loss, but not one which makes prestressing impracticable. If the cable is tensioned to 130 kg/mm² at the anchor, the friction loss is 38 kg/mm², so that the residual prestress is 92 kg/mm². Once deductions have been made for other losses, the prestress might be a little low at the mid-point, but it should nevertheless be acceptable, especially if limited prestressing is envisaged or if prestressed reinforced concrete (in which complementary reinforcement is permitted) is used.

9. Losses due to anchorage deformations at transfer
When the cable is anchored, the tension at the ends of the wires is transferred from the jack to the anchor. As a result, the anchor is deformed and this causes a shortening (termed ‘pull-in’) of the cable. This can be an embarrassment with short cables; it results in a reduction of tension local to the anchor, which can be estimated in the following manner.

Having drawn the diagram of tensile force or stress along the cable upon completion of tensioning, as explained in Fig. 2, it is necessary to estimate the loss $\Delta T_e$ at the anchor resulting from the re-entry $u$; $u$ is a known characteristic of the equipment. During the pull-in, the direction of the relative movement between the cable and the sheath, local to the

![Diagram](image)

**Fig. 12.** Losses due to re-entry of male cone.
anchor, is of opposite sense to the movement during tensioning. The stress diagram between the anchor A and a certain point N (as yet unknown) is therefore symmetrical with the diagram before transfer, with respect to the horizontal through N. The diagram is therefore NB'A' after anchorage, symmetrical with NBA about the horizontal line Na.

As explained in Chapter II, the shortening of the cable is elastic. Consequently, the strain at abscissa $\xi$ (measured from the anchor), where the loss of tension is $\Delta T$, is $\varepsilon = \Delta T/E_a$ in which $E_a$ is the elastic modulus of the steel.

The magnitude of the pull-in $u$ is therefore equal to

$$\int_A^N \varepsilon \, d\xi = \frac{1}{E_a} \int_A^N \Delta T \, d\xi$$

But $\int_A^N \Delta T \, d\xi$ is the area of the surface ABNB'A', which is twice the area ABNa. This area must be equal to $E_a u/2$.

The position of N can be found through simple calculation, resulting in a quadratic equation if the stress diagram is composed of straight lines. In the general case, it is usually expedient to proceed by trial and error.

Consider a cable, for example, curved along a length of 6 m between A and B, and then straight. The tangent to the curve at the point of anchorage has a slope of $30^\circ$. The distance from the anchor to mid-span is 20 m. Let the stress in the cable at the point of anchorage be equal to 125 kg/mm$^2$. From formula 2a, the loss at point B is

$$0.4 \times \frac{125}{100} \left( 30 + \frac{6}{3} \right) = 16 \text{ kg/mm}^2.$$

The loss at point C is

$$0.4 \times \frac{125}{100} \left( 30 + \frac{20}{3} \right) = 18 \text{ kg/mm}^2.$$

The stress is therefore

at B, and

125 - 16 = 109 kg/mm$^2$

125 - 18 = 107 kg/mm$^2$

at C.

Hence the stress diagram shown in Fig. 13 is obtained.

Assuming that the pull-in of the cone is equal to 6 mm, it is required to determine the resulting reduction in the stress at A. The area bounded by
the horizontal line through \( N \) and the stress diagram must be equal to \( E_a(u/2) = 20000 \text{ kg/mm}^2 \times 3 \text{ mm}. \)

If stresses are expressed in kg/mm\(^2\) and lengths in metres, then the area to be determined must be equal to:

\[
\frac{20000 \times 3}{1000} = 60
\]

If \( N \) were coincident with \( B \), then the area between the horizontal line through \( N \) and the stress diagram would be equal to:

\[
16 \times \frac{6}{2} = 48
\]

![Diagram](image)

**Fig. 13.**

If the horizontal line through \( N \) is dropped by a distance equivalent to 1 kg/mm\(^2\), the corresponding position of \( N \) is at \( N_1 \), midway between \( B \) and \( C \). The area becomes:

\[
48 + (1 \times 6) + \left(1 \times \frac{7}{2}\right) = 57.5
\]

If the horizontal through \( N \) is dropped by 2 kg/mm\(^2\), \( N \) coincides with \( C \), and the area becomes:

\[
48 + (2 \times 6) + \left(2 \times \frac{14}{2}\right) = 74
\]
The horizontal through N must therefore be dropped a distance relative to B which is equal to:

$$1 + 1 \times \frac{60 - 57.5}{74 - 57.5} = 1.15 \text{ kg/mm}^2$$

Half the loss of stress is represented by the distance between A and this horizontal line. It is therefore equal to $16 + 1.15 = 17.15 \text{ kg/mm}^2$, and the loss of stress at the anchor is $34.3 \text{ kg/mm}^2$.

The ordinate of the horizontal through N is $125 - 17.1 = 107.9$; the loss of stress at B is therefore $2 \times (109 - 107.9) = 2.2 \text{ kg/mm}^2$.

The distance of N from the anchorage is $6 + 1.15/2 \times 14 = 14 \text{ m}$.

The loss due to pull-in of the cone therefore occurs throughout a considerable distance, but it is low local to B.

If the pull-in were greater, the loss could obviously extend as far as mid-span; the method of calculation would be identical. It would be necessary to determine the horizontal line such that the area contained between the line and the stress diagram is equal to $E_a(u/2)$. The line would be below the horizontal through C.

For example, if the pull-in were 8 mm then the area required for ABNa would be $(20000 \times 8)/2 = 80000 \text{ kg/mm}$, or 80, with the previous units.

With N at C, the area was found to equal 74. This must be increased by 6. The horizontal should therefore be lowered by a further $0.3 \text{ kg/mm}^2$ ($0.3 \times 20 = 6$). Half the stress loss would be $18 + 0.3 = 18.3$, and $\Delta T_o = 36.6 \text{ kg/mm}^2$.

These examples illustrate that the loss of stress at the anchorage can be greater than the loss due to friction at the neutral point (34 to 36 kg/mm$^2$ at the anchor, 18 kg/mm$^2$ at mid-span).

The importance of this need not be very great in the case of beams, because a high prestress is usually necessary primarily at mid-span, and the loss of stress decreases rapidly as the distance from the anchor increases.

The effect is far more important in the case of pipes or reservoirs subjected to internal pressure, because the hoop stress due to the pressure is uniform. However, the cables are usually duplicated at the anchors, where the outgoing and incoming cables cross, and this double prestress offsets the loss of stress at the anchors.

If the stress diagram is a single straight line between the anchor and the neutral point (as opposed to two straight lines in the case of Fig. 13), the calculation of the loss of stress at the anchor is simplified. This applies when the cable profile is curved throughout its entire length, at constant radius (parabolic profile in a beam, circular profile in a reservoir).
Consider again the beams in the previous example, 40 m long. Suppose that the cable profile is parabolic. If the levels are as before at the anchorage and at mid-span, the deviation is 10°. The friction loss over the length of 20 m to mid-span is

\[ 0.4 \times \frac{T_o}{100} \left( 10 + \frac{20}{3} \right) = 0.067T_o \]

In order to obtain a stress of 107 kg/mm² at mid-span, a stress of \( T_o = \frac{107}{(1 - 0.067)} = 115 \) kg/mm² is required at the anchorage.

![Diagram showing the stress distribution and calculation of the loss](image)

**Fig. 14.**

The loss is therefore 115–107 = 8 kg/mm² over 20 m.

Let \( x \) be the abscissa of point N.

Assuming a pull-in of 4 mm, the area of triangle ANa must equal \( E_o(u/2) = 20000 \times 2 \) kg/mm²; or, with the same units as previously (kg/mm² × m), the area must be equal to 40. But \( Aa = 8 \times x/20 \).

It is therefore necessary to have:

\[ 8 \times \frac{x}{20} + \frac{x}{2} = 40 \]

or:

\[ x = \left( \frac{1600}{8} \right)^{\frac{1}{2}} = 14.1 \text{ m} \]

Hence:

\[ \frac{\Delta T_o}{2} = 8 \times \frac{14.1}{20} = 5.6 \text{ kg/mm}^2 \]

The loss is therefore 11.2 kg/mm².
10. Uniformity of prestress in circular cross-sections (pipes, reservoirs)
In some cases, excessive variations in stress along the circumference are not acceptable. Each cable may then be replaced by several cables. These are anchored on equally spaced vertical ribs arranged along the generatrices of the cylinder.

Figure 15 shows three ribs A, B and C, spaced 120° apart. Cables type 1 are anchored at A and C; type 2 at B and A; and type 3 at C and B. The stress diagram (Fig. 15b) is not uniform because of anchorage losses, but this disadvantage is almost entirely compensated by crossing the cables over each other at each one of the ribs.

![Stress diagram along the cables](image)

**Fig. 15.** Prestressing of circular reservoirs.

11. Strain measurement errors on circular cross-sections with a small radius of curvature
Before tensioning, the cable is concentric with the sheath. When tension is applied, the centripetal force due to curvature displaces the wires towards the inside generatrix of the sheath (Fig. 16b). The centroid of a section at right angles to the axis of the cable is displaced towards the centre of the circle. Let \( a \) be this displacement.

Only a small amount of tension is needed in the cable to cause this to occur. If \( r \) is the mean sheath radius and \( 2\alpha \) the subtended angle between the two ends of the cable, the length of the inside generatrix of the sheaths is \( (r - a) \times 2\alpha \). The cable, of length \( r \times 2\alpha \), is therefore too long in relation to the generatrix by an amount \( a \times 2\alpha \). This represents a relative elongation which occurs without any real force being applied. Let \( T \) be the stress induced in the cable. If friction is neglected, the elongation due to \( T \) is:

\[
\frac{T}{E_a} r \times 2\alpha
\]
The total elongation, measured by means of reference marks on the cable, is:

\[ a = \frac{T}{E_a} r \times 2\alpha + a \times 2\alpha = \frac{T}{E_a} r \times 2\alpha \left( 1 + \frac{a}{r} \times \frac{E_a}{T} \right) \]

If \( d \) is the diameter of the cable, \( a = d/2 \). If \( T = 100 \) kg/mm\(^2\) and \( E_a = 20000 \) kg/mm\(^2\), the increment coefficient of elongation is:

\[ 1 + \frac{d}{2r} \times \frac{20000}{100} = 1 + 100 \frac{d}{r} \]

![Diagram](image)

**Fig. 16**

When \( r \) is small this coefficient can be much higher than 1; if \( d = 4 \) cm and \( r = 3 \) m, or 300 cm, the coefficient is:

\[ 1 + \frac{100 \times 4}{300} = 2.35 \]

The measured elongation is thus, in this case, twice as high as the calculated elongation, without taking into account the centripetal displacement \( a \). The elongation can therefore no longer be used to control the pressure in the jack, and only the pressure gauges can be relied upon. This requires frequent and thorough inspection and control of the gauges.
II. OTHER STRESS LOSSES

It is at this stage possible to sum up the various losses which have to be taken into consideration when deciding upon the stress which is to be applied at the anchorage in order to obtain the required permanent stress in the most affected portion of the cable.

Only the effects occurring at normal temperatures are considered here. For the effects at higher temperatures, refer to Chapter II (Section 7).

12. Evaluation of partial losses

These losses will first be evaluated on the assumption that they occur separately. The fact that they occur simultaneously effectively reduces the losses, and this effect is dealt with in Chapter II, Section 16.

(i) Loss due to shrinkage ($\Delta T_r$)

Under normal conditions, and in a temperate climate, it can be assumed that the residual shrinkage strain after prestressing is of the order of 2/10 000. If the modulus of elasticity of the steel is 20 000 kg/mm$^2$, the resulting loss of cable stress is:

$$\frac{2}{10 000} \times 20 000 = 4 \text{ kg/mm}^2$$

(ii) Losses due to long-term deformation† ($\Delta T_d$)

If $\sigma_c$ is the concrete stress local to the cable at the section under consideration, and if the cable is bonded‡ by grouting after stressing, then the shrinkage strain in the concrete after anchoring is equal to $\sigma_c/E_d$, $E_d$ being the long-term deformation modulus of elasticity. The loss of stress in the cable is therefore $(E_a/E_d)\sigma_c$, or $m_d\sigma_c$, where $m_d$ is the modular ratio for long-term deformation.

It can be assumed that $E_d = 225 000 \text{ kg/cm}^2$ and $E_a = 2 000 000 \text{ kg/cm}^2$. Therefore $m_d = 2 000 000/225 000 = 9$.

The stress $\sigma_c$ in the concrete is the stress at which the deformation occurs: it is therefore the stress under permanent loading, without live loads.

† This applies to post-tensioning, where instantaneous concrete deformation occurs as soon as the cable is anchored, except for the loss dealt with in Section 12 (iii).

‡ If the cable is not bonded, the loss results only from the mean shortening of the cable between the anchor and the section under consideration.
Calculation of Cable Losses

If, for example, \( \sigma_c = 100 \text{ kg/cm}^2 \), which is quite typical, the loss through long-term deformation is of the order of 9 kg/mm\(^2\). It must, of course, be assessed for each particular case.

(iii) Losses due to cables being tensioned separately (\( \Delta T_c \))

The prestress is usually applied by means of several cables. If these were all tensioned simultaneously, only the loss (see previous subsection, ii) would need to be taken into account, since the instantaneous deformation which occurs before anchoring would not affect the reduction in length of the cables. However, when the cables are tensioned one after the other, each cable is subjected to the effects of the instantaneous deformation caused by the tensioning of the succeeding cable.

If \( N \) is the number of cables, the first cable will be subjected to the deformation caused by the \( N - 1 \) cables which are yet to be tensioned, the second will feel the effect from \( N - 2 \) cables, and so on.

Once the \( N \) cables are tensioned, the stress in the resultant cable\(^\dagger\) would be the stress \( \sigma_c \) considered in subsection (i) above, if the permanent stress \( \sigma_c \) were already established; it is in fact greater than \( \sigma_c \) since the deferred losses have not yet occurred. The stress \( \sigma_c \) which is subsequently established is the resultant of the prestress \( \sigma_{oc} \) and the self-weight stress \( \sigma_{pc} \). (The suffix \( c \) indicates that stresses in the resultant cable are implied.)

With the correct algebraic signs:

\[ \sigma_c = \sigma_{oc} - |\sigma_{pc}| \]

If \( \sigma_{oc} \) is increased by \( 20\% \) initially, so as to compensate for subsequent losses, the initial value of \( \sigma_c \) is \( 1.2\sigma_{oc} - |\sigma_{pc}| \).

It can be assumed, in order to obtain an idea of the orders of magnitude involved, that \( \sigma_{oc} = 2|\sigma_{pc}| \), and therefore that \( \sigma_c = |\sigma_{pc}| \).

In the initial state, therefore:

\[ \sigma_c \text{ initial} = 2.4|\sigma_{pc}| - |\sigma_{pc}| = 1.4|\sigma_{pc}| \]

or:

\[ \sigma_c \text{ initial} = 1.4\sigma_c \text{ permanent} \]

\(^\dagger\) The resultant cable is a fictitious cable which exerts at all sections of the beam a force which is equal to the resultant of the forces exerted by all the separate cables, and which has the same eccentricity as this resultant.
Furthermore, it can be considered that tensioning of each of the N cables produces the same increase in stress $\Delta \sigma_c = 1.4 \sigma_c / N$ local to the cable.\(^\dagger\)

Tensioning of each cable therefore produces a shortening $\Delta \sigma_c / E_i$, where $E_i$ is the elastic modulus of concrete for instantaneous deformations. The loss in stress in the cables previously anchored would therefore be $(E_a / E_i) \Delta \sigma_c$ if the cables were bonded.

Since the cables are not usually grouted in at this stage, the amount of shortening to be considered is the mean shortening over the total length of the cable. Calculation shows that this mean shortening is about 80% of the maximum loss of strain.

If $m_i$ is the modular ratio corresponding to instantaneous deformation, tensioning of each cable causes a loss of stress of

$$0.8m_i \times \Delta \sigma_c = 0.8m_i + 1.4 \frac{\sigma_c}{N}$$

Also, $E_i$ is approximately equal to $E_a/2$.

Partial losses are therefore of the order of:

$$0.8 \frac{m_d}{2} \times 1.4 \frac{\sigma_c}{N} = 0.56m_d \frac{\sigma_c}{N}$$

There are $(N - 1)$ partial losses in the first cable.

The last cable is not subject to any instantaneous deformation loss. The mean loss of stress is therefore equal to:

$$\frac{N - 1}{2} \times 0.56m_d \frac{\sigma_c}{N} = 0.28 \frac{N - 1}{N} m_d \sigma_c$$

The term 0.28 can be slightly reduced because part of the long-term deformation occurs during tensioning, and this reduces the stresses in the

\(^\dagger\) This is not strictly exact, and the increases $\Delta \sigma_1$ due to the first $N_1$ cables should be considered separately, $N_1$ being the number of cables which have to be tensioned to balance the deadweight moment $[(N_1/N)F_e = -M_p]$ and the increases due to $N - N_1$ remaining cables. An exact solution shows that only a small error is introduced by assuming that the increments $\Delta \sigma_c$ are equal, and that this approximation is sufficiently accurate for the determination of the correction to be applied due to the cables not being all tensioned simultaneously.
cables which are already anchored. The loss can be rounded off to:

\[
\frac{1}{4} \frac{N - 1}{N} m_d \sigma_c
\]

But \( m_d \sigma_c \) is loss (b) due to long-term deformation (see Section 13).

The loss of stress due to the cables not being tensioned simultaneously is therefore equal to approximately a quarter of the loss of stress which is due to the long-term deformation, with a multiplication correction factor of \( 1 - 1/N \), \( N \) being the number of cables.

(iv) Losses due to steel relaxation (\( \Delta T_\rho \))
It has been shown in Chapter II that these losses are of the order of 8% of the initial stress. If this stress is 125 kg/mm², the loss due to relaxation of the steel is approximately 10 kg/mm².

III. TOTAL LOSSES

13. Calculation of the initial stress at the anchorage
The losses due to deformation and relaxation must first be assessed for the worst condition (generally at mid-span in the case of a simply supported beam), and, more generally, at the section which is to be detailed. Let \( \Delta T_1 = \Delta T_r + \Delta T_d + \Delta T_c + \Delta T_\rho \) be the total loss resulting from the sum of the partial losses.

Given the permanent stress \( T \), which is required at a given section, the initial value \( T_i \) of the stress to be applied is obtained (\( T_i = T + \Delta T_1 \)).

To achieve the stress \( T_i \), a certain stress \( T_{ol} \) must be applied at the anchorage, taking into account friction losses \( \Delta T_f \) between the anchorage and the section under consideration.

If the value of \( T_{ol} \) which is thus obtained is higher than the maximum allowable stress, the required stress \( T \) is too high. The determination of the value of permanent stress which is possible is then obtained by the reverse process; if \( T_{ol} \) is the maximum allowable stress, then \( T_i = T_{ol} - \Delta T_f \) and \( T = T_i - \Delta T_1 \).

However, re-tensioning can sometimes be used to overcome this problem (Section 16).

Assume, for example, that the partial losses have the values previously quoted.
(i) Calculation of $\Delta T_1$.

(a) Shrinkage loss $\Delta T_r = 4\, \text{kg/mm}^2$

(b) Long-term deformation loss $\Delta T_d = 9\, \text{kg/mm}^2$

(c) Loss due to tension not being applied simultaneously in all the cables

$$\left[\frac{1}{2} + \Delta T_d(1 - (1/N))\right]$$

$$\Delta T_c = 2\, \text{kg/mm}^2\, \text{approx.}$$

(d) Relaxation loss $\Delta T_r = 10\, \text{kg/mm}^2$

$$25\, \text{kg/mm}^2$$

It is shown in Chapter II (Section 18) that the shrinkage and long-term deformation losses are not straightforward and direct additions to the relaxation loss, and that they are reduced by a factor $K$. The value of $K$ can be estimated as $1 - (2/3)(E_{T1}/E_{ao})$, $E_{T1}$ being the slope of the stress-strain diagram at $T = T_1$ and $E_{ao}$ being the slope of the diagram at the origin. If $E_{T1} = \frac{1}{3}E_{ao}$, then $K = \frac{1}{3}$.

The losses $\Delta T_r + \Delta T_d$ can be thus reduced to $\frac{1}{3} \times (4 + 9) = 9\, \text{kg/mm}^2$, and the total loss at mid-span reduced to $9 + 2 + 10 = 21\, \text{kg/mm}^2$.

(ii) If the required permanent stress at mid-span is $90\, \text{kg/mm}^2$, the stress required under initial conditions is $T_1 = 90 + 21 = 111\, \text{kg/mm}^2$.

Let $T_{oi}$ be the stress required at the anchor, let the deviation between the anchor and mid-span be equal to $\alpha = 15^\circ$ and let the length $l/2$ be equal to $20\, \text{m}$.

Assuming a friction loss $\Delta T_f$ equal to $0.4\%$ $(15 + 20/3) = 8.66\%$, say $9\%$, then $T_1 = (1 - 0.09)T_{oi} = 0.91T_{oi}$.

The initial stress at the anchorage must therefore be equal to:

$$T_{oi} = \frac{111}{0.91} = 122\, \text{kg/mm}^2$$

14. Evaluation of stresses local to the anchorage

In the same manner previously considered for the section subject to the worst conditions, the following losses are assessed:

(i) The loss due to shrinkage (this is the same as given above).

(ii) The losses due to long-term deformations and to non-simultaneous stressing. (Their magnitudes are about half the values given above under (b) and (c), because the value of $\sigma_c$ is about half the value at the section which is subject to the worst conditions.)
(iii) The relaxation loss, which is of the same order of magnitude as that at the section with the worst conditions, since the initial tension after anchoring is of the same order of magnitude as $T_i$, because of the re-entry loss.

By giving to $\Delta T_d$ and $\Delta T_e$ the values corresponding to the section at the anchorage, the stress in the immediate neighbourhood of the anchorage is obtained:

- during tensioning: $T_{ol}$—cone loss;
- under permanent conditions:

$$T_{ol} = [\Delta T_p + \Delta T_c + K(\Delta T_r + \Delta T_d)]$$—cone loss

**Remarks**

(i) The calculation of losses in separate cables can only be approximate because of the scatter associated with empirical rules, as indicated in Chapter II; but, since each beam comprises a number of cables, the mean accuracy obtained is sufficient if correct experimental data are used.

(ii) In cases of doubt, it is obviously best to overestimate the losses. It should be borne in mind, however, that a gross overestimation of losses can cause considerable embarrassment (due to excessive compression and reverse bending, especially in the unloaded condition).

15. **Successive operational conditions for the steel**

It has been shown in the foregoing that the loss of stress in the cables is of the order of 20 kg/mm². As a result, the representative point on the stress–strain diagram (Fig. 17) varies during the life of the structure along a straighter line of elastic slope ($E_m$), provided that the point does not again reach its initial value $A_p$; that is, as long as the additional stress due to live loads does not exceed about 20 kg/mm².

With total prestressing, the stress increases are always below this limit. The variation of compressive stress in the concrete adjacent to the cable does not in general exceed 100 kg/cm², and the additional stress in the cable does not exceed $m \times 100$ kg/cm², or $m \times 1$ kg/mm², say 15 kg/mm² with permanent loading ($m = 15$) and 5 kg/mm² with transient loading.

In the case of limited prestress, the change of stress in the concrete adjacent to the cable can be as high as 120 kg/cm² with consequent increases in the cable stress of 18 kg/mm² in the case of permanent loading, and of 6 kg/mm² in the case of transient loading.

When the cables are grouted, the stresses in the cables are increased in the same way as the stresses in the ordinary reinforcement, since they
remain in the elastic region for stress increases which are of the same order of magnitude as the stresses in the reinforcement.

With reinforced prestressed concrete, the elastic margin is again of the same order of magnitude as the stresses in the ordinary reinforcement. The subject merits closer study, however, because of certain shrinkage effects caused by the presence of a relatively high percentage of reinforcement. It is considered in Chapter XI.

16. Reduction of losses by re-tensioning

French regulations allow high values of stress during tensioning. In certain countries, the maximum permissible stress values are much lower. Losses which occur after initial tensioning can be partly recovered by re-tensioning after part of the losses have occurred.

If re-tensioning is carried out after 6 months, for example, it can be reckoned that the remaining shrinkage loss is reduced to 2 kg/mm² in lieu of 4 kg/mm², that the remaining long-term deformation loss is only a quarter of its initial value, or 2·5 kg/mm², and that the relaxation loss is reduced to zero. The loss due to the cables not being tensioned at the same time stays approximately the same. For the previous example, the total loss at the section with the higher loss becomes:

\[ 1 + 2·5 + 2 = 5·5 \text{ kg/mm}^2 \]

instead of 21 kg/mm².
Calculation of Cable Losses

To obtain the stress of 90 kg/mm² at this section, the stress required at the anchor is \( (90 + 5.5)/0.91 = 105 \) kg/mm².

This procedure seems to offer some advantage, therefore. However, it does mean that prestressing operations have to be repeated, and, what is more important, it leaves the cables without protection for a considerable time. Bitumen could be applied for temporary protection, but difficulties would then arise with bonding after final tensioning.

These are not simple solutions, and French experience shows that such complications are unnecessary.

The advantages of 'elastic margins' described in Section 15 are also reduced.
Chapter V

SIMPLE EXAMPLES OF BENDING CALCULATIONS
(RECTANGULAR BEAMS WITH CABLES OF SINGLE CURVATURE)

CALCULATION FOR SECTION OF MAXIMUM MOMENT.
CONCEPT OF THE CRITICAL SPAN.
CHECKING: DEFLECTION, COVER

1. Preliminary comment
It must be appreciated that calculations relating to prestressed concrete, even in the so-called ‘elastic phase’, can only be considered to be approximate.

As in the case of all other materials, they are based on the twin hypotheses that sections that are plane before bending remain plane after bending, and on the proportionality of stress and strain.

In the case of plane bending, and of a section with an axis of symmetry which is perpendicular to the axis of the bending moment, these two hypotheses lead in consequence to the usual formula expressing the relationship between the stress \( \sigma \) and the distance \( y \) from the centroid:

\[
\sigma = A + By
\]

in which the coefficients \( A \) and \( B \) are determined by equating the resultant internal force and the internal moment of resistance, respectively, to the resultant external force and the external bending moment applied to the section.

Experience shows that the first of these two assumptions is a good approximation. It agrees closely with measurements made with strain gauges in a large number of tests, and the results derived from this assumption are in close agreement with actual conditions, not only in the elastic phase but also beyond this phase and almost up to the point of failure.

The second assumption must be treated with much greater caution. This is because any concrete, no matter how carefully it is made in practice,
cannot be homogeneous either in all parts of the beam or in all parts of a single section. It is possible, of course, to treat it as a matter of statistics, and hence one can obtain a mean stress–strain diagram; but even this is generally inaccurate, since the concrete in relatively small members (such as a beam on simple supports), in which there are a number of cables which act as distributing and filtering agents for the passage of the wet concrete, can behave quite differently to concrete in large sections.

On the other hand, even if the concrete is considered to be homogeneous, it is still subjected to stresses which significantly exceed the ‘limit of proportionality’.

![Diagram](image)

Having been subjected to these excess stresses (Fig. 2), then on decompression the concrete has a larger elastic modulus than that of concretes which are being compressed for the first time. It follows that, at the time of prestressing, concrete at the bottom of the member (which is highly compressed), has a smaller elastic modulus than the concrete at the top of the section. It is not difficult to demonstrate that the prestress obtained at the bottom of the member is slightly less than that calculated for a homogeneous material whereas that at the top is slightly greater.

Conversely when the added load is applied for the first time, the inverse condition applies; the concrete at the top of the member is stressed to a lesser degree than calculations predict, while the concrete at the bottom is stressed to a higher extent than calculated.

The discrepancies between these effects and the calculated values based on an homogeneous material, in the two extreme cases (prestress plus minimum load, prestress plus maximum load), are not entirely self-compensating.
Following these first applications of the loading, and in any case after several cycles of loading and unloading, it is then possible to consider the action of the member as being virtually fully elastic, since the concrete is subject to recompression and is by this time fully hardened.

It can therefore be considered that the difference between the actual and calculated stresses is very small, but they are superimposed on a state of internal stress which is achieved once and for all during the first prestressing and loading cycles.

![Diagram](image)

**Fig. 2.**

However, the foregoing comments in no way imply that calculations relating to prestressed concrete are in any way inadequate. Long experience has demonstrated that structures designed in accordance with the rules and regulations considered in this and following chapters behave in a fully satisfactory manner. It is simply necessary to avoid attributing to calculations of this type a degree of exactitude which they do not in fact possess.

This remark having being made, however, the calculations are made as if they were rigorously accurate, and the likely degrees of error are noted in the following sections as they arise.

2. Notation

The following notation is used.

M: Bending moment due to external loading (the moment exerted by the left-hand part of the beam on the right-hand part; it is taken as positive when it tends to turn the member clockwise).
N: Longitudinal force due to external loading.

Sectional properties:
I: Second moment of area.
S: Cross-sectional area of the section.
v: The distance from the centroid to the top face section.
v': The distance from the centroid to the lower edge of the section.
I/v and I/v': Section moduli.
r²/v and r²/v': The limits of the central core or kern.

\[
\begin{align*}
\frac{r^2}{v} &= \frac{I/v}{S} \\
\frac{r^2}{v'} &= \frac{I/v'}{S}
\end{align*}
\]

R: Maximum permissible stresses.
R': Minimum permissible stresses.

(It will be seen later that these limiting stresses are different at the time that the prestress is applied, under normal working load conditions, and that they also vary with the passage of time, and according to whether the

![Diagram of section with labeled distances and symbols](image)

load is applied or removed; in these cases R and R' will be modified by different subscripts or indices for the various loading conditions considered.)

σ and σ': The stresses at the top and bottom of the section, which are modified by the following subscripts: o for stresses due to the prestress alone, p for the stresses due to dead loads, s for stresses due to live loads (σ₀ and σ₀' are the stresses due to the prestress alone; σ₀ and σ₀' are the stresses caused by the dead loads; σₛ and σₛ' are the stresses due to the action of the live loads).
Accepting for the moment, as noted in Chapter I, that the prestressing force $F$ does not vary during the loading process, the resultant of state stress will therefore be as follows:

\[ \sigma_0 + \sigma_p, \sigma'_0 + \sigma'_p \text{ under dead load;} \]
\[ \sigma_0 + \sigma_p + \sigma_s, \sigma'_0 + \sigma'_p + \sigma'_s \text{ under the action of the total loading.} \]

The signs of the stresses $\sigma_p, \sigma'_p, \sigma_s, \sigma'_s$ must of course be taken into account. In the case of a beam supported at each end and subjected to a vertical downward loading, $\sigma'_p$ and $\sigma'_s$ are both negative.

![Fig. 4.](image)

If necessary, the symbols $\sigma_1, \sigma'_1$ should be used to denote the stresses under the 'minimum' loading condition; that is, under the minimum moment (in general, in the unloaded condition), and $\sigma_2, \sigma'_2$ should be used to denote the stresses under the 'maximum' loading condition; that is, under the maximum bending moment (in general, in the fully loaded condition).

In this book, positive signs are used to denote compressive forces and stresses and negative signs are used to denote tensile forces and stresses.

The symbol $F$ denotes the prestressing force and the symbol $e$ denotes its eccentricity; $e$ is considered positive when it is measured vertically upwards from the centroid.

We start by considering beams of constant section with uniform cables.

### 3. Cable action in a statically-determinate beam

The cable\(^\dagger\) transmits forces to the concrete via its anchors (force $F$ at $A$, tangential to the cable), and forces $b = F/\rho$ per unit length due to its

\(^\dagger\) This relates to the resultant cable; that is, an imaginary cable which exerts a force equal in magnitude and direction to the resultant of the forces exerted by the separate cables.
curvature,‡ where $\rho$ is the radius of curvature (Fig. 5b). Conversely, reactions from the concrete exert forces on the cable; a tensile force $F$ at the anchorage and curvature forces $-b$ (Fig. 5a). Since the cable is assumed to have no stiffness, the force to which it is subjected is at any section tangential to it. Therefore, at any section of the beam, the point of action of the force acting on the cable at the section coincides with the position of the cable.

A statically-determinate beam (for example, a beam simply supported at both ends) is subjected to the forces exerted by the cable. In other words, it is subjected to an anchor force $F$ at each end and to the curvature forces $b$.

These forces, $F_{enA} + b - F_{enB}$, are in equilibrium since the cable is in equilibrium.

Since the beam is statically determinate, the external reactions from the system are zero.

‡ Approximately normal to the cables; it can be assumed in most cases that they are vertical (or, more generally, perpendicular to the axis of the beam) since the slope of the cables is small.
When it is subjected to pure prestress, therefore; that is, when it is not subjected to any external loading, the beam is subjected only to the action from the cable. Hence, at any section X the compressive force $F$ on the concrete is equal and opposite in magnitude and direction to the tensile force $F$ in the cable. Consequently, for a statically-determinate beam,† the prestressing force $F$ is a compressive force whose point of action at any section coincides with the position of the cable at that section.

4. Initial stress, permanent stress and stress under load. Extreme conditions of loading

The stress in the cable decreases with time because of the losses described and evaluated in Chapter IV (Section 13). The initial tension in the cables is denoted by $T_i$; in other words, $T_i$ is the tension in the cables at the time when they are tensioned, at any given section. The final or permanent tension is denoted by $T$. The loss of stress as a proportion of $T$, $(T_i - T)/T$, is of the order of 20 to 30% and is complete after two to three years, but the rate of loss is very high in the first few months.

The cable stresses increase under the live loading because of the reduction in compression in the region of the cables.

It has already been seen that this increase in the stresses is low when total prestressing is used. It is higher in the case of limited prestressing, and it can no longer be neglected in the case of prestressed reinforced concrete.

Theoretically, all conditions of loading combined with all values of stress which can occur with time should be examined.

This would entail an examination of the following limiting cases:

(a) unloaded beam (that is, with permanent dead loading but without live loads) with initial prestress;
(b) loaded beam with initial prestress;
(c) unloaded beam with permanent prestress;
(d) loaded beam with permanent prestress.

In general, case (b) is of no particular practical interest. Live loads are applied sufficiently late for the greatest part of the loss of stress to have occurred. After 6 months, for example, the loss has reached 75% of its

† This is not the case for a statically-indeterminate beam.
final value. If the final value is approximately 25% of $T$, or about 22 kg/mm$^2$
differs from case (b) by only 22/4 kg/mm$^2$ (6% of the prestress). This
temporary discrepancy can justifiably be neglected compared with the
other sources of error (elastic moduli, friction, etc.) which occur, and, in
general, this contributes to a higher factor of safety.

This leaves only cases (a), (c) and (d) to be examined.

The section should be designed using conditions (c) and (d) as a basis. If
the design were based on conditions (a) and (d), a higher cost would
result. The reason for this will be better understood later on, but it stems
from the necessity not to exceed certain limits of compressive stresses.
The advantages which could justify the increased cost no longer apply
after a few months, when, instead of the condition of minimum moment,
(a), the extreme minimum condition becomes condition (c).

It is often possible, without the introduction of special measures, for a
design which is based on conditions (c) and (d) to be acceptable for the
conditions which apply in the state (a), because in this state higher per-
missible compressive stresses can be temporarily accepted. The permissible
increases are discussed in Chapter IX; they can, without any difficulty, be
as high as 30%. In any case, the excess stress decreases with time. Also, a
high prestress is in itself a test of strength. If the beam can withstand it, it
will remain in the extreme state (c) indefinitely.

If the stresses exceed the increased limiting values permissible under con-
dition (a), then the three cases (a), (c) and (d) should be examined. But,
even in this case, there is usually a way of designing the section for cases
(c) and (d) only. This consists of applying only part of the prestress at
first; that is, in tensioning only some of the cables. This is called multi-stage
prestressing.

Multi-stage prestressing is almost always mandatory when the perma-
nent loading is not applied completely at the time of prestressing (for
example, bridges in which the superstructures, which are an important
part of the loading, are not completed at the time of prestressing). Com-
pressive stresses would then be generally too high on the most highly
stressed areas (and tensile stresses would be too high on the opposite face),
the prestress being insufficiently counteracted by the opposing stresses
due to external loading, and the concrete being also relatively green. The
following three cases may then have to be considered:

(a') partial permanent loading plus initial prestress;
(c) total permanent loading plus permanent prestress;
(d) total load plus permanent prestress.
In any event:

(i) the design of the section would be based on cases (c) and (d);
(ii) the design would then be checked for cases (a) or (a') on the basis of higher permissible stresses;
(iii) if these permissible increased stresses are exceeded, the prestress would be applied in stages.

If multi-stage stressing is not possible, or if it is insufficient, the worst cases become cases (a), or (a'), and (d).

The following must be taken into account when computing the stresses:

(i) the weakening effect in condition (a) of the ungrouted holes for the passage of the cables;
(ii) the increased stress pattern due to live loads in passing from condition (c) to condition (d).

The corresponding corrections are generally small in the cases of total and limited prestress, and they are of the same order of magnitude as the errors in the exact values of the prestressing forces.

They are studied later (Chapter VII, Sections 7 and 8).

Corrections in the case of limited prestress are greater than in the case of total prestress because of the higher stress variations in the concrete local to the cables.

In the case of reinforced prestressed concrete, overstressing of the cables when cracking occurs can no longer be neglected; this is one of the fundamental features of the design.

DESIGN OF A PRESTRESSED MEMBER USING FORMULAE FROM STRENGTH-OF-MATERIALS (MEMBER OF RECTANGULAR CROSS-SECTION)

It is shown in the following that the usual strength-of-materials principles are adequate for designing a prestressed section. The purpose of the more general theories, developed in Chapter VII, is only to permit a more intuitive approach to the problem, and they cannot be built up from any different bases where the elastic state is concerned.

In accordance with Section 4, the weakening effect of the cable holes in the concrete and the variation in the prestressing force are neglected. How overtensioning can be taken into account, in the case of partial prestress, is nevertheless explained in the example given in Section 11 and the savings which can result are discussed.
5. Example I (slab with a span less than the critical span)

Slab 6 m wide of 8 m span.
Unloaded condition: self-weight + 250 kg/m² of superstructure.
Live load: 2 000 kg/m².
Permissible limiting stresses:

\[ R = 120 \text{ kg/cm}^2 \quad R' = +10 \text{ kg/cm}^2 \]

The unknowns are the slab thickness \( h \), and the prestress required; that is, the force \( F \) and its eccentricity \( e \). There are therefore three unknowns.

![Fig. 7.](image)

If the maximum use is to be made of the concrete in the two extreme loading conditions (unloaded and fully loaded), four equations are obtained (two limiting stresses × two loading conditions). It therefore seems that there is one equation too many; but only three of the four equations are independent because the shape of the section is assumed to be known.

It is easier in general to substitute \( \sigma_o \) and \( \sigma'_o \), the stresses at the faces of the member, for the two unknowns \( F \) and \( e \).

Once \( \sigma_o \), \( \sigma'_o \) and \( h \) are known, \( F \) and \( e \) can be obtained, since:

\[
\sigma_o = \frac{F}{bh} \left(1 + 6 \frac{e}{h}\right) \quad (a)
\]

\[
\sigma'_o = \frac{F}{bh} \left(1 - 6 \frac{e}{h}\right) \quad (b)
\]

Adding:

\[
\frac{F}{bh} = \frac{\sigma_o + \sigma'_o}{2} \quad (c)
\]

Subtracting:

\[
\frac{F}{bh} \times \frac{6e}{h} = \frac{\sigma_o - \sigma'_o}{2} \quad (d)
\]

Therefore:

\[
e = \frac{h}{6} \times \frac{\sigma_o - \sigma'_o}{2(F/bh)} = \frac{h \sigma_o - \sigma'_o}{6 \sigma_o + \sigma'_o} \quad (d)
\]
Solution
Since it is required to make the best possible use of material, the stresses must be at their limit values in the two conditions. Or:

Unloaded condition (I) \[
\begin{align*}
\sigma_o + \sigma_p &= R' \\
\sigma'_o - \sigma_p &= R
\end{align*}
\] (1) (2)

(since \(\sigma'_o = -\sigma_p\) by symmetry).

Loaded condition (II) \[
\begin{align*}
\sigma_o + \sigma_p + \sigma_s &= R \\
\sigma'_o - \sigma_p - \sigma_s &= R'
\end{align*}
\] (3) (4)

These are not independent equations, because (2) can be replaced by combining (1) and (2) and writing for the system I:

\[
\begin{align*}
\text{I}' \quad \sigma_o + \sigma_p &= R' \\
\sigma_o + \sigma'_o &= R + R'
\end{align*}
\] (1') (2')

Similarly for system II:

\[
\begin{align*}
\text{II}' \quad \sigma_o + \sigma_p + \sigma_s &= R \\
\sigma_o + \sigma'_o &= R + R'
\end{align*}
\] (3') (4')

It can be seen therefore that there are only three independent equations for the three unknowns \(\sigma_o, \sigma'_o\) and \(h\), since equations (2') and (4') are identical.

Systems I and II are now considered.

(i) Value of \(h\)
Substitute the values of \(\sigma_o + \sigma_p\) and \(\sigma'_o - \sigma_p\) in System II:

\[
\begin{align*}
R' + \sigma_s &= R \\
R - \sigma_s &= R'
\end{align*}
\] (5) (6)

and:

\[
\sigma_s = R - R'
\]

Equations (5) and (6) are identical because of the choice of shape for the section. This enables \(h\) to be found.

\[
\sigma_s = \frac{M_s}{bh^2/6}
\]
Now
\[ M_s = s \frac{l^2}{8} = 12 \text{ t/m} \times \frac{8^2}{8} = 96 \text{ t/m} \]

Therefore:
\[ \frac{96}{bh^2/6} = 1200 - 100 = 1100 \text{ t/m}^2 \]
\[ \frac{bh^2}{6} = \frac{96}{1100} = 0.0872 \]

Since \( b = 6 \text{ m}; h^2 = 0.0872 \) and \( h = 0.295 \text{ m} \).

The self-weight per unit length of the slab is equal to the cross-sectional area multiplied by the density of concrete (2.4 t/m³).
\[ S = 6 \times 0.295 = 1.77 \text{ m}^2; \]
Self-weight = \( 1.77 \times 2.4 = 4.25 \text{ t/m} \)
Adding the weight of the superstructure: \( 1.50 \text{ t/m} \)
\[ p = 5.75 \text{ t/m} \]

Therefore: \( M_p = 5.75 \times 8^2/8 = 46 \text{ t/m} \).
Since
\[ \frac{bh^2}{6} = 0.0872 \left\{ \begin{array}{l}
\sigma_p = 46/0.0872 = 527 \text{ t/m}^2 \\
\sigma'_p = -527 \text{ t/m}^2
\end{array} \right. \]

(ii) Value of prestress
\( \sigma_o \) and \( \sigma'_o \) are obtained from eqns. (1) and (2):
\[ \left\{ \begin{array}{l}
\sigma_o = 527 = 100 \text{ t/m}^2 \\
\sigma'_o = -527 = 1200 \text{ t/m}^2
\end{array} \right. \]

Whence:
\[ \left\{ \begin{array}{l}
\sigma_o = -427 \text{ t/m}^2 \\
\sigma'_o = 1727 \text{ t/m}^2
\end{array} \right. \]

Once \( \sigma_o \) and \( \sigma'_o \) are known, F and e can be obtained from eqns. (c) and (d):
\[ \frac{F}{bh} = \frac{\sigma_o + \sigma'_o}{2} = 650 \text{ t/m}^2 \]
and

$$F = 650 \times bh = 650 \times 1.77 = 1150 \text{ t}$$

The eccentricity

$$e = \frac{h \sigma_o - \sigma'_o}{6 \sigma_o + \sigma'_o} = \frac{h}{6} \times \frac{-427 - 1727}{-427 + 1727}$$

$$= \frac{h}{6} \times \frac{2154}{1300}$$

$$= -1.657 \frac{h}{6}$$

$$= -1.657 \times \frac{0.295}{6}$$

$$= -0.081 \text{ m}$$

To apply the force of 1150 t, 28 cables of 41 t will be used (Freyssinet 12-wire cables, 7 mm diameter, stressed to 89 kg/mm$^2$). If the cables are in a single layer (Fig. 8), the pitch is equal to $600/28 = 21.5$ cm.

This spacing between cables is acceptable and it allows adequate room for concreting.

The cover is: $0.295/2 - 0.081 = 0.066 \text{ m (6.6 cm)}$.

This is adequate (see Section 8).

It is seen that the design of the slab is independent of the permanent load [$bh^2/6 = M_s/(R - R')$]; in other words, that this load has not meant an increase in the amount of concrete used, since the same design (that is, the same value for $h$) would have been obtained for $p = 0$ as the one obtained for the true value of $p$. Nor has the load meant an increase in steel, since $F = bh \times (\sigma_o + \sigma'_o)/2 = bh(R + R')/2$ and since $h$ is independent of $p$, as shown above.
The permanent load has only had an effect on determining the position of the cable (that is, the eccentricity $e$):

$$
e = \frac{h \sigma_o - \sigma'_o}{6 \sigma_o + \sigma'_o} = \frac{h}{6} \frac{R - R' + 2\sigma_p}{R + R'} = \frac{h}{6} \frac{R - R'}{R + R'} - \frac{6 M_p}{b h^2} \frac{h}{(R + R')/2}$$

$$= -\frac{h}{6} \frac{R - R'}{R + R'} - \frac{M_p}{b h[(R + R')/2]} = -\frac{h}{6} \frac{R - R'}{R + R'} - \frac{M_p}{F}$$

This is true as long as this $e$ value provides sufficient cover beneath the cable.

Above a certain span, total compensation of permanent loading is no longer possible. This is reflected in the value of $e$.

This case is studied in Chapter VIII, Section 10.

6. Alternative methods of beam design

These methods are of course equivalent to the method which is explained above, but they deal with the subject in a quicker and more intuitive manner.

The stress diagrams can be drawn as shown in Fig. 9. The unknowns are shown in brackets.

(i) Concrete

Since it is assumed that maximum use is made of the material, the extreme fibre stresses must be as shown in diagrams (c) and (e).

Diagram (d) is obtained by deduction, since it is equal to (e) \(-\) (c). Therefore, by going from the unloaded to the loaded condition, we must have:

$$\sigma_s = R - R' \quad \text{or} \quad \frac{M_s}{I/v} = \Delta R$$
Since $M_a$ is the change in moment $\Delta M$,

$$\frac{I}{v} = \frac{\Delta M}{\Delta R}$$

This formula is applicable to any shape of section, when self-weight can be compensated.

This is reflected in the formula itself, since $I/v$ is independent of $p$, and therefore has the same value for $p = 0$ and for the real value of $p$.

(ii) Prestress

The self-weight is known, since the dimensions of the section are known ($I/v$, hence $h$, since $I/v = bh/6$). Thus $p$ is obtained.

Diagram (b) is also known. The sum of diagrams (a) and (b) must be equal to diagram (c). Thus diagram (a) is obtained; that is, the prestresses $\sigma_o$ and $\sigma'_o$, hence $F$ and $e$.

The stresses are beyond their limits in the condition of pure prestress, diagram (a), but this is unimportant since permanent loading brings the stresses back within their limits, the slab becoming gradually loaded as prestress is applied (Chapter 1, Section 7).

Nevertheless, because the initial prestress is greater than the calculated permanent prestress, and also because, in the present case, self-weight is only a proportion of the total permanent load, it may be necessary to apply the prestress in stages, the first stage to balance self-weight (eventually with overstressing), the remainder being applied after the total permanent loading has occurred. This subject is discussed in Section 8.

The prestress ($F$ and $e$) can be calculated in yet another manner.

(a) Since $F$ is assumed to be constant in the various working conditions, it can be calculated from either of the resultant diagrams [(c) or (e)].

From diagram (c):

$$F = \frac{R + R'}{2} \frac{bh}{2}$$

In this case:

$$F = 650 \frac{bh}{2}$$

$$= 650 \times 1.77 = 1150 \text{ t}$$

If $p$ were equal to zero, the prestress diagram should be diagram (c); the eccentricity $e$ would be, from formula (d):

$$e = \frac{h R' - R}{6 R' + R} = \frac{h R}{6 R + R'} = \frac{h}{6} \times \frac{1100}{1300} = -0.846 + \frac{h}{6}$$

$$= -0.846 \times 0.0491 = -0.0415 \text{ m (say } 0.041 \text{ m)}$$
In order to obtain the necessary eccentricity, it is sufficient to displace the cable by an amount \(-\frac{M_p}{F}\), as shown in Chapter I, Section 8, and in Section 5 above.

As a check on the present example: \(M_p = 46 \text{ t/m (see above).}\)

Hence

\[
e = -0.041 - \frac{46}{1150} = -0.041 - 0.040 = -0.081 \text{ m}
\]

which is the value previously obtained.

(b) The force \(F\) can also be calculated by writing \(F = \Delta M/z\), where \(z\) is the lever arm; that is, the displacement of the centre of compression when the slab passes the unloaded position [diagram (c)] to the loaded condition [diagram (e)]. The eccentricity of the centre of compression must change from \(-\left(\frac{h}{6}\right)\left(\frac{R - R'}{R + R'}\right)\) in the unloaded state to \(+\left(\frac{h}{6}\right)\left(\frac{R - R'}{R + R'}\right)\) in the loaded state, or from \(-0.041.5 \text{ m to } +0.041.5 \text{ m.}\)

Therefore

\[
z = 0.083 \text{ m}
\]

and

\[
F = \frac{M_s}{0.083} = \frac{96 \text{ tm}}{0.083} = 1150 \text{ t}
\]

The eccentricities \(\pm\left(\frac{h}{6}\right)\left(\frac{R - R'}{R + R'}\right)\) are the limit eccentricities for a slab.

The distance \(A'A\) (Fig. 10) between the limit points is the limit core.

**Fig. 10.**
Expressions for limit eccentricities in the general case are given in Chapter VII, Section 3.

7. Various formulae. Critical span
It is seen from the above that when it is possible to compensate the permanent load, the design of a section is fairly rapid. Summarising:

$$\frac{I}{v} = \frac{\Delta M}{\Delta R} = \frac{M_s}{R - R'} = \frac{96 \text{ t/m}}{1100 \text{ t/m}^2} = 0.0872 \text{ m}^3$$

hence

$$\frac{bh^2}{6} = 0.0872$$

and

$$h = \left(\frac{6 \times 0.0872}{6 \text{ m}}\right)^{\frac{1}{2}} = 0.295 \text{ m}$$

Limit eccentricities: $$\pm \frac{h}{6} \frac{R - R'}{R + R'} = 0.0415 \text{ m}$$

$$F = \frac{M_s}{z} = \frac{96}{0.083} = 1150 \text{ t}$$

$$p = 6 \times 0.295 \times 2.4 + 1.5 = 5.75 \text{ t/m}$$

$$M_p = 5.75 \times \frac{8^2}{8} = 46 \text{ t/m};$$

hence

$$e = 0.0415 - \frac{M_p}{F} = -0.081 \text{ m}$$

The span below which it is possible to compensate for self-weight is termed the ‘critical span’.

Some useful formulae are obtained from the previous example, for application to beams of rectangular cross-sections with spans less than the critical span.

Available moment of resistance
This is the moment of resistance appropriate to live loads. It is therefore equal to $$M_s.$$
Therefore

\[ M_s = \frac{bh^2}{6} \times \Delta R \]

**Percentage of prestressing steel:**

\[ F = \frac{(R + R')}{2}bh \]

If \( A_c \) is the cross-sectional area of the cable and \( T \) its permanent stress:

\[ \frac{A_c}{bh} = \frac{R + R'}{2T} = \text{percentage} \]

**Weight of high tensile steel (longitudinal steel)**

1\% of steel signifies 100 cm\(^2\) of steel for 10 000 cm\(^2\), or 1 m\(^2\), of concrete. For 1 m\(^3\) of concrete and 1\% of steel, the corresponding volume is 100 cm\(^2\) \(\times\) 100 cm = 10 000 cm\(^3\) of steel, or 10 dm\(^3\), or about 80 kg/m\(^3\). Hence 1\% of steel = 80 kg/m\(^3\) (longitudinal steel).

To apply a prestress of 1 tonne over a length of 1 m, an amount of steel equal to approximately 0·1 kg is necessary. Indeed, a force of 1 tonne requires a steel cross-sectional area of 1 000 kg/T. If \( T = 90 \text{ kg/mm}^2 \), the required area is 11 mm\(^2\), or 0·11 cm\(^2\). With a density of 7·8, the weight of steel per metre is 0·11 \(\times\) 100 \(\times\) 8 = 88 gm, or nearly 0·09 kg. Taking losses into account, the approximate weight is nearly 0·1 kg per tonne/metre.

**With \( R = 1 200 \text{ t/m}^2 \), \( R' = 100 \text{ t/m}^2 \), \( T = 90 \text{ kg/mm}^2 \):**

Available moment of resistance:

\[ \frac{bh^2}{6} \times 1 100 = 183 bh^2 \]

\[ \frac{A_c}{bh} = \frac{650}{9 000} = 0.72\% \]

Weight of steel per cubic metre = 57 kg/m\(^3\).

**With \( R = 1 200 \text{ t/m}^2 \), \( R' = 0 \), \( T = 90 \text{ kg/mm}^2 \):**

Available moment of resistance:

\[ \frac{bh^2}{6} \times 1 200 = 200 bh^2 \]

\[ \frac{A_c}{bh} = \frac{600}{9 000} = 0.66\% \]

Weight of steel per cubic metre = 53 kg/m\(^3\).
8. Checks on initial conditions

(1) It has been assumed that the permanent load could be compensated. For this, the increase in eccentricity which is required must not take the cable outside the boundaries of the section, or, more precisely, the cover which remains must be adequate.

Whatever the nature of the steel, a minimum cover of 2 cm to the bare steel is necessary. If the cables are enclosed within mild steel stirrups, a minimum cover of about 4 cm is required for the sheaths, measured from the point which is nearest to the free concrete face. The sheath for a 7 mm diameter, 12-wire cable has a diameter of 4.5 cm; the minimum distance from the centroid of the sheath, and therefore of the cable, to the free face must be at least $4 + 2.2 = 6.2$ cm.

A slab is generally without stirrups, at least when total prestressing is employed, but in practice the cover is of the same order of magnitude.

In the case of the above example, it was seen that the cover was $6.6$ cm. The possibility of fully counteracting the permanent loading is thus verified.

If the amount of cover had not been sufficient, this would have been an indication that the critical span had been exceeded. It would then be necessary to proceed according to the methods of calculation given in Chapter VIII.

(2) Check on the initial conditions in the slab.

The extent to which the initial conditions can be satisfied is examined, over stressing being acceptable under these temporary conditions.

The initial losses at mid-span are first estimated. Since the distance of the cables from the bottom face is $0.066$ m, the permanent stress in the concrete (unloaded) at the level of the cables is:

$$100 + (1200 - 100) \times \frac{h - 0.066}{h}$$

$$= 100 + 1100 \times \frac{0.229}{0.295} = 950 \text{ t/m}^2, \text{ or } 95 \text{ kg/cm}^2$$

With $m_t = 9$, the loss of stress due to long-term deformation is:

$$95 \times 9 = 850 \text{ kg/cm}^2, \text{ or } 8.5 \text{ kg/mm}^2$$
In accordance with Chapter IV, Section 13, it is assumed that two-thirds of this loss will apply because of the long-term deformation and relaxation losses occurring together, or:

<table>
<thead>
<tr>
<th>Component</th>
<th>6.5 kg/mm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>loss due to shrinkage</td>
<td>4.0 kg/mm²</td>
</tr>
<tr>
<td>loss due to the cables not being tensioned simultaneously</td>
<td>2.0 kg/mm²</td>
</tr>
<tr>
<td>steel relaxation loss</td>
<td>10.0 kg/mm²</td>
</tr>
</tbody>
</table>

\[
\text{22.5 kg/mm}^2
\]

Since the permanent prestress is 89 kg/mm², the initial stress at this section must be equal to \(89 + 22.5 = 111.5\ \text{kg/mm}^2\), or \(111.5/89 = 1.29\) times the value of permanent prestress.

As a result, diagram (a) of Fig. 9 becomes:

\[-1.29 \times 427 = -550 \text{ t/m}^2\quad \text{at the top}\]

\[+1.29 \times 1727 = 2240 \text{ t/m}^2\quad \text{at the bottom}\]

Also, at the time when prestress is applied, the permanent load is only 4.25 t/m² instead of 4.25 + 1.50 = 5.75 t/m². The stresses on diagram (b) must therefore be multiplied by \((4.25/5.75) = 0.755\).

They became: \(\pm 527 \times 0.755 = \pm 387 \text{ t/m}^2\).

The diagrams representing the initial conditions are therefore those shown in Fig. 11.

Now, the quality of the concrete is determined by the values of the limit stresses under working conditions; 120 kg/cm² in the present case. If, in accordance with the ASP Recommendations, the value of the limit working stress is taken at 0.42 of the 28-day cylinder strength,† or 0.33 of the cube strength, the concrete must have a 28-day cube strength which is equal to, or greater than, \((120/0.33) = 360 \text{ kg/cm}^2\).

Assume that the concrete does, in fact, meet this requirement, that prestress occurs after 7 days and that the 7-day cube strength is 250 kg/cm².

Assume also that the value of the limit stress, at this particular stage (ASP Recommendations), is 0.55 of the cube strength (0.66 of the cylinder strength), or 137 kg/cm² (1370 t/m²). The stress of 1853 t/m² at the bottom fibre is too high, and it is therefore not possible to tension all the cables.

† This limit (Rule 4 332) is of the same order of magnitude as the limit 0.28 \(R_{\text{cube}, 90}\) of the Provisional Instructions, 1953, if the coefficient \((R_{\text{cyl}}/R_{\text{cube}})\) and the ages (28 and 90 days) of strength measurements are taken into consideration (see Chapter II, Section 1).
To meet the limit of $1\ 370$ t/m$^2$ the prestress at the bottom fibres (left-hand diagram of Fig. 11) must be equal to $387 + 1\ 370 = 1\ 757$ t/m$^2$. Therefore, only $(1\ 757/2\ 240) = 78\%$ of the cables can be tensioned, or $0.78 \times 28 = 22$ cables.

The prestress under the initial conditions at the top fibres becomes $0.78 \times (-550) = -430$ t/m$^2$, and the resultant stress on these fibres is $-430 + 387 = -43$ t/m$^2$ ($-4$ kg/cm$^2$), which is acceptable.

In practice, even fewer cables need be tensioned, and it is necessary, to tension only those cables which, under permanent stress conditions, balance the total permanent load. The moment $M_p$ being equal to 46 t/m, the eccentricity being $-0.081$ m, and the permanent force exerted by

![Diagram](image)

Fig. 11.

each cable being 41 t, each cable balances a moment of $-0.081 \times 41 = -3.3$ t/m. It is sufficient to tension only 15 cables, or 54\% of the total, to balance the permanent loads. The prestress, taking the higher initial stress into account, is $0.54 \times 1.29 = 0.70$ of the permanent final prestress, and strength requirements during construction are adequately met.

This method of first tensioning that number of cables which is capable of balancing the moment due to the permanent loading at their final permanent stress is often adopted. It is not a rigid method, and it can be seen that it offers a large amount of freedom in the particular case under consideration.

This freedom is not so pronounced with lighter sections, but it is generally possible to arrange for the design to be based on permanent conditions and to be independent of initial conditions.

(3) Deflection

The deflection $f$ must not exceed a certain fraction $e$ of the span when live loading is applied.
If \( s \) is the loading per unit length:

\[
f = \frac{5}{384} \frac{sl^4}{EI} = \frac{5}{48} \frac{sl^2v}{I_E} = \frac{5}{48} \frac{\sigma_s l^2}{E v}
\]

With a symmetrical section, \( v = h/2 \):

\[
f = \frac{10 \sigma_s l^2}{48 E h}
\]

\[
\frac{f}{l} = \frac{10 \sigma_s}{48 E h}
\]

The condition required is that \( (f/l) < \varepsilon \).
Therefore that: \((h/l) > (10/48) (\sigma_s/E) (1/\varepsilon)\).

If \( \sigma_s = 1 \, 100 \, \text{t/m}^2 \) and \( E = 350 \, 000 \, \text{kg/cm}^2 \) (3 \( 500 \, 000 \, \text{t/m}^2 \)) for loads of short duration, and if it is assumed that \( \varepsilon = (2/1 \, 000) \) in accordance with usual practice, then:

\[
\frac{h}{l} > \frac{10}{48} \times \frac{1 \, 100}{3.5 \times 10^6 \times 2 \times 10^{-3}}
\]

or

\[
\frac{h}{l} > \frac{1}{30} \text{ approximately}
\]

In the above example, \((h/l) = (0.295/8) = (1/27)\). The deflection criterion is therefore satisfied.

Remarks
(i) Only the live load deflection is considered above; or, in other words, the variation in deflection. It is this variation especially which can be troublesome, and which can render a structure inadequate for its intended purpose.

The deflection under permanent loading is usually negative (reverse deflection), the stress being higher at the bottom than at the top of the member. This deflection is seldom embarrassing, although differential variations with time can cause trouble if they have not reached their final values at the time of completion of the structure (floors supporting partitions or cladding).
This does not modify the conclusions with regard to the permissible \(h/l\) values, but it is possible that the introduction of some means of adjusting the laying or fixing of the loads carried by the slab might become necessary.

(ii) The above calculation assumes that live loads are of short duration. For long-term loading, the values of \(E\) must be decreased and this could lead to values of \(h/l\) which are higher than the minimum indicated above.

9. Variation in quantities and costs as a function of the depth of the slab
In the example of Section 5, the limiting permissible stresses were worked to. It was not, however, necessary to reach the top limit. This could have been fixed at 1 000 t/m\(^2\) instead of 1 200 t/m\(^2\). The advantages and disadvantages have to be considered.

To illustrate this by means of the previous example, consider various upper limits \(R\) for the stresses, assuming that the bottom limit \(R'\) is constant at 100 t/m\(^2\).

The fixed quantities are \(M_s = 96\) t/m, \(R' = 100\) t/m\(^2\), \(b = 6\) m.

If the hypothesis of compensation of permanent loading is accepted (signifying that the eccentricity resulting from the calculation provides sufficient cover, and this must be verified, then:

\[
\frac{bh^2}{6} = \frac{M_s}{R - R'}
\]

hence \(h\) is known.

Weight per metre of slab: 2.4 \(bh\) (the density of concrete being equal to 2.4 t/m\(^3\)).

Total permanent load per metre: \(p = (2.4 bh \times 1.5)\) t/m.

\[
M_p = p \frac{l^2}{8}
\]

\[
F = bh \frac{R + R'}{2}
\]

Eccentricities of the boundaries of the limit core: \(a = \pm (h/6) (R - R')/(R + R')\).

Eccentricity of prestress: \(e = -[a + (M_p/F)]\).

Cover: \((h/2) - |e|\).
The calculations are summarised in the following table:

<table>
<thead>
<tr>
<th>R(t/m²)</th>
<th>1 200</th>
<th>1 100</th>
<th>1 000</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>R + R'(t/m²)</td>
<td>1 300</td>
<td>1 200</td>
<td>1 100</td>
<td>1 000</td>
</tr>
<tr>
<td>R - R'(t/m²)</td>
<td>1 100</td>
<td>1 000</td>
<td>900</td>
<td>800</td>
</tr>
</tbody>
</table>

\[
bh^2 = \frac{96}{6} \frac{R - R'}{R - R'}
\]

| h (m) | 0.295 | 0.310 | 0.326 | 0.346 |

Weight of slab:

<table>
<thead>
<tr>
<th>2.4 bh (t/m)</th>
<th>4.25</th>
<th>4.46</th>
<th>4.70</th>
<th>4.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Total permanent load

| p (t/m) | 5.75 | 5.96 | 6.20 | 6.48 |

| M_p = p \times \frac{82}{8} (t/m) | 46   | 47.6 | 49.6 | 51.8 |

| F = bh \left(\frac{R + R'}{2}\right) (t) | 1 150 | 1 115 | 1 079 | 1 037 |

Limit eccentricity

<table>
<thead>
<tr>
<th>a = \frac{h R - R'}{6 R + R'} (m)</th>
<th>0.0416</th>
<th>0.0430</th>
<th>0.0446</th>
<th>0.0462</th>
</tr>
</thead>
<tbody>
<tr>
<td>\frac{M_p}{F} (m)</td>
<td>0.040</td>
<td>0.043</td>
<td>0.0461</td>
<td>0.0498</td>
</tr>
</tbody>
</table>

Prestress eccentricity

(absolute value)

| e = a + \frac{M_p}{F} (m) | 0.0816 | 0.0860 | 0.0907 | 0.0960 |

| Cover d' = (h/2) - e (m) | 0.066  | 0.069  | 0.072  | 0.077  |

The above table shows:

(i) that as the depth increases, the maximum stress decreases;
(ii) that the prestressing force, and hence the weight of the steel, decreases as the depth increases;
(iii) that, in the case under consideration, the cover increases with the depth.
The volume of concrete increases with the depth, and is therefore bigger with lower permissible compressive stresses.

But, because a concrete of lesser quality is acceptable, the result is a saving on cement.

Indeed, the depth varies as $1/(R - R')$ in accordance with the formula $(bh^2/6) = [M_u/(R - R')]$. The cement content varies approximately as $R$. The weight of cement therefore varies as $R/(R - R')$, which usually decreases with $R$.

Cost savings in steel and cement therefore result: these are equal to about 10\% when the permissible stress is lowered from 1 200 to 900 t/m$^2$. On the other hand, the quantities of aggregate increases (16\%).

Each particular case can be examined by trial and error as above, and the conclusion is usually the same; namely that an increase in depth usually results in cost savings.†

10. Example II (slab with a span greater than the critical span)

Slab bridge of 19-00 m span. Width of slab, 8-50 m. Width of roadway, 7-00 m. Weight of superstructures, 2-375 t/m. Maximum live load per metre length of slab, 11-734 t/m.

(Example of slab design with a span greater than the critical span.)

Minimum limit stress $R' = 100$ t/m$^2$.

The maximum permissible stress $R$ is not fixed. The most favourable value will be determined, with the reservation that it should not exceed 1 200 t/m$^2$.

A cover of 7 cm at the cable axis is assumed.

† Although theoretical economic comparisons should be treated cautiously, because of the impossibility of taking into consideration all the relevant factors, the following argument can be applied. Let $h$ be the depth; it is assumed that $R' = 0$. The steel percentage is $(A_c/bh) = (R/2T)$, where $T$ is the stress in the steel (Section 7). The weight of the steel is 80 kg/m$^3$ for $(A_c/bh) = 1\%$. The weight of steel per metre length of beam is therefore $80 \times 100 \times (R/2T) \times bh = 4 000(R/T)bh$.

If $A$ is the cost of the steel per kilogramme and $B$ is the cost per cubic metre of the concrete, the cost of the beam per metre length is:

$$P = bh \left( B + 4 000 \frac{R}{T} A \right)$$

Since $(bh^2/6) = (M_u/R)$, $h$ varies as $1/(R)^{1/2}$ and $P$ varies as $1/(R)^{1/2}(B + 4 000(R/T)A)$. This quantity is a minimum for $R = (BT/4 000A)$.

It is therefore advantageous to increase $R$ when the concrete cost is high, and to decrease $R$ when the cost of the steel is high.

If $T = 10 000$ kg/cm$^2$, $B = 160$ F/m$^3$ (concrete alone, without shuttering), $A = 4$ F/kg, and the optimum concrete strength is $R = 100$ kg/cm$^2$. 
The span, as seen in Chapter VIII, is greater than the critical span. As a result, the centre of compression, coincident with the position of the cable in pure prestress, and which rises a distance equal to \( M_p / F \) under the action of permanent loading, can no longer be at the lower limit position of the eccentricity \(- (h/6) (R - R')/(R + R')\) in the unloaded state; that is, under permanent loading alone. If it were, the displacement \(- M_p / F\) from the lower limit position would keep the cable within the section (with sufficient cover), and this is not true since the critical span is exceeded. Therefore, the cable must be positioned as low as possible within the member.

Let \( d' \) be the distance of the cable from the lower concrete face (minimum cover).

In the fully loaded state (permanent load plus live load), the centre of compression is at the top limit of eccentricity, \( + (h/6) (R - R')/(R + R') \). The eccentricity depends on the value of \( R \). Once \( R \) is specified, \( h \) can be determined in the following manner.

Under the action of the moment \( M_p + M_s \), the centre of compression
passes from eccentricity \(-[(h/2) - d']\) to eccentricity \(+(h/6)(R - R')/(R + R')\).

Therefore:

\[
F \left( \frac{h}{2} - d' + \frac{h}{6} \frac{R - R'}{R + R'} \right) = M_p + M_s
\]

\(p\) (and consequently \(M_p\)) is a function of \(h\) and \(F\) is equal to \(bh/2\) \((R + R')\). Equation (1) therefore provides a relation between \(R\) and \(h\).

It is therefore possible either to take several values of \(R\) and then to determine \(h\) and the prestress \((F\) and \(e)\), and to compare the economics of the various solutions so obtained; or, conversely, to assume several values of \(h\) and then to determine the values of \(R\) and the prestress, and again to compare the various solutions.

The second method of calculation is adopted below, because it is the simplest, as follows.

The moment due to the live loads is:

\[
M_s = 11.734 \times \frac{19^2}{8} = 530 \text{ tm}
\]

The moment due to permanent loading is:

\[
M_p = (2.375 + 2.4 bh) \times \frac{19^2}{8}
\]

\((2.4 \text{ t/m}^3\) is the density of the concrete).

The minimum moment is \(M_1 = M_p\); the maximum moment is \(M_2 = M_p + M_s\). The stresses due to these moments are:

\[
\sigma_1 = \frac{M_1}{(bh^2/6)}, \quad \sigma_2 = \frac{M_2}{(bh^2/6)}
\]

The absolute value of the eccentricity is \(|e| = (h/2) - d' = (h/2) - 0.07 \text{ m}.

The prestressing force \(F\) which is required to restore the stress at the bottom fibre to \(+100 \text{ t/m}^2\) in the loaded state is such that:

\[
\frac{F}{bh} \left( 1 + 6 \frac{|e|}{h} \right) - \sigma_2 = 100
\]

Hence \(F/bh\), and therefore \(F\), is obtained.
The stresses at the extreme fibres are:

\[
\sigma_0 = \frac{F}{bh} \left(1 - \frac{6|e|}{h}\right); \quad \sigma'_0 = \frac{F}{bh} \left(1 + \frac{6|e|}{h}\right)
\]

Hence the resultant stresses \(\sigma_0 + \sigma_1\), \(\sigma'_0 - \sigma_1\), \(\sigma_0 + \sigma_2\) in the two conditions of loading are determined (the fourth stress, in the fully loaded condition, is \(\sigma'_0 - \sigma_2 = 100 \text{ t/m}^2\)).

The maximum stresses are \(\sigma'_0 - \sigma_1\) (R₁, unloaded, at the bottom) and \(\sigma_0 + \sigma_2\) (R₂, loaded, at the top).

A comparison of values enables the most attractive design to be selected. The calculations are summarised in the table below. This shows:

(i) That a slab of minimum thickness 0·65 m is required in order not to exceed 1200 t/m² in the loaded condition (this requires a concrete cube strength of 360 kg/cm² at 28 days).

(ii) That an increase in slab thickness is accompanied by a reduction in the stresses and in the amount of steel required. For \(h = 0·75\) m, the maximum stress (line 19) is 93 kg/cm². This permits the use of a concrete with a reduced cube strength of 280 kg/cm² at 28 days.

With a concrete cube strength of 300 kg/cm² at 28 days, the required slab thickness is \(h = 0·73\) m. The maximum permissible stress is \(0·33 \times 300 = 100\) kg/cm².

The required prestressing force (line 15) is then 3340 tonnes.

In order to exert this total force, 7 mm diameter, 12-wire FREYSSINET cables are chosen, each capable of applying a force of 41 tonnes (89 kg/mm²). Eighty-one such cables are required. They are arranged in groups of three, and 27 groups are needed, at a pitch of 8·50/27 = 0·31 m.

The usefulness of applying the prestress in stages to satisfy the initial strength criteria (partial permanent load, relatively green cement, tensioning stress at its initial value) can be argued as in Section 8.

In general, an upwards camber is provided on the soffit of the beam (at the rate of 2%, with a parabolic connecting curve at the centre). The characteristics are slightly modified, but the principles of the above calculation are still valid if \(I/\nu\) and \(I/\nu'\) are substituted for \(bh^2/6\).

It is therefore seen that it is again possible to establish a design by the use of normal strength-of-materials procedures. The solution could have been obtained with the use of formula (1), by fixing either the depth or the stress R, but the trial and error calculation is just as expedient.
<table>
<thead>
<tr>
<th>$h$ (m)</th>
<th>0.60</th>
<th>0.65</th>
<th>0.70</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bh = 8.50 \times h$ (m$^2$)</td>
<td>5.1</td>
<td>5.5</td>
<td>5.95</td>
<td>6.38</td>
</tr>
</tbody>
</table>

**Weight of slab per m$^2$:**

| 2.4 $bh$ (t/m) | 12.2 | 13.2 | 14.3 | 15.3 |
| 2.37 (t/m)     | 2.37 | 2.37 | 2.37 | 2.37 |

**Permanent load $p$ (t/m):**

| 14.57 | 15.57 | 16.67 | 17.67 |

| $M_1 = M_p$ (tm) | 656 | 704 | 751 | 797 |
| $M_s$ (tm)       | 530 | 530 | 530 | 530 |

| $M_2 = M_p + M_s$ (tm) | 1186 | 1234 | 1281 | 1327 |

| $bh^2/6$ (m$^3$) | 0.51 | 0.596 | 0.695 | 0.797 |

**Eccentricity**

| $e = \frac{h}{2} - 0.07$ (m) | 0.23 | 0.255 | 0.28 | 0.305 |

| $\frac{e}{h}$ | 2.30 | 2.35 | 2.40 | 2.44 |

| $F/S = \frac{\sigma'_0}{1 + (6|e|/h)}$ (t/m$^2$) | 736 | 648 | 573 | 513 |

| $F = \frac{F}{S} \times bh$ (t) | 3754 | 3564 | 3409 | 3273 |

| $\sigma_0 = \frac{F}{S} \left(1 - \frac{6|e|}{h}\right)$ (t/m$^2$) | -957 | -874 | -802 | -739 |

**Resultant stresses (t/m$^2$): Unloaded**

| $\sigma = \sigma_0 + \sigma_1$ | +323 | +311 | +278 | +261 |
| $\sigma' = \sigma'_0 - \sigma_1(R_1)$ | +150 | +985 | +870 | +765 |

**Loaded**

| $\sigma = \sigma_0 + \sigma_2(R_2)$ | +1373 | +1196 | +1048 | +926 |
| $\sigma' = \sigma'_0 - \sigma_2$ | +100 | +100 | +100 | +100 |
11. Solution using partial prestressing

If a moderate value of tensile stress is assumed for the permissible stress in the preceding table, a limited prestress solution is obtained.

Assuming $R' = -200$ t/m² and limiting the value of $h$ to 0.70 and 0.75 m, the calculations from line 9 of the preceding table become:

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.70</th>
<th>0.75</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>1080</td>
<td>1000</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>1850</td>
<td>1665</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>$\sigma'_0 = \sigma_2 - 200$</td>
<td>1650</td>
<td>1465</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>$\frac{6</td>
<td>e</td>
<td>}{h}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{F}{S} = \frac{\sigma'_0}{1 + 6</td>
<td>e</td>
<td>/h}$</td>
<td>485</td>
</tr>
<tr>
<td>$F = \frac{F}{S} \times S$</td>
<td>2885</td>
<td>2717</td>
<td>(t)</td>
</tr>
<tr>
<td>$\sigma_0 = \frac{F}{S} \left(1 - \frac{6</td>
<td>e</td>
<td>}{h}\right)$</td>
<td></td>
</tr>
<tr>
<td>Resultant stresses:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unloaded $\sigma = \sigma_0 + \sigma_1$</td>
<td>401</td>
<td>387</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>$\sigma' = \sigma'_0 - \sigma_1$</td>
<td>570</td>
<td>465</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>Loaded $\sigma = \sigma_0 + \sigma_2$</td>
<td>1171</td>
<td>1052</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>$\sigma' = \sigma'_0 - \sigma_2$</td>
<td>-200</td>
<td>-200</td>
<td>(t/m²)</td>
</tr>
</tbody>
</table>

The solution with $h = 0.75$ m is the better.

By comparison with the solution using total prestress, for the same thickness, the stress under load at the top fibre of the concrete is increased because of the reduction in the absolute value of the negative prestress $\sigma_0$.

The quantity of steel is reduced by 17%. $F = 2717$ t, requiring sixty-six 7 mm diameter, 12-wire cables with a stress after losses of 89 kg/mm² (the force per cable is 461 mm² × 89 = 41 000 kg).

The section is wholly in compression in the unloaded condition. It loses its compression only when the proportion of the live loading is 465/(465 + 200) = 70% of the maximum.
It would be useless, from strength considerations, to provide non-tensioned complementary steel. In other words, the slab would not fracture even if cracking did occur. Assuming that the cables are not overstressed as a result of the cracking, the centre of compression would rise a distance of \((M_2/F) = (1327/2717) = 0.49\) m above the cable. It would then be at a depth of \(0.75 - 0.07 - 0.49 = 0.19\) m from the top concrete face. The depth of the section in compression would then be equal to \(3 \times 0.19 = 0.57\) m. The maximum stress in the concrete would be \((2 \times F)/(bx) = (2 \times 2717)/(4845) = 1121\) t/m². The section therefore has sufficient strength, without needing complementary reinforcement.

Nevertheless, the ASP Recommendations specify, quite rightly, a minimum quantity of complementary reinforcement. This restricts the opening of cracks, if they occur, and it spreads them out so that they take the form of closely spaced fine fissures, or microcracks. The following simplified calculation is assumed:

(a) The depth \(x'\) of the zone in tension is determined on the assumption that the section behaves as an homogeneous section, and the resultant tensile stress is also determined.

In this case \(x' = [200/(1052 + 200)] \times 0.75 = 0.119\) m.

The resultant tensile stress is:

\[
8.50 \times 0.119 \times \frac{200}{2} = 101\ t
\]

(b) Complementary reinforcement is provided, capable of resisting a proportion (one-half) of this resultant, when stressed to the usual conventional values (14 kg/mm² for mild steels, 21 kg/mm² for steels with improved bonding properties).

In this case the cross-sectional area of the steel required is:

\[
\frac{1}{2} \times \frac{107}{1.4\ t/cm^2} = 36\ cm^2
\]

This is the minimum requirement.

Safety is a further consideration which affects the provision of complementary reinforcement. This aspect is studied in Chapters IX and X. Suffice it to say that in the present case the statutory safety factor which is required in France is achieved without the need for complementary steel.

The 36 cm² quantity of steel indicated above is therefore satisfactory. Nevertheless, since there are 22 groups of 3 cables, 16 mm diameter,
22-wire cables would be provided, or 44 cm$^2$. These would be located at a distance of 3 cm from the bottom face of the concrete.

**Assessment of stresses with cracking**

If a crack should occur, the complementary steel is tensioned. The cables are also subjected to additional tension if they are well bonded. It is therefore possible to consider them as part of the reinforcement for a calculation based on reinforced concrete under combined bending and compression due to the effects of the moment $M_2$ and a compressive force resulting from the permanent stress in the cables ($F = 2717$ t, $M = 1327$ t.m). They are therefore subjected to a tensile stress additional to the permanent stress.

\[ \text{Fig. 13.} \]

The usual rules for the design of reinforced concrete could be followed if so required.

It can equally be assumed, in these calculations, that the depth $x$ in compression when cracking occurs is equal to that obtained above, neglecting the additional tensile stress in the cables, namely $x = 0.57$ m; hence the compressive stress in the top fibre is $R = 1121$ t/m$^2$, as previously calculated. From this, the stresses in the cables (stress increase $\Delta T$) and in the complementary steel ($\sigma'_a$) are deduced. If $h_1$ is the depth of concrete above the cables, $h'_1$ the depth of concrete above the reinforcement and $m$ the modular ratio, $\Delta T = mR \frac{(h_1 - x)}{x}$ and $\sigma'_a = mR \frac{(h'_1 - x)}{x}$.

If $A_c$ and $A_a$ are the cross-sectional areas of the cables and of the complementary steel respectively, the resultant force in the reinforcement is $A_c \Delta T + A_a \sigma'_a$. The moment with respect to the cable is $M = A_a \sigma'_a (h'_1 - h_1)$.
The resultant stress in the concrete in compression is:

\[ F + A_c \Delta T + A_a \sigma'_a \]

If these were the stresses in the steel, the distance of the centre of compression from the cable would be equal to:

\[ \frac{M + A_a \sigma'_a (h' - h)}{F + A_c \Delta T + A_a \sigma'_a} \]

hence corrected values for \( x \) and \( R \) are obtained.

With these values, the calculation for \( \Delta T \) and \( \sigma'_a \) is repeated, and the calculation for \( x \) and \( R \), and so on.

In this manner, values which approach the true values are very quickly obtained.

The following results for the successive stages are thus obtained.

for \( x \):
- 0.57;
- 0.606;
- 0.597 m,

for \( R \):
- 121;
- 1090;
- 1100 t/m²,

for \( \Delta T \):
- 202;
- 131;
- 148 kg/cm²,

for \( \sigma'_a \):
- 287;
- 207;
- 230 kg/cm².

The results of the third trial are sufficiently accurate.

As mentioned previously, the ASP Recommendations do not require these calculations.
Chapter VI

DESIGN OF A BEAM OF ANY CROSS-SECTION USING STRENGTH-OF-MATERIALS FORMULAE, WHERE PERMANENT LOADING CAN BE COUNTERACTED
(Span less than the critical span)

Since by hypothesis the permanent loads can be compensated, it is sufficient to know the live load bending moment in order to determine the section.

As seen in Chapter V, using the same assumption, the section and the prestress must be such that the stresses under working conditions are restricted to the permissible values \( R'_1 \) and \( R_1 \) in the unloaded state (when the minimum bending moment is applied), and \( R'_2 \) and \( R_2 \) in the loaded state (when the maximum bending moment is applied).\(^\dagger\) Moreover, it is necessary that in the construction stages and under initial prestress the stresses should remain within certain limits \( R'_1 \) and \( R_1 \), which are temporarily permissible and are greater than the limits \( R'_1 \) and \( R_1 \).

As in Chapter V, the design is based on the working-load condition, when the cables have reached their state of permanent stress.

A check that the stresses during construction do not exceed their allowable limits is then made.

In certain cases, particularly when the permanent loading is only partially present at the time when prestress is applied (at the initial stress level) it may be necessary to tension only a proportion of the cables, the remainder being tensioned once the whole of the permanent dead load is applied.

The acceptable stresses in this initial condition are discussed in Chapter IX.

At this stage the ASP allows the limits in compression to reach 50 to

\(^\dagger\) Different limits can be fixed for these two conditions of loading. This course is adopted in this chapter in order to render the discussion more general.
55% of the cube strength at that time.† With regard to tensile stresses, the limits which can be tolerated during construction depend upon the nature and the location of the work. If a potential crack, which is later closed anyway, is not an embarrassment, than a fairly high limiting stress is acceptable, provided that the tensile zone is reinforced. The ASP permits a limiting stress which is equal to the tensile strength at that time.

It is shown in this chapter that normal strength-of-materials formulae can be used for design, as in the case of a beam of rectangular cross-section.

The examples are developed and explored in greater detail, to bring out certain characteristics in the calculations which are peculiar to prestressed concrete design.

Chapter VII describes a more intuitive method of calculation, derived from a consideration of the displacements of the centre of compression.

As a first approximation, the variations of the prestressing force due to the variations in the bending moment under working conditions can be neglected in the cases of total prestressing and partial prestressing.

This approximation increases the resistance to cracking.

Some small savings in the quantity of steel are possible by taking this overtensioning into consideration.

The following approximate reasoning can be applied.

If Δσc is the stress variation local to the cable and if m is the modular ratio [m = (E_u/E_b)], the excess tension in the cables, assuming they are efficiently bonded, is m Δσc.

The force F to be applied can be obtained as explained in the following pages; T is the permanent stress after deduction of losses—that is, in the permanently loaded state. The stress in the loaded condition is then T + m Δσc. The required cross-sectional area is approximately F/(T + m Δσc). The saving which arises from the correction m Δσc is greater if Δσc is high: it is therefore greater in the case of limited prestress than in the case of total prestress. The saving is also greater if m is high, so that it is greater in the case of sustained live loading than in the case of rapidly applied loading, since E_b is smaller in the first of these cases. The method of allowing for the corrections is given in Section 3.

1. Principles of design
With a given depth of beam h, it is required to determine:

(i) the cross-sectional area S of the beam and the resistance moduli I/v and I/v' (three unknowns);

† 0.6 to 0.66 R_cyl.
(ii) the prestress, that is, the prestressing force $F$ and the eccentricity $e$ (two unknowns).

There are thus five unknowns. Because it is assumed that the span is less than the critical span, four equations are obtained from the requirement that the permissible stresses are reached at the two extreme fibres in the two conditions of extreme loading.

In practice, once the particular shape of section is selected (I-beam or T-section, say), the value of $S$ is virtually known when $I/v$ and $I/v'$ are known.

Therefore, the section moduli must first be determined. These lead to a particular shape of section, and thence to a value for $S$. The two unknowns related to prestress must then be found.

The problem is thus defined by the choice of the general shape of the section. This choice is, of course, not entirely arbitrary, and it should provide the most economical solution. This aspect could be considered as a fifth condition, bringing $S$ into the system of equations to be resolved.

(i) Design of section

Since the limit stresses occur in the unloaded conditions (or more generally when the minimum bending moment $M_1$ is applied), the corresponding stress diagrams are as shown in Fig. 1.

![Fig. 1.](image-url)

The solution is immediately obtained from Fig. 1. The stresses $\sigma_s$ and $\sigma'_s$ are due to the live load $s$, which is known. Under the action of the live load, the bending moment $M_1$ changes to a new value $M_2$.

Let $M_s$ be the bending moment due to the live load:

$$\sigma_s = \frac{M_s}{I/v}, \quad \sigma'_s = \frac{M_s}{I/v'}$$
From Fig. 1:

$$\sigma_s = R_2 - R'_{1}$$

$$\sigma'_s = -(R'_2 - R_1) = R_1 - R'_2$$

Therefore:

$$\frac{I}{v} = \frac{M_s}{R_2 - R'_{1}} \quad (1)$$

$$\frac{I}{v'} = \frac{M_s}{R_1 - R'_2} \quad (2)$$

These formulae are similar to those obtained in the case of slab design, and may be written as follows:

$$\frac{I}{v \text{ or } v'} = \frac{\Delta M}{\Delta R} \quad (3)$$

where $\Delta M$ is the change in the bending moment between the two extreme conditions of loading and where $\Delta R$ is the change of stress at the extreme fibre under consideration.

Formula (3) is important, and it illustrates the major part played by variations in bending moments and stresses in prestressed concrete design. The formula is general, and is valid even if the extreme bending moments
are not $M_p$ (moment with permanent loading) and $M_p + M_s$ (moment with permanent loading, plus total live load).†

Moments $M_1$ and $M_2$ can always be calculated, and thus the variation $\Delta M = M_2 - M_1$.

The moment $M_p$, due to permanent loading, is eliminated by taking the difference $M_2 - M_1$.

The values of $I/v$ and $I/v'$ are therefore obtained from formulae (1) and (2), and the ratio $v/v' = (I/v')/(I/v)$ is determined.

Since $h$ is known, $v$ and $v'$ are obtained. Then $I = I/v \times v$.

It is required to determine the shape of a section of given height $h$, with its centroid at a given height $(v, v')$, with a moment of inertia $I$. The most economical solution is required, in which the cross-sectional area $S$ is a minimum for the required value of $I$, or more exactly for the required values of $I/v$ and $I/v'$. Practical considerations of web and flange thickness must be observed.

Maximum values of the ratios $(I/v)/S$ and $(I/v')/S$ are required; that is, the ratios between the section moduli and the cross-sectional area of the section. These ratios are of considerable importance, and they recur constantly in the calculations. Since $I/S = r^2$, where $r$ is the radius of gyration of the section about the horizontal axis through the centroid, the ratios $(I/v)/S$ and $(I/v')/S$ are equal to $r^2/v$ and $r^2/v'$ respectively, and they are expressed in units of length. They are equal to the eccentricities of the top and bottom limits of the central core, within which the centre of compression must remain if the section is to remain wholly in compression.

† In the case of an overhanging beam, for example, the minimum bending moment is obtained by loading the overhangs, and the maximum bending moment by loading the span between the supports. If $M_s(+)\text{ is the positive bending moment under the action of the live loads (between the supports), and } M_s(-) \text{ is the absolute value of the negative bending moment with live loads (acting on the overhangs), then:}$

\[
M_1 = M_p - M_s(-) \quad \text{and} \quad M_2 = M_p + M_s(+);
\]

\[
\Delta M = M_s(+) + M_s(-)
\]

Fig. 3.
(ii) Prestress

(a) The prestressing force $F$ can be determined since the section is determined.

Let $y$ be the distance of any point in the section from the centroid, and let $M$ be the bending moment (including the moment due to the prestress) under any condition of loading. The stress $\sigma(y)$ at point $y$ is:

$$\sigma(y) = \frac{F}{S} + \frac{My}{I}$$

When $y = 0$, $\sigma = F/S$. Since it is assumed that $F$ is constant under all conditions of loading, the stress at the centroid is also constant. Let this stress be $\sigma_g$ ($\sigma_g = F/S$).

The stress diagrams in the various conditions of loading therefore have a constant value at a common fixed point $\sigma_g$ (Fig. 4). This point is obtained by considering two particular stress diagrams. Consider the diagrams $R_1R'_1$, $R_2R'_2$. They intersect at the required point $\sigma_g$ (Fig. 4). Therefore:

$$\frac{v}{R_2} = \frac{v'}{R'_2} = \frac{R'_1}{R_1 - R'_2}$$

This relation is the same as that obtained in subsection (i) above, since

$$\frac{v}{v'} = \frac{I/v'}{I/v} = \frac{R_2 - R'_1}{R_1 - R'_2}$$

Also:

$$\sigma_g = R'_1 + (R_1 - R'_1) \frac{v}{h} = R'_2 + (R_2 - R'_2) \frac{v'}{h}$$
or:

\[ \sigma_g = R_1 \frac{v}{h} + R'_1 \frac{v'}{h} = R'_2 \frac{v}{h} + R_2 \frac{v'}{h} \]

and

\[ F = S \sigma_g \]

(b) Eccentricity. For simplicity, consider the usual case where \( M_1 \) is the bending moment in the unloaded condition; that is, under the action of the permanent load only \( (M_1 = M_p) \).

If the beam were weightless, condition (1) of Fig. 1 would represent pure prestressing. The eccentricity of the prestress would be the eccentricity \( e_1 \) corresponding to this condition, where the stresses are \( R_1 \) in the bottom fibre and \( R'_1 \) in the top fibre.

\( e_1 \) is easily found:

\[ R'_1 = \frac{F}{S} + \frac{Fe_1}{I/v} = \frac{F}{S} \left( 1 + \frac{e_1}{r^2/v} \right) = \sigma_g \left( 1 + \frac{e_1}{r^2/v} \right) \quad (4) \]

and:

\[ R_1 = \frac{F}{S} - \frac{Fe_1}{I/v'} = \frac{F}{S} \left( 1 - \frac{e_1}{r^2/v'} \right) = \sigma_g \left( 1 - \frac{e_1}{r^2/v'} \right) \quad (5) \]

Hence:

\[ e_1 = \frac{r^2}{v'} \left( 1 - \frac{R'_1}{\sigma_g} \right) \quad (6) \]

or:

\[ e_1 = \frac{r^2}{v'} \left( 1 - \frac{R_1}{\sigma_g} \right) = -\frac{r^2}{v'} \left( \frac{R_1}{\sigma_g} - 1 \right) \quad (7) \]

The two expressions (6) and (7) for \( e_1 \) are obviously equal.† This can be checked on Fig. 4, which shows that \( (R_1 - \sigma_g)/v' = (\sigma_g - R'_1)/v \), hence the two expressions are equal.

It is preferable to work with equation (6) for reasons which are explained in Chapter VII.

The term \(-(r^2/v)\) is the eccentricity of the lower boundary of the central core. This is easily proved. If \( e_1 = -(r^2/v) \) is substituted in equation (4), then \( R'_1 = 0 \), and this condition defines the lower boundary of the central core.

† It should be noted that the values given by eqns. (6) and (7) are equal only because the limit stresses \( R_1 \) and \( R'_1 \) are effectively reached, the span being less than the critical span (see Chapter VII).
core (the position of the centre of compression for zero stress at the opposite extreme fibre).

The eccentricity $e_1$ is therefore obtained by a reduction or an increase $1 - (R'_1/\sigma_g)$ from the eccentricity of the lower boundary of the central core. It is a reduction if $R'_1 > 0$ and an increase if $R'_1 < 0$. Hence the point $E_1$ in Fig. 5.

(c) Since the beam is not weightless, the eccentricity due solely to pure prestress (namely the point of intersection $E_0$ of the resultant cable with the section) is obtained by displacing $E_1$ through a distance equal to $-(M_p/F)$, as seen in Chapters I and V.

\[ e_0 = \frac{r^2}{v} \left( 1 - \frac{R'_1}{\sigma_g} \right) - \frac{M_p}{F} \]  

Fig. 5.

The eccentricity of the prestressing force is, therefore:

Now since the cross-sectional area $S$ is known, the self-weight and hence the permanent load can be determined, and therefore $M_p, e_0$ is then found, and the analysis is complete.

If $e_0$ lies within the section, with sufficient cable cover, the solution is acceptable. In other words, it is possible to achieve the four limits $R_1, R'_1, R_2, R'_2$.

There is then no cost penalty for the permanent load, in terms either of concrete or of steel, since the permanent load appears neither in the

† More generally by a displacement $-(M_1/F)$ if $M_1$ is not equal to $M_p$. 
determination of the section (therefore of S), nor in the determination of the prestressing force (therefore of the cross-sectional area of steel). The permanent load is compensated solely by moving the cable from \( E_1 \) to \( E_0 \).

If the cable cover is not sufficient, then the solution is impossible. In other words, the materials cannot be used to their fullest advantage under both conditions of loading.

**2. Example**
Freely supported beam, 14 m span, 1·10 m deep, 5 t/m live load.

![Diagram](image)

**Fig. 6.**

The permissible stress limits are:

- **unloaded:**
  - 1 350 t/m at the bottom fibre
  - \(-150 \text{ t/m}^2\) at the top fibre
- **loaded:**
  - 1 000 t/m² at the top fibre
  - \(-100 \text{ t/m}^2\) at the bottom fibre

The bending moment due to the live load is:

\[
M_s = 5 \times \frac{14^2}{8} = 122.5 \text{ t/m}
\]

† Prefabricated beams for a heavily loaded floor (La Villette slaughterhouse). Some minor changes are introduced.
(i) Design of section

If the limiting stresses are effectively reached in the two conditions of loading, then the section moduli \( I/v \) and \( I/v' \) are such that:

\[
\frac{I}{v \text{ or } v'} = \frac{\Delta M}{\Delta R} = \frac{M}{\Delta R}
\]

Or, with reference to Fig. 6:

\[
\frac{I}{v} = \frac{122.5}{1150} = 0.1065 \text{ m}^3
\]

\[
\frac{I}{v'} = \frac{122.5}{1450} = 0.0845 \text{ m}^3
\]

Hence:

\[
\frac{v}{v'} = \frac{0.1065}{0.0845} = 1.26
\]

\[
\frac{v'}{1.26} = \frac{v}{1} = \frac{h}{2.26}
\]

\[
v = \frac{1.10}{2.26} = 0.487 \text{ m}
\]

\[
v' = 1.10 - 0.487 = 0.613 \text{ m}
\]

Therefore:

\[
I = \frac{I}{v} \times v = 0.1065 \times 0.487 = 0.0519 \text{ m}^4
\]

The solution now demands the determination of a section having the following characteristics \((v = 0.487 \text{ m}, v' = 0.613 \text{ m}, I = 0.0519 \text{ m}^4)\).

An irregular I-beam is assumed, with a broad top flange. The following characteristics are also postulated:

(1) the web thickness is 0.16 m;
(2) the mean thickness of the top flange is 0.13 m (0.11 m at the tip and 0.15 m at the root).

There are obviously several solutions to the problem. It may be resolved in several different ways. For example, the following method can be used.

It is first assumed that the flanges are rectangular in section, of constant thickness. The unknowns are the cross-sectional flange areas \( s \) and \( s' \),
shown hatched in Fig. 7, and the distances \( d \) and \( d' \) of their centroids from the known centroid \( G \) of the section.

There are therefore four unknowns and two equations, the first being an expression for the position of the centroid \( G \) and the second being an expression for the moment of inertia \( I \).

The unknown \( d \) is eliminated if the thickness of the top flange is assumed, and the three remaining unknowns are the breadth \( b \) of the top flange, and the two unknowns \( s' \) and \( d' \) for the bottom flange.

![Diagram](image)

**Fig. 7.**

Various values of \( s' \) and \( d' \) are obtained by taking several values of \( b \), and the calculation is repeated until a satisfactory section is obtained.

The web area is \( 0.16 \times 1.10 = 0.176 \text{ m}^2 \).

The centroid of the web is at a distance of 0.55 m from the bottom face, therefore 0.063 m below \( G \).

Since \( G \) is the centroid of the section, taking moments about \( G \):

\[ sd = 0.063 \times 0.176 + s'd'. \]

Therefore:

\[ sd - s'd' = 0.0111 \quad (1) \]

The moment of inertia of the web is: \( 0.16 \times (1.1^3/12) = 0.0178 \text{ m}^4 \).

Assuming that the moments of inertia of the flanges themselves can be neglected, the second equation (moment of inertia = 0.0519 \text{ m}^4) is written:

\[ 0.0178 + 0.176 \times 0.063^2 + sd^2 + s'd'^2 = 0.0519 \]

\[ sd^2 + s'd'^2 = 0.0519 - (0.0178 + 0.0007) \]
and
\[ sd^2 + s'd''^2 = 0.0334 \]  
(2)

If it is assumed that the top flange thickness is 0.13 m, then \( d = 0.422 \) m (Fig. 7).

For each value of \( b \), \( s \) is known, and therefore \( sd \) and \( sd^2 \) are known, and:
\[ s'd' = sd - 0.0111 \]  
(1a)

and
\[ s'd''^2 = 0.0334 - sd^2 \]  
(2a)

\( s' \) and \( d' \) are thus obtained, and therefore the dimensions of the bottom flange.

The most economical and practical of the solutions so obtained is chosen.

This method is only approximate, but it is sufficiently accurate for establishing the profile of the section; it is approximate because it is assumed that the flanges are rectangular (of constant thickness). Corrections can be easily applied to take the true shapes of the flanges into account, and they are applied in the final checking calculations.

With the present example, the correction for the top flange is negligible. The correction for the bottom flange can be obtained graphically, as explained later.

Assuming that \( b = 0.92 \) m is chosen, the profile is determined by means of equations (1a) and (2a).

Since the thickness of the top flange is 0.13 m and since the flange cross-sectional area does not include the web, then:

\[ s = (0.92 - 0.16) \times 0.13 = 0.099 \text{ m}^2 \]

\[ sd = 0.099 \times 0.422 = 0.0418 \text{ m}^3 \]

\[ sd^2 = 0.0418 \times 0.422 = 0.0176 \text{ m}^4 \]

Equations (1a) and (2a) give:
\[ s'd' = 0.0307 \]
\[ sd''^2 = 0.0158 \]

By division:
\[ d' = 0.515 \text{ m} \]

and
\[ s' = \frac{0.0307}{0.515} = 0.060 \text{ m}^2 \]
If the cross-section of the bottom flange is rectangular, its thickness is (see Fig. 7):

\[ 2(v' - d') = 2(0.613 - 0.515) = 0.195 \text{ m} \]

and its breadth is (see Fig. 8):

\[ 0.16 + \frac{s'}{0.195} = 0.16 + \frac{0.060}{0.195} = 0.466 \text{ m} \]

The bottom flange which is thus obtained is shown diagrammatically by the hatched area in Fig. 8.

The characteristics of the main section are not significantly affected if the actual shape of the bottom flange has the same cross-sectional area and centroid as the rectangular shape shown in Fig. 8. The true shape of the flange, shown dotted in Fig. 8, must be such that the increase and loss of area are compensated when it is superimposed onto the calculated rectangular shape (algebraic sum equal to zero, zero moment). The correction to the basic shape can be done by eye. The shape is then finally recalculated, and the differences between the values of the section moduli \((I/v, I/v')\) and the theoretical values can be accepted, if they are not too great.

The section in this example is taken to be as shown in Fig. 9. Its characteristics are:

- \( S = 0.334 \text{ m}^2 \)
- \( v = 0.486 \text{ m} \)
- \( v' = 0.614 \text{ m} \)
- \( I = 0.0518 \text{ m}^4 \)
- \( \frac{I}{v} = 0.1066 \text{ m}^3 \)
- \( \frac{I}{v'} = 0.0844 \text{ m}^3 \)
- \( r^2 = \frac{I}{Sv} = 0.319 \text{ m} \)
- \( \frac{r^2}{v'} = 0.253 \text{ m} \)
(ii) Determination of prestress

The stress variations due to the live load bending moment $M_l$ are:

$$\Delta \sigma = \frac{122.5}{0.1066} = 1149 \text{ t/m}^2$$

$$\Delta \sigma' = \frac{122.5}{0.0844} = 1451 \text{ t/m}^2$$

and they are very nearly equal to the design values.

In the unloaded condition, the stresses must be equal to $-150$ and $+1350 \text{ t/m}^2$.

The stress at the centroid is:

$$\sigma_g = -150 + (1350 + 150) \times \frac{0.486}{1.10} = 512.7 \text{ t/m}^2$$

The prestressing force is: $F = S\sigma_g = 0.334 \times 512.7 = 171.2 \text{ t}$.

![Diagram showing stress distribution](image)

Three 8 mm diameter, 12-wire cables are chosen, each with a cross-sectional area of 602 mm$^2$.

The permanent stress at mid-span must be equal to:

$$\frac{171200}{3 \times 602} = 95 \text{ kg/mm}^2$$
The eccentricity, if the beam were weightless, would be equal to the lower limit eccentricity (lower boundary of limit core), or (see Section 1):

\[ e_1 = -\frac{r^2}{v} \left( 1 - \frac{R'_1}{\sigma_g} \right) = -0.319 \left( 1 + \frac{150}{513} \right) = -0.412 \text{ m} \]

The eccentricity of the prestress (Section 1) must therefore be equal to:

\[ e_0 = e_1 - \frac{M_p}{F} \]

The weight \( p \) is equal to \( S \times 2.4 \text{ t/m}^2 \), or \( p = 0.334 \times 2.4 = 0.802 \text{ t/m} \).

\[ M_p = 0.802 \times \frac{14^2}{8} = 19.65 \text{ tm} \]

and:

\[ e_0 = -0.412 - \frac{19.65}{171.2} = -0.412 - 0.115 = -0.527 \text{ m} \]

The cable cover is equal to \( v' - |e_0| = 0.614 - 0.527 = 0.087 \text{ m} \). This is acceptable. The hypothesis that the self-weight could be compensated is therefore justified.

**Check Calculation**

\[ M_1 = 19.65 \text{ tm} \quad M_2 = 19.65 + 122.5 = 142.15 \text{ tm} \]

**Stresses due to external loading**

\[ \sigma_1 = \frac{19.65}{0.1066} = 184 \text{ t/m}^2 \]

\[ \sigma_2 = \frac{142.15}{0.1066} = 1333 \text{ t/m}^2 (\Delta \sigma = 1149 \text{ t/m}^2) \]

\[ \sigma'_1 = \frac{-19.65}{0.0844} = -233 \text{ t/m}^2 \]

\[ \sigma'_2 = \frac{-142.15}{0.0844} = -1684 \text{ t/m}^2 (\Delta \sigma' = 1451 \text{ t/m}^2) \]
Prestress

\[
\sigma_0 = \frac{F}{S} + \frac{Fe}{I/v} = \frac{F}{S} \left(1 + \frac{e}{r^2/v}\right) = 512.7 \left(1 - \frac{0.527}{0.319}\right) = -334 \text{ t/m}^2
\]

\[
\sigma'_0 = \frac{F}{S} \left(1 - \frac{e}{r^2/v}\right) = 512.7 \left(1 + \frac{0.527}{0.253}\right) = 1581 \text{ t/m}^2
\]

Resultant stresses

Unloaded:

\[
\sigma = \sigma_0 + \sigma_1 = -334 + 184 = -150 \text{ t/m}^2
\]

\[
\sigma' = \sigma'_0 + \sigma'_1 = 1581 - 233 = 1348 \text{ t/m}^2
\]

Loaded:

\[
\sigma = \sigma_0 + \sigma_2 = -334 + 1333 = 999 \text{ t/m}^2
\]

\[
\sigma' = \sigma'_0 + \sigma'_2 = 1581 - 1684 = -103 \text{ t/m}^2
\]

The check gives nearly exact values, because in this case the design and stresses have been very accurately calculated.

This accuracy is, however, illusory; in practice, many departures from theoretical values can occur; in particular, the exact value of the permanent stress is not known.

In practice, calculations to three significant figures are sufficient. Greater accuracy may be employed, but only for numerical consistency in the calculations. It does not give a closer indication of the true practical behaviour.

Since it is not worthwhile pursuing the calculations to the ultimate degree of accuracy, the final check need not be absolutely exact. The four basic limit stresses (two in the unloaded state and two in the loaded state) need not be perfectly achieved, the stress variations being numerically slightly different to the agreed values. It is preferable that they should err on the lower side.

Since the prestress depends on two parameters only (F and e), the four conditions become theoretically incompatible, and it is necessary to select the two conditions which are to be used as the basis for the calculation of prestress.

It is advantageous to satisfy the conditions corresponding to the lowest permissible stresses (R'\_1 and R'\_2 in this case), so as not to overestimate
the prestressing force. If the numerical check is not exact, therefore, the following equations for \( \sigma_0 \) and \( \sigma'_0 \) are solved:

\[
\sigma_0 + \sigma_1 = R'_1 \quad \text{and} \quad \sigma'_0 + \sigma'_2 = R'_2
\]

In the present case:

\[
\sigma_0 + 184 = -150 \text{ t/m}^2 \\
\sigma'_0 - 1684 = -100 \text{ t/m}^2
\]

Hence:

\[
\sigma_0 = -334 \text{ t/m}^2 \\
\sigma'_0 = +1584 \text{ t/m}^2
\]

Hence F and \( e \) are obtained.

The results in this case are identical with those obtained above, because of the accuracy with which the calculations have been carried out.

**Stresses at the time of cable tensioning.** For the permanent stress (after losses) to be equal to 95 kg/mm\(^2\) at mid-span, the initial tension must be 20 to 30\% greater at the time of tensioning (see Chapter IV) to compensate for subsequent losses.

In the present example, the losses are of the order of 24 kg/mm\(^2\) (shrinkage, steel relaxation, long-term deformations). The required initial tensile stress in the steel at mid-span is, therefore, 95 + 24 = 119 kg/mm\(^2\).

The stresses are then:

\[
\sigma_{oi} = -334 \times \frac{119}{95} = -415 \text{ t/m}^2 \\
\sigma'_{oi} = 1581 \times \frac{119}{95} = 1975 \text{ t/m}^2
\]

The resultant stresses are:

at the top fibre:

\[-415 + 184 = -231 \text{ t/m}^2\]

at the bottom fibre:

\[1975 - 233 = +1742 \text{ t/m}^2\]
If it is accepted that at this stage the compressive stress is not to exceed 55% of the cube strength, then the concrete strength at the time of prestressing must be equal to:

\[
\frac{1742}{0.55} = 3160 \text{ t/m}^2 (316 \text{ kg/cm}^2)
\]

But, in this example, where the permissible stress under load is only 100 kg/cm², the final concrete strength required is about 350 kg/cm².

If all the prestress is applied at once, the concrete strength must be at that time very nearly equal to its final strength. If the concrete is allowed to set and harden normally, the time for it to reach its final strength can be considerable, and this is not usually compatible with the time allowed for construction.

The various solutions are:

(a) Delaying the time of total prestress; that is, in this case, to tension only two cables initially, so that the beam can be safely handled, and to tension the third cable later, by which time the stresses have decreased in the first two cables and the concrete strength has increased.

(b) Over-designing the bottom flange, this is equivalent to considering the following as the critical cases for design; permanent load plus initial prestress and total load plus permanent prestress. This solution is uneconomical and should be avoided.

(c) Accelerating the setting of the concrete.

The latter solution was adopted, in practice, for the beams considered in this example. The setting was accelerated by heating.

The stress of \(-231 \text{ t/m}^2 (-23 \text{ kg/cm}^2)\) at the top fibre is acceptable during the construction stages, on condition that the tensile zone is provided with some complementary steel.

**Complementary steel**

Non-tensioned steel is required since the case under consideration involves limited prestressing.

(a) *Top complementary steel.* According to ASP regulations (see Chapter V, Section 11), half the total resultant tension must be resisted.

Under initial prestress, depth of tensile zone:

\[
\frac{231}{231 + 1742} \times 1.10 = 0.129 \text{ m}
\]
This is a little greater than the flange tip thickness, so that the tensile zone is reduced at its outer edge. It can, however, be assumed that the tensile resultant is the same as with a uniform flange thickness (the error is insignificant and less than 1%). The resultant is equal to:

\[
0.92 \times 0.129 \times \frac{231}{2} = 13.7 \text{ t}
\]

The total cross-sectional area of the top complementary reinforcement, to resist half the tensile resultant, with bonded steel stressed to 21 kg/mm² is:

\[
\frac{6850}{21} = 326 \text{ mm}^2
\]

(4 no. 8 mm and 2 no. 10 mm bars placed 0.03 m from the top).

(b) *Bottom complementary steel.* The depth of the tensile zone under maximum loading is \(\frac{100}{(100 + 1000)} \times 1.10 = 0.10 \text{ m.}\)

Resultant tension:

\[
100 \times \frac{0.10 \times 0.47}{2} = 2.35 \text{ t}
\]

In order to resist half this tensile force with mild steel, at a stress of 14 kg/mm², the total cross-sectional area required is:

\[
\frac{1}{2} \times \frac{2350}{14} = 84 \text{ mm}^2
\]

This is the minimum amount of steel required by the ASP Code. It is much too small to be of any value.
However, the criteria for the limitation of cracking can, in the case of limited prestressing, require an increase in the quantity of steel, and they do effectively require it in this instance.

These criteria are explained in Chapters IX and X; for the present it is sufficient to say that in order to determine the amount of steel required, the moment causing cracking is given by the following approximate formula:

\[ M_r = 0.9h_c F_r + 0.9h_a F_a \]

\( h_c \) and \( h_a \) are the ‘effective depths’ to the cables and non-tensioned reinforcement respectively, that is, their distances from the most highly compressed fibre.

\( F_r \) is the cable tensile strength and \( F_a \) is the yield stress of the complementary reinforcing steel.

![Fig. 12.](image)

Assuming, in accordance with the Provisional Instructions, 1953, that the rupture moment must be at least equal to \( 1.1M_p + 2.2M_s \) (\( M_p \), moment under permanent loading; \( M_s \), moment under maximum live loading), then:

\[ M_r \geq 1.1 \times 19.65 + 2.2 \times 122.5 = 291 \text{ tm approx.} \]

Breaking force for cables (rupture stress = 155 kg/mm\(^2\)):

\( F_r = 3 \times 602 \times 155 = 280,000 \text{ kg} \)

\( h_c = 1.013 \text{ m} \)

\( h_a = 1.07 \text{ m} \)

† In the case of total prestressing, the cable strength is usually sufficient (see Chapter IX) to satisfy the criteria of safety against cracking, without the need for complementary steel.
Therefore:

\[ 291 = 0.9 \times 1.013 \times 280 + 0.9 \times 1.07 F_a = 255 + 0.96 F_a \]

and:

\[ 0.96 F_a = 291 - 255 = 36 \text{ tm} \]

Therefore:

\[ F_a = \frac{36}{0.96} = 37.4 \text{ tm} \]

Steel with a high bond strength is used, with a 0.2% proof stress of about 50 kg/mm². The required cross-sectional area of complementary steel is, therefore:

\[ \frac{37 400}{50} = 748 \text{ mm}^2 (4 \text{ no. 16 mm bars}) \]

The required section is shown in Fig. 13. The arrangement of hoops and stirrups is also shown; their design is explained in Chapter II, Volume 2.

**FIG. 13.** Transverse section (see Chapter VI, Volume 2 for other possible arrangements of cables and stirrups).
Remarks
The 'minimum amount of steel' rule of the ASP would give a quantity of steel which would be too low for it to be of any real practical use in this case.

It should not be concluded that this rule is superfluous, however, but only that, in the present case, and in the elastic state, the behaviour of the beam is hardly affected if the complementary steel is entirely omitted.

This is easily seen. Assuming that there is no complementary steel and that cracking occurs, the section would be strong enough because it is subjected to combined bending and compression and because the centre of compression is contained within the section. The displacement of the centre of compression from the cable is \( M_2/F \); assuming that cracking has not caused the prestressing force to increase, then

\[
\frac{M_2}{F} = \frac{142.1}{171.2} = 0.83 \text{ m}
\]

The depth \( x \) in compression is obtained by equating the position of the compression resultant with that of the centre of compression. It is found that \( x = 0.95 \text{ m} \), giving a depth of crack of 15 cm instead of the depth in tension of 10 cm determined by treating the section as an homogeneous section.

The maximum concrete stress, at the top fibre, is then 102 kg/cm\(^2\), an increase of 2% on the previous calculation.

![Fig. 14.](image)
The complementary steel, at a distance of about 10 cm from the neutral fibre, can have only a very small effect on the strength of the beam and can only reduce the depth of crack by about 2 to 3 cm.

Nevertheless, the design is much improved with the provision of some complementary steel. If cracking should occur (and the risk, though small, can never be entirely ignored with limited prestress), the provision of some steel limits the cracking to microcracks instead of one large crack or several smaller but wider cracks.

Again, the steel must be there in sufficient quantity for this dispersion of the cracks to occur. It is intuitively felt that there must be a critical quantity of complementary steel, below which the steel is useless. Chapter IX endeavours to rationalise this intuition. In the present case, the requirements of the safety criteria provide a sufficient quantity of steel.

3. Corrections for cable holes in the concrete before grouting and for excess stresses in the cables

The variations of stress in the cables have so far been neglected; their influence is advantageous. The holes for the cable sheaths have also been neglected.

There are thus two corrections to apply, whose effect is to reduce the cost of the section.

One way of applying the corrections is given below for the example under consideration, and the subject is dealt with in a more general manner in Chapter VII, Sections 7 and 8.

By considering the combined section as homogeneous, as with reinforced concrete, the additional stresses in the cables caused by the application of the live loads can be taken into account. With regard to the moments $M_s$ due to the live loads, the cable cross-sectional area $A_c$ can be replaced by a cross-sectional area of concrete $mA_c$, where $m$ is the modular ratio. The additional stresses are thus taken into account, since any stress variation $\Delta \sigma_c$ (loss of compression) in the concrete local to the cables results in a stress variation $m \Delta \sigma_c$ (increase in stress) in the cables.

The section can only be treated as homogeneous in the above manner for loading conditions which occur after the cables are grouted. It cannot be considered homogeneous under a condition of loading which occurs simultaneously with prestressing, and therefore self-weight cannot be treated in this way, with post-tensioned steel.

Two sections must be considered therefore: in the unloaded condition (profile 1) and in the loaded condition (profile 2).
These two sections are penetrated by the sheaths. The sheath penetrations must be determined in order to establish the characteristics of profile 1. It might not be necessary to determine them for profile 2 if grouting is efficient, as the grout would then fill in the section. However, the grout is not prestressed and it is put in tension when the live loads are applied. In the following calculations, the penetrations are determined for both profiles 1 and 2.

Now, considering Fig. 1, it is seen that the variations $\Delta R = \sigma_z$ and $\Delta R' = \sigma_z'$ apply to profile 2. Therefore:

$$\frac{I_2}{v_2} = \frac{\Delta M}{\Delta R} \quad \frac{I_2}{v'_2} = \frac{\Delta M}{\Delta R'}$$

Consequently, profile 2 must have the previously determined characteristics, namely:

$$v_2 = 0.487 \, \text{m}; \quad v'_2 = 0.613 \, \text{m}; \quad I_2 = 0.0519 \, \text{m}^4.$$  

Profile 1, which must be determined in order to design the concrete section, differs from profile 2 only by the omission of the cables $mA_c$. If $mA_c$ is known, therefore, and also the position of the cables, the properties and design data for profile 1 can be deduced.

The cables $mA_c$ are not known exactly, however, since the cross-sectional area $A_c$ can be a little lower than that previously determined because the stress in the cables under load is greater. The corrections are relatively small, and the following approximation is acceptable.

It is assumed that the prestressing force under load (excess stresses being included, therefore) is the same as before, namely 171 t, and that the eccentricity is the same. The force is therefore applied at 0.087 m above the bottom face.

The excess stress in the cables is equal to the stress variation in the concrete at this level multiplied by $m$. For the present case, it is assumed that $m = 10$.

In the unloaded condition, since the stresses in the extreme fibres are $-150, +1350 \, \text{t/m}^2$, the stress in the concrete at the level of the cables is:

$$\sigma_{c1} = 1350 - (1350 + 150) \frac{0.087}{1.10} = 1.231 \, \text{t/m}^2$$

In the loaded condition (stresses in extreme fibres $-100, +1000 \, \text{t/m}^2$):

$$\sigma_{c2} = -100 + 1100 \times \frac{0.087}{1.10} = -13 \, \text{t/m}^2$$
Stress variation:

\[ \Delta \sigma_c = 1244 \text{ t/m}^2 (124.4 \text{ kg/cm}^2) \]

The excess tension in the cable is:

\[ \Delta T = m \Delta \sigma_c = 10 \times 124.4 = 1244 \text{ kg/cm}^2 (12.4 \text{ kg/mm}^2) \]

If it is assumed that the permanent stress (unloaded) is still 95 kg/mm\(^2\), the stress when loaded is 95 + 12.4 = 107.4 kg/mm\(^2\).

To exert the force \( F = 171 \text{ 000 kg} \) in the loaded condition, the following cross-sectional area of cables is sufficient:

\[ A_c = \frac{171 \text{ 000}}{107.4} = 1600 \text{ mm}^2 (0.0016 \text{ m}^2) \]

Assume that it is possible effectively to reduce the area of the cables to this value of 1600 mm\(^2\).†

Then \( mA_c = 10 \times 0.0016 = 0.016 \text{ m}^2 \).

In the case of profile 2, the eccentricity of the cables is \( e_2 = -0.527 \text{ m} \), so that:

\[ mA_c e_2 = 0.016 \times (-0.527) = -0.0084 \text{ m}^3 \]

\[ mA_c e_2^2 = -0.0084 \times (-0.527) = +0.0044 \text{ m}^4 \]

The cross-sectional area of the bottom flange, and the distance \( d' \) from its centroid to the centroid \( G_2 \), are determined in the same manner as in Section 2. For the top flange, \( s = 0.099 \text{ m}^2 \) and \( d = 0.422 \text{ as before} \), so that \( sd = 0.0417 \text{ m}^3 \) and \( sd'^2 = 0.0176 \text{ m}^4 \).

In Section 2 it was found, by noting that the centroid coincided with \( G_2 \) and that the moment of inertia was 0.0519 m\(^4\), that:

\[ s'd' = 0.0307 \text{ m}^3 \]

\[ s'd'^2 = 0.0158 \text{ m}^4 \]

In this case, allowing for the cables:

\[ s'd' = 0.0307 - 0.0084 = 0.0223 \text{ m}^3 \]

\[ s'd'^2 = 0.0158 - 0.0044 = 0.0114 \text{ m}^4 \]

† Since only a whole number of cables can be used, it is possible that only part of the total possible economy can be realised. The calculations are unaffected for the determination of \( s' \) and \( d' \), the area \( A_c \) being then the area of the cables which are effectively used. If the area is greater than the minimum area (1600 mm\(^2\)) indicated in the text, the solution is conservative. The indicated solution is therefore the minimum solution.
By division:
\[ d' = 0.514 \text{ m} \]

Therefore:
\[ s' = 0.043 \text{ m}^2 \]

The cross-sectional area \( s' \) is the nett area (holes deducted) of the bottom flange, shown hatched in Fig. 15, excluding the web.

Let the gross area of the flange be equal to \( s'' \) (holes included) and let the distance of its centroid from \( G_2 \) be \( d'' \).

The holes are 5 cm diameter so that the cross-sectional area of the three holes is 60 cm\(^2\) (0.006 0 m\(^2\)). Their eccentricity is \(-0.527\) m.

![Diagram](image)

**Fig. 15**

Now:
\[ s' = s'' - 0.006 \]
\[ s'd' = s''d'' - 0.006 \times 0.527 = s''d'' - 0.0032 \]

Therefore:
\[ s'' = 0.043 + 0.006 = 0.049 \text{ m}^2 \]
\[ s''d'' = 0.0223 + 0.0032 = 0.0255 \text{ m}^3 \]

and
\[ d'' = \frac{0.0255}{0.049} = 0.519 \text{ m} \]
The thickness of the flange is equal to \(2(0.613 - 0.519) = 0.188\) m and its breadth outside of the web is \((0.049/0.188) = 0.261\) m.

The total breadth of the bottom flange is, therefore; \(0.16 + 0.261 = 0.371\) m.

The flange profile which is thus obtained is then modified as explained in Section 2, in going from Fig. 8 to Fig. 9.

As mentioned above, this solution is approximate only, but it is adequate. In order to justify the approximation, consider the characteristics of profile 1. This profile is obtained from profile 2 by deducting the area \(mA_c\), with eccentricity \(-0.527\) m.

The cross-sectional area \(S_2\) of profile 2, including the cables \(mA_c\), is:

\[
S_2 = 0.176 + 0.099 + 0.043 + 0.016 = 0.334 \text{ m}^2
\]

and:

\[
v_2 = 0.487 \text{ m}, \quad v'_2 = 0.613 \text{ m}, \quad I_2 = 0.0519 \text{ m}^4
\]

The centroid of profile 1 is at a distance \(g\) above \(G_2\), given by:

\[
g = \frac{0.334 \times 0 - 0.016 \times (-0.527)}{0.334 - 0.016} = 0.026 \text{ m}
\]

And:

\[
S_1 = 0.334 - 0.016 = 0.318 \text{ m}^2
\]

\[
v_1 = 0.487 - 0.026 = 0.461 \text{ m}
\]

\[
v'_1 = 0.613 + 0.026 = 0.629 \text{ m}
\]

\[
I_1 = I_2 + 0.334 \times (0.026)^2 - 0.016 \times (0.527 + 0.026)^2
\]

\[
= 0.0519 + 0.0002 - 0.0049
\]

\[
= 0.0472 \text{ m}^4
\]

\[
\frac{I_1}{v_1} = 0.1024 \text{ m}^3, \quad \frac{I_1}{v'_1} = 0.0739 \text{ m}^3
\]

\[
\frac{r^2}{v_1} = 0.320 \text{ m}, \quad \frac{r^2}{v'_1} = 0.233 \text{ m}
\]

† This should strictly be written \((r_1^2/v_1)\) and \((r_1^2/v'_1)\) since \(r_1 \neq r_2\). The symbols explain themselves without any possibility of confusion. In any case, the difference between \(r_1\) and \(r_2\) is very small.
The cross-sectional area of the section, after grouting, is equal to: 0.318 + 0.006 = 0.324 m². The weight per metre length is 0.778 t/m.

\[ M_p(=M_1) = 0.778 \times \frac{14^2}{8} = 19.1 \text{ tm} \]

Stresses due to self-weight:

\[ \sigma_1 = \frac{19.1}{0.1024} = 187 \text{ t/m}^2 \]

\[ \sigma'_1 = \frac{19.1}{0.0739} = -258 \text{ t/m}^2 \]

The prestressing force is, with the beam unloaded:

\[ F = 1600 \text{ mm}^2 \times 95 \text{ kg/mm}^2 = 152 \text{ t} \]

\[ \frac{F}{S} = \frac{152}{0.318} = 477 \text{ t/m}^2 \]

The eccentricity is \(-0.527 + 0.026 = -0.553\) m.

Therefore, the values of the prestresses are:

\[ \sigma_0 = \frac{F}{S} \left( 1 - \frac{0.553}{0.320} \right) = 477 \times (-0.73) = -348 \text{ t/m}^2 \]

\[ \sigma'_0 = \frac{F}{S} \left( 1 + \frac{0.553}{0.233} \right) = 477 \times 3.37 = 1615 \text{ t/m}^2 \]

The resultant stresses under permanent loading are, therefore:

\[ \sigma_0 + \sigma_1 = -348 + 187 = -161 \text{ t/m}^2 \]

\[ \sigma'_0 + \sigma'_1 = 1615 - 258 = 1357 \text{ t/m}^2 \]

Since the stress variations are, as already established, 1150 t/m² at the top fibre and -1450 t/m² at the bottom fibre, the resultant stresses under load are:

\[ -161 + 1150 = 989 \text{ t/m}^2 \text{ (top fibre)} \]

\[ 1357 - 1450 = -93 \text{ t/m}^2 \text{ (bottom fibre)} \]

The differences from the design stresses are of the order of 1 kg/cm², at most.
The design stresses can be met exactly, if required, by slightly modifying the prestress and its eccentricity. Thus, the following prestresses in the unloaded condition are required:

\[
\sigma_0 = -337 \text{ t/m}^2 (-337 + 187 = -150 \text{ t/m}^2) \\
\sigma'_0 = 1608 \text{ t/m}^2 (1608 - 258 = 1350 \text{ t/m}^2)
\]

The stresses in the unloaded condition being equal to the permissible stresses, they must also be equal to the permissible stresses in the loaded condition, since the stress variations are the fixed design variations.

Since \(v_1 = 0.461 \text{ m}\), then:

\[
\frac{F}{S} = -337 + (337 + 1608) \times \frac{0.461}{1.10} = 478 \text{ t/m}^2
\]

Therefore: \(F = 478 \times 0.318 = 152 \text{ t}\), which is practically unchanged.

The eccentricity \(e_1\) with respect to \(G_1\) is given by:

\[
\frac{F}{S} \left(1 - \frac{e_1}{0.233}\right) = 1608 \text{ t/m}^2
\]

Hence:

\[e_1 = -0.550 \text{ m}, \text{ in place of } -0.553 \text{ m}\]

The correction of 3 mm is without any practical use, but it can be made for the presentation of exact numerical calculations.

Summarising: some gain is realised on the concrete (0.318 m\(^2\) instead of 0.334 m\(^2\), or 5\%), but the economy in steel is more important (1600 mm\(^2\) in place of 1800 mm\(^2\), or 11\%). It is not possible with the present example to achieve all the possible savings, because a whole number of cables must be employed. Where large numbers of cables are required, however, or where a structure comprises a large number of beams, important savings are possible by adopting the above method of design.

**NOTE**

The saving is high because the modular ratio is high \((m = 10)\). For \(m = 5\), the saving is much smaller.

A modular ratio \(m = 10\), corresponding to live loading of long duration, can be assumed only if it is certain that there can be no instantaneous loading at the maximum value of the live load, since the condition is then temporarily one of instantaneous live loading.

This aspect is dealt with again in Chapter VII (Sections 7 and 8).
Chapter VII

CENTRE OF COMPRESSION
DISPLACEMENTS OF THE CENTRE OF COMPRESSION
LIMIT CORE—LINES OF THRUST
LIMIT STATE—SECTION EFFICIENCY

In the two previous chapters, section design is based principally on stress calculations. The calculations are made for the section having the greatest variation in bending moment, but they can be done at any other section. The whole of a beam could be designed in this manner, by considering a number of sections, using strength-of-materials theory only.

By considering stresses alone, the primary aim of prestressing is not really brought to the fore; this aim is to create a favourable distribution of compression within the material. The initial distribution of compression is that due to the prestress itself, which coincides with the cable in the case of a statically-determinate beam. The curves of compression, or lines of thrust, under various loading conditions are obtained by a study of displacements. The prestress and the section profiles must be so designed that the effective lines of action of the compression under various conditions of loading lie within a certain limiting zone.

The lines of thrust are defined as the loci of the centres of compression relative to various sections of the beam.

This concept of centre of compression has already been mentioned in Chapters V and VI. It is necessary to examine it in greater detail, as well as the limits within which the centre of compression can vary, in order to satisfy the strength criteria.

This examination leads to more intuitive solutions.

1. Centre of compression
In a general way, the centre of compression in a section which is subjected to combined bending and compression is the point of application of the force normal to the surface. Its position is defined under any loading condition by its eccentricity $e$; $e$ is taken as positive (where the axial
plane of the beam is vertical) when the centre of compression is above the centroid of the section.

With a prestressed beam, the normal force is that force created by the tensioning of the cables.†

As a first approximation, the variations in this force under various conditions of loading are neglected. In the first instance, F is therefore assumed constant.

The corrections to apply to account for the stress variations in the cables are examined in Sections 7 and 8. As previously mentioned, these corrections are small in the case of total prestress, and even in the case of limited prestress, when the live loads are of short duration.

Since the normal force is assumed constant, the bending moment under the particular loading condition being considered is Fe.

Thus, when the resultant moment M is known (including the moment due to prestress), the eccentricity e can be obtained; \( e = \frac{M}{F} \).

Under various conditions of loading, the effective eccentricity assumes different values \( e_0, e_1, e_2, \ldots \).

If \( \Delta M \) denotes the variations in bending moment when passing from condition 0 to condition 1, from condition 1 to condition 2, and so on, then with F assumed constant, \( \Delta M = F \Delta e \), or \( \Delta e = \frac{\Delta M}{F} \). This relation has already been used by equating the displacement \( E_0 E_p \) [corresponding to the passage from condition 0 (prestress) to condition \( p \) (self-weight)] to \( M_p/F \).

2. Values of stresses as functions of the position of the centre of compression under any condition of loading

It is recalled that the stress \( \sigma(y) \), where \( y \) is the ordinate of the point with respect to the centroid, at any point of a section has the following value:

\[
\sigma(y) = \frac{F}{S} + \frac{My}{I}
\]

Since \( M = Fe \) (where \( e \) is the eccentricity under the loading condition being considered), then:

\[
\sigma(y) = \frac{F}{S} + Fe \frac{y}{I}
\]

Again, since \( I = Sr^2 \):

\[
\sigma(y) = \frac{F}{S} \left( 1 + \frac{ey}{r^2} \right)
\]

† There can be an external normal force \( N \) in addition (see Section 9, iii).
With the assumption that \( F \) is constant, the stress at the centroid (that is, for \( y = 0 \)) is constant and equal to \( F/S \); this stress is denoted by \( \sigma_y \).

With the above assumption, therefore, the stress diagrams for all conditions of loading (and therefore for the various values of \( e \) in these conditions) pass through a fixed point \( N_y \) of abscissa \( \sigma_y \) at the level of the centroid \( G \). The position of the point \( N_y \) (and therefore the position of the centroid and the value of \( \sigma_y \)) can be obtained as the intersection of any two particular stress diagrams. For the case considered in Chapter VI, where it is assumed that the limiting stresses are effectively obtained in the two extreme conditions of loading, the corresponding diagrams \((R_1, R_1')\) and \((R_2, R_2')\) are known, and therefore the values of \( v \), \( v' \), and \( \sigma_y \).

The following formulae are obtained in Chapter VI:

\[
\frac{v}{v'} = \frac{R_2 - R_1'}{R_1 - R_2'}
\]

\[
\sigma_y = R_1' \frac{v'}{h} + R_1 \frac{v}{h}
\]

\[
\sigma_y = R_2' \frac{v}{h} + R_2 \frac{v'}{h}
\]

These formulae are obvious from Fig. 1.

In addition, the stresses in the extreme fibres \((y = +v \text{ and } y = -v')\), under any condition of loading which is characterised by the value of \( e \), have the following values in accordance with formula \((1)\):

\[
\sigma = \frac{F}{S} \left(1 + \frac{e}{r^2/v}\right) = \sigma_y \left(1 + \frac{e}{r^2/v}\right)
\]
\[
\sigma' = \sigma_y \left( 1 - \frac{e}{r^2/v} \right)
\]

(3)

3. Limit points
The stress \(\sigma\) must lie between \(R'_1\) and \(R_2\) at the top fibre and between \(R_1\) and \(R'_2\) at the bottom fibre.† Therefore:

\[
R'_1 < \sigma_y \left( 1 + \frac{e}{r^2/v} \right) < R_2
\]

\[
R'_2 < \sigma_y \left( 1 - \frac{e}{r^2/v} \right) < R_1
\]

Or:

\[
\frac{r^2}{v} \left( \frac{R'_1}{\sigma_y} - 1 \right) < e < \frac{r^2}{v} \left( \frac{R_2}{\sigma_y} - 1 \right)
\]

\[
\frac{r^2}{v'} \left( 1 - \frac{R_1}{\sigma_y} \right) < e < \frac{r^2}{v'} \left( 1 - \frac{R'_2}{\sigma_y} \right)
\]

Since \(R'_1 < \sigma_y\), \((R'_1/\sigma_y) - 1\) is negative, and since \(R_1 > \sigma_y\), \(1 - (R_1/\sigma_y)\) is also negative. Therefore:

\[
-\frac{r^2}{v} \left( 1 - \frac{R'_1}{\sigma_y} \right) < e < \frac{r^2}{v} \left( \frac{R_2}{\sigma_y} - 1 \right)
\]

\[
-\frac{r^2}{v'} \left( \frac{R_1}{\sigma_y} - 1 \right) < e < \frac{r^2}{v'} \left( 1 - \frac{R'_2}{\sigma_y} \right)
\]

By re-arranging these inequalities:

\[
-\frac{r^2}{v} \left( 1 - \frac{R'_1}{\sigma_y} \right) < e < \frac{r^2}{v'} \left( 1 - \frac{R'_2}{\sigma_y} \right)
\]

(4)

\[
-\frac{r^2}{v'} \left( \frac{R_1}{\sigma_y} - 1 \right) < e < \frac{r^2}{v} \left( \frac{R_2}{\sigma_y} - 1 \right)
\]

(5)

† The terms top and bottom fibre are self explanatory. It may, in some cases, become necessary to adopt a different convention. Where this occurs, the convention is explained.
(a) Inequality (4) defines the limits within which the centre of compression must lie:

for the lower limit, therefore, in condition 1 (permanent loads only):

\[-\frac{r^2}{v} \left(1 - \frac{R'_1}{\sigma_g}\right)\]

for the upper limit, therefore, in condition 2 (maximum live load):

\[+\frac{r^2}{v'} \left(1 - \frac{R'_2}{\sigma_g}\right)\]

for the stresses to remain above their lower limits \(R'_1\) and \(R'_2\).

Consider the central core. The eccentricities at the boundaries \(C'\) and \(C\), are \(-\langle r^2/v \rangle\) and \(+\langle r^2/v' \rangle\).

Draw the points \(A'_1\) and \(A_2\) with eccentricities \(-\langle r^2/v \rangle[1 - (R'_1/\sigma_g)]\) and \(+\langle r^2/v' \rangle[1 - (R'_2/\sigma_g)]\) respectively. These points are deduced from the central core by the reductions \(1 - (R'_1/\sigma_g)\) and \(1 - (R'_2/\sigma_g)\) (these reductions must be considered algebraically:

if \(R'_1\) and \(R'_2\) are compressive, they are truly reductions; if \(R'_1\) and \(R'_2\) are negative, they are increases).
The centre of compression must lie between points $A'_1$ and $A_2$ for the lower stress limits not to be exceeded.

More concisely, the zone $A'_1A_2$ is said to be the limit zone corresponding to the tensile stresses.† If in condition 1 (the condition with minimum bending moment) the centre of compression were to lie below $A'_1$, the stresses would be too high (algebraically less than $R'_1$) at the opposite face; that is, at the top. If in condition 2 (the condition with maximum bending moment) the centre of compression were to lie above $A_2$, the stresses would be too high (algebraically less than $R'_2$) on the opposite face, that is, at the bottom.

(b) Inequality (5) shows that the centre of compression must lie between:

*at the lower limit*, therefore, in condition 1 (minimum moment):

$$\frac{r^2}{v'} \left( \frac{R'_1}{\sigma_g} - 1 \right)$$

*at the upper limit*, therefore, in condition 2 (maximum moment):

$$\frac{r^2}{v} \left( \frac{R'_2}{\sigma_g} - 1 \right)$$

in order that the compressive stresses should not exceed their design limits.

Draw points $B'_1B_2$ with eccentricities of $-(r^2/v')(R'_1/\sigma_g) - 1$ and $r^2/v[(R'_2/\sigma_g) - 1]$ respectively.

The centre of compression must lie between points $B'_1$ and $B_2$.

If it falls below $B'_1$ the compressive stresses would be too high at the bottom under condition 1.

If it rises above $B_2$ the compressive stresses are too high at the top due to condition 2.

The zone $B'_1B_2$ is the limit zone corresponding to compressive stresses. Finally, therefore, there are:

- two limit points $A'_1B'_1$ at the bottom;
- two limit points $A_2B_2$ at the top.

The centre of compression must lie within all of these four points under the various conditions of loading.

The limit zone is the smallest zone defined by the four points $A'_1$, $B'_1$, $A_2$, $B_2$; that is, by the smallest, in absolute value, of the lower limiting

† The expression may be incorrect if $R'_1$ and $R'_2$ are positive (compressions). It is self explanatory; when the centre of compression is beyond the limit zone $A'_1A_2$, there is a risk of having tensile stresses at the opposite face.
eccentricities ($A'_1$ or $B'_1$) and by the smallest of the upper limiting eccentricities ($A_2$ or $B_2$).

If in the two extreme conditions the stresses $R_1$, $R'_1$, $R_2$, $R'_2$ are effectively reached, then points $A'_1$ and $B'_1$ are coincident and so are the points $A_2$ and $B_2$.

This is obvious since when the limit $R_2$ is obtained (condition 2, top fibre), the limit $R'_2$ is obtained simultaneously (bottom fibre); the same position of the centre of compression therefore produces the stress $R_2$ at the top (centre of compression at $B_2$) and the stress $R'_2$ at the bottom (centre of compression at $A_2$). Therefore $A_2$ and $B_2$ are coincident. Similarly with $A'_1$ and $B'_1$.

This can be checked by calculation. $A_2$ and $B_2$ are coincident if:

$$\frac{r^2}{v'} \left(1 - \frac{R'_2}{\sigma_g}\right) = \frac{r^2}{v} \left(\frac{R_2}{\sigma_g} - 1\right)$$

or:

$$\frac{\sigma_g - R'_2}{v'} = \frac{R_2 - \sigma_g}{v}$$

Figure 3 shows that this relation is true.

![Diagram](image)

**Fig. 3.**

In the case where only some but not all of the limit stresses are obtained under the extreme conditions of loading, as happens when the span is greater than the critical span, the limit points ($A'_1$ and $B'_1$ on the one hand and $A_2$ and $B_2$ on the other) are no longer coincident.

If the prestressing force is suitably determined, $B_2$ is generally above $A_2$. If $B_2$ were below $A_2$, the stress diagram would reach $R_2$ at the top before
reaching $R'_2$ at the bottom, when it pivots about the fixed point $N_g$ (Fig. 4).

Figure 4 also shows that $\sigma_g$ (and therefore also $F$) would have a value which is higher than that required (dotted line), which is to be avoided.

With regard to points $A'_1$ and $B'_1$: if the span is greater than the critical span, $B'_1$ is generally above $A'_1$ (see Chapter VIII).

Consequently, apart from certain exceptions which are indicated later, it is generally satisfactory to consider, for positive eccentricities, the limit point $A_2$. The eccentricity of this point is obtained by deducting $1 - \frac{R'_2}{\sigma_g}$ from the corresponding eccentricity of the central core; for negative eccentricities, one or the other of the limit points $A'_1$ or $B'_1$ may have to be considered.

4. Limit core. Limit zone
The limit core is the core which is contained between the two limit points $A'_1$ (or $B'_1$) and $A_2$.

The limit zone is the zone which is described within a beam by the limit core at all sections of the beam. The line of thrust is the locus of the centres of compression along the beam.

The provision of suitable strength consists of arranging lines of thrust in such a way that they lie, under all conditions of loading, within the limit zone.
5. Calculation based on the limit core
With the particular example considered in Chapter VI, where the span is less than the critical span, the calculations cannot be simplified by a theory which is derived from the concept of the limit core. It is shown, particularly in the following chapters, that this concept can be used with advantage.

Nevertheless, it is a useful exercise to show that the results which are obtained are the same as those of Chapter VI.

When the section is utilised to its maximum (that is, when the limit stresses are reached in the two extreme conditions, 1 and 2), the centre of compression is at its limit eccentricities under the minimum bending moment $M_1$ (lower limit of eccentricity) and under the maximum bending moment $M_2$ (upper limit of eccentricity).

For the same reason (maximum utilisation of section) the limit zones A and B (Section 3) are coincident.

In the unloaded condition (condition 1), therefore:

$$e_1 = -\frac{r^2}{v} \left(1 - \frac{R_1'}{\sigma_g}\right)$$

(a)

and because zones A and B are identical:

$$e_1 = -\frac{r^2}{v'} \left(\frac{R_1}{\sigma_g} - 1\right)$$

(b)

Under load (condition 2):

$$e_2 = \frac{r^2}{v'} \left(1 - \frac{R_2'}{\sigma_g}\right)$$

(c)

and because zones A and B are identical:

$$e_2 = \frac{r^2}{v} \left(\frac{R_2}{\sigma_g} - 1\right)$$

(d)

The displacement $e_2 - e_1$ is equal to

$$\frac{M_2 - M_1}{F}$$

From eqns. (d) and (a):

$$\frac{M_2 - M_1}{F} = \frac{r^2 R_2 - R_1'}{v} \frac{1}{\sigma_g}$$
Since \( \sigma_g = F/S \) and \( S(r^2/v) = I/v, \)

\[
\frac{M_2 - M_1}{F} = \frac{I}{v} \frac{R_2 - R'_1}{F}
\]

or

\[
\frac{I}{v} = \frac{M_2 - M_1}{R_2 - R'_1}
\] \( \text{(e)} \)

From eqns. (c) and (b):

\[
\frac{I}{v'} = \frac{M_2 - M_1}{R_1 - R'_2}
\] \( \text{(f)} \)

is obtained in the same manner.

These are the same results as those previously obtained.

From eqns. (c) and (a):

\[
\frac{M_2 - M_1}{F} = \frac{r^2}{v'} \left( 1 - \frac{R'_2}{\sigma_g} \right) + \frac{r^2}{v} \left( 1 - \frac{R'_1}{\sigma_g} \right)
\] \( \text{(g)} \)

The second term of this equation is equal to the depth \( A'_1A_2 \) of the limit core (Fig. 2).

Therefore

\[
F = \frac{M_2 - M_1}{A'_1A_2}
\] \( \text{(h)} \)

Consequently, when the limit core is known, the prestressing force is obtained by writing that the centre of compression traverses the depth of the limit core under the influence of the variation \( M_2 - M_1 \) in the bending moment.

Although eqn. (h) is simply another way of writing eqn. (g), it is more suitable for an intuitive geometrical reasoning.

In particular, when the span is less than the critical span it enables the approximate prestressing force to be estimated during the development of the design, which enables the flange carrying the cables to be determined.

Indeed, the depth of the limit core is approximately known once the depth \( h \) of the beam is fixed. It is, as shown later (Section 10), about \( h/2 \) for normal sections.† If the lower stress limits \( (R'_1 \) and \( R'_2 \) ) are zero, the limit core is the central core and, consequently, \( F = 2 \times (M_2 - M_1)/h \).

† The method is applicable to other values of \( A'_1A_2/h \); that is, of the efficiency, which is defined in Section 10.
The value of \( F \) thus obtained is on the low side when \( R'_1 \) and \( R'_2 \) are positive (residual compression) and it is high when \( R'_1 \) and \( R'_2 \) are negative (tensile).

This value of \( F \) can be corrected by means of eqn. (g) which can be written:

\[
\frac{M_2 - M_1}{F} = \frac{r^2}{v} + \frac{r^2}{v'} - \left( \frac{r^2}{v} \frac{R'_1}{F/S} + \frac{r^2}{v'} \frac{R'_2}{F/S} \right)
\]

\[
= \frac{r^2}{v} + \frac{r^2}{v'} - \left( \frac{I}{v} \frac{R'_1}{F} + \frac{I}{v'} \frac{R'_2}{F} \right)
\]

Since it is assumed that \( r^2/v + r^2/v' = h/2 \), and since, from eqns. (e) and (f):

\[
\frac{I}{v} \frac{R'_1}{F} + \frac{I}{v'} \frac{R'_2}{F} = \frac{M_2 - M_1}{F} \left( \frac{R'_1}{R_2 - R'_1} + \frac{R'_2}{R_1 - R'_2} \right)
\]

then:

\[
\frac{M_2 - M_1}{F} = \frac{h}{2} - \frac{M_2 - M_1}{F} \left( \frac{R'_1}{R_2 - R'_1} + \frac{R'_2}{R_1 - R'_2} \right)
\]

or:

\[
F = 2 \frac{M_2 - M_1}{h} \left( 1 + \frac{R'_1}{R_2 - R'_1} + \frac{R'_2}{R_1 - R'_2} \right)
\]

As a check on the order of magnitude of \( F \) obtained with this equation, compare it with the value obtained in the example in Chapter VI, Section 2.

\[
h = 1.10 \quad R_1 = 1350 \text{ t/m}^2 \quad R'_1 = -150 \text{ t/m}^2
\]

\[
R_2 = 1000 \text{ t/m}^2 \quad R'_2 = -100 \text{ t/m}^2
\]

\[
M_2 - M_1 = M_4 = 122.5 \text{ tm}
\]

Equation (i) gives:

\[
F = \frac{2(M_2 - M_1)}{1.10} \left( 1 - \frac{150}{150} - \frac{100}{1450} \right) = \frac{2(M_2 - M_1)}{1.10} (1 - 0.13 - 0.07)
\]

Therefore

\[
F = \frac{1.6 \times 122.5}{1.10} = 178 \text{ t}
\]

The value of \( F \) previously obtained is \( F = 171 \text{ t} \). There is therefore an error of 4%.
6. Position of prestressing cable
Let $E_0$ be the point where the resultant cable passes through the section. It is assumed that the span is less than the critical span.

Under the influence of the moment $M_1$, the line of thrust (which is displaced from $E_0$ by an amount $M_1/F$) must coincide with $A_1'$, the lower limit point.

Therefore, the cable must be at a distance of $-(M_1/F)$ from $A_1'$.

If the two moments are positive, $-(M_1/F)$ is negative and consequently $E_0$ is below $A_1'$.

If the position which is thus determined lies within the section, at a sufficient distance from the bottom face (minimum cover), then the assumed solution, that the limit stresses are reached under both of the extreme loading conditions, is possible; that is, the span is less than the critical span.

If $E_0$ is beyond the boundaries of the section, the assumed solution is impossible. The critical span is exceeded. This case is studied in Chapter VIII.

If the extreme moments are of opposite sign ($M_1$ negative, $M_2$ positive), $-(M_1/F)$ is positive and $E_0$ is therefore above $A_1'$. The assumed solution is then always possible.

The following important conclusion is drawn:

*When the extreme moments are of opposite sign, the material can always be utilised to the maximum amount; that is, it is possible for the stresses to reach their limit values at the extreme fibres under both conditions of loading.*

$E_0$ then divides $A_1'A_2$ in the ratio of the moments $M_1$ and $M_2$. This case applies in particular to those structural elements where the moments are independent of the weight of the structure (vertical walls subjected to horizontal loading, pylons for transmission lines, etc.).

7. Corrections for the presence of holes for cables, before grouting, and for the additional stresses in the cables due to loading after grouting (applicable in the cases of total prestressing and limited prestressing)
Approximate methods for applying the correction relative to the additional cable stresses have already been indicated in Chapter V, Section 11 and Chapter VI, Section 3. More precise methods are given in the following.

(i) Holes for the passage of the cables
The presence of these holes modifies the properties of the section and also the stresses.
It is assumed that the cables are equivalent to a single cable (resultant cable) and that the holes are equivalent to a single hole of area $s$.

Usually, this hole is located in the area which is in tension under external loading; its eccentricity is therefore negative when the moments due to loading are positive. The absolute value of this eccentricity is denoted by $e'$ ($e' = -e$). The properties of the section, taking into account the holes, are denoted by $S_1, I_1, v_1, v'_1$.

Let $s/S = \varepsilon$.

The centroid of the section without holes being $G$, the centroid of the section including the holes is above $G$, at $G_1$. Let $g_1$ be the distance $GG_1$.

\[ g_1 = \frac{se'}{S - s} = e' \frac{\varepsilon}{1 - \varepsilon} \]

The eccentricity becomes:

\[ e'_1 = e' + g_1 = \frac{e'}{1 - \varepsilon} \]

The moment of inertia becomes:

\[ I_1 = I + Sg_1^2 - se'_1^2 = I + S(g_1^2 - \varepsilon e'_1^2) \]
\[ = I + \frac{Se'_1^2}{(1 - \varepsilon)^2} (\varepsilon^2 - \varepsilon) = I - Se'_2^2 \frac{\varepsilon}{1 - \varepsilon} \]
If \( r \) is the radius of gyration:

\[
I_1 = I \left(1 - \frac{e'^2}{r^2} \frac{\varepsilon}{1 - \varepsilon}\right) = I \frac{1 - \varepsilon[1 + (e'^2/r^2)]}{1 - \varepsilon}
\]  

(a)

But:

\[
v_1 = v - g_1 = v - \frac{e'e}{1 - \varepsilon} = v \left(1 - \frac{e'}{v} \frac{\varepsilon}{1 - \varepsilon}\right) = v \frac{1 - \varepsilon[1 + (e'/v)]}{1 - \varepsilon}
\]

Similarly:

\[
v'_1 = v' \frac{1 - \varepsilon[1 - (e'/v')]}{1 - \varepsilon}
\]

Hence the resistance moduli:

\[
\frac{I_1}{v_1} = \frac{I}{v} \frac{1 - \varepsilon[1 + (e'^2/r^2)]}{1 - \varepsilon[1 + (e'/v')]}
\]  

(b)

and

\[
\frac{I_1}{v'_1} = \frac{I}{v'} \frac{1 - \varepsilon[1 + (e'^2/r^2)]}{1 - \varepsilon[1 - (e'/v')]} 
\]  

(c)

If \( \sigma_{p0}, \sigma'_{p0} \) are the stresses due to permanent loads on the full section and \( \sigma_{p1}, \sigma'_{p1} \) the stresses on the section with the holes (permanent loads being the loads which are applied at the same time as the prestress, before grouting), then:

\[
\sigma_{p1} = \sigma_{p0} \frac{1 - \varepsilon[1 + (e'/v)]}{1 - \varepsilon[1 + (e'^2/r^2)]}
\]  

(d)

and

\[
\sigma'_{p1} = \sigma'_{p0} \frac{1 - \varepsilon[1 - (e'/v')]}{1 - \varepsilon[1 + (e'^2/r^2)]}
\]  

(e)

The eccentricities of the boundaries of the limit core become:

\[
\frac{r_1^2}{v_1} = \frac{I_1}{v_1(S - s)} = \frac{I_1}{v_1S(1 - \varepsilon)} = \frac{r^2}{v(1 - \varepsilon)} \frac{1 - \varepsilon[1 + (e'^2/r^2)]}{1 - \varepsilon[1 + (e'/v)]}
\]  

(f)

Similarly:

\[
\frac{r_1^2}{v'_1} = \frac{r^2}{v'(1 - \varepsilon)} \frac{1 - \varepsilon[1 + (e'^2/r^2)]}{1 - \varepsilon[1 - (e'/v')]}
\]  

(g)
The values of the prestresses are:

\[ \sigma_{01} = \frac{F}{S_1} \left( 1 - \frac{e'_1}{r_1^2/v_1} \right) \quad \text{and} \quad \sigma'_{01} = \frac{F}{S_1} \left( 1 - \frac{e'_1}{r_1^2/v'_1} \right) \]

Or:

\[ \sigma_{01} = \frac{F}{S(1 - \varepsilon)} \left( 1 - \frac{e'}{r^2/v} - \varepsilon \frac{e'}{r^2/v} \left[ 1 + \left( \frac{e'/v}{1 + (e'/r^2)} \right) \right] \right) \quad (h) \]

\[ \sigma'_{01} = \frac{F}{S(1 - \varepsilon)} \left( 1 + \varepsilon \frac{e'}{r^2/v'} - \varepsilon \left( \frac{1}{1 + (e'/r^2)} \right) \right) \quad (i) \]

Formulae (h) and (i) can be re-arranged as follows:

\[ \sigma_{01} = \frac{F}{S(1 - \varepsilon)} \left( 1 - \frac{e'}{r^2/v} + \varepsilon \frac{e'}{r^2/v} \frac{1 - [e'/(r^2/v)]}{1 - \varepsilon[1 + (e'/r^2)]} \right) \]

\[ = \frac{F}{S(1 - \varepsilon)} \left( 1 - \frac{e'}{r^2/v} \right) \left( 1 + \varepsilon \frac{e'(r^2/v)}{1 - \varepsilon[1 + (e'/r^2)]} \right) \]

\[ = \frac{F}{S(1 - \varepsilon)} \left( 1 - \frac{e'}{r^2/v} \right) \frac{1 - \varepsilon}{1 - \varepsilon[1 + (e'/r^2)]} \]

or:

\[ \sigma_{01} = \frac{F}{S} \left( 1 - \frac{e'}{r^2/v} \right) \times \frac{1}{1 - \varepsilon[1 + (e'/r^2)]} \]

Similarly:

\[ \sigma'_{01} = \frac{F}{S} \left( 1 + \frac{e'}{r^2/v'} \right) \times \frac{1}{1 - \varepsilon[1 + (e'/r^2)]} \]

If \( \sigma_{00} \) and \( \sigma'_{00} \) are the stresses due to the prestressing force \( F \) on the full section, then:

\[ \sigma_{00} = \frac{F}{S} \left( 1 - \frac{e'}{r^2/v} \right) \quad \text{and} \quad \sigma'_{00} = \frac{F}{S} \left( 1 + \frac{e'}{r^2/v'} \right) \]
Therefore:

\[ \sigma_{01} = \sigma_{00} \times \frac{1}{1 - \varepsilon[1 + (e'^2/r^2)]} \]  

\[ \sigma'_{01} = \sigma'_{00} \times \frac{1}{1 - \varepsilon[1 + (e'^2/r^2)]} \]

The prestresses at the extreme fibres are therefore derived from the prestresses at those same fibres for the full section by multiplying the latter by the same coefficient.

Consequently, the two prestresses (in the section with holes and in the full section) are proportional to each other; at any level, the prestress on a section with holes is derived from the prestress on the full section by applying to the latter a uniform incremental coefficient of

\[ \frac{1}{1 - \varepsilon[1 + (e'^2/r^2)]} \]

(ii) Corrections due to additional cable stresses after grouting

(a) First method. The section is made homogeneous by considering that the cross-sectional areas of the cables are equivalent to cross-sectional areas of concrete which are \( m \) times greater (\( m = E_a/E_b \)).

It is again assumed that the cables are equivalent to a single cable of cross-sectional area \( A_c \); its eccentricity is the same as that of the hole; its absolute value is again denoted by \( e' \). Let \( A_c/S = \lambda \).

The cables, which are bonded after grouting, modify the section properties; the new properties are determined in the same manner as in the previous calculation, the cables introducing an area \( +mA_c \), at eccentricity \( e' \), instead of the areas of the hole.

The new characteristics are therefore obtained from the above calculation, by substituting \( +mA_c \) for \(-s\), and therefore \( m\lambda \) for \(-\varepsilon\).

The corrections must apply to section \( S_1 \), because it cannot be assumed that grouting re-establishes the integrity of the section. The grout is not prestressed and, being in a tensile zone, it cracks. The cables therefore reinforce section \( S_1 \) and not section \( S \). It is, however, more expedient to use section \( S \) as the reference section. The calculation is done by considering that section \( S \) undergoes the weakening effect due to \(-s\), and then the strengthening due to \(+mA_c\); that is, a resultant strengthening due to \(+mA_c - s\). Therefore, \( m\lambda - \varepsilon \) is substituted for \(-\varepsilon\) in the previous formulae. The strengthened section is denoted by \( S_2 \).
The stresses involved are the stresses due to the live loads which are applied subsequent to grouting. They are denoted by \( \sigma_s, \sigma'_s \) and the suffix 0 is introduced when they refer to a full section \((\sigma_{s0}, \sigma'_{s0})\) and the suffix 2 when they refer to the strengthened section \((\sigma_{s2}, \sigma'_{s2})\).

If \( S_2, v_2, v'_2, I_2 \) are the properties of the strengthened section, formulae (b) and (c) become:

\[
\frac{I_2}{v_2} = \frac{1}{v} \frac{1 + (m\lambda - \varepsilon)[1 + (e'/r^2)]}{1 + (m\lambda - \varepsilon)[1 + (e'/v)]}
\]

\[
\frac{I_2}{v'_2} = \frac{1}{v'} \frac{1 + (m\lambda - \varepsilon)[1 + (e'^2/r^2)]}{1 + (m\lambda - \varepsilon)[1 - (e'/v')]} \]

Hence, the corrected stresses at the extreme fibres are:

\[
\sigma_{s2} = \sigma_{s0} \frac{1 + (m\lambda - \varepsilon)[1 + (e'/v)]}{1 + (m\lambda - \varepsilon)[1 + (e'^2/r^2)]} \quad (l)
\]

\[
\sigma'_{s2} = \sigma'_{s0} \frac{1 + (m\lambda - \varepsilon)(1 - e'/v')}{1 + (m\lambda - \varepsilon)(1 + e'^2/r^2)} \quad (m)
\]

(b) Second method. The section is not made homogeneous, but the additional stress \( \Delta T \) in the cable is calculated. When the cable is bonded, this stress is equal to \(-m \Delta \sigma_c\), where \( \Delta \sigma_c \) is the stress variation in the concrete local to the cable under the influence of the moment \( M_s \) which is applied to the section after grouting. The prestress is then considered as being applied by the cable, with increased stress \( T + \Delta T \).

The corrections to the stress result from the variations in the prestresses, \( \Delta \sigma_0, \Delta \sigma'_0 \), due to the force \( A_c \Delta T \). They must be calculated on the basis of section \( S_1 \). Let \( \sigma_{s1}, \sigma'_{s1} \) be the stresses that refer to section \( S_1 \) (with holes), and \( \sigma_{s2}, \sigma'_{s2} \) the corrected stresses, allowing for the additional stresses: \((\sigma_{s2} = \sigma_{s1} + \Delta \sigma_0, \sigma'_{s2} = \sigma'_{s1} + \Delta \sigma'_0)\).

Prior to moment \( M_s \) being applied:

\[
\sigma_c = \frac{F}{S_1} \left(1 + \frac{e'_1^2}{r_1^2}\right)
\]

The effect of moment \( M_s \) is to increase \( F \), accompanied by an increase in the stress \( \sigma_c \) equal to \( \Delta F/S_1[1 + (e'_1^2/r_1^2)] \), and it is accompanied by a bending stress equal to \(-M_s(e_1/I_1)\).
Therefore, in total:

\[
\Delta \sigma_c = \frac{\Delta F}{S_1} \left( 1 + \frac{e_1' \^2}{r_1^2} \right) - M_s \frac{e_1'}{I_1}
\]

\[= A_c \frac{\Delta T}{S_1} \left( 1 + \frac{e_1' \^2}{r_1^2} \right) - M_s \frac{e_1'}{I_1}\]

\[= -\lambda_1 m \Delta \sigma_c \left( 1 + \frac{e_1' \^2}{r_1^2} \right) - M_s \frac{e_1'}{I_1}\]

Hence:

\[\Delta \sigma_c = -\frac{M_s(e_1'/I_1)}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]

and

\[\Delta T = +\frac{mM_s(e_1'/I_1)}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]

Therefore:

\[\Delta \sigma_0 = A_c \frac{\Delta T}{S_1} \left( 1 - \frac{e_1'}{r_1^2/v_1} \right)\]

and

\[\Delta \sigma_0' = A_c \frac{\Delta T}{S_1} \left( 1 + \frac{e_1'}{r_1^2/v_1'} \right)\]

or:

\[\Delta \sigma_0 = \frac{m\lambda_1 M_s(e_1'/I_1)\{1 - [e_1'/(r_1^2/v_1)]\}}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]} = M_s \frac{v_1}{I_1} \frac{m\lambda_1(e_1'/v_1)\{1 - [e_1'/(r_1^2/v_1)]\}}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]

\[= M_s \frac{v_1}{I_1} \frac{m\lambda_1[(e_1'/v_1) - (e_1'^2/r_1^2)]}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]

\[\Delta \sigma_0' = \frac{m\lambda_1 M_s(e_1'/I_1)\{1 + [e_1'/(r_1^2/v_1')]\}}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]} = M_s \frac{v_1'}{I_1} \frac{m\lambda_1(e_1'/v_1')\{1 + [e_1'/(r_1^2/v_1')]\}}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]

\[= M_s \frac{v_1'}{I_1} \frac{m\lambda_1[(e_1'/v_1') + (e_1'^2/r_1^2)]}{1 + m\lambda_1[1 + (e_1'^2/r_1^2)]}\]
Since \( M_3(v_1/I_1) = \sigma_{s1} \) and \( -M_3(v'/I_1) = -\sigma_{s1} \), the values of the resultant stresses are:

\[
\sigma_{s2} = \sigma_{s1} \left\{ 1 + \frac{m\lambda_1 [(e'_{1}/v_1) - (e'_{1}^2/r_1^2)]}{1 + m\lambda_1[1 + (e'_{1}^2/r_1^2)]} \right\} \\
= \sigma_{s1} \frac{1 + m\lambda_1[1 + (e'_{1}/v_1)]}{1 + m\lambda_1[1 + (e'_{1}^2/r_1^2)]} \quad \text{(n)}
\]

\[
\sigma'_{s2} = \sigma'_{s1} \left\{ 1 - \frac{m\lambda_1 [(e'_{1}/v'_{1}) + (e'_{1}^2/r_1^2)]}{1 + m\lambda_1[1 + (e'_{1}^2/r_1^2)]} \right\} \\
= \sigma'_{s1} \frac{1 + m\lambda_1[1 - (e'_{1}/v'_{1})]}{1 + m\lambda_1[1 + (e'_{1}^2/r_1^2)]} \quad \text{(p)}
\]

This second method gives exactly the same corrected stresses \( \sigma_{s2} \) and \( \sigma'_{s2} \) as the first method. In order to compare the corrections relating to the additional stresses only, \( \varepsilon \) can be made equal to zero, which is the same as saying that the reference section for both methods is Section S1. Formulae (l) and (m) are then identical with formulae (n) and (p).

It is thus possible to evaluate ‘exactly’ the corrections to the stresses in the section with holes before grouting (stresses under permanent loading and prestress) by the use of formulae (d), (e), (j) and (k), and to calculate also the corrections to live load stresses applied after grouting in the section which is then strengthened by the cables, either with formulae (l) and (m) (with the stresses calculated for the full section as reference), or with formulae (n) and (p) (with the stresses calculated for the section with the holes as reference).

It is seen in Section 8 that the quantities \( \varepsilon \) and \( m\lambda \) (or \( m\lambda - \varepsilon \)) are small. It is sufficiently accurate in practice to consider them as being infinitely small. The following approximate formulae are thus obtained.

It is recalled that the stresses are identified by means of a double suffix; the first suffix characterises the loading (0 prestress, \( p \) permanent loading applied before grouting, \( s \) live loading applied after grouting), the second suffix defines the section (0 full section, 1 section with holes, 2 section strengthened by cables). It is also recalled that \( e' \) is made equal to \(-e\) (\( e' = -e \)) in order to emphasise the absolute value of the eccentricity, it being assumed negative:
\[
\begin{align*}
\varepsilon &= \frac{\text{area of holes}}{\text{area of full section}}; \\
\lambda &= \frac{A_\varepsilon}{S}; \\
m &= \frac{E_p}{E_b}
\end{align*}
\]

\[
\begin{align*}
\sigma_{p1} &= \sigma_{p0} \left[ 1 + \varepsilon \left( \frac{e''^2}{r^2} - \frac{e'}{v} \right) \right] \quad (d') \\
\sigma'_{p1} &= \sigma'_{p0} \left[ 1 + \varepsilon \left( \frac{e''^2}{r^2} + \frac{e'}{v} \right) \right] \quad (e') \\
\sigma_{01} &= \sigma_{00} \left[ 1 + \varepsilon \left( 1 + \frac{e''^2}{r^2} \right) \right] \quad (j') \\
\sigma'_{01} &= \sigma'_{00} \left[ 1 + \varepsilon \left( 1 + \frac{e''^2}{r^2} \right) \right] \quad (k') \\
\sigma_{s2} &= \sigma_{s0} \left[ 1 - (m\lambda - \varepsilon) \left( \frac{e''^2}{r^2} - \frac{e'}{v} \right) \right] \quad (l') \quad \text{Full section as reference} \\
\sigma'_{s2} &= \sigma'_{s0} \left[ 1 - (m\lambda - \varepsilon) \left( \frac{e''^2}{r^2} + \frac{e'}{v} \right) \right] \quad (m') \\
\sigma_{s2} &= \sigma_{s1} \left[ 1 - m\lambda_1 \left( \frac{e_{1}''^2}{r_1^2} - \frac{e_{1}'}{v_1} \right) \right] \quad (n') \quad \text{Section with holes as reference} \\
\sigma'_{s2} &= \sigma'_{s1} \left[ 1 - m\lambda_1 \left( \frac{e_{1}''^2}{r_1^2} + \frac{e_{1}'}{v_1} \right) \right] \quad (p')
\end{align*}
\]

(the quantity \(e''^2/r^2 - e'/v\) is usually positive).

These formulae show:

(a) That the corrections relative to the stresses due to external loading are smaller for the top fibre than for the bottom fibre.

(b) That the correction factors relative to permanent load stresses (\(\Delta\sigma_p, A\sigma'_p\)) are of opposite sign to those relative to the live load stresses (\(\Delta\sigma_s, A\sigma'_s\)); or, in other words, that the correction factors are >1 for \(\sigma_p, \sigma'_p\) and <1 for \(\sigma_s, \sigma'_s\). There is therefore a partial compensation under the total load (\(p + s\)).

(c) That the true prestress (\(\sigma_{s1}, \sigma'_{s1}\)) is always greater than the prestress which is estimated on the basis of the full section (\(\sigma_{00}, \sigma'_{00}\)).
It should also be noted that, in the present case, with positive moments:
\( \sigma_p \) and \( \sigma_s \) are positive, \( \sigma_0 \) is generally negative;
\( \sigma'_p \) and \( \sigma'_s \) are negative, \( \sigma'_0 \) is always positive.

Therefore, if corrections are not applied, the true state of stress compared with that obtained by considering the full section is expressed by the following:

\[ \Delta \sigma_p \text{ positive, } \Delta \sigma_s \text{ negative (small) and } \Delta \sigma_0 \text{ negative;} \]
\[ \Delta \sigma'_p \text{ negative, } \Delta \sigma'_s \text{ positive and } \Delta \sigma'_0 \text{ positive.} \]

Consequently, the resultant stresses in the fully loaded condition are usually smaller at the top and larger at the bottom (therefore outside the limit stresses) than when they are evaluated on the basis of a full section. To design on this basis is therefore slightly conservative.

Before examining the practical conclusions to be drawn from the formulae, the order of magnitude of the correction factors are examined in the following, with the aid of examples.

For any system of prestress, the ratio between the area of the holes and \( A_c \) is practically constant, and so is the ratio between \( \varepsilon \) and \( \lambda \). The corrections ultimately depend only on the single parameter \( \lambda \) (and of course on the eccentricity).

With the FREYSSINET system, \( \varepsilon/\lambda = 2.7 \) to 3.3, depending upon the size of the cables.

8. Application of formulae

The examples of Chapters V and VI are again considered.

*Example a [Example I of Chapter V, Section 5 (6 × 0.295 m slab)]:*

\[ h = 0.295 \text{ m} \quad e' = 0.081 \text{ m} = 0.274 \text{ h} \quad r^2 = h^2/12 = 0.083 \text{ h}^2 \]
\[ e'^2 = 0.075 \text{ h}^2 \quad e'^2/r^2 = 0.9 \quad e'/v = e'/v' = 0.548 \]
\[ A_c = 28 \times 4.61 = 129 \text{ cm}^2 \quad \lambda = (129 \times 10^{-4})/(6 \times 0.295) = 0.73/100 \]
\[ \varepsilon = 3.3\lambda = 2.4/100 \]
\[ m = 5.7 \quad m\lambda = 4.1/100 \quad m\lambda - \varepsilon = 1.7/100 \]

The following values are obtained (the numerator is the actual stress
and the denominator is the stress on the basis of a full section):

\[
\frac{\sigma_{01}}{\sigma_{00}} = \frac{\sigma'_{01}}{\sigma'_{00}} = 1 + \frac{2.4}{100} \times 1.9 = 1.045
\]

\[
\begin{align*}
\frac{\sigma_p}{\sigma_p^0} &= 1 + \frac{2.4}{100} (0.9 - 0.548) = 1.008 \\
\frac{\sigma_p'}{\sigma_p'^0} &= 1 + \frac{2.4}{100} (0.9 + 0.548) = 1.035
\end{align*}
\]

\[
\begin{align*}
\frac{\sigma_{s2}}{\sigma_{s0}} &= 1 - \frac{1.7}{100} (0.9 - 0.548) = 0.994 \\
\frac{\sigma_{s2}'}{\sigma_{s0}'} &= 1 - \frac{1.7}{100} (0.9 + 0.548) = 0.975
\end{align*}
\]

**Example b** [Example II of Chapter V, Section 10 (8.5 × 0.73 m slab)]:

\[h = 0.73 \text{ m} \quad e' = 0.295 \text{ m} = 0.4 \text{ m} \quad r^2 = h^2/12 = 0.083 \text{ m}^2\]

\[e'^2 = 0.16h^2 \quad e'^2/r^2 = 1.92 \quad e'/v = e'/v' = 0.8\]

\[\lambda = 0.59/100 \quad \varepsilon = 2/100 \quad m = 5 \quad m\lambda = 2.95/100 \quad m\lambda - \varepsilon = 0.95/100\]

The following are obtained (calculations as for Example a):

\[
\frac{\sigma_{01}}{\sigma_{00}} = \frac{\sigma'_{01}}{\sigma'_{00}} = 1.058
\]

\[
\frac{\sigma_p}{\sigma_p^0} = 1.022 \quad \frac{\sigma_p'}{\sigma_p'^0} = 1.054
\]

\[
\frac{\sigma_{s2}}{\sigma_{s0}} = 0.989 \quad \frac{\sigma_{s2}'}{\sigma_{s0}'} = 0.974
\]

**Example c** [Chapter V, Section 11 (slab as above, with limited prestress)]:

\[h = 0.75 \text{ m} \quad e' = 0.305 \text{ m} = 0.41 \text{ m} \quad r^2 = 0.083 \text{ m}^2 \quad e'^2 = 0.168 \text{ m}^2\]

\[e'^2/r^2 = 2 \quad e'/v = e'/v' = 0.82\]

\[\lambda = 0.46/100 \quad \varepsilon = 1.57/100 \quad m = 5 \quad m\lambda = 2.31/100 \quad m\lambda - \varepsilon = 0.74/100\]
And:
\[
\frac{\sigma_{01}}{\sigma_{00}} = \frac{\sigma'_{01}}{\sigma'_{00}} = 1.048
\]
\[
\frac{\sigma_{p1}}{\sigma_{p0}} = 1.019 \quad \frac{\sigma'_{p1}}{\sigma'_{p0}} = 1.044
\]
\[
\frac{\sigma_{s2}}{\sigma_{s0}} = 0.991 \quad \frac{\sigma'_{s2}}{\sigma'_{s1}} = 0.979
\]

Example d [Chapter VI, Section 2]:

\[
S = 0.334 \text{ m}^2 \quad A_c = 18.06 \text{ cm}^2 \quad s_{\text{holes}} = 48 \text{ cm}^2
\]
\[
e' = 0.53 \text{ m} \quad v = 0.486 \text{ m} \quad v' = 0.614 \text{ m}
\]
\[
r^2 = \frac{1}{S} = 0.155 \text{ m}^2 \quad e'^2 = 0.28 \text{ m}^2 \quad e'^2/r^2 = 1.8 \quad e'/v = 1.09
\]
\[
e'/v' = 0.87 \quad \lambda = 0.54/100 \quad \varepsilon = 1.45/100 \quad m = 10 \quad m\lambda = 5.4/100
\]
\[
m\lambda - \varepsilon = 4/100
\]

It is found that:
\[
\frac{\sigma_{01}}{\sigma_{00}} = \frac{\sigma'_{01}}{\sigma'_{00}} = 1.04
\]
\[
\frac{\sigma_{p1}}{\sigma_{p0}} = 1.01 \quad \frac{\sigma'_{p1}}{\sigma'_{p0}} = 1.042
\]
\[
\frac{\sigma_{s2}}{\sigma_{s0}} = 0.972 \quad \frac{\sigma'_{s2}}{\sigma'_{s0}} = 0.89
\]

The correction to $\sigma'$ in this last example is high because prolonged loading is involved ($m = 10$). With $m = 5$ the correction is 0.967 instead of 0.89.

From the example, it is seen that the errors incurred when designing on the basis of a full section are:

Small in every case at the top fibre: 1 to 3% low with permanent loading (actual stresses higher than calculated stresses); 1 to 3% high with live loading (actual stresses lower than calculated stresses).

Also small at the bottom fibre with instantaneous live loads ($m = 5$): 3 to 5% low with permanent loading (actual stresses higher than calculated stresses); 2 to 3% high with live loading (actual stresses smaller than calculated stresses).
In the case of instantaneous live-loading there is no advantage in modifying the design.

With regard to prestress in this same case, the actual values (section with holes) are about 4 to 6% higher than the values calculated on the basis of a full section. Again, there is no real advantage in modifying the prestressing force. Since there is partial compensation between the errors corresponding to permanent loading and to live loading in the fully loaded condition, however, the total error is of the same magnitude as that in $\sigma_0$, $\sigma'_0$; it can therefore be considered useful to apply the correction to the prestress.

In order to make this correction, the following are required, in principle:

(i) a re-estimation of the stresses under external loading (directly on the basis of sections $S_1$ or $S_2$, or with the use of formulae I). This provides new values for the required prestresses.

(ii) Calculation of the corresponding prestressing force and its eccentricity (on the basis of section $S_1$, or on the basis of the full section), the force obtained then being reduced in the ratio $1/[1 + \varepsilon(1 + (e''^2/r^2))]$.

In practice it may be sufficient to calculate (for the full section) the overtensioning $\Delta T$ of the cable due to live loading

\[
\Delta T = m \Delta \sigma_c \approx m\sigma'_{s0} \frac{e'}{v'}
\]

and then to reduce the prestressing force in the ratio $T/(T + \Delta T)$, without modifying the eccentricity.

In the case of live loads of long duration ($m = 10$ or 15), the errors in the stresses due to the live loads on the bottom fibre are noticeably higher than in the case of instantaneous live loads and it is generally advantageous to modify the design.

Since the errors at the top fibre are small, even in this case, it is only necessary to modify the bottom flange in the case of an I-beam, in order to reduce the value of $1/v'$ for the full section\(^\dagger\) in the ratio

\[
\left[ 1 - (m\lambda - \varepsilon) \left( \frac{e''^2}{r^2} + \frac{e'}{v'} \right) \right]
\]

in accordance with formula (m').

The prestress must also be modified. The values of the required prestresses $\sigma_0$ and $\sigma'_0$ are immediately obtained from the actual stress values

\(^\dagger\) A method of modifying the value of $1/v'$ is given in Chapter VIII, Section 7.
under external loading. As in the foregoing, the prestressing force and its eccentricity are obtained directly on the basis of the new section $S_1$, or they can be calculated on the basis of the modified full section, the force then being reduced in the ratio $1/[1 + \varepsilon(1 + (e'/e)^2)]$.

The formulae I therefore enable both the design of the section and the prestresses to be corrected, if considered necessary.

When the magnitude of the corrections is large a check calculation is necessary. The properties of the modified section are determined, taking the holes ($S_1$) and the cable strengthening ($S_2$) into account, directly or with the formulae given above.

Example with instantaneous live loading [Example of Chapter V, Section 11 (limited prestress in slab). See Example c above for the coefficients]:

$$R = 1100 \text{ t/m}^2 \quad R' = -200 \text{ t/m}^2 \quad h = 0.75 \text{ m} \quad S = 6.37 \text{ m}^2$$

$$e' = |e| = 0.305 \text{ m} \quad e'/h = 0.41$$

$$\sigma_0 = \frac{F}{S} \left(1 - 6 \frac{e'}{h}\right) = -1.44 \times \frac{F}{S} \quad \sigma'_0 = \frac{F}{S} \left(1 + 6 \frac{e'}{h}\right) = 3.44 \times \frac{F}{S}$$

<table>
<thead>
<tr>
<th>Stresses under external loading</th>
<th>Stresses in full section (see Chapter V)</th>
<th>Correction factor (Example c)</th>
<th>Actual stresses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre: $\sigma_p$</td>
<td>1000</td>
<td>1.019</td>
<td>1019</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>665</td>
<td>0.991</td>
<td>660</td>
</tr>
<tr>
<td>Total</td>
<td>1665 t/m²</td>
<td>0.991</td>
<td>1679 t/m²</td>
</tr>
<tr>
<td>Bottom fibre: $\sigma'_p$</td>
<td>-1000</td>
<td>1.044</td>
<td>-1044</td>
</tr>
<tr>
<td>$\sigma'_s$</td>
<td>-665</td>
<td>0.979</td>
<td>-650</td>
</tr>
<tr>
<td>Total</td>
<td>-1665 t/m²</td>
<td>0.979</td>
<td>-1694 t/m²</td>
</tr>
<tr>
<td>Required $\sigma'_0$</td>
<td>1465 t/m²</td>
<td></td>
<td>1494 t/m²</td>
</tr>
<tr>
<td>$=</td>
<td>\sigma'_p + \sigma'_s - 200</td>
<td>$</td>
<td></td>
</tr>
</tbody>
</table>

For the full slab:

$$\text{Required} \frac{F}{S} = \frac{\sigma'_0}{1 + 6(e'/h)} = \frac{1465}{3.44} = 426 \text{ t/m}^2$$

Hence $F = 426 \times 6.37 = 2717$ t (see Chapter V).
For the slab with holes: The actual stress of \( \sigma'_o = 1\,494 \, \text{t/m}^2 \) must be obtained.

The correction factor

\[
\frac{\text{actual prestress}}{\text{prestress for full slab}}
\]

is equal to 1.048 (see Example c above).

The calculation can be done on the basis of the full section. The following is then required:

\[
\frac{F}{S} = \frac{1\,494}{3.44} = 433 \, \text{t/m}^2
\]

Hence \( F = 433 \times 6.37 = 2\,740 \, \text{t} \).

This must be reduced in the ratio of 1/1.048, hence:

\[
F = \frac{2\,740}{1.048} = 2\,620 \, \text{t} \quad \text{(economy achieved = 3.5%)}
\]

Then \( \sigma_o = -1.44 \times 433 = -625 \, \text{t/m}^2 \).

The stresses due to the various conditions of loading are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Prestress</th>
<th>Self-weight</th>
<th>Total unloaded</th>
<th>Live loads</th>
<th>Total loaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre</td>
<td>-625</td>
<td>1 019</td>
<td>394</td>
<td>660</td>
<td>1 054</td>
</tr>
<tr>
<td>Bottom fibre</td>
<td>1 494</td>
<td>-1 044</td>
<td>450</td>
<td>-650</td>
<td>-200</td>
</tr>
</tbody>
</table>

There is no need to modify the design since the stresses under load are acceptable.

The results can be obtained more rapidly by calculating the excess stress \( \Delta T \) in the cable under the action of the live loads.

The stress variation in the concrete local to the cables is:

\[
\Delta \sigma_c = \Delta \sigma'_s \times \frac{e'}{v'} = -0.82 \times 665 = -545 \, \text{t/m}^2
\]

Hence:

\[
\Delta T = -m \Delta \sigma_c = +5 \times 545 = 2\,720 \, \text{t/m}^2 \quad (2.7 \, \text{kg/mm}^2)
\]

The value of \( T \) was 89 kg/mm². The prestressing force can be reduced in the ratio \( T/(T + \Delta T) = 89/91.7 = 0.97 \).

Therefore the required force is 0.97 \( \times 2\,717 = 2\,640 \, \text{t} \).
Example for the case of live loads of long duration (Example in Chapter VI; $m = 10$. For correction factors, see Example d above):

Limit stresses:

unloaded $-150, +1350$
loaded $+1000, -100$ t/m$^2$

$$S = 0.334 \text{ m}^2$$
$$e' = |0.53| \text{ m}$$
$$r^2/v = 0.319 \text{ m}$$
$$r^2/v' = 0.253 \text{ (full section)}$$
$$e'/v' = 0.87$$

$$\sigma_0 = \frac{F}{S} \left( 1 - \frac{0.53}{0.319} \right) = -0.66 \times \frac{F}{S}$$
$$\sigma_0' = \frac{F}{S} \left( 1 + \frac{0.53}{0.252} \right) = 3.09 \times \frac{F}{S}$$

<table>
<thead>
<tr>
<th>Stresses under external loading</th>
<th>Stresses in full section (Chapter VI)</th>
<th>Correction factor (Example d)</th>
<th>Actual stresses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre: $\sigma_p$</td>
<td>184</td>
<td>1.01</td>
<td>186</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>1149</td>
<td>0.972</td>
<td>1115</td>
</tr>
<tr>
<td>Total</td>
<td>1333</td>
<td>0.972</td>
<td>1301 t/m$^2$</td>
</tr>
<tr>
<td>Bottom fibre: $\sigma_p'$</td>
<td>-233</td>
<td>1.042</td>
<td>-243</td>
</tr>
<tr>
<td>$\sigma_s'$</td>
<td>-1451</td>
<td>0.89</td>
<td>-1294</td>
</tr>
<tr>
<td>Total</td>
<td>-1684</td>
<td>0.89</td>
<td>-1533 t/m$^2$</td>
</tr>
<tr>
<td>Required $\sigma_0'$</td>
<td>$</td>
<td>\sigma_p' + \sigma_s' - 100</td>
<td>$</td>
</tr>
</tbody>
</table>

For the full section: required value of

$$\frac{F}{S} = \frac{\sigma_0'}{3.09} = \frac{1584}{3.09} = 513 \text{ t/m}^2$$

Hence $F = 513 \times 0.334 = 171$ t (see Chapter VI).

For the section with holes: $\sigma_0' = 1433$ t/m$^2$ is required (actual value).

The correction factor

$$\frac{\text{actual prestress}}{\text{prestress in full section}} = 1.04$$

(see Example d).
The calculation can be done on the basis of the full section:

\[
\frac{F}{S} = \frac{1433}{3.09} = 464 \text{ t/m}^2
\]

is then required

Hence \( F = 464 \times 0.334 = 155 \text{ t} \).

It is now necessary to reduce this force in the ratio of 1/1.04, hence:

\[
F = \frac{155}{1.04} = 149 \text{ t}
\]

The value of the prestress \( \sigma_0 \) (actual) is then: \(-0.66 \times 462 = -308 \text{ t/m}^2\).

The stresses under the various conditions of loading are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Prestress</th>
<th>Self-weight</th>
<th>Total un loaded</th>
<th>Live loads</th>
<th>Total loaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre</td>
<td>-308</td>
<td>186</td>
<td>-122</td>
<td>1115</td>
<td>993 t/m²</td>
</tr>
<tr>
<td>Bottom fibre</td>
<td>1433</td>
<td>-243</td>
<td>1190</td>
<td>-1290</td>
<td>-100 t/m²</td>
</tr>
</tbody>
</table>

It should be noted that if only the prestressing force is modified, and not the dimensions of the section, the approximate value of the required force is again obtained by reducing the force which is required in the case of the full section (171 t), in the ratio \( T/(T + \Delta T) \), where \( \Delta T \) is the excess stress in the cable.

Again

\[
\Delta T = -m \Delta \sigma_c = -m \sigma'_0 \frac{e'}{v'}
\]

\[=-0.87 \times 10 \times (-1451)\]

\[=12600 \text{ t/m}^2 \text{ or } 12.6 \text{ kg/mm}^2\]

Now \( T = 95 \) (see Chapter VI). Therefore \( T/(T + \Delta T) = 95/107.6 = 0.88 \).

The required prestressing force is therefore \( 0.88 \times 171 = 151 \text{ t} \), and it is very nearly equal to the value obtained above. But the concrete is overdesigned, since the stress at the bottom fibre in the unloaded conditions is 1190 t/m², and 1350 t/m² is permissible.

There is therefore a slight advantage in changing the section. The relation \( I/v' = M_s/\Delta R \) is valid, but the value of \( I/v' \) refers to the homogeneous section. The correction factor relative to \( I/v' \) (which is the same as that
for $\sigma'$, being equal to 0.89 (see Example d), it is sufficient for the $I/\nu'$ value of the non-homogeneous full section to be equal to:

$$0.89 \times \frac{M_s}{\Delta R} = 0.89 \times \frac{122.5}{1450} = 0.89 \times 0.0845$$

$$= 0.075 \text{ m}^3$$

The calculation is then continued on this basis.

One method of considering the problem is dealt with in Chapter VI, Section 3. It is again examined in Chapter VIII, Section 7.

A slight economy of concrete is realised (5%), but the prestressing force remains roughly the same as that obtained by applying the prestress correction alone (152 t in place of 149 t). In many cases this justifies the correction to $F$ only, the possible economies of the concrete being neglected.

When the corrections are not made in the following chapters, it should be understood that they could be made in accordance with the methods shown above, if this is justified.

9. Effect of excess stress in the cables on the position of the line of thrust.

Internal and external lines of thrust

(i) Total prestress and limited prestress
The simple case without complementary steel, where the normal force due to external loading is zero, is dealt with first.

As stated in Section 7, there are two ways of considering a prestressed section:

(a) It can be considered that the prestressing force is the original force exerted by the cable at the time that it is bonded within the section (by grouting) and that this force, which modifies the initial state of stress, is an artificially applied normal force which subsequently remains constant. Under the effect of the moments acting on the section after bonding, the section behaves as if it were homogeneous, for the classes of prestress being considered here. The cables contribute to the section as reinforcement, their equivalent cross-sectional area being taken as $m$ times their actual area $A_c$. The various points in the section are then subjected to stress variations $\Delta \sigma$, which are additive to the stresses at the time of bonding. In particular, for as long as cracking in the section does not occur (which is the case with total and limited prestress) the equivalent cross-sectional areas $mA_c$ are subjected to stress variations $\Delta \sigma_c$, which are equal to the stress variations in the concrete at the same level.
The total force in the cable increases, therefore, and becomes $F + \Delta F$, but it is considered as being made up of two components: the constant term $F$, which is the applied normal force, and the term

$$\Delta F(\Delta F = -ma_e \Delta \sigma_e)$$

which is the overtensioning force caused by the loading, and which forms part of the internal resistance forces.

(b) It can be considered that the prestressing force is at any instant equal to $F + \Delta F$. In this case the section is not considered as being homogeneous. It retains the same properties as at the time of grouting and bonding.

It has been seen that the two concepts lead to identical results as far as the stresses are concerned (concrete and cables). They do, however, appear to lead to different interpretations of the line of thrust. In all cases, the thrust line is coincident with the cable† when the moments are zero. It ceases to be coincident under the influence of the moments. The displacements at any section are equal to $M/F$ in the first case and $M/(F + \Delta F)$ in the second.

In fact, these two definitions of the line of thrust (hereafter called 1 and 2) are not contradictory, but with the first the material is considered to be concrete plus steel and with the second the concrete is considered separately.

The difference between the two curves is illustrated by the comparison with a reinforced concrete beam subjected to composite bending under the action of a constant normal force $N$ and of vertical loads (Fig. 6). At each cross-section $X$ the resultants of the forces to the left (force $N$ and loads applied to the left of the section) have a resultant; the horizontal component of this resultant is $N$, and the resultant intersects the section at a certain point $E_e$.

As section $X$ moves along the beam, point $E_e$ describes the line of thrust due to external loading; that is, the curve along which the external forces acting on the combined materials are applied to the sections.

If the curve is outside the central core at section $X$, equilibrium in the section is maintained because of compressive stresses in the concrete, with a resultant of $B$ passing through the point $b$, and because of tensile stresses in the steel, with a resultant of $-A$ passing through the point $a$ which is

† This is the case for statically-determinate beams. In the case of statically-indeterminate beams, parasitic reactions are introduced and these modify the prestress line of thrust. The conclusions are still relevant in this latter case if the parasitic reactions are treated as external forces.
coincident with the centroid of the tensioned reinforcement. It is evident that \( B = N + A \), and the position of point \( b \) is obtained by taking moments with respect to the reinforcement:

\[
N \times aE_e = (N + A) \times ab
\]

The external line of thrust can be considered as being resisted by two internal lines of thrust: that due to the compression in the concrete (curve B) and that due to the tension in the reinforcement (curve A, whose path coincides with that of the reinforcement).

In the case of prestressed concrete, it can be considered that the material comprises a system of concrete plus steel (the latter being the bonded cables) subjected to the initial force \( F \) intersecting section \( X \) at \( E_e \) (coincident with the cable); the vertical loads which are superimposed onto this initial force \( F \) combine with it to give a certain resultant whose horizontal component is \( F \), intersecting the section \( X \) at a point denoted again by \( E_e \). The locus of \( E_e \) is the line of thrust (I) at a distance \( M/F \) from the cable.

Under the action of the moment \( M \), additional tension is put into the cable (as into the reinforcement in the case of reinforced concrete) due to the bonding forces \( t \) between the steel and the concrete, between the support point and section \( X \). These are considered to be internal forces; namely, the forces \(-t\) applied to the cable by the concrete and the forces \(+t\) applied to the concrete by the cable, cancelling themselves out when related to the global system of concrete plus cable. Consequently, the forces \( \pm \Delta F \) [negative sign for the cable, since the force is tensile (direction to the right if the forces applied by the right-hand side to the left-hand side
are considered); positive sign for the concrete, since the force is compressive (direction to the left), which are both along the tangent to the cable at section X, are also internal forces which cancel out within the overall system of concrete plus steel, and they change neither the force F nor the line of thrust (1).

![Diagram of lines of thrust in a prestressed concrete beam](image)

**Fig. 7.** Lines of thrust in a prestressed concrete beam (Classes I and II).

This curve (1) will be called the resultant line of thrust† (understood to apply to the system concrete plus cable).

If the concrete is now considered separately, it is subjected to the force F previously applied, this force having been displaced through a distance $\frac{M}{F}$ to the point $E_c$ by the moment $M$, and also to the force $+\Delta F$ which

† The term external line of thrust is reserved for the line of thrust caused by external loading, where the prestressed beam is subjected to a normal external load; that is, when it is subjected, by the external loads, to composite bending, to which the prestress is added (see iii, p. 252).
is tangential to the cable; it is therefore subjected to a total force $F + \Delta F$ acting through the point $E_i$. The cable, also considered in isolation, is subjected to the force $-F - \Delta F$, tangential to the cable.

The following is required for equilibrium at section $X$:

$$(F + \Delta F)E_oE_i = M$$

Therefore:

$$E_oE_i = \frac{M}{F + \Delta F} = E_oE_e \frac{F}{F + \Delta F} = E_oE_e \times \frac{1}{1 + (\Delta F/F)}$$

As section $X$ is displaced along the beam, the point $E_i$ describes the curve of compression (2). It is the line of thrust of the resistance forces within the concrete, or the internal line of thrust.

With total prestressing, the ratio $\Delta F/F$ is small and is equal to about 3%. The two curves $E_e$ and $E_i$ are therefore very close together.

With limited prestress, especially with live loading of long duration, the ratio $\Delta F/F$ can reach higher values, but it does not in general exceed 10%.

The differences between the resultant centre of thrust ($E_e$) and the centre of thrust in the concrete ($E_i$) were not taken into consideration in Section 5. Only the point $E_e$ was implicitly considered, with the condition that it should remain above the limit point $A_2$, since the displacement was taken as $M/F$.† The methods are therefore theoretically exact only if the position of $A_2$ is determined in the homogeneous section; that is, if its eccentricity $r^2/v'(1 - R_2/\sigma_y)$ has been calculated by giving corrected values to $r^2/v'$ (and to $\sigma_y$), taking homogeneity into account (Section 7). (The area $A_e$ of the cable, required for the corrections, is not at first known, but it can be evaluated by a preliminary calculation based on a non-homogeneous section.)

In practice, the design can be done on the basis of a full section as indicated in Section 5, and the corrections of Section 7 can then be applied, if they are considered necessary.

Since the force which is initially found is slightly too high, a certain corrective force $\Delta F$ must be subtracted from it (Section 7, (ii), b). In other words, the calculation does not yield the required force $F$, but the force $F + \Delta F$. Consequently point $A_2$, determined on the basis of a

† The difficulty does not arise for the lower limit point $A'_1$ when the minimum moment is the self-weight moment, $M_p$, because this moment is applied before bonding, and the displacement of the centre of compression is truly $M_p/F$. 
non-homogeneous section, is the limit position of the centre of thrust within the concrete, \( E_1 \).

It is not necessary in practice to dwell overmuch on these considerations, in the cases of total and limited prestress. The design method of Section 5, on the basis of a full section, is adequate. The corrections can then be applied if their magnitude justifies it.

(ii) Case of reinforced prestressed concrete (Class III)
Here the corrections can no longer be neglected, since, when cracking occurs, the strength is maintained solely by the added tension in the cables and by the tensile forces in the reinforcement. It is no longer a question of applying correction factors, but becomes a more fundamental issue.

The internal line of thrust and the resultant line of thrust are further apart than in the previous cases.

The resultant line of thrust is again deduced from the cable position by applying displacements \( M/F \) at each section, but the displacements are appreciably greater than with total (or limited) prestress because \( F \) is smaller. The rise of the resultant line of thrust is therefore greater than in the previous cases. It can lie outside the boundaries of the section.†

† It should be noted that the resultant line of thrust can also lie outside the boundaries of the section in the case of limited prestress if the top limit point \( A_2 \) is outside the section; that is, if \( (r^2/v') \left( 1 + \left| \frac{R'}{|\sigma_g|} \right| \right) > v', \left| R' \right| \) being the absolute value of the tensile limit stress and \( r^2/v' \) being the eccentricity of the top boundary of the central core in the homogeneous section; or:

\[
1 + \frac{|R'|}{\sigma_g} > \frac{vv'}{r^2}
\]

But (Section 10) \( r^2/vv' \) is approximately equal to \( \frac{1}{4} \). The above condition becomes:

\( \sigma_g < |R'| \).

The initial line of thrust \( (E_0) \) could even be outside the section because the stresses in the concrete include compression stresses, whose resultant is \( \mathcal{C} \), and tensile stresses, whose resultant is \( \mathcal{F} \). The points of intersection of \( \mathcal{C} \) and \( \mathcal{F} \) with the section are of course within the section, but the point of intersection \( (E_0) \) of their resultant could lie outside the section. This case is exceptional, however.

![Fig. 8.](image)
The internal line of thrust \( (E_i) \) stays outside the section because the stresses in the uncracked concrete are all compressive, since the tensile resistance of the concrete is assumed to be zero.

It can be said, in short, that the additional tension in the cables and the tensile force in the reinforcement restore the line of thrust from its position \( E_e \) (resultant) to the position \( E_i \) inside the material.

Consider the lines \( (C) \) and \( (A_2) \) in a simply supported beam (Fig. 9). Lines \( (C) \) and \( (A_2) \) are the loci of the top boundary of the central kern and of the top limit point, this latter point corresponding to a limit stress \(-|R'|\), equal to the tensile strength of concrete.

The internal line of thrust, with its origin at the point of intersection \( O \) of the cable with the vertical through the support, intersects line \( (C) \) at \( c \) and line \( (A_2) \) at \( a_2 \).

Between \( O \) and \( c \) the beam is in total prestress; between \( c \) and \( a_2 \) it is in limited prestress; to the right of \( a_2 \) its behaviour is that of reinforced prestressed concrete.

The internal line of thrust is close to the resultant line of thrust between \( O \) and \( a_2 \); beyond \( a_2 \) the curves are noticeably farther apart.

Let the additional tension in the cable again be \( \Delta F \) and let \( F_a \) be the tensile force in the ordinary reinforcement.

At any section \( X \) (Fig. 10), the external force acting normally is \( F \), its point of application being \( E_e \) (intersection of the resultant line of thrust with plane \( X \)). \( \Delta F \) and \( F_a \) can be calculated in accordance with the rules for reinforced concrete subjected to combined bending and compression.

\( E_t \) is the centre of compression (point of application of the resultant compressive stress) which is determined in accordance with the above rules.
Also:

\[ E_a E_t = \frac{M - F_a \times E_a E_t}{F + \Delta F + F_a} \]

where \( E_a \) is the centroid of the ordinary reinforcement (the reinforcement is assumed to be below \( E_a \)).

Curve \( E_t \) can therefore be easily plotted.

If, conversely, the position of \( E_t \) is approximately known, which often happens, the forces \( \Delta F \) and \( F_a \) can be determined.

Fig. 10.

If two neighbouring sections are considered, with abscissae \( X \) and \( X + \Delta X \), the difference \( \Delta F(x + dx) - \Delta F(x) \) between the forces \( \Delta F \) is caused by the bonding forces exerted by the concrete on the cable sheaths over the length \( dx \); the bonding force is therefore equal to the derivative of \( \Delta F \) with respect to the abscissae. The same applies for the bonding forces exerted on the ordinary reinforcement.

(iii) Case of a normal force under external loading

This case is encountered mostly in statically-indeterminate structures, when the beam is subjected to an external compressive force \( N \) and to the loads acting along its span, which are assumed to act vertically.

A line of external thrust which is the same as that for the beam without prestress corresponds to the force \( N \) and to the vertical loads. A line of
thrust due to the prestress is established by tensioning the cables and this
curve is coincident with the cable if the parasitic reactions are zero† (see
Volume II). The combination of the external line of thrust with that due to
the prestress produces the resultant line of thrust, with the same function
as before.

Suitable values must be given to the prestressing forces $F$, so that the
resultant line of thrust is contained within the beam, or within 'limiting
bands' defined in Chapter I, Volume 2 (which may lie outside the beam in
the case of Class III prestress. Example: Fig. 11).

![Diagram](image)

**Fig. 11.** Example of lines of thrust in the case of external normal loading (portal
with concentrated load. (Case of reinforced prestressed concrete.) AB: External line of
thrust (reaction $R$).

1. Resultant line of thrust (combination of $R$ and $F$ at $A$, and of $R$ and $F'$ at $B$).
The change in direction at $d$ is due to the resultant of the anchorage forces $F$ and $F'$
at the corner $D$.

2. Internal line of thrust resulting from the breakdown of (1) between (2) and cables
(with additional tension) + ordinary reinforcement (the ordinary reinforcement is not
shown).

The resultant line of thrust represents the force $N + F$ (or, more
exactly, the resultant of $N$ and $F$ at each section $X$). If need be, it can be
broken down into a line of thrust within the concrete, transmitting the
force $N + \Delta F$ ($+F_a$ eventually), and the cable (plus the reinforcement
eventually).

† If the parasitic reactions are not zero, there is still a prestressing line of thrust, but it
is no longer coincident with the cable. It can be considered that the prestress line of
thrust is coincident with the cable as long as parasitic reactions are treated as external
forces.
These breakdowns often lead to a simple treatment for portals and arches. They apply especially to statically-indeterminate structures, but can be applied to certain statically-determinate structures (portals or arches with three hinges).

To summarise, a study of the lines of thrust is essential for the understanding of the objects of prestressed concrete and of the methods of achieving them; under all conditions of loading it is necessary that the lines of thrust (resultant or internal) should be contained within the boundaries of certain limit zones.

With total and limited prestress, it is generally sufficient to consider that the internal line of thrust is obtained approximately as a displacement $M/F$ from the cable and to manipulate this curve in order to obtain the required conditions.

This method is generally adopted in the following chapters, the limit zones being defined by the limit eccentricities of Section 3 (calculated on the full section).

It is emphasised that this is only an approximation. The curve $M/F$ is in fact a resultant curve, the internal curve which must lie within the limit zone being obtained using displacements equal to $M/(F + \Delta F)$.

The approximate solution which is obtained by neglecting $\Delta F$ is acceptable in the cases of total and limited prestress. It is not acceptable with reinforced prestressed concrete; not only is it then no longer possible to neglect $\Delta F$, but the tensile forces in the ordinary reinforcement must be added to the additional tension in the cables. These aspects are examined in Chapter XI.

10. Section efficiency
The above calculations, whether they be made directly by consideration of stress variations or on the basis of the limit core, determine only the values of the section moduli $I/v$ and $I/v'$.

These moduli must be selected to give the most economical results. For this, the cross-sectional area $S$ must be as small as possible, and therefore $(I/v)/S$ and $(I/v')/S$ must be as great as possible, or, again, $r^2/v$ and $r^2/v'$ must be as great as possible.

A reduction in the value of $S$ obviously uses less concrete. The increase in $r^2/v$ and $r^2/v'$ gives a saving on cables, since the lever arm ($A_1/A_2 = M_s/F$) varies as $(r^2/v) + (r^2/v')$.

But $r^2/v$ is always less than $v$ (Fig. 12). Its greatest possible value is $v'$. The degree of economy of the section can be defined, therefore, by the ratio $(r^2/v)/v' = r^2/vv'$. This same ratio applies to $r^2/v' [(r^2/v')/v] = r^2/vv'$. 
This ratio is denoted by $\rho$ and is termed the efficiency of the section:

$$\rho = \frac{r^2}{vv'}$$

If the profile were reduced to two infinitely thin flanges with negligible web thickness, the efficiency would be equal to unity.

![Diagram](image)

**FIG. 12.**

The efficiency decreases rapidly as the thicknesses of the flanges and of the web increase. At the limit, when the thickness of each flange is equal to half the depth of the beam, or when the web thickness is equal to the breadth of the section, the profile is rectangular and the efficiency is:

$$\frac{h/6}{h/2} = \frac{1}{3}$$

A slab is therefore not economical from the point of view of utilisation of its materials.†

![Diagram](image)

**FIG. 13.**

† The cost comparison depends, of course, upon the unit prices corresponding to the various solutions.
If a rectangular section of size \( b'h' \) is removed from the inside of a rectangular profile of size \( bh \), a box section (or an equivalent I-beam) is obtained (Fig. 13).

By writing \( b' = \beta b \) and \( h' = \gamma h \), then:

\[
I = \frac{bh^3 - b'h'^3}{12} = \frac{bh^3}{12} (1 - \beta \gamma^3)
\]

\[
\frac{I}{v} = \frac{bh^2}{6} (1 - \beta \gamma^3)
\]

\[
S = bh - b'h' = bh(1 - \beta \gamma)
\]

\[
\frac{r'^2}{v} = \frac{h^2}{6} \frac{1 - \beta \gamma^3}{1 - \beta \gamma}
\]

Hence:

\[
\rho = \frac{r'^2}{v(h/2)} = \frac{1}{3} \frac{1 - \beta \gamma^3}{1 - \beta \gamma}
\]

Figure 14, drawn for symmetrical I-beams and for T-sections, shows how the efficiency varies as a function of the parameters:

<table>
<thead>
<tr>
<th>web thickness</th>
<th>breadth</th>
<th>and</th>
<th>flange thickness</th>
<th>depth</th>
</tr>
</thead>
</table>

Thicker webs in particular reduce the efficiency. There is therefore a gain to be obtained with webs which are as thin as possible. There are limits to this because of practical considerations of concrete placing, and because of the shear resistance which the web provides.†

Acceptable I-sections have an efficiency of about 0·5 to 0·6 (the efficiencies of the sections in Chapter VI, Sections 2 and 3 are 0·52 and 0·50). The efficiency of box sections is usually higher because of the reduced thickness of the sides.

The efficiency of a rectangular section can be increased by providing a hole of circular cross-section within the rectangle (Fig. 15).

† An empirical formula for the thickness \( b' \) of the web is:

\[
b' = \frac{h}{36} + 5·5 \text{ cm} + \text{sheath diameter}
\]

With proper safeguards and with care this thickness can be reduced by 10%.
Fig. 14. Efficiencies of T- and I-sections.
It can easily be shown that the efficiency of a square section of depth $h$ with a concentric hole of diameter $\gamma h$ is:

$$\rho = \frac{1}{3} \frac{1 - 0.43\gamma^4}{1 - 0.785\gamma^2}$$

For

- $\gamma = 0$ (rectangular section) $\rho = 0.33$
- $\gamma = 0.6$ $\rho = 0.44$
- $\gamma = 0.8$ $\rho = 0.55$

The efficiency therefore increases quickly with the lighter sections. This is because the moment of inertia (which varies as the numerator) reduces very slowly with $\gamma$ because of the lower index of 4, whereas the cross-sectional area (which varies as the denominator) reduces much more rapidly.

![Fig. 15.](image)

The savings are useless if they lead to more expensive shuttering. The reductions in material quantities and the cost savings can be reconciled with the use of inexpensive shuttering (waterproof cardboard tubes, expanded polystyrene, wooden framework with plastic covering, or metal shutters where they are re-used many times).
Chapter VIII

DESIGN OF A SECTION OF ANY SHAPE CASE IN WHICH THE PERMANENT LOADS CANNOT BE COMPENSATED (Span greater than the critical span)

1. Maximum span above which self-weight can no longer be compensated (critical span) — (uniform loading)

Using the same symbols as in the previous chapter, and  \( A_1A_2 \) being the limit core, it is necessary to satisfy the following relationship for the self-weight to be compensated (Fig. 1).

\[
\frac{M_p}{F} \leq v' - d' - GA'_1
\]  

(1)

in which \( d' \) is the minimum cover.

Also, with this hypothesis, the centre of compression lies within the limit core \( A'_1A_2 \), under the action of the moment \( M_s \). Thus:

\[
\frac{M_s}{F} = A'_1A_2 \quad \text{or} \quad F = \frac{M_s}{A'_1A_2}
\]

Inequality (1) can therefore be written:

\[
\frac{M_p}{M_s} \leq \frac{v' - d' - GA'_1}{A'_1A_2}
\]

and, since the beam is assumed to be uniformly loaded:

\[
\frac{p}{s} \leq \frac{v' - d' - GA'_1}{A'_1A_2}
\]  

(1a)

With a constant live load \( s \), the self-weight \( p \) increases as the span increases. Inequality (1a) becomes untrue when the span exceeds a certain value \( l_{crit} \).

This is the critical span. The point of intersection \( E_o \) of the cable with the section then lies at a distance \( d' \) from the outside face.
If symmetrical sections are concerned and if the limit stresses \( R \) and \( R' \) are the same in the two extreme loading conditions, the critical span is defined by the relation:

\[
\frac{p}{s} = \frac{(h/2) - d' - (r^2/v)[1 - (R'/\sigma_g)]}{2(r^2/v)[1 - (R'/\sigma_g)]}
\]

(i) Rectangular slab
Assume that \( d' = 0.1h \). Also, \( r^2/v = h/6 \).

The critical span is defined by:

\[
\frac{p}{s} = \frac{0.4h - (h/6)[1 - (R'/\sigma_g)]}{(h/3)[1 - (R'/\sigma_g)]} = \frac{1.2 - 0.5[1 - (R'/\sigma_g)]}{[1 - (R'/\sigma_g)]}
\]

If \( R' = 0 \),

\[
\frac{p}{s} = 0.7
\]

With this hypothesis, the calculation for a slab of unit breadth is given in the following; \( s \) is the live load per unit area of surface. For a slab which is simply supported at the ends:

\[
M_s = 1 \times \frac{s l^2}{8}
\]

Also: \([\Delta M = (I/v) \Delta R]\)

\[
M_s = 1 \times \frac{h^2}{6} \times R
\]
Therefore:

\[
\frac{sl^2}{8} = \frac{h^2}{6} R
\]

\[
h = \left(\frac{3}{4} \frac{s}{R}\right)^{\frac{1}{2}} \tag{3}
\]

Also, if \( D \) is the density of the concrete, \( p = 1 \times hD \), and condition (2) may be written:

\[hD = 0.7s\]

Therefore:

\[
l \left(\frac{3}{4} \frac{s}{R}\right)^{\frac{1}{2}} D = 0.7s
\]

The value of the critical span is:

\[
l_{\text{crit}} = 0.7 \frac{(s)^{\frac{1}{2}}}{D} \left(\frac{4}{3} \frac{R}{R}\right)^{\frac{1}{2}} = 0.7 \frac{R}{D} \left(\frac{4}{3}\right)^{\frac{1}{2}} \left(\frac{s}{R}\right)^{\frac{1}{2}}
\]

or

\[
l_{\text{crit}} = 0.805 \frac{R}{D} \left(\frac{s}{R}\right)^{\frac{1}{2}} \tag{4}
\]

With \( R = 1\,200 \text{ t/m}^2 \), \( D = 2.4 \text{ t/m}^2 \), \( R/D = 500 \text{ m} \):

\[
l_{\text{crit}} = 402 \left(\frac{s}{R}\right)^{\frac{1}{2}}
\]

If \( s \) is in \( \text{t/m}^2 \):

\[
l_{\text{crit}} = \frac{402}{(1\,200)^{\frac{1}{2}}} (s)^{\frac{1}{2}} = 11.6(s)^{\frac{1}{2}} \text{ (metres)} \tag{5}
\]

With \( s = 1 \text{ t/m}^2 \), the critical span is therefore equal to 11.60 m.

It has been assumed, however, that the value of \( R \) could be arbitrarily chosen. This is not valid if the maximum deflection under live loading is limited, as shown in Chapter V.

For example, with \( s = 1 \text{ t/m}^2 \) and \( R = 1\,200 \text{ t/m}^2 \), then, in accordance with formula (3):

\[
h = \left(\frac{3}{4} + \frac{1}{1\,200}\right)^{\frac{1}{2}} = \frac{1}{40}
\]

and the slab is not sufficiently rigid.

It may be necessary to reduce \( R \), therefore; that is, to increase \( h \).
If a deflection equal to $2/1000$ of the span is not to be exceeded in the live load condition, then the corresponding limiting condition is:

$$\frac{5}{384} \frac{s l^4}{E h^2/12} = \frac{2}{1000} \times l$$

Hence:

$$\frac{h}{l} = \left(\frac{5000}{64} \frac{s}{E}\right)^{\frac{3}{4}} = 4.28 \left(\frac{s}{E}\right)^{\frac{3}{4}}$$

(6)

The value of the maximum allowable stress $R$ is then obtained by equating (3) and (6), and:

$$R = \frac{1}{24.5} E^{\frac{3}{4}} s^{\frac{3}{4}}$$

Condition (6) replaces condition (3); condition (2) remains at $hD = 0.7s$. Therefore: $4.28[(s/E)/D]^{\frac{3}{4}} = 0.7s$

$$l_{crit} = \frac{0.7s}{4.28 D} \left(\frac{E}{s}\right)^{\frac{3}{4}}$$

Expressing $D$ in $t/m^3$, $E$ and $s$ in $t/m^2$, the value of the critical span in metres is:

$$l_{crit} = \frac{0.7}{4.28 \times 2.4} s \left(\frac{E}{s}\right)^{\frac{3}{4}}$$

If $E = 3.5 \times 10^6$ (short-term loading),

$$l_{crit} = 10.30 \times s^{\frac{3}{4}}$$

(7)

If, instead of a broad slab, a beam of rectangular cross-section of breadth $b$ is considered, under a live load $s$ per unit length, $s/b$ must be substituted for $s$ in formulae (5) and (7). Consequently, the critical spans for the same values of $E$ and $R$ are equal to:

$$l_{crit} = 11.6 \left(\frac{s}{b}\right)^{\frac{3}{4}}$$

(5a)

or:

$$l_{crit} = 10.3 \left[\left(\frac{s}{b}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}}$$

(7a)

The least of these two values is the acceptable value.

It is easily seen that the governing condition is condition (7a) when $s/b < 2 t/m^2$, and condition (5a) when $s/b > 2 t/m^2$. 
When condition (7a) is the determining factor (deflection), the value to be used for R is \( R = 945(s/b)^{3/2} \text{ (t/m}^2) \).

**Critical spans for a beam of rectangular cross-section**

\[ (E = 3.5 \times 10^6 \text{ t/m}^2) \]

<table>
<thead>
<tr>
<th>( s/b )</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>t/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{\text{crit}} )</td>
<td>10.30</td>
<td>13.50</td>
<td>16.30</td>
<td>18.30</td>
<td>20</td>
<td>m</td>
</tr>
<tr>
<td>( R_{\text{max}} )</td>
<td>945</td>
<td>1080</td>
<td>1190</td>
<td>1200</td>
<td>1200</td>
<td>t/m²</td>
</tr>
<tr>
<td>( h )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \frac{h}{l} )</td>
<td>35.4</td>
<td>30.9</td>
<td>28.2</td>
<td>25.2</td>
<td>23</td>
<td></td>
</tr>
</tbody>
</table>

The critical span for slab bridges (\( s \) of the order of 1 to 1.25 t/m²) is therefore about 10 to 12 m. It can be appreciably greater for heavily loaded structural beams of rectangular cross-section, but their size can be excessively large for such spans.

**Symmetrical I-beam**

With the previous value \( d'/h = 0.1 \), and also \( r^2/v = v'/2 = h/4 \), the critical span is obtained in the condition:

\[
\frac{p}{s} = \frac{0.4h - 0.25h[1 - (R'/\sigma_y)]}{0.5h[1 - (R'/\sigma_y)]}
\]

If \( R' = 0 \), \( p/s = 0.3 \).

With this hypothesis, \( M_s = (l/v)R = S(r^2/v)R = S(h/4)R \). \( M_s = s(l^2/8) \) for a beam which is simply supported at the ends.

\[
p = SD \quad \text{or} \quad S = \frac{p}{D}
\]

Therefore: \( s(l^2/8) = (p/D)(h/4)R \), and, since \( p = 0.3s \),

\[
l^2 = 0.3 \times 2h \times \frac{R}{D} = 0.6h \frac{R}{D}
\]

For \( R = 1200 \text{ t/m}^2 \) and \( D = 2.4 \text{ t/m}^2 \), \( R/D = 500 \text{ m} \).

Therefore

\[
l^2 = 300h
\]

\[
l_{\text{crit}} = 300 \frac{h}{l} \quad \text{(metres)}
\]
If \( h/l = 1/20 \), the critical span is of the order of \( 300/20 = 15 \) m, irrespective of the live loading.

It is easily shown that for normal values of \( h \), of the order of \( 1/20 \) or greater, deflection is not the controlling factor. For example, if the live load deflection is not to exceed \( 2/1000 \) of the span as before, the corresponding limit condition is:

\[
\frac{5}{384} \frac{sl^4}{EI} = \frac{2l}{1000}
\]

But:

\[
I = \frac{Sr^2}{v} = S \frac{h}{4} = S \frac{h^2}{8} = \frac{p}{D} \frac{h^2}{8}
\]

The condition for maximum deflection can therefore be written:

\[
\frac{5}{384} \frac{sl^4}{E(p/D)(h^2/8)} = \frac{2l}{1000}
\]

If \( E = 3.5 \times 10^6 \) t/m\(^2\), and \( D = 2.4 \) t/m\(^2\),

\[
\frac{1}{14 \times 10^6} \frac{s l^4}{p h^2} = \frac{2l}{1000}
\]

or,

\[
\frac{l^3}{h^2} = 28000 \frac{p}{s}
\]

Since \( p/s = 0.3 \),

\[
l_{crit} = 8400 \frac{h^2}{l^2}
\]

(9)

Condition (9) is controlling only if \( 8400(h^2/l^2) \) is less than \( 300(h/l) \), or \( h/l < 1/28 \), which is unusual.

**Remarks**

(i) The critical span decreases if \( R \) decreases. It is therefore less in the case of limited prestress than in the case of total prestress.

The following general expressions are obtained for the critical span if deflection is ignored:

- beam of rectangular cross-section:

\[
l_{crit} = \frac{0.337R + 0.815R'}{(R - R')^4} \left( \frac{s}{b} \right)
\]
Design of a Section with a Span Greater than the Critical Span

symmetrical I-beam:

\[ I_{\text{crit}} = \left( \frac{R}{4} + 1.08R' \right) \frac{h}{l} \]

(R and R' are in t/m² and the algebraic value of R' is used).

(ii) When the span is greater than the critical span, self-weight can no longer be wholly compensated, but only a proportion of it (0.7s in the case of a slab, 0.3s for a symmetrical I-beam). This is still of considerable advantage.

2. Orders of magnitude of moment arms for I-beams when the span is greater than the critical span

The whole of the self-weight can no longer be compensated, or the cable would lie outside the section. The cable is laid as low as possible within the section.

If \( d' \) is the minimum cable cover, the cable eccentricity is \(- (v' - d')\).

For the section to be utilised to its maximum under load, the centre of compression must rise to the top boundary of the limit core. The eccentricity of this boundary is 0.5 if it is assumed that the section efficiency is 0.5 and that R' = 0.

The centre of compression rises, under the total moment \( M_p + M_s \), by an amount \( v' - d' + 0.5v \) (Fig. 2). Let this be denoted by \( z \); \( z \) is the moment arm for the total moment (whereas in Fig. 1, \( A_1A_2 \) is the moment arm for the moment \( M_s \)).
Therefore, if \( M \) is the total moment \((M_p + M_z)\),
\[
F = \frac{M}{z}
\]

Assume \( d' = 0.1h \).

For
\[
\begin{align*}
v' &= 0.7h \\
z &= h \left( 0.7 - 0.1 + \frac{0.3}{2} \right) = 0.75h
\end{align*}
\]

\[
\begin{align*}
v' &= 0.6h \\
z &= h \left( 0.6 - 0.1 + \frac{0.4}{2} \right) = 0.70h
\end{align*}
\]

\[
\begin{align*}
v' &= 0.5h \\
z &= h \left( 0.5 - 0.1 + \frac{0.5}{2} \right) = 0.65h
\end{align*}
\]

For \( d' = 0.05h \), the moment arms are increased by 0.05\( h \). They are less if \( R' > 0 \) and greater if \( R' < 0 \).

A usual value of \( z \) is 0.7\( h \). This value can be used for a first approximation of the prestressing force, and therefore of the number of cables required.

With reinforced concrete, with the same minimum cover (0.1\( h \) or 0.05\( h \)), the value of \( z \) is between 0.8\( h \) and 0.85\( h \). Thus, for a given moment, the prestressing force is greater than the force in the reinforcement of an identical reinforced concrete section. But, since the cable stresses are between 4 and 6 higher than those in reinforcement, the cross-sectional area of the cables is much less than that of the reinforcement. The only worthwhile comparison is a comparison of costs.

3. Design of a section of any shape of given depth \( h \), with a span greater than the critical span

Only three out of the four strength limits (two in the unloaded condition and two in the loaded condition) can be realised.

There are indeed five unknowns, \( S, I/v, I/v', F \) and \( e \), but there is a relation between \( v' \) and \( e \), since \( e = -(v' - d') \).

If it is assumed that, having chosen the type of section (T-beam, I-beam, box beam, etc.), \( S \) is fixed by the dimensions which enable \( I/v \) and \( I/v' \) to be determined, there are then three unknowns. Only three of the limit stresses can therefore be realised.

If possible, the most advantageous out of the possible combinations shown in Fig. 3 should be chosen (the stress which does not reach its limit
is denoted by $O$). In practice, the choice is not arbitrary, and trial and error is reduced to a minimum by attempting to obtain combination (d) of Fig. 3. This corresponds to the case where the moments under external loading are positive, but the rule is valid whatever the sign of the moments; the aim is to obtain maximum utilisation under load (working to the two limit stresses) and to obtain the maximum compression limit in the unloaded condition at the fibre in which the precompression is later reduced by the live load.

![Fig. 3.](image)

The proof of this is difficult, and so is the economic comparison between the various combinations of Fig. 3 when they are possible. It is only a practical rule which may not apply in every case. An attempt at its justification is given in Section 9.

4. Choice of limit stresses
To achieve the limits $R_2$, $R'_2$ and $R_1$ means of course that the selected limiting stresses are achieved. To a point, this is a free choice, and to each choice is related a solution ($I/v$, $I/v'$, $F$).

A complete justification should therefore include a justification of this choice. This is attempted in Section 11.

With regard to $R'_2$, it is desirable to use a value which is as low as possible, and negative if possible. By reducing $R'_2$, the stress variation $R_1 - R'_2$ is increased. Consequently $I/v'$ is decreased, since

$$\frac{I}{v'} = \frac{M_s}{R_1 - R'_2}$$

(solution (d) of Fig. 3 being adopted).

Conclusions are not so obvious for the limits $R_1$ and $R_2$. It is shown in Chapter V (Sections 9, 10 and 11) that there is generally an economic advantage in reducing the limit stress $R_2$. There is a loss on concrete, but a gain on prestressing force.
A slab is, however, a particular case, where the shape is previously assumed. The true reason for the resulting economy is the increase in the moment arm, and not the reduction in $R_2$, which is merely a consequence which cannot be avoided.

This is also often the case when certain properties of the shape are fixed.

If, for example, the breadth $b$ of the top flange cannot be varied, as well as the minimum thicknesses, it is not possible to reduce the thickness below a ‘mean thickness’ $e_m = S/b$ of the transverse section, or to go below a certain fictitious density $D' = SD/bh = (e_m/h)D$. The conditions are the same as with a slab of breadth $b$, of reduced density $D'$ and of improved efficiency; it is again advantageous to increase the depth $h$, and consequently to decrease $R_2$, if the resulting economy (increase in concrete, reduction in prestress) is positive. This is generally the case.

If complete freedom is allowed in the choice of the shape of the section, it is still advantageous to increase the depth, but it is possible that $R_2$ need not be reduced, and the economy will in general be greater.

An attempt is made in Section 10 to justify the various rules given above, regarding the conditions to be satisfied and the magnitude of the stresses. This section may be omitted on a first reading if so desired, and only the conclusions need be noted. These are summarised as follows:

(i) As a general rule, every effort should be made to:

(a) make the maximum use of the section when loaded ($R_2$ and $R_2'$ realised);
(b) achieve, if possible, the limit stress $R_1$ in the unloaded condition (this condition cannot be obtained for slabs).

(ii) The maximum possible depth of section should be chosen.

(iii) Concerning the choice of stresses:

(a) the maximum limit stress $R_1$ for the particular type of concrete should be used;
(b) the lower limit stress $R_2'$ should be made as small as practicable, negative if possible;
(c) use the highest possible limit for $R_2$, consistent with the chosen depth of section. If the chosen depth does not permit $R_2$ to be the maximum for the particular type of concrete in use, preference should be given to maintaining the value of the depth $h$ as high as possible [condition (ii)]. The possible value of $R_2$ is also limited if the top flange or the minimum thicknesses are specified and fixed.
If, for example, the dimensions of the top flange are fixed, as well as the web thickness in terms of the section depth, there are only two arbitrary unknowns. The solution for any particular value of $h$ is given by

$$\frac{I}{v'} = \frac{M_s}{R_1 - R'_2}$$

$F$ being now a function of $I/v'$ only.

It is possible that the limit $R_1$ itself cannot be achieved, since the dimensions of the bottom flange must be such that they can accommodate the required number of cables.

In short, the solution depends upon a number of practical considerations. The main purpose of the discussion in Section 10 is to indicate the optimum conditions, as a matter of interest, as they can rarely be satisfied.

Solutions without trial and error are possible, as shown later, but if they do not take practical considerations into account they are purely of academic interest.

Solutions by trial and error are only a little more tedious, and they have the advantage that, apart from taking specific requirements into account, they enable the best or the only solution to be deduced, by consideration of the effects of small variations in the parameters.

Although the depth of beams should be as great as possible, there are obvious practical limits (weight and minimum thicknesses, transverse stability, appearance, etc.). It is impossible to give an absolute ruling. As an indication in the case of bridges, however, the maximum practical values of $h/L$ are of the order of 1/20 for values of $L$ up to 35 m, 1/17 for $L = 50$ m, and 1/15 for $L = 100$ m. $L$ is the distance between the piers; that is, the span for a simply supported beam, and twice the overhang for cantilever beams.

The formula

$$\frac{h}{L} = \frac{1 + 4(L/100)}{11 \frac{3 + 4(L/100)}{}}$$

($L$ in metres) gives a good guide to the maximum practical values. However, the choice of depth depends essentially on experience and good judgement. Greater depths can be used with beams in buildings. They are generally continuous beams with a variable depth, and the ratio $h/L$ then refers to the depth at the supports.

5. Design of a slab with a span greater than the critical span

The problem is very simple since $r^2/v = r^2/v' = h/6$; if $s$ is the live load per unit area, consideration of Fig. 4 shows that, by writing that the
centre of compression rises from $E_o$ to the top boundary of the limit core under the influence of the total loading moment:

$$F \left[ \frac{h}{2} - d' + \frac{h}{6} \left( 1 - \frac{R'_2}{\sigma_g} \right) \right] = M_p + M_s = bhD \frac{l^2}{8} + bs \frac{l^2}{8}$$

The limit stresses $R_2$ and $R'_2$ being achieved in the loaded condition (even if the best value has to be chosen for $R_2$, as stated in Section 3), then:

$$F = bh \frac{R_2 + R'_2}{2}$$

![Diagram](image)

**Fig 4.**

Hence an equation in $h$ is obtained:

$$h \frac{R_2 + R'_2}{2} \left( \frac{h}{2} - d' + \frac{h}{6} \frac{R_2 - R'_2}{R_2 + R'_2} \right) = (hD + s) \frac{l^2}{8} \quad (1)$$

If it is assumed that $d' = 0.1h$,

$$h^2(0.566R_2 + 0.234R'_2) - hD \frac{l^2}{4} - s \frac{l^2}{4} = 0 \quad (2)$$

Example: for $R_2 = 1200 \text{ t/m}^2$, $R'_2 = 0$, then:

$$bh^2 \times \frac{1200}{2} \times 0.566 = (bhD + bs) \frac{l^2}{8} = M \quad (3)$$

or:

$$M = 340bh^2$$

The moment of resistance of such a slab is therefore $340bh^2$. It is related to the total moment ($M_p + M_s$) and, when the span exceeds the
critical span, it must replace the formula \(200bh^2 = M_p\), given in Chapter V, Section 7, for the same limit stress of 1 200 t/m².

With \(D = 2.4\) t/m³,

\[ M_p = bh \times \frac{2.4l^2}{8} = 0.3bh^2 \]

and:

\[ 340h^2 = 0.3l^2h + \frac{l^2}{8} \]  \(\text{(4)}\)

from which \(h\) is determined.

For all other values of the limit stresses \(R_2\) and \(R'_2\), it is necessary to change the value of the moment of resistance appearing in eqn. (4).

It can be immediately checked that an increase in depth reduces the prestressing force. If \(R'_2 = 0\), then \(F = bh(R_2/2)\), and eqn. (2) can be written:

\[ 0.566h^2 \times \frac{2F}{bh} = \frac{l^2}{4}(hD + s) \]

or

\[ \frac{F}{b} = \frac{l^2}{4.528} \left(D + \frac{s}{h}\right) \]

As mentioned in Chapter V (note in Section 9), an economic assessment is required, comparing the increased cost of concrete against the saving in the cost of the steel. The trial and error methods given in Chapter V (increasing values of \(h\)) provide the best illustration of the comparisons. It is not proposed to discuss these again or to give further examples.

6. Design of a section of any shape

Formulae which determine the solution with the least amount of trial and error are given below. Choosing solution (d) of Fig. 3, the problem would be the same as that in Chapter VI if the fourth stress \(R^*_1\) were known (Fig. 5). This stress could be taken as the unknown, but this is equal to taking \(\sigma_g\), the stress at the centroid, as the unknown; \(R^*_1\) can be determined when \(\sigma_g\) is known, by constructing Fig. 5.

Since the limit stress \(R_1\) (maximum compressive) is obtained in the unloaded condition at the bottom fibre, the centre of compression coincides with the lower limit point \(B'_1\), under the action of the bending moment \(M_p\). The eccentricity of \(B'_1\) is \(-r^2/v'[(R_1/\sigma_g) - 1]\).

Therefore:

\[ \frac{M_p}{F} = v' - d' - \frac{r^2}{v'} \left(\frac{R_1}{\sigma_g} - 1\right) \]  \(\text{(a)}\)
In the case of a simply supported beam, if \( D \) is the concrete density:

\[
M_p = SD\frac{l^2}{8}
\]

and

\[
F = S\sigma_g
\]

Therefore:

\[
\frac{M_p}{F} = \frac{DL^2}{8\sigma_g}
\]

![Diagram](image)

Fig. 5.

If the efficiency \( \rho \) is assumed: \( \sigma^2/v' = \rho v \).

From Fig. 5:

\[
v = h\frac{R_2 - \sigma_g}{R_2 - R'_2} \quad \text{and} \quad v' = h\frac{\sigma_g - R'_2}{R_2 - R'_2}
\]

Substituting these values in eqn. (a), an equation in \( \sigma_g \) is obtained:

\[
D\frac{l^2}{8\sigma_g} = h\frac{\sigma_g - R'_2}{R_2 - R'_2} - d' - \rho h\frac{R_2 - \sigma_g}{R_2 - R'_2} \frac{R_1 - \sigma_g}{\sigma_g}
\]

which can be written:

\[
\left(\frac{\sigma_g}{R_1}\right)^2 (1 - \rho) + \frac{\sigma_g}{R_1} \left[ \rho \left(1 + \frac{R_2}{R_1}\right) - \frac{R'_2}{R_1} - \frac{d'}{h} \frac{R_2 - R'_2}{R_1} \right] - \left(\rho \frac{R_2}{R_1} + \frac{DL^2}{8hR_1} \frac{R_2 - R'_2}{R_1}\right) = 0
\]
Design of a Section with a Span Greater than the Critical Span

\[ \sigma_\theta/R_1, \] and therefore \( \sigma_\theta \), being thus obtained, \( R'_1 \) can be determined, and the calculation is concluded as in Chapter VI:

\[
\frac{I}{v} = \frac{M_s}{R_2 - R'_1} \quad \text{and} \quad \frac{I}{v'} = \frac{M_s}{R_1 - R'_2}
\]

Hence \( v \) and \( v' \) (\( v + v' = h \)) and I are obtained.

Then: \( I/v' = S\rho v \), hence \( S = (I/v')/\rho v \).

It is then required to find the section whose depth \( h \) is known, as well as the position of the centroid and the values of I and S. As in Chapter VI it is possible to find a section with the required values of \( v \), \( v' \) and I. It will not necessarily have the required cross-sectional area \( S \) and therefore the assumed efficiency \( \rho \). The value given to \( \rho \) must then be modified in the equation for \( \sigma_\theta/R_1 \), and the calculations done again. The method is one of successive approximations, therefore. Direct trial and error methods are just as quick. The procedure is given in the following section.

7. Design by trial and error of a beam of any cross section

Example. It is required to design a beam of 37 m span, 2 m deep, with a live load bending moment at mid-span of \( M_s = 261 \) t/m².

Limit stresses:

- \( R_1 = 1400 \) t/m²
- \( R_2 = 1200 \) t/m²
- \( R'_1 = -150 \) t/m²
- \( R'_2 = 100 \) t/m²

The span is clearly greater than the critical span. The four limit stresses cannot all be realised, therefore. \( R_1 \) and \( R'_2 \) at the bottom flange are chosen, and \( R_2 \) in the loaded condition on the top flange. The solution will only be correct if the fourth stress is greater than \( R'_1 \), and this will have to be checked. The cable must be located as low as possible in the section. As a starting point, \( d' \) is made equal to 0.10 m, but this may need to be modified. A web thickness of 0.15 m is assumed.

A first condition to be met by the section profile is, in accordance with Fig. 6:

\[
\frac{I}{v'} = \frac{\Delta M}{\Delta R} = \frac{M_s}{R_1 - R'_2} = \frac{261}{1300} = 0.201 \text{ m}^3
\]
A profile with this $I/v'$ value must therefore be found, such that the limit stresses $R_2$ and $R'_2$ are obtained in the loaded condition.

The second condition is expressed by equating the moment of the internal forces, with respect to the cable (resultant cable), to $M_p + M_s$ when the limits $R_2$ and $R'_2$ are reached. Assume the flange thicknesses, even if they need to be modified subsequently. Let the thickness of the top flange be 0.15 m and that of the bottom flange 0.16 m.

If the flanges are assumed to be rectangular, the only remaining unknowns are the breadth of the flanges, or the flange cross-sectional areas $A$ and $A'$ (outside of the web; shown hatched in Fig. 7). The two conditions $I/v' = 0.201$ m³ and the achievement of $R_2$ and $R'_2$ under load would suffice for the determination of $A$ and $A'$, by means of two equations.

It is easier to operate by trial and error, first satisfying the second condition which provides a relation between $A$ and $A'$, and then varying $A$ and $A'$ to satisfy the first condition ($I/v' = 0.201$ m³).
With the chosen thicknesses:
The stress at the level of the centroid of the bottom flange is:

\[
100 + 1 \times 100 \times \frac{0.08}{2} = 144 \text{ t/m}^2
\]

The force in the bottom flange \((A')\) is therefore 144 \(A'\) \((A'\) is in square meters).
The normal moment due to this force, with respect to the centroid of the cable, is:

\[
144A'(0.08 - 0.10) = 144A' \times (-0.02) = -2.88A' \text{ (tm)}
\]

The mean stress in the web is:

\[
\frac{100 + 1 \times 200}{2} = 650 \text{ t/m}^2
\]

Force in the web:

\[
0.15 \times 2 \times 650 = 195 \text{ t}
\]

Moment due to this force with respect to the cable:

\[
0.15 \times 2 \times 100 (1.00 - 0.10) + 0.15 \times 2 \times 550 (\frac{1}{3} \times 2.00 - 0.10) = 230 \text{ tm}
\]

The stress at the centroid of the top flange is equal to:

\[
100 + 1 \times 100 \times \frac{2 - 0.075}{2} = 1160 \text{ t/m}^2
\]

Force in the top flange: \(1160 \times A\) (\(A\) in square metres).

Moment due to this force, with respect to the cable:

\[
1160A \times (1.925 - 0.10) = 2120A \text{ (tm)}
\]

The moment due to the internal forces, with respect to the cable, is, therefore:

\[
-2.88A' + 2120A + 230 \text{ (tm)}
\]

It must be equal to the moment due to the external forces, \(M_p + M_v\).
The cross-sectional area of the section is equal to:

\[
0.15 \times 2 + A + A' = 0.3 + A + A'
\]
The weight per metre length of the beam is, in t/m (density of concrete = 2·4 t/m³):

\[ p = 2·4(0·3 + A + A') = 0·72 + 2·4(A + A') \]

The moment due to self-weight is:

\[ M_p = p \frac{l^2}{8} = p \times \frac{37^2}{8} = 171p \]

Therefore:

\[ M_p = 171[0·72 + 2·4(A + A')] = 123 + 410(A + A') \]

Also,

\[ M_s = 261 \text{ tm} \]

Hence (moments in tm, areas in m²):

\[ -2·88A' + 230 + 2 \times 120A = 123 + 410(A + A') + 261 \]

or:

\[ 1710A - 412·88A' = 154 \quad (1) \]

A and A' must also be such that \( I/v' = 0·201 \text{ m}^3 \).

First, the bottom flange dimensions are assumed approximately, even though the dimensions may have to be modified subsequently. The flange can be dimensioned such that it is just sufficient to accommodate the required cables. It is the minimum bottom flange size, used as the starting point for trial and error calculations. The moment arm for the moment \( M_p + M_s \) is approximately 0·7h (Section 2) or 1·40 m.

For the first approximation, \( p \) can be estimated either by referring to previous calculations of a similar nature, or by estimating the cross-sectional area of the section. Now: \( I/v' = 0·201 \text{ m}^3 \); also \( r^2/v' \) is equal to \( 0·5v \) approximately, if the efficiency \( \rho \) is taken as \( \frac{1}{4} \). Therefore \( S(r^2/v') = 0·5vS \).

\( v \) is approximately equal to 1 m. Therefore \( I/v' = 0·5 \times 1 \times S \); therefore \( S = 0·201/0·5 = 0·402 \text{ m}^2 \).

With a concrete density of 2·4 t/m³, the weight \( p \) is about 1 t/m; so the approximate value of \( M_p \) is \( 1 \text{ t} \times \frac{37^2}{8} = 171 \text{ tm} \).

\[ M_p + M_s = 171 + 261 = 432 \text{ tm} \]

\[ F = \frac{M_p + M_s}{1·40} = \frac{432}{1·40} = 308 \text{ t} \]
Six, 8 mm diameter, 12-wire cables can be used. They must be placed over a width (Fig. 8) of about 0·36 m. Thus the approximate breadth of the bottom flange, outside the web, is 0·36 − 0·15 = 0·21 m.

Hence $A' = 0·16 \times 0·21 = 0·034$.

With this value for $A'$, eqn. (1) becomes:

\[ 1710A - 14 = 154 \]

and

\[ A = 0·098 \text{ m}^2 \]

say: 0·15 m × 0·652 m.

![Fig. 8. Minimum breadth of bottom flange (cables without central helix).](image)

The profile of the section is shown in Fig. 9. This profile will not in general have the required $I/v'$ value (0·201 m$^3$) and some changes are required in the design.

The section shown in Fig. 9 has the following characteristics:

\[ S = 0·432 \text{ m}^2 \quad v = 0·87 \text{ m} \quad v' = 1·13 \text{ m} \quad I = 0·205 \text{ m}^4 \]

\[ I/v = 0·236 \text{ m}^3 \quad I/v' = 0·180 \text{ m}^3 \]

$I/v'$ is therefore not large enough; the size of the bottom flange must be increased, and consequently the top flange must be increased as well in order to satisfy eqn. (1).

Various methods are possible:

(a) An approximate method comprises finding the increase $\Delta A'$ to the cross-sectional area of the bottom flange, without modifying the top flange, in order to obtain the required value of $I/v'$.

Once $\Delta A'$ is obtained, $A$ is increased by an amount $\Delta A$ in order to satisfy eqn. (1). A strictly exact solution will not be obtained because the
increase $\Delta A$ changes the properties of the section, but it will be nearly exact; a second correction can be made in a similar way, if necessary, and this will enable an even closer approximation to be obtained.

Consider that the centroid of the area $\Delta A'$ is at the same level as that of $A'$. Let $S_o, v_o, v'_o$ and $I_o$ be the section properties before correction, and let $\delta$ be the distance of the area $\Delta A'$ from the centroid $G_o$. After correction, the properties are $S, v, v'$ and $I$.

![Diagram showing a section with dimensions and distances labeled.]

**Fig. 9.**

Taking moments about $G$:

$$ (v'_o - v')S_o = [\delta - (v'_o - v')] \Delta A' $$

(2)

The moment of inertia is:

$$ I = I_o + S_o(v'_o - v')^2 + \Delta A'[\delta - (v'_o - v')]^2 $$

This can be written, from eqn. (2):

$$ I = I_o + S_o(v'_o - v')^2 + S_o(v'_o - v')[\delta - (v'_o - v')] $$

$$ = I_o + S_o\delta(v'_o - v') $$
Hence:

\[ I + S_o \delta v' = I_o + S_o \delta v'_o \]

or:

\[ v' \left( \frac{I}{v'} + S_o \delta \right) = v'_o \left( \frac{I_o}{v'_o} + S_o \delta \right) \]

Hence:

\[
\frac{v'}{v'_o} = \frac{(I_o/v'_o) + S_o \delta}{(I/v') + S_o \delta}
\]

(3)

\[
\begin{align*}
\Delta A' \\
\delta \\
v' \\
v'_o \\
\end{align*}
\]

In this case:

\[ S_o = 0.432 \text{ m}^2 \quad \delta = 1.13 - 0.08 = 1.05 \text{ m} \quad S_o \delta = 0.454 \text{ m}^3 \]

\[ I_o/v'_o = 0.180 \text{ m}^3 \quad \text{and} \quad I/v' = 0.201 \text{ m}^3 \]

are required.

Therefore it is necessary that:

\[
\frac{v'}{v'_o} = \frac{0.180 + 0.454}{0.201 + 0.454} = \frac{0.634}{0.655} = 0.97
\]

or:

\[ v' = 0.97 \times 1.13 = 1.09 \text{ m} \]

\[ v'_o - v' = 1.13 - 1.09 = 0.04 \text{ m} \]

\[ \Delta A' \] is given by eqn. (2):

\[
\Delta A' = \frac{v'_o - v'}{\delta - (v'_o - v')} S_o = \frac{0.04}{1.05 - 0.04} S_o = 0.04 \times 0.432 = 0.017 \text{ m}^2
\]
A' becomes equal to \( A' + \Delta A' = 0.034 + 0.017 = 0.051 \text{ m}^2 \), or \( 0.16 \text{ m} \times 0.32 \text{ m} \).

The total breadth of the bottom flange is \( 0.32 + 0.15 = 0.47 \text{ m} \).

To satisfy eqn. (1) it is necessary that:

\[
1710A - 412.88 \times 0.051 = 154 \text{ tm}
\]

\[
A = \frac{154 \times 21}{1710} = 0.1025 \text{ m}^2
\]

or: \( 0.15 \text{ m} \times 0.684 \text{ m} \).

The total breadth of the top flange is \( 0.684 + 0.15 = 0.834 \text{ m} \).

The profile which is obtained is shown in Fig. 11.

Its properties are:

\[
S = 0.453 \text{ m}^2 \quad v = 0.895 \text{ m} \quad v' = 1.105 \text{ m} \quad I = 0.226 \text{ m}^4
\]

\[
I/v = 0.252 \text{ m}^3 \quad I/v' = 0.204 \text{ m}^3
\]

![Fig. 11.](image)

The condition that the limit stresses can be reached in the loaded state is satisfied if the necessary prestressing force is applied. The section profile is slightly over-designed at the bottom fibre. This can be shown to be a direct consequence of the method; that is, \( I/v' \) increases slightly because of the increase \( \Delta A \) in the top flange.\(^\dagger\) It is not necessary to correct for this in this particular case.

\(^\dagger\) Consequently, repeated corrections bring the solution closer to the exact solution, alternating about the exact values.
It is now required to check that the resultant stresses are acceptable. The prestress eccentricity \( e = -(v' - 0.10) = -1.005 \) m.

\[
\frac{r^2}{\nu} = 0.555 \text{ m} \quad \frac{r^2}{\nu'} = 0.45 \text{ m}
\]

\[
\sigma_o = \frac{F}{S} \left(1 - \frac{1.005}{0.555}\right) = -0.81 \frac{F}{S}
\]

\[
\sigma'_o = \frac{F}{S} \left(1 + \frac{1.005}{0.45}\right) = 3.24 \frac{F}{S}
\]

\[
p = 2.4 \times 0.453 = 1.088 \text{ tm}
\]

\[
M_p = 1.088 \times 171 = 186 \text{ tm}
\]

\[
M_p + M_s = 186 + 261 = 447 \text{ tm}
\]

<table>
<thead>
<tr>
<th>Stresses under external loading</th>
<th>Unloaded ( M = 186 \text{ tm} )</th>
<th>Loaded ( M = 447 \text{ tm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre ( M/(I/\nu) )</td>
<td>787</td>
<td>1775</td>
</tr>
<tr>
<td>Bottom fibre - ( M/(I/\nu') )</td>
<td>-911</td>
<td>-2190</td>
</tr>
</tbody>
</table>

To make the stress at the bottom fibre equal to \(+100 \text{ t/m}^2\), \( \sigma'_o \) must be equal to \(2290 \text{ t/m}^2\). Hence:

\[
\frac{F}{S} = \frac{2290}{3.24} = 707 \text{ t/m}^2
\]

\[
\sigma_o = -0.81 \times 707 = -572 \text{ t/m}^2
\]

The resultant stresses are then:

<table>
<thead>
<tr>
<th></th>
<th>Unloaded</th>
<th>Loaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top fibre</td>
<td>+165</td>
<td>+1203</td>
</tr>
<tr>
<td>Bottom fibre</td>
<td>+1379</td>
<td>+100</td>
</tr>
</tbody>
</table>

Also: \( F = 707 \times S = 707 \times 0.453 = 320 \text{ t} \) (six, 8 mm diameter, 12-wire cables tensioned to 89 kg/mm\(^2\)).
The arrangement shown in Fig. 8 can be retained, as well as the distance \( d' = 0.10 \), although this could be slightly reduced. It is not worthwhile to make the corresponding correction.

(b) A different method consists of giving various values to \( A' \), and in calculating the corresponding values of \( A \) in accordance with eqn. (1):

\[
A = \frac{154 + 412.88A'}{1710}
\]

The following table of results is then obtained. The table immediately shows that for \( I/v' \) to be equal to 0.201, \( A' \) must be equal to 0.05 approximately; the complete solution is obtained by using the section properties corresponding to this value.

| \( A' \) \( (m^2) \) | 0.03 | 0.04 | 0.05 | 0.06 |
| \( A \) \( (m^2) \) | 0.0972 | 0.0995 | 0.102 | 0.1045 |
| \( S \) \( (m^2) \) | 0.427 | 0.439 | 0.452 | 0.464 |
| \( v \) \( (m) \) | 0.855 | 0.878 | 0.89 | 0.91 |
| \( v' \) \( (m) \) | 1.145 | 1.1222 | 1.11 | 1.09 |
| \( I \) \( (m^4) \) | 0.1992 | 0.2112 | 0.2245 | 0.2361 |
| \( I/v \) \( (m^3) \) | 0.233 | 0.241 | 0.253 | 0.260 |
| \( I/v' \) \( (m^3) \) | 0.174 | 0.188 | 0.203 | 0.216 |
| \( r^2/v \) \( (m) \) | 0.522 | 0.549 | 0.561 | 0.561 |
| \( r^2/v' \) \( (m) \) | 0.408 | 0.428 | 0.450 | 0.466 |
| \( \rho \) | 0.455 | 0.488 | 0.505 | 0.515 |

The above, or similar, tables also indicate whether the best conditions have been chosen. It could be decided, for example, to achieve \( R_1' \) in the unloaded state at the top fibre, and \( R_2R_2' \) in the loaded conditions (solution (c) of Fig. 3). It would then be necessary for \( I/v \) to equal \( M_x/(R_2 - R_1) = 261/1350 = 0.193 \text{ m}^3 \). \( I/v' \) would then be necessarily greater than 0.201 \text{ m}^3 \), because, the limit stress \( R_1 \) not being reached on the bottom fibre, the variation \( \Delta R \) at this bottom fibre would be less than that considered above. The table shows immediately that such a solution is impossible.

Again, having decided to obtain the limits \( R_2R_2' \) in the loaded condition, it may be necessary to investigate whether there is any advantage to be gained by reducing the limit stress \( R_1 \). This is equal to \( 100 + M_x/(I/v') \), and can be calculated for each column in the table; it is also required
that \( \sigma'_o = (M_p + M_s)/(I/v') + 100 \); therefore
\[
\frac{F}{S} = \frac{\sigma'_o}{1 + [e/(r'^2/v')]}\]

and hence \( F \) can be obtained.

All the factors can be calculated by means of the table.

The following results are obtained:

\[
\begin{align*}
R_1 &= 1 \, 600 \quad 1 \, 490 \quad 1 \, 385 \quad 1 \, 310 \, t/m^2 \\
S &= 0.427 \quad 0.439 \quad 0.452 \quad 0.464 \, m^2 \\
F &= 311 \quad 315 \quad 320 \quad 324 \, t
\end{align*}
\]

It is seen, in accordance with Section 3, that it is advantageous to take for \( R_1 \) the maximum value permissible for the quality of the concrete which is used.

When there is any doubt, confirmation that the correct choice has been made can be obtained in all cases using trial and error calculations.

8. Use of the equation for \( \sigma_g \) in designing the section

Equation (c) of Section 6 can be used, after deciding upon a value for \( \rho \):

\[
\left( \frac{\sigma_g}{R_1} \right)^2 (1 - \rho) + \frac{\sigma_g}{R_1} \left[ \rho \left( 1 + \frac{R_2}{R_1} \right) - \frac{R'_2}{R_1} - \frac{d'}{h} \frac{R_2 - R'_2}{R_1} \right] - \left( \rho \frac{R_2}{R_1} + \frac{Dl^2}{8hR_1} \frac{R_2 - R'_2}{R_1} \right) = 0
\]

Assume \( \rho = \frac{1}{4} \). Then:

\[
1 + \frac{R_2}{R_1} = 1 + \frac{1 \, 200}{1 \, 400} = 1.86 \quad \frac{R'_2}{R_1} = \frac{100}{1 \, 400} = 0.0715
\]

\[
\frac{d'}{h} \frac{R_2 - R'_2}{R_1} = 0.005 \times \frac{1 \, 100}{1 \, 400} = 0.0393 \quad \frac{R_2}{R_1} = 0.86
\]

\[
\frac{Dl^2}{8hR_1} \frac{R_2 - R'_2}{R_1} = \frac{2 \times 1 \, 71}{2 \times 1 \, 400} \times \frac{1 \, 100}{1 \, 400} = 0.1255
\]

The equation is written:

\[
\left( \frac{\sigma_g}{R_1} \right)^2 (1 - \rho) + \frac{\sigma_g}{R_1} (1.86 \rho - 0.1108) - (0.86 \rho + 0.1255) = 0 \quad (a)
\]
With $\rho = \frac{1}{2}$:
\[
\frac{1}{2} \left( \frac{\sigma_g}{R_1} \right)^2 + 0.819 \frac{\sigma_g}{R_1} - 0.5555 = 0
\]
and
\[
\frac{\sigma_g}{R_1} = 0.516
\]
Hence $\sigma_g = 0.516 \times 1400 = 722 \text{t/m}^2$.

Hence (Fig. 12):
\[
\frac{722 - 100}{v'} = \frac{1200 - 100}{h} \cdot \frac{v'}{h} = 0.565 \cdot 4v' = 1.131 \text{ m}
\]
Since $I/v' = 0.201 \text{ m}^3$, it is required to find a profile with a moment of inertia of: $I = 0.201 \times 1.131 = 0.227 \text{ m}^4$, $v = 0.869$, $v' = 1.131$.

If, with the chosen thickness, it is not possible to arrive at a suitable profile, it is because the value chosen for $\rho$ is not suitable. From the table in Section 7, this is the case with the present example, and the required moment of inertia can only be obtained with $\rho$ equal to 0.51. Equation (a) must therefore be considered again with $\rho = 0.51$, and the solution is obtained using successive approximations. It is seen, therefore, that the method of using the equation for $\sigma_g$ enables the order of magnitude of the required properties to be rapidly obtained, but that, in the end, trial and error methods are easier.
9. Practical section from theoretical profile

Refer to Chapter VI, Section 2, Fig. 8.

The rectangular flanges shown diagrammatically in Fig. 11 are replaced by ‘equivalent’ trapezoidal flanges with an adequate slope for satisfactory concreting, and with sufficient edge cover for satisfactory cable protection (4 to 5 cm cover). Figure 13 shows the modified profile shape.

The properties are:

\[ S = 0.454 \text{ m}^2 \quad v = 0.89 \text{ m} \quad v' = 1.11 \text{ m} \quad I = 0.228 \text{ m}^4 \]

\[ I/v = 0.256 \text{ m}^3 \quad I/v' = 0.205 \text{ m}^3 \]

It is nearly equivalent to the profile shown in Fig. 11.
10. Justification of the use of the limit stresses \((R_1, R_2 \text{ and } R'_2)\) and their values

As seen in Section 3, since all four limits cannot be reached, the limit \(R'_1\) is in principle disregarded. In other words, the effect is to obtain the maximum utilisation in the loaded condition \((R_2 \text{ and } R'_2 \text{ reached})\), and maximum utilisation of the bottom flange \((R_1 \text{ and } R'_2 \text{ reached at this flange})\). More generally: maximum utilisation in the loaded condition, and maximum utilisation of the flange subjected to tension.

Without giving a complete proof, some simple cases can be examined.

A. Profile of depth \(h\), with flanges and web of small thickness, of uniform cross-section.

It is assumed that the web can be neglected. Let \(A\) and \(A'\) be the cross-sectional areas of the top and bottom flanges. It is assumed that the cable is in the bottom flange, therefore no separate allowance is made for cover.

Under the effect of a moment \(M\) due to the loads, the forces in the flanges are \(\pm M/h\), and the stresses are \(M/HA\) and \(- (M/HA')\).

Consider the case of a span of length \(L\). This length is equal to the span in the case of a beam freely supported at its ends; it is equal to the distance between supports for a double-cantilever (Fig. 15), each overhang having a length of \(L/2\).
In both cases, under the action of the two extreme conditions of loading, 
$p$ (self-weight) and $p + s$ (self-weight plus live load), the absolute values of the maximum bending moments are $p(L^2/8)$ and $(p + s)L^2/8$. In the case of the simply supported beam, the maximum is positive and it occurs at mid-span; in the case of the cantilever, the maximum is negative and it occurs at the support point. The case of the simply supported beam is dealt with in the following. The conclusions are valid for the cantilever case, the roles of the flanges being then reversed.

Let $F$ be the prestressing force. The unit prestresses are $\sigma_o = 0$ in the top flange, and $\sigma'_o = F/A'$ in the bottom flange. Let $M_p$ and $M_s$ be the maximum loading moments under the loading conditions $p$ and $p + s$. Since the loading is uniform, $M_p = M_s(p/s)$.

The resultant stresses in the two extreme loading conditions ($p$ and $p + s$) are given by:

<table>
<thead>
<tr>
<th></th>
<th>Unloaded</th>
<th>Loaded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top flange</td>
<td>$\frac{Ms}{hA} \frac{p}{s}$</td>
<td>$\frac{Ms}{hA} \left(1 + \frac{p}{s}\right)$</td>
</tr>
<tr>
<td>Bottom flange</td>
<td>$\frac{Fh - Ms(p/s)}{hA'}$</td>
<td>$\frac{Fh - Ms[1 + (p/s)]}{hA'}$</td>
</tr>
</tbody>
</table>

It may be decided to reach the top limit $R_2$ in the loaded condition in the top flange, namely:

$$\frac{M_s}{hA} \left(1 + \frac{p}{s}\right) = R_2$$  \hspace{1cm} (a)

In that case, it is not generally possible to achieve also the limit $R_1'$ in the top flange, since it would be necessary to have

$$R_1' = R_2 \frac{p/s}{1 + (p/s)}$$

This relation cannot be satisfied if $R_1'$ is negative.

If $R_1'$ is positive, $R_2/R_1'$ is equal to 10 at least, and it would be necessary for $1 + p/s$ to equal 10(p/s), or $p/s = 1/9$. This is not obtained, in general, the self-weight being nearly always greater than $s/9$ when the span is greater than the critical span.
Of the four limits, it is therefore necessary to disregard $R'_1$. The limit stresses $R_1$ and $R'_2$ are reached in the bottom flange if:

\[ \frac{Fh - M_s(p/s)}{hA'} = R_1 \]
\[ \frac{Fh - M_s[1 + (p/s)]}{hA'} = R'_2 \]

By subtraction:

\[ \frac{M_s}{hA'} = R_1 - R'_2 \]

or

\[ A' = \frac{M_s}{h(R_1 - R'_2)} \] (b)

There is no reason for not determining $A'$ in this manner, and it obviously results in the most economical design of bottom flange.

The area $A$ is determined from condition (a). If $D$ is the concrete density, $p = (A + A')D$, and condition (a) can be written:

\[ \frac{M_s}{hA} \left( 1 + \frac{A + A'}{s} D \right) = R_2 \] (c)

Hence:

\[ A = \frac{(M_s/h)[1 + A'(D/s)]}{R_2 - (M_s/h)(D/s)} \] (d)

Then $F = AR_2 + A'R'_2$ since the stresses $R_2$ and $R'_2$ are obtained in the loaded condition. Therefore:

\[ F = \frac{M_s}{h} \left[ \frac{1 + A'(D/s)}{R_2 - (M_s/h)(D/s)} R_2 + \frac{R'_2}{R_1 - R'_2} \right] \]

The solution is completely defined, and the rule stating that the most economical design is brought about by choosing the limits $R_1$, $R_2$, $R'_2$ is confirmed, both for the concrete and the steel. The solution is possible only if $R_2 > M_s/h(D/s)$, or, since $M_s = sL^2/8$, if

\[ R_2 > \frac{L^2 D}{8h} \] (e)

This places a minimum value on $h$, which depends upon the span and the value of $R_2$, but which is independent of the live load.
For \( R_2 = 1200 \, \text{t/m}^2 \), and \( D = 2.4 \, \text{t/m}^3 \), condition (e) becomes:

\[
h > \frac{L^2 \times 2.4}{9600} \quad \text{or} \quad \frac{h}{L} > \frac{L}{4000} \quad \text{(L in metres)}
\]

With simply supported beams, this condition is almost always satisfied, even for large spans.

For cantilever beams, if \( l \) is the overhang \( (l = L/2) \), the above condition can be written \( (l \) in metres):

\[
\frac{h}{l} = \frac{1}{1000}
\]

In general, this condition can be satisfied.

It is advantageous with this type of construction and for large spans to specify a high stress \( R_2 \) in the loaded condition (for the bottom flange, which is now in compression). Formula (d) does indeed show that the area of the bottom flange increases very rapidly as \( L^2D/8h \) approaches \( R_2 \).

These conclusions only apply to a very elementary and simple case; in practice, the cross-sectional area of the web is far from being negligible in relation to the area of the flanges, when large spans are concerned, and the moment arm is less than \( h \). Limiting factors are studied in Volume 2 Chapter V, where cantilevered constructions are discussed.

**Note**

Formula (e) can be interpreted in the following manner:

If the depth of the beam is variable, so that the flange in compression is parabolic, under uniform loading, \( L^2/8h \) is the radius of curvature \( r \) of the flange (not to be confused with the radius of gyration). If both sides of inequality (e) are multiplied by \( A \), \( AR_2 \) is the resisting force in the flange and \( ADr \) is the force to which it is subjected due to its self-weight.

In the limit, if (e) was to become an equality, the flange would have just enough strength to carry its own weight. The more nearly this limit is approached, the smaller is the useful fraction of the resistance of the flange in compression.

B. Case of an I-section. Assume that the lower limits for stresses \( R_1' \) and \( R_2' \) are zero, and that \( d' = 0 \) (zero cover).

(i) Calculate the critical span in terms of the stresses. At the critical span, the four limits are reached. Therefore (Fig. 16):

\[
v' = h \frac{R_1}{R_1 + R_2} \quad \text{and} \quad \sigma_y = R_2 \frac{v'}{h} = \frac{R_1R_2}{R_1 + R_2}
\]
Under the self-weight, the centre of compression is displaced from the lower boundary to the limit point A₁, of eccentricity \(-(r^2/v)\).

If \(\rho\) is the efficiency: \(r^2/v = \rho v\).

Therefore:

\[
\frac{M_p}{F} = v' - \rho v' = v(1 - \rho) = \frac{hR_1}{R_1 + R_2} (1 - \rho)
\]

Also:

\[
\frac{M_p}{F} = \frac{SD(L^2/8)}{8\sigma_g} = \frac{L^2D}{8\sigma_g} = \frac{L^2D}{8} \frac{R_1 + R_2}{R_1R_2}
\]

\((D = \text{concrete density})\)

With the critical span, therefore:

\[
L_{\text{crit}}^2 \frac{D}{8} \frac{R_1 + R_2}{R_1R_2} = \frac{hR_1}{R_1 + R_2} (1 - \rho)
\]

Let \(R_2 = kR_1\) \((k \text{ is } < 1 \text{ usually})\). Then:

\[
L_{\text{crit}}^2 = \frac{8h(1 - \rho)}{D} \frac{k}{(1 + k)^2} = \frac{2h(1 - \rho)}{D} R_1 \times \frac{4k}{(1 + k)^2}
\]

Therefore:

\[
L_{\text{crit}} = \frac{2h R_1}{L} \frac{4k}{D (1 + k)^2} (1 - \rho)
\]

\(\dag\) Formula (a) differs from formula (8) of Section 1 because the hypotheses are different.
(ii) Assume that the critical span is exceeded. Therefore \( L^2 > L^2_{\text{crit}} \). Of the four limit stresses, only three can be reached.

Therefore, in the extreme loading condition, there can be one of the stress combinations shown in Fig. 17.

![Fig. 17.](image)

On each of the sketches, the centroid is at the intersection of the diagonal lines. Therefore:

\[
\text{cases 1 and 2: } v = h \frac{\sigma_g}{R_1} \quad \quad v' = h \frac{R_1 - \sigma_g}{R_1} \tag{b}
\]

\[
\text{cases 3 and 4: } v = h \frac{R_2 - \sigma_g}{R_2} \quad \quad v' = h \frac{\sigma_g}{R_2} \tag{c}
\]

Since at least one of the limit stresses is reached in each of the two conditions of loading, the centre of compression is coincident with one of the limit points defined in Chapter VII in each of these conditions.

In the unloaded state, the limit point \( A' \) for cases 1, 2 and 3, since the limit \( R' \) is reached. The limit point is \( B' \) for case 4 since limit \( R \) is reached and \( R' \) is not.

The eccentricities of \( A' \) and \( B' \) are respectively \(- (r^2/v)\) and \(- (r^2/v')[(R_1/\sigma_g) - 1]\). For cases 1, 2 and 3, therefore:

\[
\frac{M_p}{F} = \frac{L^2D}{8\sigma_g} = v' - \frac{r^2}{v} = v'(1 - \rho)
\]

For case 4:

\[
\frac{L^2D}{8\sigma_g} = v' - \frac{r^2}{v'} \left( \frac{R_1}{\sigma_g} - 1 \right) = v' - \rho v \left( \frac{R_1}{\sigma_g} - 1 \right)
\]

By giving the values (b) and (c) to \( v \) and \( v' \), an equation in \( \sigma_g \) is obtained for each of the cases. It is preferable to denote the unknown \( \sigma_g/R_1 \) by \( N \).
After calculation, the following equations are obtained:

For cases 1 and 2:

\[ N^2 - N + \frac{L^2D}{8hR_1(1 - \rho)} = 0 \]

For case 3:

\[ N^2 = \frac{L^2DR_2}{8h(1 - \rho)R_1^2} \]

For case 4:

\[ N^2(1 - \rho) + \rho(1 + k)N - k\rho \left( 1 + \frac{L^2D}{8h\rho R_1} \right) = 0 \]

In terms of the critical span, these equations can be written:

\[
\begin{aligned}
\text{cases 1 and 2:} & \quad N^2 - N + \left( \frac{L}{L_{\text{crit}}} \right)^2 \frac{k}{(1 + k)^2} = 0 \\
\text{case 3:} & \quad N^2 = \left( \frac{L}{L_{\text{crit}}} \right)^2 \frac{k^2}{(1 + k)^2} \\
\text{case 4:} & \quad N^2(1 - \rho) + \rho(1 + k)D - k\rho \left[ 1 + \left( \frac{L}{L_{\text{crit}}} \right)^2 \frac{k(1 - \rho)}{\rho(1 + k)^2} \right] = 0
\end{aligned}
\]

With the value of N, and therefore of \( \sigma_g \), given by these equations, the fourth stress can be determined. The solution is acceptable only if this stress is within its specified limits. This condition is written (see Fig. 17):

For case 1:

\[ \sigma_g > R_2 \frac{v'}{h} = kR_1 \frac{R_1 - \sigma_g}{R_1} \quad \text{or} \quad N > \frac{k}{1 + k} \]

For case 2:

\[ \sigma_g < R_2 \frac{v'}{h} \quad \text{or} \quad N < \frac{k}{1 + k} \]

For case 3:

\[ \sigma_g < R_1 \frac{v}{h} = R_1 \frac{R_2 - \sigma_g}{R_2} \quad \text{or} \quad N < \frac{k}{1 + k} \]

For case 4:

\[ \sigma_g > R_1 \frac{v}{h} \quad \text{or} \quad N > \frac{k}{1 + k} \]
Cases 1 and 2: The solution to eqn. (d) can be found only if:

\[ 1 - 4 \left( \frac{L}{L_{\text{crit}}} \right)^2 \frac{k}{(1 + k)^2} > 0 \]

or: \( L/L_{\text{crit}} < (1 + k)/2(k)^{\frac{1}{2}} \), and, since by hypothesis \( L > L_{\text{crit}} \), only if \( L \) satisfies the following inequality: \( L_{\text{crit}} < L < L_{\text{crit}}[(1 + k)/2(k)^{\frac{1}{2}}] \).

The interval is very small for normal values of \( k \). For \( k = 1 \),

\[ \frac{1 + k}{2(k)^{\frac{1}{2}}} = 1 \quad \text{for} \quad k = 0.6 \quad \frac{1 + k}{2(k)^{\frac{1}{2}}} = 1.03 \]

In this instance, \((L_{\text{crit}}, 1.03 L_{\text{crit}})\), the solution is:

\[ N = \frac{1 \pm \{1 - 4(L/L_{\text{crit}})^2[k/(1 + k)^2]\}^{\frac{1}{2}}}{2} \]

The negative sign must be taken in the numerator, as it is easily seen that the smaller the value of \( \sigma_p/R_1 \), the smaller are the values of \( F \) and \( S \). For case 1, it is necessary that [condition (g)]:

\[ 1 - \frac{\{1 - 4(L/L_{\text{crit}})^2[k/(1 + k)^2]\}^{\frac{1}{2}}}{2} < \frac{k}{1 + k} \]

This condition is equivalent to:

\[ \left( \frac{L}{L_{\text{crit}}} \right)^2 > 1 \]

By hypothesis, this condition is achieved. Solution 1 is therefore possible within the small interval of spans considered.

For case 2, it is necessary that:

\[ 1 - \frac{\{1 - 4(L/L_{\text{crit}})^2[k/(1 + k)^2]\}^{\frac{1}{2}}}{2} > \frac{k}{1 + k} \]

This condition is equivalent to:

\[ \left( \frac{L}{L_{\text{crit}}} \right)^2 < 1 \]

which is contrary to the hypothesis. Case 2 is therefore impossible.

Case 3: Eqn. (e) gives \( D = L/L_{\text{crit}}k/(1 + k) \).

Since \( L/L_{\text{crit}} > 1 \), \( N > k/(1 + k) \); condition (i) is therefore not satisfied and case 3 is impossible.
Case 4: The solution of eqn. (f) is always real.

\[ N = \frac{-\rho(1 + k) + \{\rho^2(1 + k)^2 + 4k\rho(1 - \rho) \times [1 + (L/L_{\text{crit}})^2(k(1 - \rho))/((\rho(1 + k)^2))\}}{2(1 - \rho)} \]

This solution is possible [condition (j)] if \( N > k/(1 + k) \).
It can be shown that this is equivalent to: \( L/L_{\text{crit}} > 1 \).
Solution 4 is therefore always possible.
The above discussion demonstrates that, within the terms of these hypotheses, \( (d' = 0, R'_1 = R'_2 = 0) \), and for spans greater than the critical span:

(i) Solution 4 is always possible;
(ii) Within a restricted interval \( \{L_{\text{crit}} < L < L_{\text{crit}}[(1 + k)/2(k)^{1/2}]\} \), solution 1 is possible;
(iii) Solutions 2 and 3 are excluded.

The interval \( L_{\text{crit}} < L < L_{\text{crit}}[(1 + k)/2(k)^{1/2}] \) has no real practical value, because the span is very close to the critical value; consequently, solutions 1 and 4 are very nearly equivalent.

It can nevertheless be shown that, even within its narrow band, solution 4 is more economical than solution 1.
For the lower limit \( (L_{\text{crit}} = L) \), solutions 1 and 4 are identical. For the top limit, \( L = L_{\text{crit}}[(1 + k)/2(k)^{1/2}] \), \( N = \frac{1}{2} \) in the case of solution 1.
In the case of solution 4, it is found that, with \( \rho = \frac{1}{2} \):

\[ N = \frac{1 + k}{2} \left\{ \left[ 1 + \frac{4k}{(1 + k)^2} \left( 1 + \frac{1}{4} \right) \right]^{1/2} - 1 \right\} \]

Since \( 4k/(1 + k)^2 \) is very nearly equal to 1,

\[ D = \frac{1 + k}{2} \left[ (2 + \frac{1}{4})^{1/2} - 1 \right] \]

\[ \approx \frac{1 + k}{4} \]

By denoting the prestressing forces and the concrete cross-sectional areas corresponding to the two solutions as \( F_1, S_1, F_4, S_4 \), and assuming \( \rho = \frac{1}{2} \), it is found that:

\[ F_1 = 2 \frac{M_s}{hk} \quad F_4 = 2 \frac{M_s k(1 + k)}{h} \frac{3k - 1}{3k - 1} \]

\[ S_1 = 4 \frac{M_s}{hR_1 k} \quad S_4 = 4 \frac{M_s}{hR_1} \frac{2k}{3k - 1} \]
It can easily be shown that $F_1$ and $S_1$ are respectively greater than $F_4$ and $S_4$ for normal values of $k$ ($0.5 < k < 1$). In every case, therefore, a design according to solution 4 should be attempted, although it must be remembered that in the above proof it is assumed that $R' = 0$, $d' = 0$ and $k < 1$.

C. There are only two unknowns, $F$ and $h$, in the case of a slab with a span greater than the critical span, and it is only possible to reach two of the limit stresses.

If the span were less than the critical span, the stress variation at the bottom fibre would be $R_1 - R'_2 = 6(M_s/bh^2)$ since the limit stresses would be reached in the extreme conditions of loading. Thus, the value of $h$ could be determined. The depth of the slab now under consideration is greater, because the critical span is exceeded. The stress variation is therefore less than $R_1 - R'_2$.

Hence, if $R'_2$ is reached in the loaded condition, the stress in the unloaded condition is less than $R_1$; conversely, if $R_1$ is obtained in the unloaded condition, the stress in the loaded condition is greater than $R'_2$. There are therefore two groups of possible solutions.

The first group is obviously more economical, because the corresponding prestressing force is less. If it is assumed that this group is chosen for the design, only the two cases illustrated diagrammatically in Fig. 18 are possible.

![Diagram](image)

**Fig. 18.**

It is shown in Chapter V, Section 5, that maximum concrete economy is obtained by choosing a limit stress at the top fibre in the loaded condition which is as high as possible, and therefore in accordance with diagram (b).

If concrete economy is the determining factor, then the above rule is applicable. If, on the other hand, maximum economy of steel is required, a low value of the stress must be chosen, on the basis of diagram (a).

Generally, for overall economy, solution (a) is preferable. This is not contradictory to the previous statements: it is a consequence of having a section of the particular specified shape.
11. Discussion of limiting values of stress

It is shown in Section 3 that it is always advantageous to work with values of $R'_2$ (or, more generally, with limit stresses at the fibre in tension under the action of external loads) which are as low as possible, and preferably negative.

If is now shown that it is advantageous to work with values of $R_1$ and $R_2$ which are as high as possible, except where the shape of the section is specified (see Section 4).

For the diagrammatic profile illustrated in Fig. 14 (web and flanges of small thickness), it was shown that:

$$A' = \frac{M_s}{h(R_1 - R'_2)} \quad A = \frac{(M_s/h)[1 + (A'D/s)]}{R_2 - (M_s/h)(D/s)}$$

The cross-sectional areas $A$ and $A'$ are smaller for higher values of $R_1$, $R_2$ and $h$.

The prestressing force $F$ is equal to $AR_2 + A'R'_2$; the term $A'R'_2$ is smaller for higher values of $h$ and $R_1$ and for lower values of $R'_2$.

The term

$$AR_2 = \frac{M_s(R_2/h)[1 + (A'D/s)]}{R_2 - (M_s/h)(D/s)} = \frac{(M_s/h)[1 + A'(D/s)]}{1 - (M_s/hR_2)(D/s)}$$

is smaller for higher values of $R_2$ and $h$.

There is therefore a definite advantage in making $R_1R_2$, and $h$, as great as possible in the diagrammatic case illustrated in Fig. 14.

The discussion is more involved when realistic profiles are examined, where the efficiency is no longer equal to 1. Only case 4 of Fig. 17 is considered here; as mentioned above, this is the solution which must be attempted. Assume again that $R'_2 = 0$ and $d' = 0$ (no cover).

Then $I/v' = M_s/R_1$ since $R'_2 = 0$.

But:

$$\frac{I}{v'} = S \frac{v'^2}{v'} = S \rho v = Sh\rho \frac{R_2 - \sigma_g}{R_2}$$

Writing $R_2 = kR_1$:

$$S = \frac{M_s}{hR_1} \times \frac{1}{\rho} \times \frac{1}{1 - (\sigma_g/kR_1)}$$

and

$$F = S\sigma_g$$
Therefore

\[ F = \frac{M_s}{h} \times \frac{1}{\rho} \times \frac{\sigma_g/R_1}{1 - (\sigma_g/kR_1)} \]

Assuming \( \rho = \frac{1}{2} \), then:

\[ S = \frac{2M_s}{hR_1[1 - (1/k)(\sigma_g/R_1)]]} \quad F = \frac{2M_s}{h} \frac{\sigma_g/R_1}{1 - (1/k)(\sigma_g/R_1)} \]

As previously seen, for \( \rho = \frac{1}{2} \):

\[ \frac{\sigma_g}{R_1} = \frac{-(1 + k) + \{(1 + k)^2 + 4k[1 + (L^2D/4hR_1)]\}^{\frac{1}{2}}}{2} \]

By differentiating, it is found that:

(a) When \( R_2 \) is fixed, \( \sigma_g/R_1 \) decreases as \( k \) decreases, therefore as \( R_1 \) increases. Also, \( S \) and \( F \) vary directly with \( \sigma_g/R_1 \), and inversely as \( k \). It is advantageous therefore to make \( R_1 \) as high as possible.

(b) When \( k \) is fixed, \( S \) and \( F \) decrease as \( R_1 \) increases, therefore as \( R_1 \) and \( R_2 \) increase. Also, \( S \) and \( F \) decrease as \( h \) increases.

This latter conclusion presupposes complete freedom in the choice of shape. This freedom does not exist, because of the requirements for minimum thickness, especially for the webs, which can become inordinately thick if depth is excessive, with a possible loss in efficiency.

Also, the risk of transverse instability increases with the ratio \( h/L \), and it must be kept within a certain limit. The formula which is indicated in Section 3,

\[ \frac{h}{L} = \frac{11}{3 + 4(L/100)} \]

provides a good indication of the order of magnitude of the maximum values of the ratio \( h/L \), compatible with practical considerations.
Chapter IX

SAFETY IN BENDING OF STATICALLY-DETERMINATE BEAMS
PERMISSIBLE STRESSES—LIMIT STATES

I. VARIOUS SAFETY CONCEPTS

1. Permissible stresses and safety factors
The traditional safety concept consists of checking, in absolute terms, that the stress $\sigma$, calculated in accordance with the standard methods of the theory of elasticity and under working conditions, is less than a fixed proportion $1/\gamma$ of the mean strength of the material, $R_m$ ($R_m$ is general, and denotes the resistance, or strength, of any one of the component materials).†

This concept is expressed by the inequality:

$$\sigma \leq \frac{R_m}{\gamma}$$

(1)

It is the concept of permissible stresses, applied in Chapters V to VIII.

The factor $\gamma$ is an overall factor, including all unknowns: uncertainties of loading, stress distribution, strength variations, etc. Its value is chosen in such a manner that, with the worst possible combination of uncertainties and unknowns, the material remains in the elastic state, the boundary of which is a certain 'elastic limit', beyond which plastic phenomena occur. This, at least, is the classical concept.

In order to highlight the various causes of uncertainty, the following can be written, if $S$ is the loading:

$$\gamma S \sigma (ks) \leq \frac{R_m}{\gamma_m}$$

(2)

† The strength $R_m$ is not necessarily the rupture strength. It can be the stress which corresponds to a characteristic of the material such that its effect is to render that material useless (example: elastic limit for a mild steel with a definite yield point).

If several materials are involved, inequality (1) must be checked for each of them.
$k$ is a coefficient which is applied to the loading, and $\gamma_s$ and $\gamma_m$ take into account the possible variations in stress distributions and strength respectively, relative to mean values.

When the stress is proportional to the load, inequalities (1) and (2) are equivalent, since $\sigma(ks) = k\sigma$. In this case, inequality (2) can be written:

$$\sigma \leq \frac{R_m}{k\gamma_s\gamma_m}$$

and it is equivalent to inequality (1).

In the case of prestressed concrete, where the stress is not proportional to the load, but where it includes a constant quantity $\sigma_o$ and a variable quantity $\sigma(s)$, the following inequality replaces inequality (2):

$$\gamma_o\sigma_o + \gamma_s\sigma(ks) \leq \frac{R_m}{\gamma_m}$$

This inequality is no longer equivalent to inequality (1), indicating that the method of permissible stresses is not logical when dealing with prestressed concrete. It is akin to writing:

$$\sigma_o + \sigma_s \leq \frac{R_m}{\gamma}$$

or:

$$\gamma_s(\sigma_o + \sigma_s) \leq \frac{R_m}{\gamma_m}$$

which gives a common factor to the two stresses, although their variations can be quite different.

The concept of permissible stresses is nevertheless adopted in the following, when the calculations are concerned with the elastic region under normal conditions of service, in order to comply with the existing codes.

It must be remembered, however, that inequality (3) represents the true conditions, and it may be necessary to apply it in certain cases.

Since the method of permissible stresses is only concerned with the elastic state—that is, the state in which the stresses due to the loading are proportional to the loading—it is necessary to make a distinction between the factors $\gamma_s$ and $k$ in inequality (3), because $\gamma_s\sigma(ks)$ is equal to $k\gamma_s\sigma(s)$. The inequality can therefore be re-written as follows:

$$\gamma_o\sigma_o + \gamma_s\sigma(s) \leq \frac{R_m}{\gamma_m}$$

(3a)
The factor $\gamma_s$ accounts for both the possible increase in the loading and the variations of the stress distribution from the mean.

It can be necessary to distinguish between the types of loading, and to give different factors to the stresses due to each type of load. For example, if the permanent loads ($p$) and the live loads ($s$) are segregated, then:

$$\gamma_o \sigma_o + \gamma_p \sigma_p + \gamma_s \sigma_s \leq \frac{R_m}{\gamma_m}$$

(4)

In the above formula, $R_m$ is the strength of one of the materials, and $\sigma_o$, $\sigma_p$, and $\sigma_s$ are the stresses induced in this material under the different conditions of loading.

2. Method of limit states
The aim of the permissible stress method is to provide a design which is in the elastic state under all conditions.

This concept can be too exacting.

The method of limit states consists of defining certain conditions at which the structure would be unsuitable if they were attained, and then designing the structure in such a way that the probability of these conditions being realised is sufficiently remote for it to be acceptable.

The limit states which can be envisaged are numerous: rupture, cracking, excessive opening-up of cracks (in the case of reinforced prestressed concrete), excessive deflections, buckling, and so on.

Only the limit states of cracking and failure are considered in this chapter.

Each of these is characterised by limit strains: limit tensile strain of concrete for cracking, limit compressive strain of concrete and limit tensile strain of steel for rupture.

To each limit state corresponds a certain stress distribution within the section, resulting from the stress–strain diagram of the material and from the fact that a limit strain is reached.

Consider the stress–strain relationship for concrete. As seen in Chapter II, the stress–strain diagram follows approximately an elastic law within a certain range of stresses, or better, within a certain band $a'a'$ of strains (Fig 1). Beyond $a$, the diagram becomes curved, and the compressive stress increases more and more slowly with the strain, and terminates at a limit where the strain increases at constant stress, equal to $R_b$, the failure stress; failure occurs only when the compressive strain reaches a certain limit value, $e_r$, of the order of $3.5/1000$ (point $r$).
With regard to tensile stresses, the existence of a similar curvature and of a bounding value can be argued. It is assumed for the present that these plastic phenomena do exist; they are discussed in Section 4. With this assumption, the tensile strain continues to increase at constant stress once the stress \( R'_b \) is reached, until the tensile rupture strain is reached (point \( f' \)).

Consider any particular state of a section (for simplicity, it is first assumed that cracking has not occurred). It is wholly defined if the strains \( \varepsilon' \) and \( \varepsilon \) on the bottom and top fibres are known, since it is assumed that the strain variation is linear between the two fibres (principle of conservation of plane sections, proved with sufficient accuracy in practice).

The stresses corresponding to these strains are \( \sigma' \) and \( \sigma \) (Fig. 1). They are the stresses at the extreme fibres of the section. Since the strains vary linearly with the ordinate, the stress distribution diagram to a given scale in the section is the hatched portion of Fig. 1, where the length \( \varepsilon' \varepsilon \) corresponds to the depth of the section (rotated through 90°).

To each state which is thus defined there corresponds, at a given section, a definite resultant of the forces in the concrete and a definite position of this resultant, which can be obtained by evaluating on Fig. 2 the stress at each level \( y \).

It is seen that the converse applies: that is, knowing the resultant and its position, the portion of the diagram (Fig. 1) which represents the stress distribution in the section can be determined.
Cracking occurs when the tensile rupture strain $\varepsilon'_{f}$ in the concrete is exceeded at the bottom fibre (point $f'$ in Fig. 1). The state of the section is still defined by the tensile strain $\varepsilon'_{f}$ at the bottom fibre and the compressive strain $\varepsilon$ at the top fibre, but $\varepsilon'_{f}$ occurs at a certain point $f''$ within the depth of the beam (Fig. 3). Stresses therefore exist in the concrete only above the level of $f''$; that is, above a certain depth $x$.

Also, the cables, which are assumed to be in the cracked area are stretched. The increase $\varepsilon'_{a}$ in the strain in the cables can be determined when $\varepsilon'_{1}$ and $\varepsilon$ are known (Fig. 3). If the initial prestressing strain in the
cables is $\varepsilon'_a$, their total strain is $\varepsilon'_o + \varepsilon'_a$ and the stress $T$ in the cable can be found by means of the stress–strain diagram for the steel. If the cross-sectional area $A_c$ of the cables is known, the force exerted by the cables is $A_cT$ (it is assumed for simplicity that there is no non-tensioned reinforcement).

If there is no normal external force, the resultant of the internal forces in the concrete is equal to the force in the cables. Since the depth $x$ is known, the moment arm $z$ is known, equal to the distance between the cable and the resultant of the internal forces. Hence the moment is:

$$M = zA_cT$$

Therefore, having fixed the state of strain ($\varepsilon'_c\varepsilon$), the required cross-sectional area $A_c$ of the cables can be determined at any section in the member and so can the moment which must act on the section so that the true state of strain agrees with the value originally required.

If, on the contrary, the moment and the dimensions are given (concrete and $A_c$), the state of strain can be determined (two unknowns $\varepsilon'_c\varepsilon$ and two equations, the one expressing the equality of the compressive and tensile resultants and the other expressing the fact that these equal resultants give a couple equal to $M$).

Conversely, if a particular limit state is required under a moment $M$ ($Me'_c\varepsilon$ are known), the appropriate design can be determined (concrete and steel).

To be more explicit, consider the particular limit state corresponding to failure due to crushing of the concrete. The crushing strain occurs at the top fibre. The depth of the stress diagram in the concrete is equal to $x_r$ (less than in Fig. 3), containing the entire diagram of Fig. 1 (from $f'$ to $r$). The strain in the cable is again $\varepsilon'_a$ (Fig. 4) and it is assumed that the stress in the cable is then equal to the failure stress $T_r$ in view of the high value of $\varepsilon'_a$ and the shape of the stress–strain diagram for steel. (The corrections to qualify this assumption are discussed in Section 7.)

$x_r$ is determined from the condition that the resultant of the forces in the concrete is equal to $A_cT_r$; $z_r$ is then found, and the moment $M_r$ at rupture is equal to $z_rA_cT_r$.

This moment of resistance at failure is therefore a function of the concrete dimensions $b, h, \text{etc.}$, of the cable cross-sectional area $A_c$ and of the position of the cables, of the concrete compressive strength $R_c$, and of the cable rupture strength $T_r$. It can be written $M_r(bh, R_c, A_cT_r)$.

In this case, the method of limit states consists of designing the section and the cables in such a way that the probability of the moment $M$ under
load reaching the limit moment is sufficiently small for it to be acceptable. For failure, a probability of $1/100\,000$ or of $1/1\,000\,000$ is generally accepted.

The resulting design is represented by the inequality:

$$\gamma_s M \leq M_r \left( bh, \frac{R_b}{\gamma_b}, \frac{A_c T_r}{\gamma_{cr}} \ldots \right)$$

(5)

The stresses $R_b, T_r, \ldots$ and the coefficients $\gamma_s, \gamma_b, \gamma_p, \gamma_{cr}$, must be chosen so that the agreed probability is not exceeded.

Such is the method of limit states for failure. The concept of permissible stresses no longer exists. It is replaced by the concepts of a limit value of one of the parameters (the moment) corresponding to strains at failure, and by a probability of reaching this limit value, which can be made as small as required by the appropriate choice of the factors $\gamma$.

![Fig. 4.](image)

A similar argument can be developed for any other limit state. If plastic phenomena are not present in a particular limit state, the portion of the stress–strain diagram (Fig. 1) representing the stress distribution in the section is a straight line. The same results are then obtained with the limit state method as with the permissible stress method, since the strains are proportional to the stresses. The maintenance of a fixed probability of reaching the limit strain is expressed by an inequality of the form: $\varepsilon < (\varepsilon_{lim}/\gamma)$, which is equivalent to inequality (1), $\sigma < (\sigma_{lim}/\gamma)$, of Section 1.

Whether or not plastic phenomena occur in the case of fracture is debatable. Is cracking preceded by such phenomena? In other words, is the failure point of Fig. 1 realistic? If not, a design on the basis of the method of limit states is the same as a design on the basis of permissible stresses.
If the point does exist, a limit state of cracking can be defined, as for rupture. This state is then characterised by a stress distribution in the section which is represented by the hatched sections in Fig. 5. Apart from differences in the scale, this diagram represents a portion of Fig. 1, starting from the point $f'$ at the bottom fibre, where the tensile strain is the rupture strain $e'_r$; the stress is then uniform over a depth $f'd'$ and equal to $R'_b$, corresponding to the maximum of Fig. 1. The position of the neutral point 0, and consequently the value of the compressive strain $e$ at the top fibre, is obtained by equating the resultant of the forces in the concrete to the force $A_cT$ in the cables.

![Fig. 5.](image)

By taking moments of the internal forces about the cable position, the moment which must be exerted after prestress is applied (that is, the moment to be exerted by the external forces) is determined, such that this limit state is reached.

This moment, $M_f$, depends on the properties of the concrete section (cross-sectional area, section moduli), on the cross-sectional area $A_c$ of the cables and on their position, on the tensile strength $R'_b$ of the concrete, and on the stress in the cables, which is known, at least approximately.

There corresponds to any design, therefore, a limit moment causing cracking, which can be expressed as $M_f(B, R'_b, A_cT, \ldots)$, where B is a general symbol representing the various concrete properties which are involved.

† It is seen in Section 6 that a slightly different diagram can be taken in practice.
The design is therefore prepared on the basis that the probability of
the moment \( M \) reaching the limit value \( M_f \) is sufficiently small to be
acceptable. In general, a fairly high probability is accepted for cracking
(1/10,000 and even greater).

This is expressed by the inequality:

\[
\gamma_s M \leq M_f \left( \frac{B}{\gamma'_b}, \frac{R'_b}{\gamma'_b}, A_c \gamma_o T \ldots \right)
\]  \hspace{1cm} (6)

where the values given to the factors \( \gamma_s, \gamma'_b, \gamma_o \ldots \) correspond to the
desired probability.

It is shown in Section 4 that, even if there is strictly speaking no plastic
defformation in tension, equivalent phenomena occur if small-diameter,
well-distributed, non-tensioned steel is introduced into the section.

Under these conditions, the limit state of cracking can be envisaged for
the design of sections with limited prestress (Class II of the FIP–CEB
Committee), where a limited risk of cracking is acceptable.

**Note**

Figures 4 and 5 do not contradict each other, despite the fact that they
refer to the same section under the same moment \( M \), because they corre-
spond to different values of probability, and consequently to different values
of the factor \( \gamma_s \), much higher in the case of Fig. 4 than in the case of Fig. 5.

3. Practical consideration of probabilities in the method of limit states.

**Definitions and symbols**

An exact probability calculation, taking into consideration the various
combinations of safety factors, is not possible in practice. On the whole,
the simplified methods of the CEB and of the Joint FIP–CEB Committee
are adopted here.

(i) *Assessment of the parameters which affect the limit states (such as
\( M_r \) and \( M_f \) in the preceding section)*

The following are defined.

(a) A reduced strength for the various materials, such that the prob-
ability of obtaining test results of a lower order has a value which is
accepted *a priori*, instead of the mean strength of the materials obtained
from test results. This reduced value is called the characteristic strength
\( R_k \). It is defined as \( R_k = R_m(1 - \chi \delta) \).

\( R_m \) is the mean value, obtained from test results, \( \delta \) is the mean square
relative deviation, and \( \chi \) is a coefficient which depends on the accepted
probability of obtaining test results which give values less than $R_k$, and on
the number of tests defining $R_m$.

In principle, the risk of obtaining strengths which are less than $R_k$
for 5\% of the test results is accepted. The corresponding factor $\chi$, for a
normal statistical distribution, is then equal to 1.64.

In this manner, the following are defined:

compressive strength of concrete: \[ R_{bk} = R_{bm}(1 - 1.64\delta) \]
tensile strength of concrete: \[ R'_{bk} = R'_{bm}(1 - 1.64\delta) \]
rupture strength of prestressing steel: \[ T_{rk} = T_{rm}(1 - 1.64\delta) \]
elastic limit of ordinary steels: \[ \sigma'_{ek} = \sigma'_{em}(1 - 1.64\delta) \]

and so on.

(b) reduction factors $1/\gamma_m$ which are applied, for the various materials,
to the characteristic strengths, taking into consideration, according to
the CEB: ‘those variations which cannot be easily evaluated or measured’.

For a given material, the values of the factors $\gamma_m$ depend on the particular
limit state which is concerned, and are related to the seriousness of the
risk which is involved.

The calculated strength $R^*$ is the strength which is thus reduced,
relative to the characteristic strength, or:

\[ R^* = \frac{R_k}{\gamma} \]

The various factors are hereafter identified by means of suffixes, and the
strengths, for calculation purposes, are defined as:

\[ R^* = \frac{R_{bk}}{\gamma_b} \quad R'_{*b} = \frac{R'_{bk}}{\gamma'_b} \quad T^*_{*a} = \frac{T_{rk}}{\gamma_{ao}} \quad \sigma^*_{*a} = \frac{\sigma'_{ek}}{\gamma_a} \]

and so on.

Should parameters other than strength affect the definition of a partic-
ular limit state, such as strains or moduli of elasticity, the characteristics
and design values of these parameters are defined in a similar way.

† In principle, the strength of concrete in compression is the 28-day cylinder strength.
It should be remembered that the ratio $R_{cyl}/R_{cube}$ is not necessarily constant, and that
it can increase with strength (see Chapter II, Section 1). This may need to be taken into
consideration when defining the value of $\gamma_b$. 

(ii) Loads
A. For loads which can be considered as ill-defined, and where a possible increase could be detrimental, the following are defined:

(a) a characteristic value \( Q_k = Q_m (1 + \chi \delta) \);
(b) a design value \( Q^* = \gamma_s Q_k \) \( (\gamma_s > 1) \).

B. If, on the other hand, a reduction in the value of these loads is detrimental, the following are defined:

(a) a characteristic value \( Q'_k = Q'_m (1 - \chi \delta) \);
(b) a design value \( Q'^* = \gamma_s Q'_k \) \( (\gamma_s < 1) \).

C. For causes other than loads, as a consequence of shrinkage, creep, temperature, and so on, characteristic and design values must also be defined.

(iii) Prestress
The following are defined:

(a) a characteristic value of the prestressing force:
\[
F_k = A_c T_k = A_c T_m (1 \pm \chi \delta)
\]
(b) a design value:
\[
F^* = \gamma_o F_k = A_c \gamma_o T_k \quad (\gamma_s \geq 1)
\]
In these formulae, the positive sign (and as a corollary \( \gamma_s > 1 \)) or the negative sign (and \( \gamma_s < 1 \)) is chosen according to whether an increase or a decrease in the prestress is detrimental.

(iv) Check on safety
This consists of checking that, for all the limit states considered, the effects of the design loads and stresses are at worst equal to the absolute values which are permitted by the design strengths of the materials.

The characteristic formulae (5) and (6) may be written in the following manner, denoting the various loads which can occur simultaneously by \( Q_1, Q_2 \):

for compressive failure of the top flange:
\[
M(\gamma_1 Q_1, \gamma_2 Q_2, \text{etc.}) \leq M_r \left( bh, \frac{R_{bk}}{\gamma_b}, A_c \frac{T_{rk}}{\gamma_{or}}, A'_a \frac{\sigma'_{ek}}{\gamma_a}, \text{etc.} \right)
\]
for cracking:
\[
M(\gamma_1 Q_1, \gamma_2 Q_2, \ldots, A_c \gamma_o T_k) \leq M_f \left( B, \frac{R'_{bk}}{\gamma'_b}, A'_a \frac{\sigma'_{ek}}{\gamma_a} \right)
\]
The values of the factors $\gamma$ are those which correspond to one or the other of the limit states.

In practice, the left-hand sides of the inequalities can be simplified. The designer should not be concerned with the assessment of the characteristic loads. They should be defined by the specification, based on statistical data. The suffix $k$ for the loads is therefore unnecessary in these circumstances, and it is not used hereafter.

It is also necessary to distinguish between the various types of loading. Permanent loads are designated by $p$ and live loads by $s$, without indices. If other types of loading have to be considered (wind, earthquake shock, etc.), their characteristic values should also be given in the specification.

All the loads must be given their appropriate scaling factors $\gamma$, which depend on the nature of the loading ($\gamma_p$, $\gamma_s$, etc.).

With regard to the prestressing force, the specification need only give the characteristic force (or the tensile stress). It is then the designer's responsibility to make sure that the mean force ($F_n$) enables the specified characteristic force ($F_k$) to be obtained, taking into consideration the variations in the material properties and in the system which he proposes to adopt.

With this convention, the index $k$ qualifying $F$ or $T$ is omitted from now on. It is retained for strength symbols, however, in order to conform generally with the symbols of the FIP–CEB Committee used in the determination of the quantities defining limit states.

The formulae can be written in the following form, therefore (some of the parameters need not necessarily be present in either the left-hand side or the right-hand side of the inequality):

$$M(\gamma_p p, \gamma_s s, \gamma_o F, \text{etc.}) \leq M_{11\text{im}} \left( b, h, \frac{R_{bk}}{\gamma_b}, \frac{R'_{bk}}{\gamma'_b}, \frac{A_c}{\gamma_{or}}, \frac{T_{rk}}{\gamma_r}, \frac{A'_a}{\gamma_a}, \frac{\sigma'_{ek}}{\gamma_a}, \text{etc.} \right)$$

With statically-determinate structures, $M(\gamma_p p)$, $M(\gamma_s s)$, etc. are equal to $\gamma_p M(p)$, $\gamma_s M(s)$, and so on. By denoting the moments due to loading by $M_p$, $M_s$, etc., and the algebraic prestressing moment by $M_o$, the following can be written:

for compressive failure of the top flange:

$$\gamma_p M_p + \gamma_s M_s \leq M_r \left( b, h, \frac{R_{bk}}{\gamma_b}, \frac{A_c}{\gamma_{or}}, \frac{T_{rk}}{\gamma_r}, \frac{A'_a}{\gamma_a}, \frac{\sigma'_{ek}}{\gamma_a}, \text{etc.} \right) \quad (7)$$

for cracking:

$$\gamma_p M_p + \gamma_s M_s + \gamma_o M_o \leq M_f \left( B, \frac{R'_{bk}}{\gamma'_b}, \frac{A'_a}{\gamma_a}, \frac{\sigma'_{ek}}{\gamma_a}, \text{etc.} \right) \quad (8)$$
where the factors $\gamma$ have the value appropriate to the particular limit state.

The expressions for the right-hand sides of the inequalities are developed in the following paragraphs and in the following chapters.

**COMMENT**

Prestressing steel has a dual function: in the case of cracking, it creates a force, and therefore a load; the applied force (or the moment $M_o$ due to this force) is therefore coupled with a multiplying factor $\gamma_o$ (which can be greater or less than unity).

For failure of the top flange, the strength of the prestressing steel contributes to the strength of the section; the strength of the steel is therefore coupled with a reducing factor $1/\gamma_{or}$.

Confusion because of this dual function cannot in practice arise with the formulae.

It is seen in Chapter X that the limit state which corresponds to a failure of the bottom flange under the action of the initial prestressing force $F_t$ may need to be considered. The load due to this force is then introduced with a multiplying factor $\gamma_{oi}$.

**4. Comparison between permissible stress and limit state methods**

In general, whether prestressed concrete or any other material is concerned, permissible stress methods can be considered as particular cases of limit state methods, the limit states being obtained when one of the materials reaches the 'elastic limit'.

Considering only the most simple case, comprising only one type of loading and the strength of only one material, the safety criterion with both methods can be expressed as an inequality of the form

$$M(\gamma_o) \leq M_{lim}(\frac{R_m}{\gamma_m})$$

but with different values for the factors $\gamma(\gamma_o$ and $\gamma_m)$.

It can also be considered that, with a given loading (working load), a given design, and given stresses ($R_m$), a limit state can be reached by a progressive increase in the loading (that is, in the factor $\gamma_o$) or by a progressive reduction in the strength of the material (that is, in the factor $1/\gamma_m$); in other words, by a progressive decrease in the effective safety margin.

During the course of this decrease, the section will pass through the elastic limit state before reaching the ultimate limit state.
If, from a given design, the factors $\gamma_s$ and $\gamma_m$ under which the elastic limit state and the ultimate limit state are successively reached are respectively equal to those which have been defined for either method of checking, the same design is obtained with either method by applying these factors, with given mean loads and a given mean strength.

There is no reason, however, for choosing the factors in such a way that the two designs are the same. It would be impossible, anyway, because different phenomena are involved.

With statically-determinate structures, $M(\gamma_s)$ is equal to $\gamma_s M_s$ with either of the two methods.

But, whereas $M_{\text{lim}}(R_m/\gamma_m) = (1/\gamma_m)M_{\text{lim}}(R_m)$ with the elastic theory, it is no longer the case with the limit state method, because of the plastic phenomena which increase the strength of the section (redistribution of stresses within the section).

With statically-indeterminate structures, $M(\gamma_s)$ at a given section is no longer necessarily equal to $\gamma_s M_s$, since, because of the redistribution of stress within the section, there is also a redistribution of the moments.

With statically-determinate structures, designs which are based on limit states are generally a little better than those obtained on the basis of limit stresses. Consequently, the probability of reaching the ‘elastic limit’ is a little greater.

If the occurrence of plastic phenomena, in exceptional circumstances, has no practical consequence on normal working, there is no valid reason for imposing a more costly design.

A particularly obvious case is that of a bridge comprising prefabricated beams in which in-filling concrete is poured between the top flanges after positioning. The top flanges, which are loaded first, reach their ‘elastic limit’ under increasing loads, or with a reduction in strength, sooner than the in-filling concrete. This is important, because plastic deformations would then occur within the beams, during which the stresses would increase much more slowly than in the in-filling concrete, which continues to shorten elastically. There is therefore a tendency for the stresses to equalise, or a transfer of the compression in the prefabricated beams to the in-filling concrete. This transfer is irreversible, the deformations being more or less elastic when the loading is removed; if total equalisation has occurred during loading, the two concretes will retain equal amounts of compression when unloaded, and will act together when loading again occurs. This example of stress redistribution in the section is not covered by the elastic theory; there is no risk attached to it, and it is automatically taken into account in designing on the basis of the limit state at failure.
With most applications, but in a less obvious way, moderate plastic strains in compression are not an overriding disadvantage, and the true safety criterion is the limit state at failure. Naturally, if plastic deformation was in effect the most dangerous phenomenon (certain fatigue phenomena with cumulative plastic deformations, for example), the design check on the elastic basis could become the determining condition.

With cracking, the discussion is somewhat different, being concerned with the presence or otherwise of tensile plastic strains.

Although the top flange is of the same order of size whether the design is carried out on the basis of elastic calculations or limit state calculations, this is not the case for the bottom flange, or for the prestress, when these are determined for conditions where cracking is not allowed. This concerns mainly Class II prestress (limited prestress), and it is discussed in Section 6.

One particular design case follows logically from the method of limit states. This is the design of the bottom flange under the action of the initial prestressing force. This occurs only once, and the stress can then only decrease. Plastic deformations are not generally troublesome, and the determining condition is that of safety against failure.

With statically-indeterminate structures, the difference between elastic and limit state designs may be considerably greater. Also, plastic deformations can be less significant than may be expected, because of the interaction of the redistributions in the sections, and because of the redistributions of the moments.

II. SAFETY AGAINST CRACKING

A. Elastic theory

5. Allowable stresses
It is often said, in principle, that concrete remains elastic until cracking occurs. This hypothesis is discussed in Section 6.

Assuming that this is so, the proof of safety against cracking, that is, the proof that the tensile strain causing rupture is not exceeded, is equivalent to proving that the absolute value of the resultant stress $\sigma'$ remains lower than the tensile strength $R'$; let $|\sigma'| \leq |R'|$, or, algebraically, if $R'$ is the absolute value of tensile strength:

$$\sigma' \geq -R'$$

(1)

But $\sigma'$ is the sum of the prestress $\sigma'_o$ (positive) and of the stress $\sigma'_s$ (negative) under external loading.
The design stress is $\sigma'_o - \sigma'_s$. Let it be denoted by $\overline{\sigma}$. It is to this design value that a permissible limit must be given.

In accordance with Section 1, the different terms in expression (1) must be multiplied by increasing or reducing factors. The safety criterion is therefore written:

$$\gamma_o \sigma'_o - \gamma_s \sigma'_s \geq -\frac{R'_bm}{\gamma'_b}$$

or:

$$\sigma'_o - \frac{\gamma_s}{\gamma_o} \sigma'_s \geq -\frac{R'_bm}{\gamma'_b \gamma'_o}$$

The permissible stress limit $\overline{\sigma}$ is the stress at which the cracking stress is reached, when the two terms in expression (3) are equal.

Therefore:

$$\overline{\sigma} = \sigma'_o - \sigma'_s = \frac{\gamma_s}{\gamma_o} \sigma'_s - \sigma'_s - \frac{R'_bm}{\gamma'_b \gamma'_o}$$

The lower permissible limit stress is therefore defined by:

$$\overline{\sigma} = \sigma'_s \left(\frac{\gamma_s}{\gamma_o} - 1\right) - \frac{R'_bm}{\gamma'_b \gamma'_o}$$

(a) Total prestress

For the most unfavourable conditions $\gamma_s > 1$ and $\gamma_o < 1$. If it is also assumed that $R' = 0$, the lower permissible limit stress from equation (4) is positive (compression).

This satisfies the Instructions Provisoires de 1953 (French regulations for bridges and roads) which specify that there should exist a residual compression under load (equal to 8% of the stress variation).

In fact, this specification is too stringent for normal designs,† where cracks can occur only under exceptional loading, and where they close when the load is removed. The aim is to protect the design against cracking, with an acceptable probability. The mean tensile strength is never zero, providing that the necessary precautions against cracking during shrinkage have been observed (in particular, a fraction of the prestress being applied as soon as possible).

† In cases where cracking must be avoided (railway bridges and generally structures which are subjected to alternating load, possibility of corrosion of the reinforcement, possibility of aggressive atmosphere, etc.), it is safer to decide on a lower value of the compressive residual stress, and in some cases to be more conservative than the cited specification.
Under normal conditions, and with the usual types of concrete, \( R'_{hm} \) is generally above 25 to 30 kg/cm\(^2\).

There is, however, a large amount of scatter. For safety, with total prestress, \( R' \) is taken as being equal to zero, but it is expression (3) which gives the controlling value.

To take \( R' = 0 \) already implies a large safety factor. To take very unfavourable values of \( \gamma_o \) and \( \gamma_s \) at the same time is equivalent to taking probabilities which are less than those indicated as being reasonable in Section 2.

With regard to \( \gamma_o \), the variable causes which can affect the value of prestress (errors in the pressures in the tensioning jacks, errors in the position of the cables, differences between the actual and the calculated values of the losses, and so on) tend to cancel out as soon as the number of cables is sufficiently large. On the other hand, systematic causes always tend to increase the prestress \( \sigma_o \) (additional tensioning of the cables due to loss of compression in the concrete,\(^\dagger\) and eventual supplementary strains). It would seem therefore that \( \gamma_o > 1 \) should be used.

However, it is reasonable to use \( \gamma_o = 1 \), providing that \( \gamma_s = 1 \) is also used. This has been agreed by the FIP-CEB Committee. Under these conditions, the lower limit strain can be taken as being equal to zero in the case of total prestress.

**COMMENT**

The factor of safety of a section is greatly increased by adding ordinary reinforcement, even in the case of total prestress (for example, 0.3% of the zone which would be in tension if the stress in the fibre under tension were accidentally to become equal to 8/100 of the theoretical stress variation in this fibre, without any change in the stress in the fibre under compression; at centres of about 10 cm for a normal type of structure).

It is certain that, taking the limit as zero, a better design is obtained by including some reinforcement; without this, some residual compression has to be provided.

(b) *Limited prestress*

Assuming \( \gamma_o = \gamma_s = 1 \) as above, the permissible stress from eqn. (4), is, \( R'_{hm}/\gamma'_{b} \). The permissible tensile stress given in the ASP is: 0.5 \( R'_{28} \) at

\(^\dagger\) This excess tension is taken care of if the section is considered as homogeneous by multiplying the cable cross-sectional area by \( m \). The new French Codes (1965) permit this; the clause requiring 8% residual compression is no longer included.
the level of the cables nearest to the tensile face, and \( R'_{28} \) on the tensile face itself, \( R'_{28} \) being the 28-day tensile strength \([3.6 \text{ M/(a}^3\)]\) obtained from prism tests.

It also specifies a minimum quantity of ordinary reinforcement local to the tensile face. If \( F'_b \) is the tensile stress resultant in the tensile zone, the amount of reinforcement must be capable of balancing \( F'_b/2 \) when stressed to 60\% of the elastic limit (yield point or 0.2\% proof stress).

These requirements are reasonable, in view of the addition made by the ordinary reinforcement to the ultimate safety (see Section 6). The spacing of the reinforcement should not usually exceed 8 to 10 cm.

(c) Prestressed reinforced concrete

Since cracking is considered to be a normal risk, a limit stress is not imposed; the widths of the cracks must, however, be limited by a judicious choice of diameter for the complementary reinforcement (see Chapter XI).

B. Design on the basis of a limit state of cracking

6. Plastic phenomena as cracking is approached

Bending tests on prestressed beams show that the cracking strength can be much higher than that suggested by the elastic theory, when suitably spaced reinforcement is present local to the tensile face.

The author advances (Prestressed Concrete, Volume 1, Chapter XVII) an explanation based on the hypothesis of the plasticity of concrete in tension; that is, by assuming a stress distribution of the form shown in Fig. 5. He shows that, in agreement with practice, the increase in strength is considerable for rectangular sections and small for flanges and webs of small thicknesses, and that it increases with the value of the mean prestress \( F/S \). BRICE has given a different explanation,† based on the ‘threshold of visibility’: it would seem that the plastic state was reached, but it would be due to the effect of microcracks, made possible through the presence of the reinforcement.

Although tests have demonstrated important plastic phenomena in bending, they probably cannot be relied on implicitly, and BRICE’s explanation is not impossible. The author nevertheless considers that the effect is due not so much to a ‘threshold of visibility’ but to a discontinuity in the behaviour of the bond between the concrete and the steel, and that a

† Prestressed Concrete, Volume 1, Chapter XVII, Second Edition.
crack opens suddenly as a consequence of sudden relative movement, when the bond ceases to be elastic and becomes dependent on friction, due to the failure of the concrete surrounding the reinforcement bar.

This hypothesis leads to consequences which are similar to those corresponding to the hypothesis of plasticity, including the laws of stress distribution between two microcracks.

Consider a section, provided with reinforcement local to the tensile face, which is subjected to a constant normal force F (prestressing force), and to an increasing moment M. For simplicity, it is assumed that the moment is uniform over the portion of the beam being considered.

Let \( M' \) be the moment under which the tensile stress reaches its limit value \( R' \) (absolute value) (Fig. 6), at the tensile face, and consider an increment \( \Delta M \) of the moment.

\[ \text{FIG. 6.} \]

Without tensile plasticity, cracks will appear, but, because of the reinforcement which prevents concentrations of tensile strain (if the reinforcement is suitably spaced), the cracks will remain very small, of the order of \( 1/100 \) mm as shown later.

At first, the cracks occur at random, depending on the scatter of the tensile rupture strains of the concrete in the area of constant bending moment. Then preferential planes of microcracking are formed, when any crack which has appeared at any point in an area makes that area more vulnerable to cracking than an area which is still intact as the bending moment increases.

Consider such an area of microcracking (X). By hypothesis, the stress diagram is a straight line (Fig. 7), and the concrete is still at its tensile limit \( R' \) at the head of the crack.
From the conditions of equilibrium of the stress diagram, the stress $\sigma'_a$ in the steel can be calculated, the equivalent cross-sectional area $mA_a$ being subjected to the stress $\sigma'_a/m$.

Let

$$\frac{\sigma'_a/m}{R'} = \phi$$

Hence:

$$\sigma'_a = \phi mR'$$

Let $2l_o$ be the distance between two microcracks (Fig. 8).

At a section Y, half way along this distance, the material is subjected to a certain pattern of stress, without cracks; the stress distribution diagram is therefore a 'full' diagram. It cannot be linear, or the stress on the tensile face would be $R'' > R'$ (in absolute value), and this contradicts the hypothesis.†

† Because of scatter, the tensile strength may be greater at section (Y) than at section (X), but, when $\Delta M$ reaches a sufficiently high value, the tensile strength is reached at section (Y) and the reasoning applies.
Consider, nevertheless, the imaginary straight line diagram (Fig. 9) which would correspond to the stress \( R'' \) (apparent tensile strength). Let \( M'' \) be the corresponding moment.

The true diagram must be such that the internal forces have the same resultant \( F \) and the same moment \( M'' \) with respect to the cable. At the tensile face, the stress is \( -R' \); in the upper portion, it can be assumed that the diagram is straight between \( a \) and \( b \), the stress at \( b \) being also equal to \( -R' \); between \( b \) and \( c \), at the limit, conditions tend to be uniform, as will be shown.

The hatched portion in Fig. 9 is thus obtained, superimposed on the diagram (\( R'' \)). It has the same appearance as a ‘plastic’ diagram, although, as shown later, it results only from elastic phenomena.

![Fig. 9.](image)

At section (Y), the stress in the reinforcement, which is assumed to be located on the bottom face, is \( mR' \), the steel being subjected to the same strain as the concrete in this median zone.

Between (X) and (Y) the stress in the reinforcement passes from \( \sigma'' \) to \( mR' \). The difference between \( \sigma'' \) and \( mR' \) is small for small values of \( \Delta M \); the bond between the reinforcement and the concrete is then elastic (\textit{Prestressed Concrete}, Volume 1, Chapter VII). The encasing concrete distorts, as indicated in Fig. 10, and a meniscus is formed at the extreme faces of the block \( 2l_0 \) (that is, in the planes of the microcracks), whose diameter is of the order of 6 cm + \( d \) (\( d \) being the diameter of the bar). Let \( w \) be the depth of the meniscus. The material remains knitted together by the menisci which are joined to each other (Fig. 10).†

† Figure 10 is obviously only diagrammatic; the widths of the microcracks are in practice about 1/1 000 the size of those shown.
From tests carried out by Bichara and Brice, the surrounding concrete does not break providing the depth of the meniscus (that is, the displacement of the bar relative to the concrete) does not exceed a limit value \( w_1 \), of the order of 0.02 mm.

![Fig. 10. Knitting together of microcracks by the bars, before rupture of the encasing concrete.](image)

During the whole of the elastic phase, the bonding stresses are at a maximum at right angles to the crack and they decrease rapidly away from the crack (exponential law with negative exponent). A linear law can be substituted for the true law, passing from a maximum to zero within a very short length \( a \) (Fig. 11), which is characteristic of the bond.

![Fig. 11. Elastic bond between the bars and the surrounding concrete.](image)

If \( 2l_o = 2a \), the stress in the steel reaches \( \sigma'_a \) in the two cracks defining this interval of length.

If \( a < 2l_o < 2a \), the stress is \( \sigma'_a \) at only one of the cracks, and it has a smaller value in the other.
Because of the lack of symmetry, certain cracks develop whilst others are arrested. This is confirmed with electric strain gauges: there are always some zones which ‘absorb’ all the strain, the neighbouring zones remaining static, or even losing some of the strain which they had previously acquired.

At a certain stage, the sheathing concrete fails; the crack then opens, and the bond is maintained by means of friction, with the concrete sliding backwards relative to the bar, towards the centre of the length interval $2l_o$.

The friction stress (bonding stress) is independent of the stress $\sigma'_a$; therefore, as $\sigma'_a$ increases (that is, as the difference $\sigma'_a - mR'$ increases), the bonding length must increase, and this closes up some of the micro-cracks (if the length which is required for bonding by friction is greater than the length $2l_o$ of Fig. 10, the friction forces will close the meniscus on the right-hand side if the crack which is opened is the left-hand crack).

During the whole process of elastic attachment, the menisci have acted as springs, and the length $2l_o$ has increased without the appearance of any visible cracks, solely by the action of the meniscus with a value of $2w$.

The effect is therefore the same as if there existed, in addition to the elastic strain $R'/E_b$, an additional strain $\Delta \varepsilon' = 2w/2l_o = w/l_o$; $w$ is very small, but $l_o$ is also small. The smaller the value of $l_o$, the greater the value of $\Delta \varepsilon'$; but, in design, $\Delta \varepsilon'$ must be given its smallest value for safety, so that $l_o$ must have its greatest value. This value is the value which corresponds to bonding by friction at the time when cracking becomes visible.

The minimum value of $\Delta \varepsilon'$ would therefore be obtained by writing:

(i) That the depth of the meniscus reaches the limit value $w_1$ (about 0.02 mm), and that the opening of the crack is therefore $2w_1$; this width is calculated by BRICE’s theory, since it is the start of the friction phase.

(ii) That the value of the stress in the steel at right angles to the crack is equal to $\sigma'_a$ of Fig. 7.

Let $d$ be the diameter of the bar, $2l$ the length of the block, $R'_{adh}$. the bonding stress.

Then:

$$\frac{\pi}{4} d^2(\sigma'_a - mR') = \pi dl R'_{adh}.$$ 

Hence:

$$l = \frac{d (\sigma'_a - mR')} {4 R'_{adh}}.$$ 

As above, let $\sigma'_a = \varphi m R'$; let $R'_{adh} = \chi R'$, $\chi$ being the bonding coefficient, depending upon the nature of the surface of the bar, and the distance of the bar from the free faces.
Then:
\[
l = \frac{d}{4} (\varphi - 1) \frac{m}{\chi}
\]

(a)

Also, the depth \( w_1 \) of the meniscus is equal to the difference between the increase in length of the concrete and of the steel over the length \( l \).

The mean stress in the steel is
\[
\frac{mR' + \sigma'_a}{2} = mR' \frac{(1 + \varphi)}{2}
\]

The mean concrete stress is \( R'/2 \). Therefore:
\[
w_1 = \left( \frac{mR'(1 + \varphi)}{2E_a} - \frac{R'}{2E_b} \right) l
\]

Since \( E_a = mE_b \),
\[
w_1 = \left[ \frac{R'(1 + \varphi)}{2} - \frac{R'}{2} \right] \frac{l}{E_b} = \frac{\varphi R'l}{2E_b}
\]

(b)

By giving \( l \) its value (a):
\[
w_1 = \frac{\varphi R'}{2E_b} \times \frac{d}{4} (\varphi - 1) \frac{m}{\chi}
\]

or:
\[
\varphi^2 - \varphi - \frac{8E_b \chi}{mR'} \frac{w_1}{d} = 0
\]

(5)

Hence \( \varphi \) can be obtained.

Then:
\[
\Delta \varepsilon' = \frac{w_1}{l} = \frac{\varphi}{2} \frac{R'}{E_b}
\]

It can therefore be considered that, for the calculation of the moment which causes visible cracking, there is effectively an additional strain equal to \( \varphi/2 \) times the elastic strain \( R'/E_b \), \( \varphi \) being given by eqn. (5).

The total available strain before visible cracking is:
\[
\frac{R'}{E_b} \left( 1 + \frac{\varphi}{2} \right)
\]

Until this value of strain is reached, microcracking is not only invisible, but it is not dangerous, since the steel is protected by the meniscus (Fig. 10).
Consider the case of loads of short duration. Let:

\[ m = 6 \quad E_b = 350,000 \text{ kg/cm}^2 \quad R' = 30 \text{ kg/cm}^2 \quad w_1 = 0.02 \text{ mm} \]

Assume \( \chi = 1 \) and \( d = 12 \text{ mm} \).

Then:

\[
\frac{8E_b \chi w_1}{mR'} \frac{w_1}{d} = \frac{8 \times 350,000 \times 1}{6 \times 30} \times \frac{0.02}{12} = 26
\]

From eqn. (5), \( \varphi = 5.6 \). Therefore \( \Delta \varepsilon' = 2.8 \) \( (R'/E_b) \) \( (2.8 \text{ times the elastic strain}) \).

In the case of sustained loading \( m \), \( E_b \) and \( R' \) can be given the following values:

\[ m = 15 \quad E_b = 150,000 \text{ kg/cm}^2 \quad R' = 20 \text{ kg/cm}^2 \]

Then:

\[
\frac{8E_b \chi w_1}{mR'} \frac{w_1}{d} = \frac{8 \times 150,000 \times 1}{15 \times 20} \times \frac{0.02}{12} = 6.7
\]

Equation (5) gives \( \varphi = 3.1 \). Therefore

\[ \Delta \varepsilon' = 1.5 \frac{R'}{E_b} \]

\( (1.5 \text{ times the elastic strain}) \).

The total strain, equal to \( R'/E_b[1 + (\varphi/2)] \), is always less for sustained loading [in this case 20/150,000 \( (1 \times 1.5) = 3.3/10,000 \text{ instead of } 30/350,000 \]

\[ (1 + 2.8) = 3.8/10,000 \text{ for instantaneous loading}] \).

The bending moment at cracking is also less, because \( \varphi \) is less; conditions are therefore worst in this respect for sustained loading.

Since the limit state of cracking is of interest, especially for Class II (limited prestress), and since there is generally no tension under permanent loading for this class of work, the effects of temporary loading are the most significant.

With the above values of those parameters which determine \( \varphi \), it is found that \( \varphi = 5.6 \) \( (m = 6, \ E_b = 350,000 \text{ kg/cm}^2, \ R' = 30 \text{ kg/cm}^2, \ w_1 = 0.02 \text{ mm}, \ \chi = 1, \ d = 12 \text{ mm}) \). For other values of the parameters, different values of \( \varphi \) are obtained.

Now, \( \varphi \) can be neglected compared to \( \varphi^2 \) for guidance in eqn. (5), when assessing the effect of the parameters upon \( \varphi \).

\( \varphi \) varies approximately as

\[
\left( \frac{E_b \chi w_1}{mR'} \frac{1}{d} \right)^{1/2}
\]
If it is assumed that \( E_b/m \), \( R' \) and \( w_1 \) do not vary excessively, the principal parameters which affect \( \varphi \) are \( \chi \) and \( d \), and the order of magnitude of \( \varphi \) is given by \( 5.6(\chi/1 \times 1.2/d)^{1/2} \), where \( d \) is in cm; or \( \varphi \approx 6(\chi/d_{\text{cm}})^{1/2} \).

It can then be assumed that the total strain before cracking is approximately equal to:

\[
e' + \Delta e' = \frac{R'}{E_b} \left[ 1 \times 3 \left( \frac{\chi}{d_{\text{cm}}} \right)^{1/2} \right]
\]

The term \( R'/E_b \) is the elastic strain and the term \( R'/E_b \times 3(\chi/d_{\text{cm}})^{1/2} \) is the pseudo-plastic strain, for which an attempt to describe the physical significance is made in the foregoing.

This highlights the very significant effect of all the causes which can reduce it (in particular, the absence of links or stirrups when the reinforcement is very close to the face of the concrete). It also highlights the influence of the bar diameter, which must be kept as small as possible (see Chapter XI).

The formulae which are given above do not give any indication as to the minimum percentage of ordinary reinforcement which should be present. There is obviously a condition which controls this quantity, because the moments of resistance in section X (cracked) and section Y (not cracked) must be identical, and this implies a relation between the tensile force exerted by the reinforcement at section X and the resultant tensile stress in the concrete at section Y.

With the additional tension in the cables, and the fact that the moment arm is greater in the cracked section being considered, the requirement indicated in Section 5(b) for the minimum quantity of reinforcement seems reasonable, namely that the force in the reinforcement, stressed to 0.6 \( \sigma'_{ek} \), should be equal to half the concrete tensile stress resultant, calculated on the basis of an homogeneous section.

In certain cases, however, this requirement could be an underestimate (see Chapter VI, Section 2), and it is difficult to define a threshold where the reinforcement definitely starts to be useful. It obviously depends on the ratio between the moment which can be balanced by the reinforcement alone, which can be evaluated by assuming that the moment arm is the same as in reinforced concrete under simple bending (0.8\( h \) approximately), and the moment \( (I/v')R'' \), the additional moment in the section with respect to the moment caused by loss of compression.

Consequently, the reinforcement loses much of its usefulness if the ratio \( 0.8hA_d\sigma'_{ek}/(I/v')R'' \) is too low. Since \( I/v' = S(r^2/v') \), the ratio \( A_d/S \times 0.8h/(r^2/v') \times \sigma'_{ek}/R'' \) must have a minimum value.
Assuming that this ratio should not fall below \( \frac{1}{2} \) (for example), then the minimum percentage would be:

\[
\frac{A_a}{S} \text{min} = \frac{1}{2} \times \frac{r^2}{v'} \times \frac{0.8h}{R''} \times \frac{\sigma'_{ek}}{\sigma_{ek}}
\]

If \( r^2/v' = 0.25h, \ R'' = 40 \ \text{kg/cm}^2, \) and \( \sigma'_{ek} = 3600 \ \text{kg/cm}^2, \) then \( A_a/S \text{min} = 0.15\% \) approximately. This percentage could be lowered if \( R'' \) is reduced.

These are of course only empirical rules. They can be useful if it is obvious that the results obtained on the basis of the resultant stress are blatantly inadequate.

Consequently, a second rule for the minimum amount of reinforcing steel is taken to be as follows, in its general form (not expressed as a percentage), where \( z_{BA} \) is the reinforced concrete moment arm:

\[
z_{BA}A_a\sigma'_{ek} \geq \frac{1}{2} \frac{I}{v'} R''
\]

\( (I/v')R'' \) is a quantity which is hereafter called the specific cracking moment.

The rules applying to the minimum quantity of reinforcement required are therefore akin to taking for the cross-sectional area of the reinforcement the larger of the quantities defined by:

\[
A_a \times 0.6\sigma'_{ek} \geq \frac{1}{2} \times \text{tensile stress resultant}
\]

or:

\[
z_{BA}A_a\sigma'_{ek} \geq \frac{1}{2} \times \text{specific cracking moment}
\]

The spacing of the bars must be limited to about \( d + 6 \ \text{cm}, \) diameter of the meniscus carried by a bar. This figure is indicative, and is open to quite large tolerances; a spacing of 8 to 10 cm is acceptable.

However approximate, these minimum percentage and spacing rules determine very clearly the diameters and the arrangements of the reinforcement under normal conditions.

In special cases, such as complementary reinforcement in very thick walls, the rules are too costly or too limiting and the solutions are out of proportion with the general design. In such cases, good judgement is required.
Secondary Phenomena Accompanying Pseudo-plasticity

Consider a portion of concrete contained between two planes X containing microcracks and a plane Y at mid-distance between the planes X.

The stress diagrams at an X-plane are those shown in Fig. 7. The stress diagram at the Y-plane, with moment $M'$, is that shown in Fig. 6. It must now be modified so that the portion X-Y remains in equilibrium; the forces at plane Y must therefore correspond to the difference between diagrams (X) and (Y). These forces form a couple, since the normal force is not changed—equal to $M'' - M'$. The forces consist mainly of a tensile force equal to $A_d(\sigma' \bar{a} - mR')$, therefore equal to the bonding force between the steel and the concrete, and of a compressive force in the upper portion, since the compressive resultant at plane X is greater than at plane Y.

The steel/concrete bonding force gives rise to a tensile distribution, which must be resisted at plane Y. It cannot be resisted in the bottom portion, since the bottom fibre is already at its strength limit $R'$; it must therefore be resisted higher up within the beam. If the whole of the strength potential is absorbed, the whole of the lower portion of plane Y is at the uniform stress $R'$, as assumed in Fig. 9. In any event, the section at plane Y tends towards this condition.

![Diagram](image)

**Fig. 12.**

If the reinforcement is very close to the bottom face, the lines of tension have an upward resultant giving rise to vertical compressive forces local to the section at plane X. Towards the top, these forces cause the compression exerted by section X on section Y to deviate.

At section Y, the addition of these forces (tensile and compressive) to those which correspond to Fig. 6, results in a diagram (Fig. 13) which is wavy in appearance, but which can be diagrammatically replaced by the ideal shape shown in Fig. 9.
At section Y, the meeting-point of the lines of flow which emanate from planes X bounding the portion of concrete under consideration gives rise to vertical forces of opposite sign to the above, which are therefore generally tensile. Stirrups and links are useful for resisting these forces.

*Effects of shrinkage*

Shrinkage places the concrete in longitudinal tension, because of the resistance of the reinforcement to free contraction. The resulting tensile strain $\varepsilon'_{\rho}$ in the concrete must be deducted from the available strain as evaluated above.

If $\lambda$ is the percentage of reinforcement ($\lambda = A_u/S$), all of which is assumed to be local to the tensile face, it is seen in Chapter XI that:

$$\varepsilon'_{\rho} = \frac{50 K \lambda}{1 + 50 K \lambda} \eta \text{ approximately}$$

$\eta$ is the shrinkage strain, and $K$ is a factor of the form $[K = 1 + (v^2/r^2)]$, which is numerically equal to 4 with rectangular cross-sections and to 3 for an I-beam of normal proportions. Since only approximate values are being considered here, a mean value of $K$ equal to 3·5 can be assumed, and:

$$\varepsilon'_{\rho} = \frac{170 \lambda}{1 + 170 \lambda} \eta$$

In the case of Class II (limited prestress), now being considered, the percentage is low; it can nevertheless reach 0·1% ($\lambda = 0·001$). If $\eta = 4/10\,000$, the available strain is reduced by:

$$\varepsilon'_{\rho} = \frac{1·7 \times 0·1}{1 + 1·7 \times 0·1} \times \frac{4}{10\,000} = \frac{0·6}{10\,000}$$
representing about $\frac{1}{6}$ of the total strain $\varepsilon' + \Delta \varepsilon'$. For $m = 6$, $E_b = 350 \, 000$, $R' = 30 \, \text{kg/cm}^2$, $\chi = 1$ and $d = 12 \, \text{mm}$, the order of magnitude of the total strain $\varepsilon' + \Delta \varepsilon'$ is approximately $3.8/10 \, 000$.

There are considerable advantages therefore in using for reinforcement high strength steels with special bonding properties. The strain $\Delta \varepsilon'$ is increased [proportionately to $(\chi)^2$] and the percentage of steel can be reduced, so that the shrinkage effect is reduced.

With such steels, it is possible to have $\chi = 2$ and $\lambda = 0.07\%$ and the total strain $\varepsilon' + \Delta \varepsilon'$ can be as high as $4.5/10 \, 000$ before cracking occurs, and the proportion of the total strain which is offset by shrinkage can be reduced to $0.4/10 \, 000$; therefore, the available strain can be increased by about $30\%$, compared with previous results. It can then be equal to more than four times the elastic strain, of the order of $30/350 \, 000 = 0.85/10 \, 000$.

7. Limit state of cracking

From the foregoing, a better appreciation of microcracking phenomena is obtained. Due to these phenomena, complementary reinforcement permits a higher strain to be accommodated in the tensile zone, without risk. The phenomena are elastic, and they are therefore reversible. Microcracks, which are caused by exceptional loading, close up when the loading is removed.

Including the effect of the deformations of the concrete around each reinforcement bar, the total tensile strain before failure can be three or four times greater than the elastic strain. In order to define the limit state of cracking, it is necessary to apply a safety factor to the strain, and to define a stress diagram.

The Joint Committee of the FIP-CEB gives the symbol $C$ to the ratio of the design strain at tensile failure (where the safety factor is included) to the elastic strain $R'/E_b$.

The Soviet Code puts $C = 2$.

The FIP-CEB Committee recommends $C = 2.5$.

The most logical of the stress diagrams for the section is the one shown in Fig. 9. For simplification, rectangular diagrams are chosen to represent the tensile zone.

The Soviet Code uses the diagram shown in Fig. 14a, where the whole of the tensile zone is at the uniform stress $R'$.

The FIP-CEB Committee also takes a rectangular diagram, but with a uniform tensile stress $R'$ to a depth $y' = \frac{2}{3}x'$ for slabs and T-sections, and
to a depth $y' = x'$ for I-beams, where $x'$ is the depth of the tensile zone (Fig. 14b).

It is assumed that the coefficient C takes shrinkage into account.

Whichever definition is taken (Figs. 14a or 14b), the stress diagram is completely defined (by the depth $x'$ of the tensile zone) by equating the concrete stress resultant to the forces exerted by the steel (cables and reinforcement). For Class II (limited prestress), which this limit concerns, the forces exerted by the reinforcement can in any case be neglected in practice, so that the concrete stress resultant is effectively equal to the prestressing force. In order to calculate this force, a higher stress than the permanent stress can be taken, taking into consideration the increase in strain in the cables between the permanent state and the limit state of cracking.

![Fig. 14. Limit state of cracking. Recommended stress diagrams.](image)

The limit moment of cracking, $M_f$, whose origin is taken as at the prestressed condition (that is, the moment to be exerted by the external loading to achieve the limit state), is equal to the moment, taken about the cable, of the stresses so defined.

The limit state is thus completely defined by the value of the moment $M_f$, which is itself dependent on the concrete dimensions ($B$), on its tensile strength $R'$ and on the prestress.

The following inequality must be obtained for the section design:

$$\gamma_p M_p + \gamma_s M_s \leq M_f \left( B, \frac{R_{hk}'}{\gamma_b}, A_c \gamma_o T \right)$$

The FIP–CEB Committee accepts $\gamma_p = \gamma_s = \gamma_o = 1$. The inequality then becomes:

$$M_p + M_s \leq M_f \left( B, \frac{R_{hk}'}{\gamma_b}, A_c T \right)$$
The expression for the limit moment $M_f$ can be derived from the concept of the centre of compression, which is the point of action of the resultant of the stresses in the concrete.

Let the algebraic eccentricity of the centre of compression be $e_f$, and let that of the cable be $e_o$. Then $M_f = F(e_f - e_o)$ (it is assumed for simplicity that the prestressing force remains equal to the permanent force, without additional tension†).

It is useful to introduce the apparent strength given by the stress $R''$ when calculating $e_f$. It is the tensile strength which the concrete should possess for the moment $M_f$ to be reached elastically; that is, in the section which is assumed to be homogeneous, and with the normal stress diagram (see Fig. 9).

Then (Chapter VII):

$$e_f = \frac{r^2}{v'} \left(1 + \frac{|R''|}{\sigma_g}\right)$$

Let $R'' = kR';$ as is shown later, $k$ is a known coefficient, which depends on the shape of the section and on the mean prestress $\sigma_g$. The limit moment may then be calculated in the following manner.

Let $|e_o|$ be the absolute value of the prestress eccentricity.

Then:

$$M_f = F[|e_o| + e_f] = F \left[|e| + \frac{r^2}{v'} \left(1 + \frac{k|R'|}{\sigma_g}\right)\right]$$

$$= F \left(|e_o| + \frac{r^2}{v'}\right) + k \frac{|R'|}{\sigma_g} \times F \frac{r^2}{v'}$$

But $\sigma_g = F/S$; therefore:

$$k \frac{|R'|}{\sigma_g} F \frac{r^2}{v'} = k|R'| \frac{r^2}{v'} = k \frac{I}{v'} |R'|$$

Hence:

$$M_f = F \left(|e_o| + \frac{r^2}{v'}\right) + k \frac{I}{v'} |R'|$$

$F(|e_o| + r^2/v')$ is the moment due to loss of compression, bringing the centre of compression to the top boundary of the central core. $k(I/v')|R'|$ is the additional moment which initiates cracking. At first sight, formula

† See Chapter VII, Section 9, for the corrections to apply for the additional tensioning. They can be taken into account by calculating $r^2/v'$ in the section which is first made homogeneous.
(7) does not seem to provide any advantages over formula (6), but its practical interest is considerable, for the following reasons.

For a given section profile, \( k \) increases with \( \sigma_y \), but the variations of \( k \) are relatively small. It can therefore be assumed in practice that \( k \) is constant for any given profile. Then the value of \( k \) is that which corresponds to the case \( F = 0 \). \( k(I/v')|R'| \) is then the moment of resistance of the non-prestressed section, due account being taken (factor \( k \)) of pseudo-plastic phenomena.

Denoting the ratio \( \sigma_y/R' \) by \( \mu \):

(i) to assume that \( k \) is constant is an approximation, but in the expression for \( M_f \), the term \( k(I/v')|R'| \) is only a relatively small contribution to the total value of \( M_f \), so that an error in \( k(I/v')|R'| \) has only a small effect on \( M_f \);
(ii) instead of using the value of \( k \) for \( \sigma_y = 0 \), the value which corresponds to a mean value of \( \sigma_y \), and therefore of \( \mu \), can be taken (for Class II under consideration, \( \mu \) in practice varies between 1 and 3).

The complementary term \( k(I/v')|R'| \) is called the specific moment of cracking. This is the title which is used in the Soviet Code, where it is defined in a slightly different manner.

It is preferable to define a specific modulus of cracking, \( k(I/v') \), denoted by \( \overline{W}' \). It is then possible to tabulate the values of \( \overline{W}' \) for various sections.

It is now possible to write: \( M_f = (\text{moment due to loss of compression}) + (\text{specific moment of cracking}) \), or, with the safety factors:

\[
M_p + M_s \leq \text{moment due to loss of compression} + \overline{W}' \frac{R'_{bk}}{\gamma'_{b}} \quad (8)
\]

This can also be written:

moment of cracking Class II = moment of cracking Class I

\[
+ \overline{W}' \frac{R'_{bk}}{\gamma'_{b}} \quad (9)
\]

The limit state which is thus defined implies that a minimum quantity of complementary reinforcement is provided (capable of balancing half the resultant of the concrete tensile stresses when stressed to 60\% of the elastic limit of the steel).

Example: Rectangular cross-section. Code of the FIP–CEB Committee (\( C = 2.5 \), \( y' = \frac{1}{3}x' \)).

Determine the position of the neutral axis by equating the resultant of the
stresses in the concrete to the prestressing force F (see Fig. 14b). If $\sigma$ is the top fibre stress, then

$$\frac{x'}{x} = \frac{CR'/E_b}{\sigma/E_b}$$

or: $\sigma = CR'(x/x')$.

In this case, therefore, $\sigma = 2.5 \times R'(x/x')$ (R' has its absolute value).

Also:

$$-\frac{3}{4} bx'R' + \frac{1}{2} bx \times 2.5 R' \frac{x}{x'} = F$$

Let

$$x = \alpha h, \quad x' = \alpha' h \quad (\alpha + \alpha' = 1)$$

Then:

$$R_{bh}' \left( -0.75 \alpha' + 1.25 \frac{\alpha^2}{\alpha'} \right) = F$$

Also:

$$F = bh \sigma_g \quad \text{Let} \quad \sigma_g = \mu R'$$

Hence:

$$1.25 \alpha^2 - 0.75 \alpha'^2 = \mu \alpha'$$

or:

$$1.25(1 - \alpha')^2 - 0.75 \alpha'^2 = \mu \alpha'$$

The position of the neutral axis is determined by this equation, and it depends only on the ratio $\mu$ between the mean prestress $\sigma_g$ and the tensile strength $R'$. The equation can be written:

$$\alpha'^2 - 2(\mu + 2.5) \alpha' + 2.5 = 0$$

or

$$\alpha' = \mu + 2.5 - [(\mu + 2.5)^2 - 2.5]^\frac{1}{2} \quad \text{(a)}$$

The point of application of the resultant of the stresses in the concrete is now determined; let $u$ be its distance from the bottom fibre. The moment due to the stresses about the bottom fibre is:

$$F_u = -\frac{3}{4} bx'R' \times \frac{3}{8} x' + \frac{2.5}{2} R' \frac{x}{x'} bx \left( h - \frac{x}{3} \right)$$

$$= bh^2 R' \left[ \frac{2.5}{2} \frac{\alpha^2}{\alpha'} \left( 1 - \frac{\alpha}{3} \right) - \frac{9}{32} \alpha'^2 \right]$$

$$= bh^2 R' \left[ \frac{2.5}{2} \frac{(1 - \alpha')^2}{\alpha'} \frac{2 + \alpha'}{3} - \frac{9}{32} \alpha'^2 \right]$$
Hence:
\[
u = \frac{bh^2R'}{F} \left[ 2.5(1 - \alpha')^2(2 + \alpha') \frac{9}{32} \alpha'^2 \right]
\]

\[
\frac{bh^2R'}{F} = \frac{bh^2R'}{b\sigma_g} = \frac{h}{\mu}
\]

It is finally found that:
\[
u = \frac{h}{6\mu} \left[ 2.5 \left( \frac{2}{\alpha'} - 3 \right) + 0.812 5\alpha'^2 \right]
\] (b)

Also, if \( R'' \) is the apparent tensile strength (absolute value), the eccentricity of the centre of compression is:
\[
e_f = \frac{h}{6} \left( 1 + \frac{R''}{\sigma_g} \right) = \frac{h}{6} \left( 1 + k \frac{R'}{\sigma_g} \right) = \frac{h}{6} \left( 1 + \frac{k}{\mu} \right)
\]

Hence:
\[
u = \frac{h}{2} + \frac{h}{6} \left( 1 + \frac{k}{\mu} \right) = \frac{h}{6\mu} (4\mu + k)
\] (c)

From (b) and (c):
\[
k = 2.5 \left( \frac{2}{\alpha'} - 3 \right) + 0.812 5\alpha'^2 - 4\mu
\]

For various values of \( \mu (\mu = \sigma_g/R') \), it is found that:

<table>
<thead>
<tr>
<th>( \mu = \frac{\sigma_g}{R'} )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>0.564</td>
<td>0.451</td>
<td>0.377</td>
<td>0.287</td>
<td>0.232</td>
</tr>
<tr>
<td>( k )</td>
<td>1.62</td>
<td>1.75</td>
<td>1.88</td>
<td>2</td>
<td>2.10</td>
</tr>
</tbody>
</table>

By substituting these values of \( k \) in formula (7), the 'exact' values of the limit moment of cracking would be obtained.

It can now be shown that, as stated above, the value of \( k \) can be considered as constant for practical purposes. In practice, \( \mu \) varies between 1 and 3. Therefore, take \( k = 1.88 \) (approximately, and on the safe side).
The corresponding error in $M_f$ is small. The expression for $M_f$ can be written:

$$M_f = F \left( |e_o| + \frac{h}{6} \right) + k \frac{bh^2}{6} R'$$

Since $F = bh\sigma_o = bhR'\mu$,

$$M_f = \frac{bh^2R'}{6} \left[ \mu \left( \frac{6|e_o|}{h} + 1 \right) + k \right]$$

For a given value of $\mu$, the lower the value of $|e_o|$, the greater is the relative error in $M_f$. The worst case is with central prestressing ($e_o = 0$). The relative error is then equal to the relative error of the term in brackets, namely:

For $\mu = 1$, $\mu + 1.88 = 2.88$; exact value 2.88
- $\mu = 2$, $\mu + 1.88 = 3.88$; exact value 4, error 3%
- $\mu = 3$, $\mu + 1.88 = 4.88$; exact value 5.10, error 4.5%

For a beam in bending, where $|e_o|$ is usually greater than $h/6$, the errors in $M_f$ do not exceed 2%.

The constant value $k = 1.88$ can therefore be used, the specific modulus of cracking being equal to $\overline{W'} = 1.88(bh^2/6) = 0.314bh^2$.

**Comments**

(i) Algebraically,

$$M_f = F \left( \frac{h}{6} - e_o \right) + k \frac{bh^2}{6} R'$$

In certain exceptional cases, such as reversal of bending moments, $h/6 - e_o$ could be small; the relative error through using a value of $k$ which is constant would be higher than that quoted above. The 'exact' $k$ values could then be used.

(ii) With the Soviet Code ($C = 2$, $y' = x'$), it is found that:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha'$</td>
<td>0.5</td>
<td>0.333</td>
<td>0.25</td>
<td>0.2</td>
</tr>
<tr>
<td>$k$</td>
<td>1.75</td>
<td>1.89</td>
<td>1.94</td>
<td>1.96</td>
</tr>
</tbody>
</table>
The variations in the values of $k$ are less, but the values of the specific moment of cracking (or of the modulus $W'$) are very close to the values obtained by the FIP–CEB Code.

(iii) The additional tension in the cables (and the presence of complementary steel) can be taken into consideration by making the section homogeneous by using a suitable $m$-value, which changes with the values for $e_o$ and $r^2/v'$; they may also be taken into consideration by calculating the increase in the tensile strain in the concrete local to the steel between the permanent state and the limit state of cracking.

(iv) For $\mu = 0$ (section without prestress), it has been seen that $k = 1.62$. $R'_{b}$ being the true tensile strength, the tensile bending strength is then $R''_{b} = 1.62R'_{b}$. This is in agreement with the normal practice of taking $R''_{b} = 0.6 \times R'_{b}$, and justifies the factors adopted by the FIP–CEB Committee.

If $k$ were given the constant value 1.62 (instead of 1.88), the limit moment of cracking $M_f$ would be equal to:

$$(\text{moment due to loss of compression}) + \frac{bh^2}{6} \times (\text{tensile bending strength})$$

This formula is very nearly self-evident.

In fact, the apparent tensile strength increases with $\sigma_g$. For rectangular cross-sections, with the $k$-values given above, it can be approximately written that:

$$k = 1.6 + \frac{\mu}{6} \quad \text{or} \quad R'' = 1.6R' + \frac{\sigma_g}{6}$$

With $\sigma_g = 60$ kg/cm², the apparent bending strength would therefore increase by 10 kg/cm². Similar calculations can be considered for a cross-section of any profile (see Chapter X).

The ratio $k$ decreases as the depth of the section increases, and it tends towards 1 when the thicknesses of the web and the flange are very small.

8. Experimental checks

Numerous bending tests, where extensometer measurements on the tensile concrete face have been taken, provide a qualitative check of the pseudoplastic phenomena which precede cracking. It is found that the strains are much greater than the elastic strain $R'/E_b$ when well-distributed complementary reinforcement is present in the beam.

It is particularly recommended that reference should be made to the author’s book *Prestressed Concrete*, Volume 1 (Chapter XVI: Cracking
Tests) and to a paper by CESTELLI-GUIDI (Report of the Giornale del Cemento Armado Precompression, Venice 1963), where descriptions of tests can be found, with many strain/load diagrams.

With the strains on the abscissa and the loads as ordinates, the diagrams, which are at first linear (strain proportional to load), change their shape under a load which corresponds approximately to the stress R' in the elastic theory. This change of slope provides a check on the change in behaviour, without any visible cracking occurring when viewed through instruments with very large magnification (capable of detecting 1/100 mm). This continues for some time, then the slope \( \Delta \sigma / \Delta \varepsilon \) decreases very rapidly until it is nearly zero as cracking becomes imminent.

Unfortunately, it is extremely difficult to observe the 'exact' value of the strain \( \varepsilon' f \) at which cracking occurs, because of the inherent error which exists at the threshold of visibility; the shorter the effective length of the extensometer, the greater is the error. The C-values which are deduced from the tests are therefore open to discussion.

On the other hand, the value of the tensile strength \( R'' \) is obtained with sufficient accuracy, for the converse reason (namely, small increases in the loading as cracking is approached). This strength \( R'' \) is the tensile stress which would exist in the concrete at the time of crack initiation if the phenomena were elastic. These values are generally much greater than the prism strength \( R'_p \), provided that the beams contain complementary reinforcement. This provides indirect confirmation of the pseudo-plastic phenomena.

The values obtained from \( R'' \) with and without complementary steel are summarised in the author's book *Prestressed Concrete*, Chapter XVII. With beams of rectangular cross-section, dealt with in Chapter XVI of the book, the results given below were obtained with beams provided with complementary steel, where every effort was made to obtain precise values of \( R'' \) by taking various corrections into consideration, the details of which can be found in *Prestressed Concrete*, Chapter XVI.

<table>
<thead>
<tr>
<th>Beams</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>A₀</th>
<th>D₀</th>
<th>kg/cm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean prestress ( \sigma_0 )</td>
<td>65·5</td>
<td>43·5</td>
<td>43·5</td>
<td>65·5</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( R'' )</td>
<td>99·6</td>
<td>76·8</td>
<td>66·2</td>
<td>91·3</td>
<td>51·4</td>
<td>43·9</td>
<td>kg/cm²</td>
</tr>
<tr>
<td>( k = R''/R' )</td>
<td>2·76</td>
<td>2·13</td>
<td>1·83</td>
<td>2·53</td>
<td>1·43</td>
<td>1·22</td>
<td></td>
</tr>
</tbody>
</table>

It should be noted that beams \( A₀ \) and \( D₀ \) are the same beams as beams \( A \) and \( D \) previously tested, which were then re-tested after removal of
Fig. 15. Cracking tests under varying prestress and percentage reinforcement. (Extract from Cestelli-Guido and Radogna, *Giornale del Precompreso*, Venice 1963.) For each beam group, the diagrams corresponding to the greatest and to the smallest strains have been taken.
prestress, at a zone which had not yet cracked; there were, however, some cracks very near to the loading point, which throws doubt on the value of the test and which may explain the reason for the small \( k \)-values which were obtained (see Fig. 2 of Chapter XVI mentioned above). It should also be noted that the values of \( R' \), measured on prisms [by applying the formula \( 3-6(M/bh^2) \)] gave a spread of \( \pm 10\% \) relative to the mean value.

With particular reference to tests A, B, C, D:

\[
\text{for } \sigma_y = 65.5 \text{ kg/cm}^2 \left( \mu = \frac{\sigma_y}{R'} = 1.8 \right) \quad k = 2.53, 2.76
\]

\[
\text{for } \sigma_y = 43.5 \text{ kg/cm}^2 \ (\mu = 1.2) \quad k = 1.83, 2.13
\]

Compare these values with those resulting from Section 7:

- if it is assumed that \( C = 2.5 \), then, for \( \mu = 1.8, k = 2 \text{ approx.} \)
  - for \( \mu = 1.2, k = 1.9 \)

- if it is assumed that \( C = 2 \), then, for \( \mu = 1.8, k = 1.93 \)
  - for \( \mu = 1.2, k = 1.9 \)

These values are very close to the values obtained in the test.

For \( \mu = 0 \), then \( k = 1.62 \) or \( 1.75 \) with \( C = 2.5 \) or \( 2 \); the test values are a little low, but a probable reason for this has been given above.

A reasonable estimate of the possibility of cracking is therefore possible by using \( C = 2.5 \) (\( C = 2 \) only slightly affects the results). It is understood that this relates to visible cracking; harmless and reversible microcracking is not excluded in Class II structures. This is the only aim when designing to Class II standards; by using an adequate safety factor for \( R' \), structures which are so designed behave satisfactorily; USSR applications designed on these lines are a good example.

The apparent strength against cracking increases with the mean prestress \( \sigma_y \), and this is in agreement with the theoretical analysis. This influence, however, was overestimated in the author's book *Prestressed Concrete*, Chapter XVI.

Experimental confirmation is limited here to these few remarks, and the definition of the limit state given in Section 7 is considered to be justified.

### III. SAFETY AGAINST FAILURE IN BENDING

9. Limit state of failure under maximum loading. General formulae
The following formulae and procedures are based on the assumption that the cables are well bonded with good-quality grout. When the cables are
not bonded, the bending moment at failure is reduced by at least 20%, and often by more.

For simplicity, it is assumed that the fibre in compression is the top fibre.

Consider first the case of a beam which is subjected to simple bending; that is, with no external axial loading and without any non-tensioned complementary steel.

The concrete is heavily cracked at the time of failure. Let \( x \) be the depth of concrete above the crack at the section of rupture. It is assumed that the tensile strength is zero; the only stresses which are to be considered in the concrete are the compressive stresses throughout the depth \( x \).

![Diagram of Compressive Stress](image)

**Fig. 16.** Compressive stress diagram. Parameters affecting the bending moment at failure.

The compressive strain in any fibre is proportional to the distance between that fibre and the neutral fibre. The stress diagram over the depth \( x \) therefore assumes, to a certain scale, as explained in Section 2, the same shape as the stress–strain diagrams of Chapter II, which are approximately parabolic.† It can be shown that it is approximately equivalent to a rectangular diagram of depth \( g = \frac{3}{4}x \) of uniform stress \( R \) (the failure stress), without changing the compressive resultant or its position.

Let it also be assumed that the width of the section is uniform over the depth \( x \) (T or I-beams where the neutral axis lies within the compression zone). If the width is \( b \), the compressive stress resultant force is equal to \( byR = \frac{3}{4}bxR \).

† The apex of the parabola is just below the top fibre (by about 0.4\( x \)) with the type of concrete which is used in prestressing work.
Let $T$ be the stress in the prestressing steel (if this steel is in several layers, it is assumed that their mean stress can be substituted for their separate stresses),† and let $A_c$ be the total cross-sectional area of this steel. The force which is exerted is $A_cT$.

Let $T_r$ be the steel breaking stress, and let $\lambda = T/T_r$ (stress coefficient).

Let $M_r$ be the moment at rupture. This moment is balanced by the resultant of the concrete compressive forces, $byR$, and by the tensile force $A_cT$. The distance between these two forces is $h_1 - (y/2)$, where $h_1$ is the effective depth (distance of the steel from the top fibre).

Hence two equations can be written:

$$byR = A_cT = \lambda A_cT_r \quad \text{(equilibrium of resultants)}$$

and‡

$$M_r = A_cT \left( h_1 - \frac{y}{2} \right) = h_1 \lambda A_cT_r \left( 1 - \frac{y}{2h_1} \right)$$

From the first equation:

$$y = \lambda \frac{A_cT_r}{bR}$$

or:

$$\frac{y}{h_1} = \lambda \frac{A_cT_r}{bh_1R} \quad (1)$$

$A_cT_r/bh_1R$ is the ratio of the steel rupture strength to the bending compression strength of the concrete. This ratio is denoted by $\sigma$ and is called the 'mechanical percentage' to distinguish it from the geometrical percentage.

Therefore:

$$\begin{align*}
\frac{h}{y_1} &= \lambda \sigma \\
&= (2)
\end{align*}$$

whence:

$$M_r = \lambda h_1 A_cT_r \left( 1 - \frac{\sigma}{2} \right) \quad (3)$$

† See Section 12 for the case in which the cables are located at widely different levels.

‡ Since $A_cT = byR$, it could equally well be written that:

$$M_r = byR \left( h_1 - \frac{y}{2} \right) = bh_1^2R \frac{y}{y_1} \left( 1 - \frac{y}{2h_1} \right) = bh_1^2R \lambda \sigma \left( 1 - \frac{\sigma}{2} \right) \quad (3a)$$

This formula yields less information than formula (3). The latter brings out the fact that the important factors are the cable breaking forces $A_cT_r$ and the useful depth. In most cases $\lambda$ is very nearly equal to 1, and $\sigma$ is small, so that $M_r$ is approximately equal to $h_1A_cT_r$. This is more evident from formula (3) than from formula (3a).
The value of $\lambda$ is determined by consideration of the deformations.

Let $T_o$ be the permanent stress under working conditions, after all losses have occurred.

Let: $\varepsilon'_o$ be the corresponding tensile strain in the steel,

$\varepsilon'$ be the strain due to the stress $T$,

$\varepsilon_o$ be the compressive strain in the concrete when the external moment is zero, that is, due to the effect of the prestress alone,

$\varepsilon_r$ be the compressive strain in the concrete at failure.

It is assumed that, effectively, the concrete compressive strain is reached. This is justified in practice by the fact that it is always the concrete which fails and never the steel, except where low-strength steel is used, and this is not recommended.†

![Diagram](image)

**Fig. 17.**

Due to the moment $M_r$, the increase in the concrete compressive strain is $\varepsilon_r - \varepsilon_o$; the increase of tensile strain in the steel is $\varepsilon' - \varepsilon'_o$. Since the deformations are assumed to be linear:

\[
\frac{\varepsilon_r - \varepsilon_o}{\varepsilon' - \varepsilon'_o} = \frac{x}{h_1 - x}
\]  

(4)

† Except for beams with alternating loads, where cracks would open and close many times (Section 15); fatigue failure of the steel could then occur. However, only the moment of rupture for a statically-loaded beam is concerned here.

Where alternating loading is applied, very strict precautions are necessary to safeguard against cracking (Section 5). See also Section 16 for sudden failure; this, however, is a totally different case.
The stress in the steel is not strictly $T_o$ (permanent stress) when the moment is zero, since it is achieved only after all the losses have taken place, and since, under zero moment, the long-term strain corresponding to the compression local to the cable due to the moment $M_p$ has not yet occurred. With zero bending moment, or virtually with pure prestress, which cannot be considered in isolation as stated in Chapter I, the tension would be $T_o \times m\sigma_{cp}$, $m$ being the modular ratio for long-term deformations ($m = 10$). If $\sigma_{cp} = 100$ kg/cm$^2$, $m\sigma_{cp} = 1000$ kg/cm$^2$. The corresponding increase in strain $\Delta\varepsilon'_o$ is of the order of 0.5/1000, which can generally be neglected when compared to $\varepsilon'$.

The term $\varepsilon_o$ is generally small in comparison with the compressive rupture strain $\varepsilon_r$; it is most often a tensile strain, caused by a relatively small tensile stress.

It has been shown experimentally (tests by Billiet and Appleton), that the value of the moment at rupture is practically unaffected by variations of $\varepsilon_r$ about its mean value of 3.5/1000. Under these conditions, there is no advantage in considering the term $\varepsilon_o$.

By neglecting $\varepsilon_o$ and $\Delta\varepsilon'_o$, therefore:

$$
\frac{\varepsilon_r}{\varepsilon' - \varepsilon'_o} = \frac{x}{h_1 - x}
$$

Since $x = \frac{4}{3}y$, then

$$
\frac{y}{h_1} = \frac{3}{4} \frac{\varepsilon_r}{\varepsilon_r + \varepsilon' - \varepsilon'_o}
$$

(5)

If the ratio $T_o/T_r$ is denoted by $\lambda_o$ (coefficient of initial stress), $\varepsilon'$ and $\varepsilon'_o$ are obtained as a function of $T_o$ and $T$ with the stress-strain diagram for the steel under consideration. If the relation between $\varepsilon'$ and $T$ is written as $\varepsilon' = F(\lambda)$, eqn. (5) can be written:

$$
\frac{y}{h_1} = \frac{3}{4} \frac{\varepsilon_r}{\varepsilon_r \times F(\lambda) - F(\lambda_o)}
$$

(6)

The problem is solved with eqns. (2), (3) and (6), written as follows:

$$
\left\{
\begin{array}{l}
\frac{y}{h_1} = \lambda \sigma \\
M_r = \lambda h_1 A_c T_r \left(1 - \frac{\sigma}{2}\right)
\end{array}
\right.
$$

(2)

$$
\left\{
\begin{array}{l}
y = \frac{3}{4} \frac{\varepsilon_r}{\varepsilon_r + F(\lambda) - F(\lambda_o)}
\end{array}
\right.
$$

(6)
Fig. 18.

Fig. 19.
The value 3·5/1 000 can be assumed for $\varepsilon_r$.

$y/h_1$ and $\lambda$ are given by eqns. (2) and (6); the moment at rupture is given by eqn. (3).

Equations (2) and (6) can be solved by a series of approximations,† or graphically.

The method suggested by the author is given below (Prestressed Concrete, Volume II, Chapter XXIX, Section 3).

It is required to find the values of $\lambda$ and $y/h_1$. Take these properties as co-ordinates.

† The method which is summarised in Fig. 20 is given by the ASP. By giving various values to $x$, the position of the neutral fibre is obtained with successive approximations. The value of $x$ is indeed such that the resultant of the compressive forces in the concrete is equal to the total force which is applied by the steel (cables and complementary reinforcement). This method presupposes that the diagram shown in Fig. 21 consists of two straight lines.

N.B. The coefficient 0·87 is $(1/ya)(ya = 1·15)$.

\[
\Delta \sigma_a' = 70 \left( \frac{h_a}{x} - 1 \right) \text{ (kg/mm}^2) \text{ (Limit } 0·87 \sigma_{ok}' \text{)}
\]
With System I, a relation between $\lambda$ and $y/h_1$ is defined by eqn. (6). From the stress–strain diagram for the steel, a curve $\lambda = f(y/h_1)$ can be drawn, since $\lambda_c$ is known and $y/h_1$ can be calculated from eqn. (6) for any value of $\lambda$. Let this curve be called the ‘characteristic curve’ of the steel. Such a curve can be drawn for any type of steel (that is, for any stress–strain diagram). Point $(y/h_1, x)$ lies on this curve. Also, a straight line which passes through the origin is defined by eqn. (2). Its slope $\beta$ with respect to the $\lambda$-axis is such that $\tan \beta = (y/h_1)/\lambda = w$.

On the horizontal $\lambda = 1$, a metric graduation $w$ is marked to the same scale as that taken for $y/h_1$ on the bottom horizontal scale. The calculated value of $w$ for the beam under consideration is marked off. The line joining this value with the origin is the straight line (2). It intersects the characteristic curve at a point $m$, whose co-ordinates are the required values of $y/h_1$ and $\lambda$. The amount at rupture is then given by eqn. (3).

In the case of the stress–strain diagram shown by a full line in Fig. 21,

\[ \lambda = \frac{T}{T_r} \]

Fig. 21. Stress–strain diagrams for high-strength steels [$\varepsilon' = F(\lambda)$].

for example, for a steel with a rupture strength of 160 kg/mm², which is currently used in France, the graph shown in Fig. 22 enables the moment at rupture to be determined.
Introduction of safety factors for the calculation of the moment at failure
The design values for \( R \) and \( T \) are:

\[
R^* = \frac{R_{bk}}{\gamma_b} \quad T^* = \frac{T_{rk}}{\gamma_{or}}
\]

In general, \( \gamma_{or} = \gamma_a \) is used (same reduction coefficient as for ordinary steels).

The ratio \( \varpi \) is evaluated with the design values. Therefore

\[
\varpi = \frac{A_c T^*_r}{b h_1 R^*}
\]

Variation of the moment at failure with \( \varpi \)
Tables showing the moment at failure as a function of the mechanical percentage \( \varpi \) for a given steel can be drawn up by means of the graph shown in Fig. 22.

Fig. 22. Nomogram for the moment at failure.

Use of the nomogram. Calculate \( \varpi = A_c T^*_r/bh_1 R^* \). Mark off \( \varpi \) on the top horizontal scale (or on the right-hand side vertical scale for large values
of \( \omega \). Draw a straight line joining the point so obtained to the origin. This line cuts the characteristic curve for the steel at a point \( m \) of which the co-ordinates are the required values of \( y/h_1 \) and \( \lambda \).

**Example:** \( \omega = 0.24 \); \( \lambda_o \) (permanent tension) = 0.6 (i.e. \( T_o = 0.6T^* \)).

From the graph: \( \frac{y}{h_1} = 0.22 \), \( \lambda = 0.95 \), \( \lambda \omega = 0.21 \).

\[
M_r = h_1 A_c T^* r \times 0.95 \times \left( 1 - \frac{0.21}{2} \right) = 0.85h_1 A_c T^* r
\]

For the steel considered in Fig. 21 (diagram in full lines), Fig. 23a represents the variation of the moment at failure (the quantity \( M_r/h_1 A_c T^* r \)) with the mechanical percentage \( \omega \).

The straight line \( M_r/h_1 A_c T^* r = 1 - (\omega/2) \) is drawn, representing the approximate law, and valid on the safe side for values of \( \omega \leq 0.25 \).

Representative points obtained from 44 tests† are marked in the figure. The scatter is fairly wide, but, on the whole, the results agree fairly well with theory. Even closer agreement is found from tests conducted by Billiet and Appleton (*Journal of the American Concrete Institute*, June 1954) (Fig. 23b); the steels which were used correspond to the dotted line in Figure 21.‡

From these comparisons, it can be concluded that the proposed method provides a good assessment of the moment at failure.

Provided \( \omega \leq 0.25 \), the following law is satisfactory:

\[
M^* r = h_1 A_c T^* r \left( 1 - \frac{\omega}{2} \right)
\]  

(7)

Mechanical percentages do not usually exceed the value of 0.25 with statically-determinate beams.

The percentages \( \omega \) can attain higher values in the case of beams subjected to reverse bending moments, the cables being relatively close to the fibre under tension under the maximum absolute value of the bending moment,

† For comments on these tests, see *Prestressed Concrete*, Volume II, Chapter XXIX, Section 1. Only those tests where the characteristics of the steel and the strength of the concrete were reasonably well established are represented in Fig. 23. Even so, the information in many cases was not absolutely precise (tests on site), which partly explains the rather wide scatter band. The scatter is more pronounced with Billiet and Appleton's tests where the material characteristics were known precisely.

‡ Only the diagram for one particular steel is shown in Fig. 21 (steel II); the others are similar.
Fig. 23. Moments at failure—test results. (a) Various beams; (b) tests conducted by BILLIET and APPLETON.
and this fibre is then put into compression when the sign of the bending moment changes. In any event Figs. 22 and 23 enable the moment at failure to be evaluated for any value of the mechanical percentage.

10. Case of an external normal force $N$

The equation representing equilibrium of the resultants becomes:

$$byR = A_cT + N = \lambda A_cT_r + N$$

Hence:

$$\frac{y}{h_1} = \lambda \frac{A_cT_r}{bh_1R} + \frac{N}{bh_1R} = \lambda \sigma + \frac{N}{bh_1R}$$

Let

$$\sigma' = \frac{N}{bh_1R}$$

$$\frac{y}{h_1} = \lambda \sigma + \sigma'$$

(8)

Equation (6) applies:

$$\frac{y}{h_1} = \frac{3}{4} \varepsilon_r = \frac{\varepsilon_r}{4 \varepsilon_r + F(\lambda) - F(\lambda_o)}$$

(6)

It denotes that the point $[(y/h_1)\lambda]$ is on the characteristic curve. Equation (8) denotes that this point is on a straight line which is parallel to the line $0\sigma$ and which passes through the point $\sigma + \sigma'$ on the $\sigma$ scale (Fig. 24); in other words, for $\lambda = 1$. The following construction is adopted:

Points $\sigma$ and $\sigma + \sigma'$ are marked off on the horizontal $\sigma$ axis; through the point $\sigma + \sigma'$, a line parallel to $0\sigma$ is drawn. It intersects the characteristic curve at the required point $m$.

In general, it is sufficient to know the position of the point of intersection of the line of action of the compressive force $A_cT + N$ with the plane of the section (centre of compression at failure), and this enables the necessary reasoning and construction to be carried out; the centre of compression is at the mid-depth of the distance $y$.

If the concept of the moment of resistance is preferred, this moment is that which causes the compressive force $A_cT + N$ to pass through the preceding centre of compression. The condition of pure prestress acting alone is used as the datum for assessing the displacements, so that the moment $M = 0$. Therefore the displacement of the force $F = A_cT$ must
Fig. 24. Use of the failure nomogram in the case of a normal external force $N$:

$$\sigma' = \frac{N^*}{bh_1R^*}$$

be reckoned from its initial position (centroid of the cable), and the displacement of force $N$ must be measured from the centroid of the section. Therefore, using design values:

$$M_{*r} = \lambda A_c T_{*r} \left( h_1 - \frac{y}{2} \right) + N^* \left( v - \frac{y'}{2} \right)$$

If $\sigma + \sigma' \leq 0.25$, $\lambda = 1$ can be used. Therefore $y/h_1 = \sigma + \sigma'$, and:

$$M_{*r} = h_1 A_c T_{*r} \left( 1 - \frac{\sigma + \sigma'}{2} \right) + N^* \left[ v - \frac{h_1}{2} (\sigma + \sigma') \right]$$

Fig. 25.
11. Presence of non-tensioned complementary steel
This problem is met especially in the case of limited prestress or with reinforced prestressed concrete (Classes II and III). Let $A'_a$ be the cross-sectional area of the non-tensioned complementary steel and let $h_a$ be the distance from its centroid to the fibre in compression.

![Diagram](image)

**Fig. 26.**

It is assumed that the stress in the complementary steel reaches the elastic limit. This assumption is generally true, and the limits of the validity of the assumption are discussed later.

The mechanical percentage relative to the complementary steel is denoted by the ratio $\sigma_a = A'_a \sigma'_e / bh_a R$.

The equilibrium and deformation equations become:

\[ b y R = A_c T + A'_a \sigma'_e \]

\[ \epsilon' - \epsilon'_o = \epsilon_r \frac{h_1 - x}{x} \]

\[ M_r = A_c T \left( h_1 - \frac{y}{2} \right) + A'_a \sigma'_e \left( h_a - \frac{y}{2} \right) \]

Dividing both sides of the first of the above equations by $bh_1 R$:

\[ \frac{y}{h_1} = \frac{A_c T}{bh_1 R} + \frac{A'_a \sigma'_e}{bh_1 R} \]
Since
\[
\frac{A'_a \sigma'_e}{bh_1 R} = \omega_a \frac{h_a}{h_1}
\]
the equations may be written in the form:
\[
\begin{align*}
\frac{y}{h_1} &= \lambda \omega + \omega_a \frac{h_a}{h_1} \\
\varepsilon' - \varepsilon'_o &= \varepsilon_r \frac{h_1 - x}{x} \\
M_r &= \lambda A_c T_r \left( h_1 - \frac{y}{2} \right) + A'_a \sigma'_e \left( h_a - \frac{y}{2} \right)
\end{align*}
\]
(9) (10) (11)

This system of equations can be solved graphically in the same way as in Section 9.
Equation (10) again shows that the point \((y/h_1, \lambda)\) is on the characteristic curve.

\[\text{FIG. 27. Use of the failure nomogram when complementary steel is present:}\]
\[
\bar{\omega} = \frac{A'_a \sigma'_e e^*}{bh_a R^*}
\]
Equation (9) expresses that the point is on the straight line (D), whose equation is:

\[ \frac{y}{h_1} = \lambda \sigma + \frac{\sigma_a}{h_1} \]

If point \( \sigma \) is marked on the \( \sigma \) scale, as well as point \( \sigma + \sigma_a(h_a/h_1) \), this straight line is parallel to the line \( 0\sigma \) and intersects the line \( \lambda = 1 \) (the scale being drawn through the ordinate \( \lambda = 1 \)) at point \( \sigma + \sigma_a(h_a/h_1) \).

The point \( m \) is then obtained, and the values of \( y/h_1 \) and \( \lambda \) (Fig. 27) are deduced.

(The constructions must be done using design values.)

Discussion

It has been assumed that the stress in the complementary steel is \( \sigma'_{\text{ci}} \). The conditions under which this assumption is valid must be defined.

(i) Normal case. The cables and the complementary reinforcement are close to each other, local to the bottom fibre (fibre under tension).

In the permanent load condition, the bottom fibre is in compression. Let \( \sigma' \) be the resultant stress. The compressive strain is \( \sigma'/E_b \); the value of \( \sigma' \) can be of the order of 200 kg/cm², \( E_b = 150 000 \) kg/cm²; the compressive strain can be of the order of 200/150 000 = 1.3/1 000.

Under the same loading condition, the stresses at the top fibre are small, and the strain can be neglected when compared with \( \varepsilon_r \) (Fig. 28).

The strain diagram throughout the depth of the beam can be represented by line 1 of Fig. 28.

```
Fig. 28.
```
At failure, the compressive strain at the top fibre is \( \varepsilon_r \); the neutral fibre is at a distance \( x \) below the top fibre. The strain diagram at rupture is shown by line 2 of Fig. 28.

The strains in the complementary steel are equal to \( Aa_1 \) and \( Aa_2 \).

(a) If the complementary steel is mild steel, the value of the compressive strain in the steel in the permanent condition is very close to that of the strain at the bottom fibre. With a strain of the order of 1.3/1 000, to which a strain of about 0.3/1 000 due to shrinkage must be added, the limit of ductility in compression is reached and the stress is \(-\sigma'_e\) (the minus sign corresponding in this case to compression in the steel).

The strain to be taken up by the steel in going from condition 1 to condition 2, or \( a_1a_2 \), must cause the stress to go from \(-\sigma'_e\) to \(+\sigma'_e\).

Approximately,

\[
a_1a_2 = \varepsilon_r \frac{h_a - x}{x} + \frac{\sigma'}{E_b}
\]

It is required that

\[
a_1a_2 \geq \frac{2\sigma'_e}{E_a}
\]

Hence:

\[
\varepsilon_r \frac{h_a - x}{x} + \frac{\sigma'}{E_b} > \frac{2\sigma'_e}{E_a}
\]

or:

\[
\frac{h_a}{x} > 1 + \frac{2\sigma'_e}{E_a \varepsilon_r} - \frac{\sigma'}{E_b \varepsilon_r}
\]

Since \( y = \frac{3}{4}x \), and assuming \( h_a = h_1 \):

\[
\frac{y}{h_1} < \frac{0.75}{1 + 2(\sigma'_e/E_a \varepsilon_r) - (\sigma'/E_b \varepsilon_r)}
\]

\[
2\sigma'_e = \frac{2 \times 24 \text{ kg/mm}^2}{20 000 \times (3.5/1 000)} = 0.68
\]

\[
E_b \varepsilon_r = 150 000 \times \frac{3.5}{1 000} = 525 \text{ kg/cm}^2
\]

It is therefore necessary that, with \( \sigma' \) in kg/cm\(^2\):

\[
\frac{y}{h_1} < \frac{0.75}{1.68 - \sigma'/525}
\]
For:

\[
\sigma' = 150 \text{ kg/cm}^2, \quad \frac{y}{h_1} < 0.54
\]

\[
\sigma' = 180 \text{ kg/cm}^2, \quad \frac{y}{h_1} < 0.56
\]

\[
\sigma' = 200 \text{ kg/cm}^2, \quad \frac{y}{h_1} < 0.58
\]

Now, Fig. 27 shows that \( y/h_1 \) is less than \( \sigma' + \sigma_a(h_u/h_1) \). Therefore, providing \( \sigma' + \sigma_a(h_u/h_1) < 0.54 \), the elastic limit \( \sigma'_e \) is reached.

(b) If the complementary steel is a steel with good bonding characteristics, it does not exceed the elastic limit in compression in condition 1 (permanent load), since the compressive strain, including shrinkage, is of the order of \( 1.3/1000 + 0.3/1000 = 1.6/1000 \) and the elastic limit is about 40 to 50 kg/mm\(^2\) (corresponding strain = \( 40/20000 = 2/1000 \)).

As a result the portion of the strain \( a_1A \) in the concrete when the live load is applied restores a very low or zero stress in the steel.

It is only necessary to show that the tensile strain \( Aa_2 \) is greater than \( \sigma'_e/E_a \).

Now:

\[
Aa_2 = \varepsilon_r \frac{h_a - x}{x}
\]

The elastic limit is therefore reached in the complementary steel if:

\[
\frac{h_a}{x} - 1 > \frac{\sigma'_e}{E_a\varepsilon_r}
\]

Let

\[
\sigma'_e = 50 \text{ kg/mm}^2 \quad E_a\varepsilon_r = 20,000 \times \frac{3.5}{1000} = 70 \text{ kg/mm}^2 \quad \frac{\sigma'_e}{E_a\varepsilon_r} = \frac{50}{70} = 0.71
\]

The original assumption is therefore valid if \( x/h_a < 1/1.71 \), or, since \( y = \frac{1}{4}x \) (and taking \( h_a \) equal to \( h_1 \)), if:

\[
\frac{y}{h_1} < \frac{0.75}{1.71}
\]

\[
\frac{y}{h_1} < 0.44
\]

For the same reasons as with mild steel this condition is realised if \( \sigma' + \sigma_a(h_u/h_1) < 0.44 \).
These total values of the mechanical percentage (0·54 and 0·44) are high, greater than the usual values for statically-determinate beams (at least when the beams are subjected to bending moments of constant sign).

Usually, therefore, it is certain that the elastic limit will be reached in the complementary reinforcement.

(ii) Case in which the cables and the complementary reinforcement are at widely different levels. In practice, this case is met only with sections subjected to moments of differing sizes.

Consider, for example, a section which can be subjected to a positive moment \(+M\) and to a negative moment \(-M'\). Assume that the absolute value of \(-M'\) is much greater than the absolute value of \(+M\). It is necessary to locate the cable fairly close to the top fibre, in order to provide sufficient resistance to the moment \(-M'\).

![Diagram](image)

**Fig. 29.**

The section may be inadequately prestressed as far as \(M\) is concerned; when rupture occurs under the positive moment (\(M\)), the cable does not extend significantly, and it is necessary to increase the moment of resistance by introducing some complementary steel located as low as possible within the section; \(h_a\) and \(h_1\) are then quite different, and it is required to determine the condition for which the elastic limit \(\sigma'_e\) is reached in the complementary steel.

In this case, contrary to Fig. 28, the strains in the permanent condition are small at the bottom of the section.

Figure 30 shows that the assumption that \(\sigma'_e\) is reached at failure is valid if:

\[
\frac{h_a - x}{x} \frac{\sigma'_e}{E_a} > \epsilon_r
\]
This can be written with \( y = \frac{3}{4}x \):

\[
\frac{y}{h_a} < \frac{0.75}{1 + (\sigma'_e/E_a \varepsilon_r)}
\]

It is found that the following is necessary:

with mild steel \((\sigma'_e = 24 \text{ kg/mm}^2)\): \(\frac{y}{h_a} < 0.55\)

with high-strength deformed steel \((\sigma'_e = 50 \text{ kg/mm}^2)\): \(\frac{y}{h_a} < 0.44\)

Steels of the latter type are most generally used in this application. Construction of Fig. 27 enables \(y/h_1\), and therefore \(y\), to be determined, so that the check is simple.

It is in any event extremely rare, even in this case, for the elastic limit not to be reached in the complementary steel; \(h_a\) is, in fact, roughly equal to \(h\), the total depth of the section. It would be necessary, for the elastic limit not to be reached, to have \(y > 0.55h\) in the case of mild steel, or \(y > 0.44h\) in the case of high-strength deformed steel, which is exceptional.

**Approximate formula**

In most cases, \(\sigma + \sigma_a(h_a/h_1)\) is less than 0.25, and the value of \(\lambda\) is approximately unity (that is, the ultimate stress in the cable is reached).
In general, it is sufficiently accurate to use the following formula for the moment at failure:

$$M^* = h_1 A_c T^* r \left[ 1 - \frac{1}{2} \left( w + w_a \frac{h_a}{h_1} \right) \right] + h_a A'_a \sigma'_e \left[ 1 - \frac{1}{2} \left( w \frac{h_1}{h_a} + w_a \right) \right]$$  \hspace{1cm} (12)

where:

$$w = \frac{A_c T^* r}{b h_1 R^*}, \quad w_a = \frac{A'_a \sigma'_e}{b h_a R^*}, \quad R^* = \frac{R_{bk}}{\gamma_b}, \quad T^* = \frac{T_{rk}}{\gamma_a}, \quad \sigma'_e = \frac{\sigma'_e}{\gamma_a}$$

12. Case in which the cables are arranged at significantly different levels

Assume that the cables are arranged at levels $h_1$ and $h_2$. If $\varepsilon'_1$ and $\varepsilon'_2$ are the respective strains:

$$\frac{x}{h_1} = \frac{\varepsilon_r}{\varepsilon_r + \varepsilon'_1 - \varepsilon'_0}, \quad \frac{x}{h_2} = \frac{\varepsilon_r}{\varepsilon_r + \varepsilon'_2 - \varepsilon'_0}$$

The representative points are on the characteristic curve; if they were known, the coefficients $\lambda_1$ and $\lambda_2$ would be known, and:

$$byR = A_{c1} T_1 + A_{c2} T_2$$  \hspace{1cm} (13)

Therefore:

$$M_r = \lambda_1 A_{c1} T_r \left( h_1 - \frac{y}{2} \right) + \lambda_2 A_{c2} T_r \left( h_2 - \frac{y}{2} \right)$$  \hspace{1cm} (14)

A value of $y$ can be tried which is close to the value which would be obtained by considering the mean cable coinciding with the centroid of the cables, and applying the method of Section 9; hence values of $y/h_1$ and $y/h_2$ are obtained. By taking these values as abscissae in Fig. 22, values of $\lambda_1$ and $\lambda_2$ are obtained, and $y$ is varied until eqn. (13) is solved, by trial and error.
13. Section of any shape

The moment at rupture $M_r$ can be determined by a series of approximations, using the method illustrated in Fig. 20.

When the percentages $[(A_c T_r + A'_a \sigma'_e)/S]$ are small, an approximate method can be used by assuming that $\lambda = 1$.

Assuming also that the true stress diagram can be replaced with a rectangular diagram,† over a depth $y = \frac{3}{4}x$, and denoting by $S_b$ the area of the section in compression limited by the depth $y$, it is then necessary to have:

$$S_b R^* = A_c T^* + A'_a \sigma'_e$$

Hence $S_b$ is found and $y$ is determined.

Then, denoting the moment arms by $z_c$ and $z_a$ (distances of the cables and the complementary reinforcement from the centroid of $S_b$):

$$M^*_r = A_c T^*_c z_c + A'_a \sigma'_e z_a$$

The solution (either as above or through a series of approximations) is easily extended to the case where ordinary reinforcement is present in the concrete.

It is then assumed that the elastic limit is reached in the reinforcement. If $A_a$ is the cross-sectional area of the reinforcement:

$$S_b R^* + A_a \sigma^*_e = A_c T^*_r + A'_a \sigma'_e$$

† If the shape of the section does not permit the approximation $y = \frac{3}{4}x$ to be made, the parabolic stress diagram should be used (see Fig. 16).
If \( A_a = A'_a \) and \( \sigma_e = \sigma'_e \) (section with symmetrically arranged reinforcement), the forces in the complementary reinforcement are in equilibrium.

Then \( S_b R^* = A_c T^*_r \), and the moment of resistance is equal to the sum of the moment of resistance of the section without complementary reinforcement \( (A_c T^*_r z_c) \) and of the moment of resistance corresponding to the complementary reinforcement \( A'_a \sigma'_e \times (h - 2d) \), where \( h \) is the total depth of the section and \( d \) is the distance of the complementary reinforcement from the outside faces.

The method can be extended to the case where a normal external load \( N \) is present. Then, in the general case:
\[
S_b R^* + A_a \sigma^*_e = \lambda A_c T^*_r + A'_a \sigma'_e + N^*
\]

With a series of approximations, by giving different values to \( \lambda \) until both sides of the equation are equal, \( S_b \) is determined. Alternatively, \( S_b \) is found by using the approximate method (small percentages) by assuming that \( \lambda = 1 \).

**Case of a T or I-section**

This case is an example of the case considered previously. Again, with small percentages:
\[
S_b R^* = A_c T^*_r + A'_a \sigma^*_e
\]

Let \( u \) be the thickness of the flange, \( b \) its breadth, and \( b' \) the thickness of the web.

If \( S_b < bu \), the neutral fibre lies inside the flange and the problem is similar to that of a beam in which the flange in compression is of constant breadth, dealt with in the preceding sections.

If \( S_b > bu \), then \( S_b = b'y + (b - b')u \); hence \( y \), and \( M_r \), are obtained.

![Fig. 33.](image-url)
With higher percentages ($\sigma > 0.2$), the method of successive approximations must be used. If preferred, the characteristic curve of Fig. 22 may be used.

Considering only the case where there is no complementary reinforcement:

$$\frac{x}{h_1} = \frac{\varepsilon_r}{\varepsilon_r + \varepsilon' - \varepsilon'}_o \text{ (characteristic curve)}$$

and:

$$b'yR^* + (b - b')uR^* = A_cT^* = \lambda A_cT^*_r$$

or:

$$\frac{y}{h_1} = \frac{\lambda A_cT^*_r}{b'h_1R^*} - \frac{(b - b')u}{b'h_1}$$

If the quantity $\omega_b' = A_cT^*_r/b'h_1R^*$ (Fig. 34) is marked on the $\omega$ scale, and if the quantity $\omega_u = -[(b - b')u]/b'h_1$ is marked from the point $\omega_b$, so obtained (negative; that is, towards the left), then a point E is obtained.

By arguments similar to those developed in Sections 9 and 10, the point $m$ with co-ordinates $y/h_1$ and $\lambda$ is obtained by drawing a straight line (D) through E, parallel to $O\omega_b$. Point $m$ is the intersection of the line (D) with the characteristic curve.
Hence:

\[ M^*_r = b'\gamma R^* \left( h_1 - \frac{y}{2} \right) + (b - b')uR^* \left( h_1 - \frac{u}{2} \right) \]

14. Check on safety. Overall safety factors

The section must be designed in such a way that, if \( p \) and \( s \) represent the conditions of loading:

\[ \gamma_p M_p + \gamma_s M_s + \text{etc.} \leq M^*_r \]  \hspace{1cm} (15)

**CEB and FIP-CEB Committee safety factors**

As regards loading, these committees give certain rulings for the choice of the factors \( \gamma \); but the usual value which is adopted for vertical loads is 1.4 for all types of loads. It is supposed in the following that the factors \( \gamma_p, \gamma_s, \) and so on, all have this uniform value.

As previously stated, characteristic strengths are considered for the evaluations of \( M^*_r, \) and they are reduced by applying to them a factor of the form \( 1/\gamma_m; \) the strengths which enter into the design value of \( M^*_r \) are thus:

\[ R^* = \frac{R_{bk}}{\gamma_b}, \quad T^*_r = \frac{T_{rk}}{\gamma_a}, \quad \sigma'_{ek} = \frac{\sigma'_{ek}}{\gamma_a} \ldots \]

The values adopted by the FIP-CEB Committee for the limit state of rupture are \( \gamma_b = 1.5 \) and \( \gamma_a = 1.15. \) The introduction of these reduced strengths reduces the value of the moment of resistance \( M_r \) when compared to its value when calculated on the basis of mean effective strengths; that is, under normal conditions.

If \( M_{rm} \) denotes the normal moment of resistance and if \( 1/\gamma_M \) denotes the reduction factor for this moment, resulting from the reductions in the strengths of the materials, then \( M^*_r = (1/\gamma_M)M_{rm}. \)

If \( \gamma_s \) is the uniform multiplying factor for the loads and if \( M \) is the moment due to the loads under normal conditions \((M = M_p + M_s + \text{etc.}), \) inequality (15) can be written:

\[ \gamma_s M = \frac{M_{rm}}{\gamma_M} \]

The overall safety factor can be denoted by \( \Gamma = \gamma_s \gamma_M. \)

This factor \( \Gamma \) is now evaluated, using the values given by the FIP-CEB Committee for the factors \( \gamma. \)
The values of the effective moments of resistance obtained in tests are very nearly equal to those that would be obtained by substituting in the expressions for $M_r$ in the preceding sections the mean strengths of the materials, the strength of concrete being that which is measured on cubes at the time of the tests.

With regard to these strengths, the following are introduced into the formula in order to obtain the design value $M^{*}_{r}$:

On the one hand, a reduced value for the concrete strength, this strength being evaluated from:

(a) the cylinder strength (reduction factor of the order of $0.83$);†
(b) the characteristic value for this strength, which introduces a further reduction factor $1 - \chi \delta$, of the order of $0.85$;
(c) the reduction factor $1/\gamma_b = 1/1.5$.

The concrete strength is in total reduced by a factor of

$$\frac{R^*}{R_{\text{cube}}} = \frac{0.83 \times 0.85}{1.5} = 0.47$$

on the other hand, a reduced value for the strength of the steel, in the ratio $1/1.15 = 0.87$ (it is accepted that the strength $T_r$ which is introduced into the calculation for the effective moment of resistance $M_{rm}$ is equal to $T_{rk}$).‡

As a result, the mechanical percentages $\sigma = A_c T_r / b h_1 R$ and $\sigma_a = A'_e \sigma'_e / b h_a R$ are increased; they are multiplied by $0.87/0.47$, or $1.9$ approximately.

Figures 22, 24, 27, 34 show that these increases in the mechanical percentages reduce the stress factor $\lambda$ relative to its effective value.

Since the formula for the moment at failure is of the form:

$$M_r = \lambda h_1 A_c T_r \left[ 1 - \frac{1}{2} \left( \lambda \sigma + \sigma_a \frac{h_a}{h_1} \right) \right] + h_a A'_e \sigma'_e \left[ 1 - \frac{1}{2} \left( \lambda \sigma + \frac{h_1}{h_a} + \sigma_a \right) \right]$$

the term $\lambda h_1 A_c T_r$ is affected by this reduction in $\lambda$ and by the reduction factor ($0.87$) which is applied to $T_r$ and $\sigma'_e$.

The effects of these reduction factors on the value of the moment at failure for different values of $\sigma$ (calculated from non-reduced values of cube strength), are shown in the table below, on the basis of Fig. 22.

† See Chapter II, Section 1.
‡ In other words, it is assumed that the factor for scatter is already included in the mean values of $T_r$. 
Values of \( \sigma \) (without reductions) & 0.2 & 0.4 & 0.6 \\
Corresponding values of \( \frac{M_r}{h_1A_cT_r} \) & 0.885 & 0.725 & 0.605 (1) \\
The percentages are multiplied by 1.9; they become & 0.38 & 0.76 & 1.14 (2) \\
Figure 22 is applicable: \( \frac{M_r}{h_1A_cT_r} \) becomes & 0.742 & 0.525 & 0.400 (3) \\
Reduction in \( \frac{M_r}{h_1A_cT_r} \) (line 3) becomes & 0.84 & 0.725 & 0.66 (4) \\
Applying the reduction factor 0.87 on \( T_r \), the total reduction factors are: & 0.73 & 0.63 & 0.57 (5) \\
Or, denoting the total reduction factor by \( \frac{1}{\gamma_M} \), & & & \\
\( \gamma_M = \) & 1.37 & 1.59 & 1.75 (6) \\

Therefore, with the FIP–CEB recommendations (\( \gamma_s = 1.4 \)):

for \( \sigma = 0.2 \), \( \Gamma = 1.37 \times 1.4 = 1.92 \)

\( \sigma = 0.4 \), \( \Gamma = 1.59 \times 1.4 = 2.22 \)

\( \sigma = 0.6 \), \( \Gamma = 1.75 \times 1.4 = 2.44 \)

(the values of \( \sigma \) being those corresponding to the cube strength, without reduction).

As expected, overall safety factors vary according to the various codes.

For example, the French Instructions of 1953 are generally more advantageous [except for very high values of the ratio (live load/permanent load)]. A reduction factor of 0.7 on mean concrete strength is specified, and 0.9 on the strength of steel. This increases the percentages \( \sigma \) in the ratio 0.9/0.7, or by nearly 1.3. The loading is increased in the ratio of 1 for permanent loads, and in the ratio of 2 for live loads.† The ‘safety factor’ is thus related to the ratio (live load/permanent load). The following comparison with the results given above for the FIP–CEB recommendations is obtained (intermediate calculations are not detailed):

† The Provisional Instructions of the 12th August 1965 reduce the margin factor for live loads from 2 to 1.8. The overall safety factors are therefore slightly less than those given in the text for the Instructions of 1953 (by 6 to 7% under normal conditions). On the whole, the conclusions are still valid.
\begin{table}
\centering
\begin{tabular}{lccc}
\hline
\( \omega = \) & 0.2 & 0.4 & 0.6 \\
\hline
Overall safety factors in accordance with FIP-CEB & 1.92 & 2.22 & 2.44 \\
\hline
Overall safety factors in accordance with 1953 instructions & s/p = 0.5 & 1.57 & 1.63 & 1.76 \\
\hline
s/p = 1 & 1.77 & 1.84 & 1.99 \\
\hline
s/p = 2 & 1.95 & 2.03 & 2.20 \\
\hline
\end{tabular}
\end{table}

The ASP specifies the same reduction factors for strength as the FIP-CEB Committee.

With regard to loading, it distinguishes between permanent loads \( (p) \), live loads \( (s) \) and wind loads \( (v) \), but it considers three types of live load \( s \): \( S_1 \) (static loading), \( S_2 \) (variable loading), \( S_3 \) (dynamic loading), and it takes \( S = S_1 + 1.15S_2 + 1.15\alpha S_3 \), where \( \alpha \) is a coefficient for dynamic effects which is defined by the relevant codes.

It is then necessary to check that the reduced moment of resistance \( (M_r) \) exceeds the greatest increased moment obtained from the following combinations:

\begin{equation}
1.4(M_p + M_s)
\end{equation}

\begin{equation}
0.9M_p + 1.4M_s
\end{equation}

\begin{equation}
0.9 \times 1.4(M_p + M_s + M_v)
\end{equation}

and, if \( M_p = 0 \),

\begin{equation}
1.4(M_s + M_v)
\end{equation}

On the whole, the results are in fair agreement with those of the CEB; combination (b) is brought in because, in some cases, with prestressed concrete, a reduction in the permanent loading can be a disadvantage; combination (c) introduces the factor 0.9 to take into account the low probability that all the loads will assume their maximum values simultaneously.

Most regulations introduce similar combinations for the determination of the overall safety factor, but the coefficients used vary widely between one country and another. They are not analysed in this context, since it is intended to indicate only basic principles.

There is, however, one consideration which must be borne in mind: it would obviously be desirable for safety factors to be such that to design
to the failure criterion is sufficient, so that the designer can be confident that safety against cracking under normal working conditions is assured, for the particular class of prestress under consideration.

In the case of total prestress, for example, it would be required that under the stated loading the stress at the bottom fibre should be zero. It has been seen (Chapter VIII) that, in the case of a beam with a span greater than the critical span, the lever arm \(z\) is of the order of \(0.7h\). If \(T\) is the permanent tensile stress, the cross-sectional area of the steel which is necessary to resist a moment \(M\), with the condition that \(R' = 0\) at the bottom fibre, is given approximately by \(A_cT = M/0.7h\). It would be desirable for the overall safety factor \(\Gamma\) (resulting from the preceding increases and reductions) to be such as to give the same area of steel when designing to the limit state of failure. If the lever arm at failure is \(0.95h\) and if \(h_1 = 0.95h\), then \(z_1 = 0.9 \times 0.95h = 0.86h\).

It is then necessary to have \(A_cT_r = M_r/0.86h = \Gamma M/0.86h\).

The overall safety factor from the practical aspect is therefore obtained by:

\[
\frac{\Gamma M}{0.86hT} = \frac{M}{0.7hT}
\]

hence \(\Gamma = 0.86/0.7(T_r/T)\). If \(T_r = 160\ \text{kg/mm}^2\) and \(T = 90\ \text{kg/mm}^2\), the required factor \(\Gamma\) would be:

\[
\Gamma = \frac{0.86}{0.7} \times \frac{160}{90} = 2.19
\]

From this brief calculation, which could be gone into in greater detail by evaluating more precisely the value of the lever arm, it is seen that, in the case of total prestress, a reduction in the values of the safety factors has no advantage, because the quantity of steel cannot be reduced below that quantity which is required to provide safety against cracking. With the codes considered above, the conditions at cracking are generally the governing requirements in the case of total prestress.

No attempt is made here to investigate this question in detail, by considering the actual values of the lever arm \(z\), which depend on the shape of the section, or by considering the various classes of prestress. In practice, safety against cracking and safety against failure are independent. It has been the intention here to draw attention to the fact that certain advantages which might seem possible at the expense of the safety factor against failure are of little value if safety against failure is not the determining factor from the point of view of material quantities.
15. Repeated alternating loading
Repeated alternating loads which can give rise to fatigue phenomena are considered (machine foundations, railway bridges, etc.), and not random occasional alternations.

Practical experience shows that, with loading of this type, the moment of resistance is sharply reduced, by 30% or more, if cracking can occur at each cycle. In this case, limited prestress is to be entirely discouraged; it is even necessary to adopt more stringent conditions than with normal total prestress, with minimum residual compressions. The coefficient \( \gamma_s \) must also be increased (by about 20%). This increase in \( \gamma_s \) does not usually require more material, relative to that which is required for desirable values of residual compression.

*Loads of long duration*
The same increase in \( \gamma_s \) (about 20%) should be applied in the case of structures which are subjected for long periods of time to maximum loads. This case is exceptionally rare, since maximum loads are in principle unusually high loads which are applied for a relatively short time.

16. Sudden fracture
If the resistance to fracture is less than the resistance to cracking, the section fails suddenly, without a preliminary stage of cracking.

This is a dangerous phenomenon because not only does failure occur without warning but also because it is accompanied by a high release of energy which can be great enough to cause disintegration of the member, with the possibility of injury or damage from flying fragments.

Sudden fracture could occur in the case of prestress if the steel percentage is low and the permanent tensile stresses are very high; practically, this is possible only in the case of bonded-wire beams (when the phenomenon mentioned in Section 6 can delay cracking), should the wires be too highly stressed and, consequently, of insufficient section (giving a relatively low moment of failure).

In the interest of safety, it should be checked that the moment causing failure is greater than the moment causing cracking, by at least 30%.

17. Limit state of failure with minimum loading (failure of prestressed zone under compression)
As in the case of failure under maximum loading, it is assumed that the stress distribution can be represented by a rectangular diagram (uniform
stress) over a depth $y$ equal to $\frac{3}{4}x$, where $x$ is the depth of the zone in compression.

The strain at the fibre in compression is equal to the rupture strain (3.5/1000), and the uniform stress is equal to the rupture strength $R$.

The centre of compression $E_p$ is obtained from the position of the point of application $E_o$ of the prestressing force by the displacement $E_o E_p^* = M_p/F$, where $M_p$ is the moment under the minimum loading conditions.

If the limit state is reached, $E_p$ coincides with the centroid of the partial area $S_y$ which is limited by the depth $y$ (shown hatched in Fig. 35), because

![Fig. 35. Limit state of failure of the bottom flange under conditions of minimum loading.](image)

of the hypothesis of the uniformity of the stress over this depth, and the stress $F/S_y$ is equal to $R$.

The check on safety consists of applying a multiplication factor $\gamma_o$ to the prestressing force as well as reduction factors $\gamma_p(<1)$ and $1/\gamma_p$ to the moment $M_p$ and to the characteristic strength $R_{bk}$.

Therefore, if the calculated centre of compression is denoted by $E_p^*$, its position is determined by the displacement $E_o E_p^* = \gamma_p M_p/\gamma_0 F$. The partial area $S_y$ with $E_p^*$ as centroid is considered separately from the remainder of the section. It is necessary to have

$$S_y \geq \frac{\gamma_0 F}{R_{bk}/\gamma_b} \quad \text{or} \quad S_y \geq \gamma_0 \gamma_b \frac{F}{R_{bk}}$$

The area $S_y$ is the area which is rendered homogeneous, by replacing the cross-sectional area $A_u$ of the complementary reinforcement†, if any, by

† There is no need to consider the cross-sectional area of the cables when assuming the section to be homogeneous. More exactly, the cables, which are assumed to be grouted in, only contribute to reinforcing the section for the differences between the moment $M_p$ and the moment which existed prior to the initial prestress.
an area $mA_w$ (very high values of $m$ can be used, at least equal to 15, since the elastic modulus of concrete becomes low if the limit state is reached), and by deducting the holes for the cables.

Consideration of this limit state enables a given section to be checked, or it enables the size of the bottom flange to be found,† when the section is designed on the bases of limit states.

If trial and error methods are used in the design by first assuming a size of section, and gradually modifying it in order to obtain the required result, the method of design is the same as the method of checking.

For this check, a certain portion of the section is considered separately, this portion being limited by the bottom fibre and being such that its area $S_y$ is equal to $\gamma_0 \gamma_b \left( F/R_{sk} \right)$. Its centroid $G_p$ is determined, as well as the displacement $E_o E_p \left[ = \left( \gamma_p/\gamma_o \right) \left( M_p/F \right) \right]$; hence $E^*_p$ is obtained.

If $E^*_p$ is above $G_p$, the safety criteria are amply satisfied.

The size of the bottom flange can be reduced (thereby reducing $M_p$ and $S_y$) until $G_p$ coincides with $E^*_p$, if this is possible.‡

If $E^*_p$ is below $G_p$, it is necessary to change the design by increasing the size of the bottom flange. The section so obtained is generally overdesigned with regard to cracking, but this is unavoidable.

In order to achieve designs comparable to those obtained with the elastic theory,†† safety factors of the order of $\gamma_p = 0.9$, $\gamma_0 = 1.3$ to 1.35, $\gamma_b = 1.5$ must be used.

In addition to safety against failure, the safety factor against cracking at the top fibre at the minimum loading conditions must of course be checked separately, in relation to the requirements corresponding to the given class of prestress (see Chapter X, Section 3). Complementary reinforcement may be required in the tensile zone (top zone).

18. Limit state of failure under initial prestress and attendant loading

This limit state is defined in the same manner as the preceding limit state, the prestressing force being the force $F_i$ which is exerted by the cables (usually by a fraction of the total number of cables) at their initial stress

† The web thickness is generally determined from considerations of shear resistance (Volume 2, Chapter II), or from considerations of minimum thickness.

‡ This is not always possible, because of the web thickness, which is fixed, and because of the minimum dimensions required for accommodating the cables.

†† As stated in Section 4, the condition that the design should agree with a design prepared in accordance with the elastic theory is neither mandatory, nor is it justified logically.
$T_i, M_p$, being the moment which is due to the accompanying loads (which may be only a fraction of the permanent loads), the concrete strength being the resistance $R_f$ which is reached at the time of prestress.

Values of the factors $\gamma$ which are a little less stringent than those given previously may be used ($\gamma_p = 0.9, \gamma_{oi} = 1.2$).†

As stated in Chapter V (Section 4), the design should be based on long-term conditions, and not on initial conditions. If possible, the design of the bottom flange which is obtained by a consideration of the early stage should not be changed, and this may be achieved by tensioning only a fraction of the cables, if necessary.

19. Allowable compressive stresses when the section is checked on the basis of elastic behaviour

(i) Allowable stresses in the most highly compressed fibre under conditions of maximum loading

The recommendations of the ASP and the joint FIP-CEB Committee consider that the failure criterion is sufficient, and an elastic check is not required.

However, most present-day codes require this check and they specify a permissible limit. The order of magnitude is usually $0.33R_{cube}$.

In the author’s opinion, this is conservative.

(ii) Allowable compressive stresses at the most highly compressed fibre in the case of minimum loading (fibre compressed by prestress)

The ASP permits a limit stress of $0.33R_{cube} (0.42R_{cylinder})$ (28-day strengths).

(iii) Permissible compressive stress at the time of initial prestress

This case involves the same fibre as in case (ii), but the prestressing force is the initial force, and the moment due to external loading is usually less than in the previous case.

The case is not recurrent, since the compression diminishes due to the loss in tension. It is therefore needless to consider the effects of irreversible plastic deformation on the particular fibre due to alternating loading and unloading. In these circumstances, the permissible stress can be raised.

The ASP allows a value of $0.5R_{cube}$ for this stress, and, in the case where there is no risk of personal danger or serious material risk, a limit of $0.55R_{cube}$ and $0.66R_{cylinder}$ respectively) is allowed.

† Refer to the note in the preceding section concerning 'agreement' with elastic theory calculations.
The value of R which is used must, of course, be the value which applies at the time of prestress.

20. Allowable tensile stresses in the steel
Consideration of the safety of the steel against failure as a separate item is meaningless; with a composite material, as in the case of prestressed concrete in the failure stage, the important factors which affect safety are the maximum strains in the constituent materials in the context of their mutual interdependence (resulting in the stresses reached, and the moments at failure). Now, apart from fatigue failures, it has been seen that the tensile failure strain is never reached in the steel if the steel conforms to the relevant specifications, some details of which are given in Chapter II.

There is, however, some disagreement in the various codes as to the permissible stresses to use under normal service conditions.

The questions which arise are the following:

(i) What is the limit to use at the time of prestress in order to reduce the risk of failure to an acceptable probability?

(ii) As opposed to reinforced concrete, the prestressing steel is permanently highly stressed, but the variations in the stress are small if cracking is avoided. Is there a limit beyond which the steel could eventually reach a dangerous condition?

(iii) By increasing the stress in the steel, the available margin of strain \((\varepsilon' - \varepsilon''_a)\) is reduced. Can this be accompanied by a reduction in the moment at rupture?

It seems that the following answers can be given to these questions:

(i) The tensile strains at the time of tensioning must not exceed too high a proportion of the rupture strain, account being taken of scatter. The permissible limit therefore depends on the shape of the diagram and on its reproducibility or reversibility. The characteristic basis of the diagrams can be that of a proof stress of 0.1 or 0.2%.

Although French regulations do not specify any limit for the initial stress, it has become accepted practice to adopt as a reasonable limit the guaranteed characteristic \(T_G\). As stated in Chapter II, this limit corresponds approximately to the guaranteed proof stress at 0.1%. Shape and scatter are thus effectively introduced. As stated in Chapter II also, it is incumbent on the wire manufacturers to ensure that the guaranteed limit is not too low.
(ii) The danger which could result from maintaining the steel at a high stress can really only be related to that of corrosion, which is itself related to the risk of permanent cracking. If such cracking is avoided and if the steel is of the required quality (Chapter II), and if grouting is to the required standard, this question need not arise.

(iii) For a reduction in the moment of resistance at failure, it would be necessary for the initial strains to be much higher than those which are envisaged in (i); the monitoring of strains at the time of this operation excludes this risk.

This being the case, it can be seen that, on the contrary, the moment of resistance increases as the initial tension is increased. This can be understood by considering Fig. 22. The ‘characteristic’ curve in this figure rises as $e'_a$ increases (permanent strain after deduction of relaxation). This stems from the fact that $y/h_1 = \frac{3}{2} e'/\lambda_o (\varepsilon_e + e' - e'_a)$. Characteristic curves corresponding to $\lambda_o = 0.7$ and $\lambda_o = 0.5$ are drawn (in dotted lines) on Fig. 22. The lines in the figure show that for a given value of $w$, $\lambda$ increases as $\lambda_o$ increases, and consequently, so does the moment $M$, at failure. Nevertheless, variations are small for the usual values of $w$.

Therefore, with steel of the required quality, the only risk which has to be guarded against is the risk during application of the prestress. It is a risk for which a fairly high probability can be accepted, since it corresponds to a temporary phase, prior to putting the structure to its intended use.

21. Safety of the anchor

In principle, the cables are grouted in; in this case the safety of the anchor is of small importance, since good quality grouting is sufficient to anchor the cable (at the failure stress) over a relatively short length which is of the order of 70 to 100 times the diameter of the wire or unit strand.†

In certain constructions, however, for example pressure vessels for nuclear reactors, the cables are not grouted in. It is then a requirement that the anchor should not fail before the cable, and that the cable should not fail prematurely, during tests to failure, inside the anchor. In fact, it is usually difficult to avoid failure occurring in the anchor before it occurs in the free cable since that part of the cable which is inside the anchor is

† As a result, where cables cross each other (portal angles for example), the anchors must be located sufficiently far away from the cross-over points for the anchoring to be effected entirely by the grout before the intersection.
subjected to the supplementary loading which is mentioned in Chapter III, in addition to its longitudinal stress. Nevertheless, the resulting reduction in strength must be sufficiently low, and the cable must be almost entirely in its plastic state at the moment of failure. If this were not so, the anchorage would be a weak spot, as dangerous as an error in length with risks of sudden failure, without warning.

The surface condition of the cable within the anchor is also critical, in the case of cables without grout, because of the risk of corrosion. In this case, particular care must be taken over the quality of the seal.
Chapter X

DESIGN OF SECTIONS ON THE BASIS OF FAILURE AND LIMIT STATE CALCULATIONS

1. General
In Chapters V to VII, sections are designed on the basis of the elastic theory, using permissible stresses, determined in such a way as to ensure that the actual stresses do not exceed optimum values. This is dealt with in Chapter IX, Section 1, where the considerations leading to the choice of permissible stresses are explained.

In addition, most codes require a check on the conditions at failure under maximum loading; that is, a check with respect to one of the limit states described in the preceding chapter.

Conversely, it is possible to design on the basis of the limit state of failure, and then to check, if necessary, that the design is satisfactory for normal service conditions.

For such a design to satisfy all the required conditions, it must include all the limit states which could render the section unsafe or unsuitable should they occur under working conditions.

To design against failure under maximum loading conditions is therefore inadequate by itself, since it can only be a guarantee of safety with regard to those parts of the section which have been considered, namely the flange in compression due to the external loading, and the cables and reinforcement which provide the strength against failure. It does not take into consideration the flange which is put into tension by external loads, nor does it examine the number of cables and the amount of reinforcement necessary to satisfy the criteria of cracking and deflection.

These complementary calculations can of course be carried out on the basis of permissible stresses (which does not exclude the pseudo-plastic phenomenon prior to cracking, if the apparent tensile strength R" is taken as the lower limit stress), and therefore on the basis of elastic behaviour.

This mixed method can be used, and it is seen in the following pages
that it results in relatively simple calculations; it is not, however, a very satisfactory method, nor is it homogeneous.

The logical method consists of designing entirely on the basis of optimum limit states; the end result should be such that it does not require checking with a method based on the elastic theory.

In order to illustrate these facts, an example is considered where both methods are used: a design against failure under maximum loading, with complementary elastic theory calculations, and a design which is based entirely on limit states.

In both cases, it is assumed that the web thickness has been determined from considerations of shear strength (elastic or limit state conditions), or from minimum thickness considerations (Volume 2, Chapter XIII).

Only the cases of total prestress or partial prestress are dealt with in the following. The case of reinforced prestressed concrete is dealt with in Chapter XI. To conform to the nomenclature advocated by the Joint Committee of the FIP–CEB, Class I denotes total prestress and Class II denotes limited prestress.

2. Example
Beam of 48 m span, depth of 2.8 m, width b at top = 1.75 m. In addition to its self-weight, it is subjected to:

\[
a \text{permanent load } q = 0.208 \text{ t/m } \left( M_q = 60 \text{ t/m} \right) \\
a \text{live load } s = 2.91 \text{ t/m } \left( M_s = 840 \text{ t/m} \right)
\]

The 28-day cube strength of the concrete is 450 kg/cm², or 360 kg/cm² cylinder strength.

The ultimate strength of the high-tensile steel is 160 kg/mm².

The 0.2% proof stress of the complementary reinforcement is 50 kg/mm².

In this case, the beam is designed against failure of the top flange, and, in accordance with Section 1, the design is prepared for elastic conditions; the given permissible stresses for this latter case are:

under minimum loading:

- top fibre: \(-15\text{kg/cm}^2 \ (-150 \text{ t/m}^2)\)
- bottom fibre: \(+140 \text{ kg/cm}^2 \ (1400 \text{ t/m}^2)\)

under maximum loading:

- bottom fibre: 0 for total prestress (Class I)
- \(-30 \text{kg/cm}^2 \ (-300 \text{ t/m}^2)\), in the case of limited prestress (Class II)
Under self-weight and initial prestress loading, the permissible compressive stress at the bottom fibre is 158 kg/cm² (0·66 of the cylinder strength at the compression reached during this operation).

The web thickness is taken as 0·18 m in accordance with the empirical rule for minimum thickness:

$$b' = \frac{h}{36} + 5 \text{ cm} + \text{sheath diameter}$$

(a) Design against failure under maximum loading

The rules of the Joint Committee of the FIP–CEB are used. The design value for the strength of high tensile steels† is (160/1·15) = 139 kg/mm². The design value for the 0·2% proof stress of the complementary steel is (50/1·15) = 43·6 kg/mm².

The characteristic strength of concrete in compression, with a factor of deviation of \((1 - \chi \delta)\) equal to 0·83, is:

$$R_{bk} = 0·83 \times 360 = 298 \text{ kg/cm}^2$$

The design value for strength is \(R^* = (298/1·5) = 200 \text{ kg/cm}^2\) approximately (2 000 t/m²).

† It is not necessary in the case of steels to apply the deviation factor 1 − \(\chi \delta\), because this is already included in the guaranteed strength (ASP Specifications).
Seven millimetres diameter, 12-strand cables are chosen (461 mm$^2$). The rupture strength (design value) of each cable is $139 \times 461 = 64$ t.

It is provisionally assumed that the centroid of the cables is 0·10 m above the bottom fibre. This may have to be modified after the first calculation.

The effective depth

$$h_1 = 2·80 - 0·10 = 2·70 \text{ m}.$$ 

It is assumed as a first approximation that the lever arm at failure (distance of the cables from the centre of compression at failure) is equal to 0·95$h_1$, or $0·95 \times 2·70 = 2·56$ m.

If $F_r$ is the breaking force in the cables (in tonnes), and assuming that the failure stress is effectively reached (a hypothesis which must be subsequently checked), the value of the moment of resistance in tonne-metres is:

$$M_r = 2·56 F_r$$

It is assumed as a first trial (to be checked subsequently) that

$$M_p = 640 \text{ t/m}$$

The moment under maximum loading is then:

$$M = 640 + 60 + 840 = 1\,540 \text{ tm}$$

Applying the factor $\gamma_s = 1·4$, it is necessary to have

$$M_r \geq 1\,550 \times 1·4 = 2\,160 \text{ tm}$$

hence $F_r \geq (2\,160/2·56) = 845$ t. Therefore 13 cables are required.

To equal the breaking force in the cables, the cross-sectional area of concrete in compression must be:

$$\frac{F_r}{R^*} = \frac{845}{2\,000} = 0·422 \text{ m}^2, \text{ or } 1·75 \times 0·24 \text{ m}$$

For the neutral axis not to lie outside the top flange† (which, as seen in Chapter IX, Section 13, reduces the moment of resistance), it would appear that the thickness of the top flange should be 0·24 m. In fact, it is seen later than this thickness is excessive.

† It is more correct to say: for the depth $y$ (over which it is assumed that compression is uniform) not to lie outside the top flange.
It is now necessary to check the assumptions concerning the lever arm and the stress in the cables.

Now:

$$\sigma = \frac{F_r}{bh_1R^*} = \frac{850}{1.75 \times 2.7 \times 2000} = 0.089$$

The approximate formulae for the moment at rupture (Chapter IX, Section 11) are applicable for this value of $\sigma$. Therefore $\lambda = 1$ (and the stress reached in the cables is indeed the breaking stress), and the lever arm is $h_1 \times (1 - \sigma/2) = 0.955 \times h_1 = 2.57$ m, which is very nearly equal to the assumed value.

The hypotheses are therefore verified (were it not so, it would be necessary to re-calculate using the modified values).

(b) Thirteen cables to be arranged in the bottom flange

It is certain that in the case of Class I prestress (total prestress) this number of cables is inadequate to satisfy the conditions at cracking; thirteen cables could be sufficient for Class II prestress (partial prestress). The dimensions of the bottom flange to accommodate the cables are therefore minimum dimensions for Class I prestress. These dimensions can be used as the starting point for arriving at the complementary elastic design.

By locating the cables as shown in Fig. 2, the breadth of the flange so obtained is 0.51 m. The flange can be approximated to a rectangle of dimensions 0.26 $\times$ 0.51 m.
The properties of the section are then:

\[ S = 0.967 \text{ m}^2 \quad v = 1.00 \text{ m} \quad v' = 1.80 \text{ m} \quad I = 0.946 \text{ m}^4 \]
\[ I/v = 0.946 \text{ m}^3 \quad I/v' = 0.525 \text{ m}^3 \]

Now, the section modulus \( I/v \) which is definitely required is equal to \( M_s/\Delta R \).

\[ M_s = 840 \text{ tm} : \]
\[ \Delta R = 1400 \text{ t/m}^2 \text{ for Class I (total prestress)} \]
\[ \Delta R = 1700 \text{ t/m}^2 \text{ for Class II (partial prestress)} \]

The required section moduli are therefore equal to \( (840/1400) = 0.600 \text{ m}^3 \) for Class I and \( (840/1700) = 0.495 \text{ m}^3 \) for Class II.

The bottom flange is therefore inadequate for Class I and overdesigned for Class II.

![Fig. 3. Corrected design for Class I prestress.](image)

In the case of Class I, it is necessary to increase the size of the flange, and it can be reduced in the case of Class II, provided that it is still possible to accommodate the cables.

**Class I (total prestress)** If the breadth of the bottom flange is 0.66 m, Fig. 3, the section has the following properties:

\[ S = 1.006 \text{ m}^2 \quad v = 1.07 \text{ m} \quad v' = 1.73 \text{ m} \quad I = 1.05 \text{ m}^4 \]
\[ I/v = 0.975 \text{ m}^3 \quad r^2/v = 0.97 \text{ m} \quad r^2/v' = 0.602 \text{ m} \]
The section is therefore adequate \((I/v' \approx 0.60)\), and the cables are easily accommodated.

The nett weight is \(p = 2.4\) t/m, hence the corrected value of \(M_p\) is 690 tm.

The eccentricity of prestress, assuming that the centroid of the cables is 0.10 m above the bottom fibre, is \(e = 1.63\) m.

Hence the prestress at the bottom fibre is:

\[
\sigma'_o = \frac{F}{S} \left(1 - \frac{e}{r^2/v'}\right) = \frac{F}{S} \left(1 + \frac{1.63}{0.602}\right) = 3.7 \frac{F}{S}
\]

The stresses at the bottom fibre due to external loading are:

**due to moment** \(M_p = 690\) tm, \(\sigma' = \frac{690}{0.605} = -1140\)

\(M_q = 60\) tm, \(-99\)

\(M_s = 840\) tm, \(-1390\)

\(-2629\) t/m²

In order to restore the resultant stress to zero, \(\sigma'_o\) must be equal to +2629 t/m², or:

\[
3.7 \frac{F}{S} = 2629
\]

\[
\frac{F}{S} = 711\ t/m^2
\]

\(F = 711 \times 1.006 = 715\) t

In the determination of the number of cables, consideration is taken of their additional stress \(\Delta T\), due to the loss of compression in the adjacent concrete under the action of the loading which is applied subsequent to the grouting (Chapter VII, Section 7). If \(\Delta \sigma_c\) is the stress variation in the concrete local to the cables, due to the loads \(q + s\), then \(\Delta T = m\Delta \sigma_c\).

\[
\Delta \sigma_c = (M_q + M_s) \frac{e}{I} = -900 \times \frac{1.63}{1.05} = -1390\ t/m^2 (139\ kg/cm^2)
\]

With \(m = 5\), \(\Delta T = 5 \times 139 = 700\) kg/cm² (7 kg/mm²).

Therefore, assuming that the permanent stress in the cables is equal to 93 kg/mm², the stress in the cables can be taken as:

\(T + \Delta T = 93 + 7 = 100\) kg/mm²
Therefore, the force which is exerted by each cable is equal to:

\[ 461 \times 100 = 46100 \text{ kg} \]

The required number of cables is then \((715/46.1) = 15.5\) cables, say 16 cables.

**Class II (partial prestress).** In order to obtain the required \(I/v'\) value, 0.495 m\(^3\), the breadth of the bottom flange need be only 0.45 m, with a depth of 0.26 m as before.

The properties of the section are:

\[
\begin{align*}
S &= 0.951 \text{ m}^2 \\
v &= 0.98 \text{ m} \\
v' &= 1.82 \text{ m} \\
I &= 0.904 \text{ m}^4 \\
I/v &= 0.924 \text{ m}^3 \\
I/v' &= 0.495 \text{ m}^3 \\
r^2/v &= 0.97 \text{ m} \\
r^2/v' &= 0.52 \text{ m}
\end{align*}
\]

It is necessary to modify the arrangement of the cables, as shown in Fig. 4.

Assuming again a distance of 0.10 m between the centroid of the cables and the bottom face of the flange, then:

\[
e = -1.72 \text{ m}
\]

\[
\sigma'_o = \frac{F}{S} \left(1 + \frac{1.72}{0.52}\right) = 4.31 \frac{F}{S}
\]
The self-weight becomes \( p = 0.951 \times 2.4 = 2.28 \) t/m. Hence

\[
M_p = 655 \text{ tm}
\]

The stresses under external loading are:

\[
(M_p = 655) \sigma'_p = \frac{655}{0.495} = -1320
\]

\[
(M_q = 60) \sigma'_q = -121
\]

\[
(M_s = 840) \sigma'_s = -1700
\]

\[
-3141 \text{ t/m}^2
\]

In order to restore the resultant stress under maximum loading to 
\(-300 \text{ t/m}^2\), it is necessary for \( \sigma'_o \) to be equal to \(+2841 \text{ t/m}^2\).

Hence

\[
4.31 \frac{F}{S} = 2841 \text{ t/m}^2
\]

\[
\frac{F}{S} = 660 \text{ t/m}^2
\]

\[
F = 660 \times 0.951 = 625 \text{ t}
\]

The excess tension in the cables is equal to \(-m \Delta \sigma_c\), \( \Delta \sigma_c \) being the stress variation \( M(e/I) \) local to the cables, under the action of the loads \( q + s \):

\[
\Delta \sigma_c = -(60 + 840) \times \frac{1.72}{0.904} = -1710 \text{ t/m}^2 (171 \text{ kg/cm}^2)
\]

With \( m = 5 \), \( \Delta T = 5 \times 171 = 855 \text{ kg/cm}^2 \), or \( 8.5 \text{ kg/mm}^2 \).

The stress in the cables is, therefore:

\[
T + \Delta T = 93 + 8.5 = 101.5 \text{ kg/mm}^2
\]

The force in each cable is \( 461 \times 101.5 = 46.9 \text{ t} \).

Therefore the required number of cables is \( (625/46.9) = 13.3 \), say 14 cables.

However, with this number of cables and with the arrangement shown in Fig. 4, the distance of their centroid from the bottom face becomes equal to 0.13 m instead of the assumed value 0.10 m. The eccentricity becomes equal to \(-1.69 \text{ m} \).
The correction of the calculation is made as follows:

\[ \sigma'_o = \frac{F}{S} \left(1 + \frac{1.69}{0.52}\right) = 4.25 \times \frac{F}{S} \]

\[ 4.25 \frac{F}{S} = 2841 \text{ t/m}^2 \]

\[ \frac{F}{S} = 670 \text{ t/m}^2 \]

\[ F = 670 \times 0.951 = 636 \text{ t} \]

And \((636/46.9) = 13.6\) cables are required, say 14 cables.

*Final calculations.* The preceding arithmetic shows that the number of cables required to resist cracking is greater, in the case of total and partial prestress, than the number which is required to resist failure.

Under these conditions, it is possible to reconsider the limit state of failure, since the number of cables is excessive for this limit state. A reduction in the lever arm can be accepted, and it is possible to relax the requirement that the neutral fibre should lie within the top flange at the moment of failure, which previously led to a top flange thickness of 0.24 m.

Thus this thickness can be reduced, but the design of the bottom flange and of the cables must be checked, and corrections made if necessary.

The following are the final calculations:

(i) Class I. It was found that 16 cables are required to satisfy the working conditions. the strength of these cables is \(16 \times 64 = 1024 \text{ t}\).

In order to resist this force, the cross-section of the concrete under compression at the moment of failure must be equal to

\[ \frac{1024}{2000} = 0.51 \text{ m}^2 \]

The maximum moment, with \(M_p = 690 \text{ tm}\), is:

\[ 690 + 60 + 840 = 1590 \text{ tm} \]

It is required to have \(M \geq 1.4 \times 1590 = 2230 \text{ tm}\). Assuming that the cables reach their breaking stress,† the required lever arm (distance

† In practice the breaking stress would not be reached since 13 cables would be sufficient to provide safety against failure. The calculation is therefore on the safe side. A more correct calculation is given in Section 5.
of the cables from the centroid of the area in compression) is:

\[
\frac{2230}{1024} = 2.19 \text{ m}
\]

The centroid of the area in compression must therefore be at a distance from the top face which is less than, or equal to:

\[
2.80 - (2.19 + 0.10) = 0.51 \text{ m}
\]

![Diagram](image)

**FIG. 5.** Correction to the top flange for Class I.

With a top flange thickness of 0.20 m, the calculation for the required centroid of the area of 0.51 m² is as follows:

\[
(1.75 - 0.18) \times 0.20 = 0.314 \text{ at 0.10 from the top face: 0.0314}
\]
\[
1.09 \times 0.18 = 0.196 \text{ at 0.545 from the top face: 0.1070}
\]

\[
0.510 \text{ m}^2, 0.271 \text{ m}, 0.1384 \text{ m}^3
\]

This centroid is suitable.

The calculation can therefore be done again with a top flange thickness of 0.20 m. Assuming the same bottom flange as in Fig. 3, the properties of the section are:

\[
S = 0.944 \text{ m}^2, \quad v = 1.13 \text{ m}, \quad v' = 1.67 \text{ m}, \quad I = 0.999 \text{ m}^4
\]

\[
I/v = 0.884 \text{ m}^3, \quad I/v' = 0.600 \text{ m}^3, \quad r^2/v = 0.935 \text{ m}
\]

\[
r^2/v' = 0.635 \text{ m}
\]

With the same reasoning as before, it is found that 15 cables could be sufficient, with a slight increase in the permanent stress.

As a check, and in order to compare the results with calculations which are based on the limit states (Section 4 *et seq.*), the properties of the nett
section are calculated below, the holes being allowed for, as well as the properties of the homogeneous section, with 15 cables and \( m = 5 \):

<table>
<thead>
<tr>
<th>S(m(^2))</th>
<th>( v(m) )</th>
<th>( v'(m) )</th>
<th>I(m(^4))</th>
<th>( \frac{I}{V} ) (m(^3))</th>
<th>( \frac{I'}{V} ) (m(^3))</th>
<th>( r^2 ) (m)</th>
<th>( \frac{r^2}{V} ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nett section</td>
<td>0.921</td>
<td>1.10</td>
<td>1.70</td>
<td>0.941</td>
<td>0.855</td>
<td>0.554</td>
<td>0.925</td>
</tr>
<tr>
<td>Homogeneous section</td>
<td>0.955</td>
<td>1.15</td>
<td>1.65</td>
<td>1.025</td>
<td>0.890</td>
<td>0.620</td>
<td>—</td>
</tr>
</tbody>
</table>

The self-weight is \( p = 2.21 \) t/m, and \( M_p = 636 \) tm.
With 15 cables tensioned to \( 43.6 \) t \times (94.5 kg/mm\(^2\)) (permanent tension), \( F = 655 \) t.

\[
\sigma_o = \frac{655}{0.921} \left( 1 - \frac{1.60}{0.925} \right) = -520 \text{ t/m}^2
\]

\[
\sigma'_o = \frac{655}{0.921} \left( 1 + \frac{1.60}{600} \right) = 2600 \text{ t/m}^2
\]

The stresses during successive states (\( p \) acting on the nett section, \( q \) and \( s \) on the homogeneous section) are the following, in t/m\(^2\):

<table>
<thead>
<tr>
<th>Top fibre</th>
<th>Bottom fibre</th>
</tr>
</thead>
<tbody>
<tr>
<td>cumulative</td>
<td>cumulative</td>
</tr>
</tbody>
</table>

Prestress due to self-weight \( p \)
(636 tm)

| Prestress due to load \( q \) (60 tm)
| Prestress due to load \( s \) (840 tm)|
| 741 | 67 | 945 |
| 221 | 288 | 1233 |

With initial prestress, the stress in the cables is 120 kg/mm\(^2\) at mid-span.
Force in each cable: \( 461 \times 120 = 55.4 \) t.
During tensioning, a compressive stress of \( 0.66 \times 240 = 158 \) kg/cm\(^2\) (1 580 t/m\(^2\)) is not to be exceeded at the bottom fibre.
Since the self-weight stress is \( -1 \) 150 t/m\(^2\), \( \sigma'_o \) must not exceed

\[
1150 + 1580 = 2730 \text{ t/m}^2
\]
or

\[
\frac{F_t}{S} \left( 1 - \frac{e}{r^2/v} \right) \leq 2.730
\]

Therefore:

\[
\frac{F_t}{S} \left( 1 + \frac{1.60}{0.60} \right) = \frac{F_t}{S} \times 3.66 \leq 2.730
\]

\[
\frac{F_t}{S} \leq 745, \quad F_t \leq 745 \times 0.921 = 685 \text{ t}
\]

A maximum of 12 cables can be tensioned at the first application of prestress.

(ii) Class II. It has been seen that 14 cables are required to satisfy the working conditions; the breaking strength of these cables is 14 \times 64 = 895 \text{ t}.

To equal this force, the area of concrete which is in compression at the moment of failure must be equal to \((895/2000) = 0.447 \text{ m}^2\).

The maximum bending moment, with \(M_p = 655 \text{ tm}\), is:

\[
655 + 60 + 840 = 1555 \text{ tm}
\]

It is required to have \(M_p \geq 1.4 \times 1555 = 2170 \text{ tm}\). The required lever arm (distance of the cables from the centroid of the concrete in compression) is \((2170/895) = 2.43 \text{ m}\). Therefore the centroid of the cross-sectional area of the concrete in compression must be at a distance which is less than or equal to \(2.80 - (2.43 + 0.13) = 0.24 \text{ m}\) from the top face.

With a top flange thickness of 0.20 m, the calculation for the required centroid of the area of 0.447 \text{ m}^2\) is as follows:

\[
(1.75 - 0.18) \times 0.20 = 0.314 \text{ at 0.10 from the top face: 0.0314}
\]

\[
0.18 \times 0.74 = 0.133 \text{ at 0.37 from the top face: 0.0492}
\]

\[
0.447 \quad 0.18 \quad 0.0806
\]

The position of this centroid is suitable (0.18 < 0.24).

The calculation has to be done again with this top flange thickness of 0.20 m, and, if necessary, the calculation of the prestress has to be corrected.

A preliminary trial shows that it is necessary slightly to increase the size of the bottom flange, compared with that shown in Fig. 4, its width
being increased to 0.46 m. With this flange, and with a top flange thickness of 0.20 m, the properties of the section are:

\[
\begin{align*}
S &= 0.892 \, \text{m}^2 \\
v &= 1.045 \, \text{m} \\
v' &= 1.755 \, \text{m} \\
I &= 0.868 \, \text{m}^4 \\
I/v &= 0.829 \, \text{m}^3 \\
I/v' &= 0.494 \, \text{m}^3 \\
r^2/v &= 0.926 \, \text{m} \\
r^2/v' &= 0.551 \, \text{m}
\end{align*}
\]

The self-weight \( p \) is equal to 2.14 t/m, and \( M_p = 615 \, \text{tm} \).

Maximum moment: \( 615 + 60 + 840 = 1515 \, \text{tm} \).

Induced stress at the bottom fibre:

\[
\frac{-1515}{0.494} = -3070 \, \text{t/m}^2
\]

To restore the resultant stress to \(-300 \, \text{t/m}^2\), the prestress required is \( \sigma'_o = 2770 \, \text{t/m}^2 \).

The eccentricity of prestress is \( e = -(1.755 - 0.13) = -1.625 \, \text{m} \)

\[
\sigma'_o = \frac{F}{S} \left( 1 + \frac{1.625}{0.551} \right) = 3.95 \frac{F}{S}
\]

\( \sigma'_o = 2770 \, \text{t/m}^2 \) is required. Therefore:

\[
\frac{F}{S} = 702 \, \text{t/m}^2
\]

and \( F = 702 \times 0.892 = 626 \, \text{t} \).

With each cable applying a force of 46.9 t (see above), taking into consideration the excess tension \( \Delta T \) due to the loss of compression in the concrete, the number of cables required is:

\[
\frac{626}{46.9} = 13.3 \text{ cables}
\]

(a) With a design which is based solely on elastic considerations, 13 cables are acceptable only if an increase in the permanent tension in the strands is acceptable, previously taken as 93 kg/mm^2.

Each cable would be required to exert a force of \( (626/13) = 48.2 \, \text{t} \), at a stress of \( 48 \times 200/461 = 104.5 \, \text{kg/mm}^2 \), with the same amount of excess tension. Since \( \Delta T = 8.5 \, \text{kg/cm}^2 \), the permanent stress is 96 kg/mm^2.

If this stress is considered excessive, 14 cables are required.

(b) If the basis is the limit state of cracking (Chapter IX, Section 6), 13 cables only need be provided, since the apparent tensile strength can be increased.
With 13 cables tensioned to 46·9 t (including the excess tension $\Delta T$), then:

$$F = 13 \times 46.9 = 610 \text{ t}$$

$$\frac{F}{S} = 680 \text{ t/m}^2$$

$$\sigma' = 3.95 \times 680 = 2690 \text{ t/m}^2$$

The apparent strength against cracking must then be equal to:

$$3070 - 2690 = 380 \text{ t/m}^2 (38 \text{ kg/cm}^2)$$

Now, the factor $k = (R''/R')$, where $R''$ is the apparent strength, is of the order of 1·6 (see Section 6),† and so this stress is acceptable.

Nevertheless, safety against failure would be insufficient with 13 cables, since it has previously been shown that 13 cables are necessary with a top flange thickness of 0·24 m. With a thickness of 0·20 m, the lever arm is reduced and the moment of resistance $M_r$ becomes too small; that is, less than $1.4 \times 1515 = 2130 \text{ tm}$. It has to be increased by adding reinforcement local to the bottom face. For this purpose, five 12 mm diameter bars of steel with high bonding properties are required, with a design elastic limit value of $(50/1.15) = 43.6 \text{ kg/mm}^2$, as shown by the following calculation:

**Forces in the steel at failure:**

13 cables at 64 t................................. 830 t
five, 12 mm diameter bars = 5.65 cm$^2$ at 4.36 t/cm$^2$ .... 24 t

$$\frac{830 + 24}{5} = 854 \text{ t}$$

In order to balance this force, the required area of concrete in compression is:

$$\frac{854}{2000} = 0.427 \text{ m}^2$$

or

**Moment with respect to the top face**

$$(1.75 - 0.18) \times 0.20 = 0.314 \text{ at 0.10 from the top face: } 0.0314$$

$$0.18 \times 0.63 = \frac{0.113}{0.427} \text{ at 0.315 from the top face: } \frac{0.0354}{0.0668} \text{ m}^3$$

† For the section under consideration in Section 6, it is found that $(l/v) = 0.431 \text{ m}^3$, and the specific section modulus is found to be 0.719 m$^3$. Hence $k = (0.719/0.431) = 1.66$. 
The centroid of the area in compression is at a distance of
\[
\frac{0.0668}{0.427} = 0.16 \text{ m}
\]
approximately from the top face, or at a distance of
\[
2.80 - (0.16 + 0.13) = 2.51 \text{ m}
\]
from the cables and \(2.80 - (0.16 + 0.04) = 2.60\) m from the reinforcement.

Then:
\[
M_r = 830 \times 2.51 + 24 \times 2.60 = 2080 + 62 = 2142 > 2130 \text{ tm}
\]

In any event, the complementary reinforcement is necessary in order to satisfy the ASP regulation whereby it must resist, under a stress which is equal to \(\frac{2}{3}\) of the elastic limit (or 26 kg/mm²), half the resultant tensile stress of the tensile zone.

For a solution based entirely on elastic conditions, then:
\[
\frac{F}{S} = 702, \quad \text{and} \quad \sigma' = -300 \text{ t/m}^2
\]

The depth of the tensile zone is equal to
\[
v' \times \frac{300}{300 + 702} = 0.3v' = 0.53 \text{ m}
\]

Resultant of the tensile stresses:
\[
0.18 \times 0.53 \times \frac{300}{2} + 0.26 \times 0.28 \times 300 \times \frac{0.53 - 0.13}{0.53} = 31.8 \text{ t}
\]

The reinforcement must therefore withstand a force of 15.9 t so that the cross-sectional area is
\[
\frac{15900}{26} = 610 \text{ mm}^2
\]

Similarly, it is found for hypothesis (b) that the minimum quantity of reinforcement is eight 12 mm diameter bars, which is therefore greater than the quantity (five 12 mm diameter bars) which is required for safety against failure.
To compare the results with the limit state calculations given later (Section 4), the properties of the nett section (allowing for the holes) and of the homogeneous section (cables and complementary reinforcement taken into account with $m = 5$) are given below. (The total cross-sectional area of cables and reinforcement is the same for $a$ and $b$.)

![Diagram](image)

**FIG. 6.**

<table>
<thead>
<tr>
<th></th>
<th>$S$</th>
<th>$v$</th>
<th>$v'$</th>
<th>$I$</th>
<th>$rac{I}{v}$</th>
<th>$rac{I}{v'}$</th>
<th>$r^2$</th>
<th>$r'^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net section</td>
<td>0.874</td>
<td>1.01</td>
<td>1.79</td>
<td>0.821</td>
<td>0.81</td>
<td>0.458</td>
<td>0.925</td>
<td>0.523</td>
</tr>
<tr>
<td>Homogeneous section</td>
<td>0.904</td>
<td>1.06</td>
<td>1.74</td>
<td>0.910</td>
<td>0.85</td>
<td>0.516</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

1. By tensioning 13 cables to 43 t,† or $F = 560$ t, with an eccentricity equal to $1.79 - 0.13 = 1.66$ m, then:

$$\sigma_o = \frac{560}{0.874} \left(1 - \frac{1.66}{0.925}\right) = 641 \times (-0.8) = -510 \text{ t/m}^2$$

$$\sigma'_o = \frac{560}{0.874} \left(1 + \frac{1.66}{0.523}\right) = 641 \times 4.18 = 2680 \text{ t/m}^2$$

The self-weight stresses are calculated on the basis of the nett section,

† It is no longer necessary to consider the excess tension in the cables since the live load stresses are calculated on the basis of the homogeneous section.
and the stresses due to the other types of loading are calculated on the basis of the homogeneous section. The stresses are as follows:

<table>
<thead>
<tr>
<th>Resultant stresses</th>
<th>Resultant stresses</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_o = -510$</td>
<td>$\sigma'_o = +2,680$</td>
</tr>
<tr>
<td>$M_p = 615$ t/m, $\sigma = 675$</td>
<td>165</td>
</tr>
<tr>
<td>$M_q = 60$</td>
<td>70</td>
</tr>
<tr>
<td>$M_s = 840$</td>
<td>990</td>
</tr>
</tbody>
</table>

It is therefore necessary in this case to accept an apparent tensile strength of 40 kg/cm$^2$.

2. With 14 cables, the stresses are increased in the ratio $\frac{1}{14} = 1.08$, the stress at the bottom fibre becomes equal to $2\,680 \times 1.08 - 3\,086 = 186$ t/m$^2$. A choice can be made between these two solutions, and the final design of the complementary reinforcement can be completed.

Under initial prestress, the cables exert a force of 55.4 t (see above, Class I).

In order not to exceed a stress of 1 580 t/m$^2$ at the bottom fibre, with a self-weight stress of $-1\,340$ t/m$^2$, $\sigma'_o$ must not exceed

$$1\,580 + 1\,340 = 2\,920$$

that is, $(F_i/S) \times 4.18 = 2\,920$. $F/S = 700$ t/m$^2$, or $F_i = 700 \times 0.874 = 610$ t. A maximum of 11 cables must therefore be tensioned initially.

To summarise, the following results are obtained by designing against failure under maximum loading conditions, with additional check calculations based on the elastic theory (or limit calculation against cracking in the case of Class II):

**Class I:**
- top flange $1.75 \times 0.20$
- bottom flange $0.66 \times 0.26$
- 15 cables (permanent stress 94.5 kg/mm$^2$)

**Class II:**
- top flange $1.75 \times 0.20$
- bottom flange $0.46 \times 0.26$
- 13 cables (assuming an apparent tensile strength $R''$ which corresponds to the limit cracking state of Chapter IX, Section 6), or 14 cables according to calculations on a purely elastic basis (these cables are excessive).
- Six (or eight) 12 mm diameter bars of high-bond steel.†

† The design of the complementary steel should be further developed to make it conform to ASP regulations, but the tubulated sections are adequate.
3. Design on the basis of limit states

Two cases of loading must be considered: maximum and minimum loads, and, for each case, the following must be checked: the resistance to failure of the compressive zone and the resistance to cracking of the tensile zone.

If the safety factors were the same for both failure and cracking, it would be necessary only to consider a single limit state for each case of loading, corresponding to the same moment $\gamma M$, under the action of which the safety of the two extreme fibres would be checked. However, since different safety factors are chosen, because of the difference in the degree of risk which is involved, two limit cases must be considered for each loading case, one of failure and the other of cracking.

In addition, initial prestress and the attendant loading must be examined, and two limit states must again be considered for this condition (failure and cracking).

In total, six limit states must therefore be considered, as defined in the following table, each one enabling a particular aspect to be checked or designed, and the whole making up the complete design.

<table>
<thead>
<tr>
<th></th>
<th>To obtain designs for: $^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>Limit state of failure under maximum loading</td>
</tr>
<tr>
<td>b.</td>
<td>Limit state of maximum strain at the bottom fibre, under minimum loading</td>
</tr>
<tr>
<td>c.</td>
<td>Limit state of failure of the bottom flange under minimum loading at working conditions</td>
</tr>
<tr>
<td>d.</td>
<td>Limit state of strain at the top fibre under minimum loading at working conditions</td>
</tr>
<tr>
<td>e.</td>
<td>As c, but under initial prestress with its attendant loading</td>
</tr>
<tr>
<td>f.</td>
<td>As d, but under initial prestress with its attendant loading</td>
</tr>
</tbody>
</table>

$^a$ For simplification, it is assumed that the moments due to external loads are positive.
The three ruling conditions are those corresponding to states $a$, $b$ and $c$; the other three should not require any corrections.

Denoting generally by $\gamma_s$ the multiplication factors for loads, by $1/\gamma_m$ the reduction factors for strength, and by $\gamma_o$ the multiplication factors (or eventually reduction factors) for the prestressing force, the following remarks apply to the limit states.

Limit state $a$

This is the limit state of failure of the top flange in compression, under maximum loading, already examined in the previous sections. The safety factors $\gamma_s$, $\gamma_m$ are appreciably greater than unity. The 'overall' safety factor $\Gamma$ (Chapter IX, Section 14) is of the order of 2 with the $\gamma$-factors used by the FIP–CEB Committee.

Limit state $b$

This relates to the prevention of cracking at the bottom flange under maximum loading. The loading is therefore the same as in state $a$, but the safety factors $\gamma_s$, $\gamma_m$ are different, and approximately equal to 1, as is $\gamma_o$. The FIP–CEB Committee uses $\gamma_s = \gamma_o = 1$; $\gamma_a = 1$; $\gamma_b = 1.2$.

The quantities calculated on the bases of limit states $b$ and $c$ (this latter is studied further on) are interdependent. Indeed, the bottom flange (limit state $c$) can only be designed once the number of cables is known; conversely, the number of cables can only be determined once the complete design of the section is known, including therefore the design of the bottom flange. The two designs (cables and flange) can only proceed in parallel, therefore, through a series of approximations.

However, the following observations enable the prestressing force to be obtained immediately, with a high degree of accuracy, and this gives the design for case $c$. Subsequent refinements to the calculation are usually of a minor nature.

(i) In the case of Class I, the required prestressing force $F$ is practically independent of the size of the bottom flange, and it can be immediately determined. For this class of prestress, the strain $\varepsilon'$ at the tensile face is zero, and since the factors $\gamma_s$ and $\gamma_b$ are nearly equal to unity, the stress distribution law is elastic. The stress diagram is therefore of triangular shape (zero stress at the base).

The moment due to the internal forces relative to the cable is equal to the moment due to the external loads. But the internal forces with respect to the bottom flange are low, because the stresses are low in the neighbourhood of the tensile face. Their distances from the cables are also small.
Therefore the moment, relative to the cable, of the stresses within the boundaries of the bottom flange, shown hatched in Fig. 7, can be neglected. The calculation can then be conducted on the basis of a T-section comprising the top flange (known from state $a$) and the web, whose thickness is assumed to be known.

![Diagram](image)

**Fig. 7.** Limit state of compression at the bottom fibre.

In other words, the centre of compression $E_o$, under the maximum loading conditions, is practically coincident with the top boundary $G_s$ of the central core of the T-section. It is therefore possible to write, denoting by $F_f$ the force which is exerted by the cables at the time of cracking (the suffix $f$ is a reminder that the stress in the cables is then greater than the permanent stress, because of the loss of compression in the concrete local to the cables): $F_f = (M^*/E_oG_s)$, where $M^*$ is the moment under maximum loading, including the effect of the multiplying factors which have to be applied.

It is therefore necessary to determine only the central core of the T-section (outwith the hatched area of Fig. 7) in order to obtain the force $F_f$ (hence the permanent force $F$, as shown later).

A second consideration even eliminates the necessity for calculating the central core. It can indeed be noticed in numerical examples that, with T-beams of uniform flange width, the point of application of the stress resultant, the shape of the stress diagram being triangular (with zero stress at the bottom), is almost exactly coincident with the centroid of that part of the section which is limited to the top two-thirds of the depth (shown hatched in Fig. 8). Coincidence is exact in the case of a rectangular section, and the deviation does not exceed 4 to 5% of $E_oG_s$ with the usual T-sections.
The centroid of the top two-thirds therefore coincides very nearly with the top boundary $G_s$ of the central core. The lever arm $E_o G_s$ can thus be easily evaluated, and the force $F_f$ obtained. If the tension $T + \Delta T$ is known, the cable cross-sectional area can be determined; $T$ denotes the permanent stress which is known (as a fixed parameter), and $\Delta T$ is the increase in the stress due to the loss of compression in the concrete.

It is seen later that $\Delta T$ can easily be determined.

(ii) The case of Class II. Several approximate methods enable the pre-stressing force to be obtained from that which is required if Class I were being considered.

The two methods given below (the second being a different form of the first) seem to be the best.

(a) If tables are available giving section moduli $W'$ specific to cracking (Chapter IX, Section 7), it is seen that, for a given profile:

moment at cracking (Class II) = moment due to loss of compression (Class I) + $W'R'^*$

Therefore if $M^*$ is the moment under maximum loading (safety factors on loading included if necessary), and if $M_{f1}$ is the limit moment of loss of compression for the section treated as Class I, then:

$$M^* = M_{f1} + W'R'^*$$

(this assumes that the limit state is effectively reached; in other words, that the inequality for checking has become an equality).

The problem then resolves itself into finding the prestressing force which would be necessary in Class I prestress to obtain the limit state of loss of compression under the reduced moment $M^* - W'R'$.
The centre of compression which corresponds to this moment is the same point \( G_s \) as determined above, since the T-section is the same as before. Therefore, denoting by \( F_{fI} \) and \( F_{fII} \) the forces exerted by the cables in the limit states of loss of compression and cracking in Class I and Class II:

\[
\frac{F_{fII}}{F_{fI}} = \frac{M^* - \bar{W}'R''}{M^*}
\]

(1)

Therefore, if the section were known, and hence \( \bar{W}' \), the required force \( F_{fII} \) could be determined from this equation, and the permanent force \( F \) obtained.

A value \( \bar{W}' \) is assumed. Knowing \( F \), the bottom flange is designed by means of limit state \( c \) (see above). The section is then entirely determined, and its \( \bar{W}' \) value is checked against the tables. If \( \bar{W}' \) is not the same as the value first assumed, the calculation is done again, using a value lying between the original value and the value last obtained. In general, a small number of trials is sufficient before obtaining the correct solution.

(b) An approximate value of the ratio \( F_{fII}/F_{fI} \) can also be obtained, as a first shot, on the basis of the value of this ratio for a design in accordance with the elastic theory.

Let \( R'' (=kR') \) be the ‘apparent’ tensile strength. The value of \( k \) is known approximately from tables \([k = \bar{W}'/(I/v')]\); Chapter IX, Section 7 also gives a method for evaluating \( k \) (see also Section 6 of this chapter).

Designing on the elastic basis:

in the case of Class I: \( \sigma'_{oI} = \sigma'_p + \sigma'_s \)

in the case of Class II: \( \sigma'_{oII} = \sigma'_p + \sigma'_s - R'' \)

If the section profile were the same for both classes, then:

\[
\frac{F_{fII}}{F_{fI}} = \frac{\sigma'_{oII}}{\sigma'_{oI}} = \frac{\sigma'_p + \sigma'_s - R''}{\sigma'_p + \sigma'_s} = 1 - \frac{R''}{\sigma'_p + \sigma'_s}
\]

Also:

\[
\sigma'_s = R_1 \quad \text{and} \quad \sigma'_p + \sigma'_s = \sigma'_s \left( 1 + \frac{M_p}{M_s} \right) = R_1 \left( 1 + \frac{M_p}{M_s} \right)
\]

Therefore:

\[
\frac{F_{fII}}{F_{fI}} = 1 - \frac{R''}{R_1[1 + (M_p/M_s)]}
\]
But the profiles are dissimilar.

It could easily be found that the 'correct' value of $F_{fII}/F_{fI}$ is:

$$\frac{F_{fII}}{F_{fI}} = 1 - \frac{R''}{(R_1 + R'')[1 + (M_p/M_s)]}$$  \hspace{1cm} (b)

These results are indicative only, and practical applications show that the value which is nearest to the final value is obtained by the use of eqn. (a).

\[ \text{Fig. 9.} \]

It is therefore recommended that the value (a) should be used as the starting point. Denoting the variation in moment by $\Delta M (\Delta M = M_{\text{max}} - M_{\text{min}})$, eqn. (a) can be written more generally in the form:

$$\frac{F_{fII}}{F_{fI}} = 1 - \frac{kR_b'}{R_1} - \frac{\Delta M}{M_{\text{max}}}$$  \hspace{1cm} (2)

It is reasonable to use $R_1 = 0.33R_{bm\text{cube}} = 0.4R_{bm\text{cyl}}$. Therefore, with formula (2) a first trial value of $F_{fII}$ is obtained, from the force $F_{fI}$ for which the calculation is known, by giving to $k$ the value corresponding to the range of section profiles considered. The dimensions of the section are roughly known. (The value of $k$ can be taken as that which corresponds to the profile obtained for Class I.)

The approximation is usually adequate, and it is in any case eventually corrected in the final check calculation.
(iii) The forces $F_f$ being thus determined, for either Class I or Class II, the cross-sectional area of the cables is obtained:

$$A_c = \frac{F_f}{T + \Delta T}$$

where $T$ is the permanent stress given as a fixed parameter, and where $\Delta T$ is the increase in the tension, caused by the loss of compression in the concrete local to the cables.

In the case of Class I, let it again be assumed that the stress at the bottom fibre is equal to the permissible limit $R_1$ under minimum loading. $R_1$ can be taken as equal to $0.4R_{bm\; cyl}$. The stress variation local to the cable, between the minimum and maximum loading conditions, is equal to $\Delta \sigma_c = R_1(1 - d'/h)$, $d'$ being the distance from the cable to the bottom fibre. Then:

$$\Delta T_1 = mR_1 \left( 1 - \frac{d'}{h} \right)$$

(3)

$m = E_a/E_b$ is usually of the order of 5.

In the case of Class II, the stress resulting from the strain in the concrete before cracking must be added to the excess tension. $C$ being the factor defined in Chapter IX, Section 7, equal to the ratio of the total strain, including pseudo-plastic phenomena, to the elastic strain at a stress $R_b^*$, the strain at the bottom fibre is $\varepsilon'_f = C(R_b^*/E_b)$.

The strain local to the cable is a little less. It can be taken that $\varepsilon'_c = 0.8\varepsilon'_f$. A value of 2.5 has been taken for $C$ (Chapter IX, Section 7).
Therefore \( \varepsilon'_c = 2(R'_b*/E_b) \) can be used. This results in a supplementary excess stress equal to \( 2(R'_b*/E_b)E_a = 2mR'_b* \).

Therefore:

\[
\Delta T_{ll} = m \left[ R_1 \left( 1 - \frac{d'}{h} \right) + 2R'_b* \right]
\] (4)

These evaluations are approximate only and, if considered necessary, they can be evaluated more precisely in particular applications.

**Limit state c**

This is the limit state of failure of the bottom flange under compression of the concrete with minimum loading.

Failure occurs in this manner if the stress at the bottom flange reaches the compressive strength \( R_b* \), under the action of a force equal to the prestressing force \( F \), exerted at a point \( E_p \) which is coincident with the centre of compression in this particular loading condition. (Minimum loading is denoted here by the symbol \( p \). More often than not, at least with statically-determinate beams, this is the permanent loading.)

Point \( E_p \) is obtained from the point of application \( E_o \) of the prestressing force (that is, the point of intersection of the section with the cable, in the case of statically-determinate beams) by the displacement \( E_oE_p = M_p/F \).

It is assumed that, in the case of limit state \( c \), the compressive stress is uniform on the part of the section having the centre of compression \( E_p \) as its centroid.

In order to obtain the most favourable conditions, the loads must be reduced (\( \gamma_p < 1 \)) and the prestressing force must be increased (\( \gamma_o > 1 \)).

The coefficients \( \gamma_p \) and \( \gamma_o \) must be taken into consideration when evaluating the displacement \( E_oE_p \). The design value for this displacement is:

\[
E_oE_{p*} = \frac{\gamma_sM_p}{\gamma_oF}
\]

Hence the design point \( E_{p*} \) is obtained.

Let \( S_b \) be the cross-sectional area of the section with \( E_{p*} \) as its centroid (hatched in Fig. 11).

For the bottom flange to satisfy exactly the safety criteria in this limit state, it is necessary to have \( \gamma_p F/S_b = R_{bb}/\gamma_b \), or

\[
S_b = \frac{\gamma_o \gamma_b (F/R_{bb})}{\gamma_p}
\]

Those parts of the bottom flange not comprised within the web of the section (shown cross-hatched in Fig. 11) must be designed such that:

(a) \( S_b \) has the value given in the foregoing;
(b) \( E_{p*} \) is the centroid of the area.
Fig. 11. Limit state of failure at the bottom fibre under minimum loading conditions.

As will be seen in Section 5, this determines the depth \( y' \) of the compressive zone, and the size of the flange.

The design must allow for the cable holes, and the complementary steel; the cross-sectional area of the latter must be multiplied by a suitable modular ratio \( m \).

**Limit state d (cracking of the top flange under minimum loading)**

This state corresponds to the same loading conditions as for state \( c \), but with different safety factors. Generally, \( \gamma_o \) and \( \gamma_p \) are taken as being equal to unity.

The centre of compression \( E^*_p \) is obtained in the same way as for state \( c \) \( (E_oE^*_p = \gamma_pM_p/\gamma_oF) \).

With \( \gamma_o = \gamma_p = 1 \), the design position of the centre of compression coincides with the effective position under nominal loading. Nevertheless, the position of the centre of compression must be such that the conditions which correspond to the limit state under consideration are satisfied. The centre of compression \( E^*_p \) must not be too low, so that the top fibre is not subjected to excessive tensile strain.

With Class I prestress, \( E^*_p \) must lie above the lower boundary of the central core \( G_ch. \)
With Class II prestress, $E_p^*$ must lie above a limit point $G'_b$ which is below $G_b$.

Again, since $M^t$ cracking Class II = $M^t$ cracking Class I + I + $\overline{WR}'_b$ ($\overline{W}$ being the specific modulus of resistance to cracking corresponding to the top flange), the point $G'_b$ is obtained from $G_b$ by the displacement $\overline{WR}'_b*/\gamma_o F$. Hence the position of $G'_b$ can be obtained if tables giving the values of $\overline{W}$ and $\overline{W}'$ are available.

Fig. 12. Limit state of cracking at the top fibre under minimum load.

It is seen in Section 6 how $\overline{W}$ can be calculated directly, thus obtaining the position of $G'_b$.

The position of $G_b$ is easily determined, since it is the lower boundary of the central core which is now defined by the design to satisfy states $a$, $b$ and $c$.

With sufficient accuracy, it is also possible to consider that $G_t$ coincides with the centroid of the bottom portion of the section, limited to two-thirds of the depth. Should the centre of compression $E_p^*$ lie below $G_b$ or $G'_b$ (according to the class considered), the design should be corrected, or the level of the cable raised.

Limit state $c$

This is the limit state of failure of the bottom flange, as in the case of $c$, but under initial prestress (with accompanying loads), and with different values of the safety factors $\gamma_{ob}$, $\gamma_p$ and $\gamma_b$. 
If $F_i$ is the initial prestressing force, the area of concrete required to resist it is:

$$S_i = \frac{\gamma_{ci}F_i}{R_{bk} \gamma_b} = \gamma_{ci} \frac{F_i}{R_{bk}}$$

This area $S_i$ is marked off from the lower part of the section (due account being taken of the holes and of the complementary steel, whose cross-sectional area is multiplied by a suitable modular ratio $m$). The centroid of this area, shown hatched in Fig. 13, is $G_i$.

![Fig. 13.](image)

The displacement $E_o E^*_{i}$ (the design displacement) from the centre of compression is also calculated ($E_o E^*_{i} = \gamma_p M_p / \gamma_{ci} F_i$, $M_p$ being the moment due to the attendant loading). Hence the centre of compression $E^*_{i}$ is obtained. $E^*_{i}$ must lie above $G_i$.

Very often, if all the cables are tensioned initially, $E^*_{i}$ lies below $G_i$. The safety criterion is then not satisfied (because the area $S'_i$, with $E^*_{i}$ as its centroid, is less than $S_i$). Prestress has then to be applied in stages, and only some of the cables tensioned such that the above condition is satisfied ($E^*_{i}$ above $G_i$, the displacement $E_o E^*_{i}$ and the area $S_i$ being evaluated by taking as $F_i$ that force which is exerted by the number of the cables which are tensioned).
Limit state \( f \) (cracking of the flange under initial prestress)
This is checked in the same way as for limit state \( d \), but under conditions of initial prestress with its attendant loading, and with safety factors which are appropriate to this particular limit state.

*Design of the complementary reinforcement.* Having determined the profile of the section, the ASP regulation may be applied (complementary steel sufficient to balance half the tensile stress resultant, at a stress equal to 60\% of its elastic limit).

This rule has the disadvantage of referring to elastic theory calculations, whereas the design is conducted on the basis of limit states. It is, however, accepted here because of the simplicity of the calculations involved (see Section 6).

*Choice of safety factors.* In the example studied below, the values chosen for the various factors \( \gamma \) are as shown in the following table: \([\gamma_p\) applies to minimum loading, \( \gamma_s \) to the maximum load, \( \gamma_o \) to the prestressing forces, \( \gamma_b \) to the ultimate compressive strength of the concrete, \( \gamma'_b \) to the tensile strength of the concrete, \( \gamma_a \) to the strength of the steels (prestressing steel and complementary steel)].

<table>
<thead>
<tr>
<th>Limit state</th>
<th>( \gamma_s )</th>
<th>( \gamma_p )</th>
<th>( \gamma_o )</th>
<th>( \gamma_b )</th>
<th>( \gamma'_b )</th>
<th>( \gamma_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Failure of top flange under maximum loading</td>
<td>1.4</td>
<td>1.4</td>
<td>1.5</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cracking of bottom flange under maximum loading</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Failure of bottom flange under minimum loading</td>
<td>1 ± 0.1</td>
<td>1.3</td>
<td>1.5</td>
<td>1.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cracking of top flange under minimum loading</td>
<td>1</td>
<td>1</td>
<td>1.2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Failure of bottom flange under initial prestress</td>
<td>1 ± 0.1</td>
<td>1.15</td>
<td>1.5</td>
<td>1.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cracking of top flange under initial prestress</td>
<td>1</td>
<td>1</td>
<td>1.2</td>
<td>1.15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Other values could be chosen, but the design methods would remain the same.†

† See Section 7 for the effect of the values of \( \gamma \) on the design.
4. Example. Known values

Consider the same beam designed by the ‘mixed’ method given in Section 2.

Span: 48 m, depth: 2·80 m, width at top: 1·75 m, web thickness: 0·18 m.
Permanent load, additional to self-weight: \( q = 0\cdot208 \text{ t/m} (M_q = 60 \text{ t/m}) \).
Live load: \( s = 2\cdot71 \text{ t/m} (M_s = 840 \text{ t/m}) \).

Concrete: 28-day compressive strength: \( R_{bm\ cube} = 450 \text{ kg/cm}^2 \), or
\( R_{bm\ cyl.} = 360 \text{ kg/cm}^2 \).

Characteristic strength \( R_{bk} = 360 \times 0\·83 = 300 \text{ kg/cm}^2 \).
Design strength \( R^*_b = 300/1\cdot5 = 200 \text{ kg/cm}^2 \) (2 000 t/m²).

Compressive strength at time of prestress: two-thirds of the above, or
\( R^*_{bL} = 133 \text{ kg/cm}^2 \) (1 330 t/m²).

Tensile strength measured on prism \([3\cdot6(M/b^3)]\): \( R'_{bk} = 30 \text{ kg/cm}^2 \).
With \( y'_b = 1\cdot2 \), design strength \( R'^*_b = 25 \text{ kg/cm}^2 \) (250 t/m²).

Cables: Rupture strength (guaranteed): 160 kg/mm²; design strength: 160/1·15 = 139 kg/mm².

Twelve 7 mm diameter cables are used (461 mm²). Design strength of one cable: 64 t.

Reinforcement: Elastic limit \( \sigma'_e = 50 \text{ kg/mm}^2 \); design elastic limit
\( \sigma'^*_e = 50/1\cdot15 = 43\cdot6 \text{ kg/mm}^2 \).

The total force in all the cables at failure is denoted by \( F_r \); \( F \) denotes the permanent prestressing force; \( F_f \) denotes the force at the time of cracking; and \( F_i \) denotes the initial prestressing force.

For each cable:

- permanent force 43 t approx. (93 kg/mm²).
- initial prestressing force 55·4 t (120 kg/mm²).

From the results of preliminary calculations, the moment due to self-weight \( p \) is taken as \( M_p = 650 \text{ tm} \) if the beam is designed for Class I, and as \( M_p = 615 \text{ tm} \) if the beam is designed for Class II.

5. Calculation for Class I (total prestress)

(i) Limit state of failure under maximum loading (design of the top flange and of the number of cables required to prevent failure). (The calculation is the same as in Section 2)

The distance of the cables from the bottom fibre is taken as \( d' = 0\cdot10 \text{ m} \),
so that the effective depth is \( h_1 = 2\cdot80 - 0\cdot10 = 2\cdot70 \text{ m} \). The lever arm at failure is taken as \( z_r = 0\cdot95h_1 = 2\cdot56 \text{ m} \) (these values must be confirmed in the final check calculation).
Moment under maximum loading:

\[ M_p + M_q + M_s = 650 + 60 + 840 = 1550 \text{ t/m} \]

It is therefore necessary to have \( M_r \geq \gamma_s \times 1550 = 1.4 \times 1550 = 2160 \text{ t/m} \).

Hence:

\[ F_r = \frac{M_r}{z_r} = \frac{2160}{2.56} = 845 \text{ t} \]

To resist this force, the area of concrete in compression must be equal to \( 845/2000 = 0.422 \text{ m}^2 \), where 2000 t/m² is the design compressive strength of the concrete.

For the neutral fibre to lie within the top flange, the thickness of the flange must be equal to 0.24 m (1.75 \( \times \) 0.24 \( \approx \) 0.422 m²). However, it was seen in Section 2 that this condition is not necessary, and that, by accepting a slight reduction in the value of the lever arm, the thickness could be reduced to 0.20 m.

The cross-sectional area of the flange is then: 1.75 \( \times \) 0.20 = 0.350 m². To obtain the compression area of 0.422 m², the additional area needed is 0.072 m². Therefore part of the web must be utilised, equal to 0.40 m (0.40 \( \times \) 0.18 = 0.072 m²) (Fig. 14).
The distance of the centroid of the area in compression from the top fibre is equal to:

\[
\frac{0.350 \times 0.10 + 0.072 \times 0.40}{0.422} = 0.15 \text{ m}
\]

The lever arm is: \(h_1 - 0.15 = 2.70 - 0.15 = 2.55 \text{ m}\), which is very close to the estimated value.

The force \(F_r\) stays very nearly equal to 845 t. In any event, it is not this condition which determines the number of cables required in the case of Class I.†

(ii) Limit state of loss of compression at the bottom fibre under maximum loading (determination of the required prestressing force)

It is this limit state which replaces, for Class I, the limit state of cracking under maximum loading. It is characterised by the fact that the centre of compression lies at the top boundary of the central core.

![Fig. 15. Limit state of loss of compression.](image)

As mentioned above (Section 3, Fig. 8), this top boundary very nearly coincides with the centroid \(G_s\) of the area of the top two-thirds of the section, which in this case lies at a distance of \(\frac{2}{3} \times 2.80 = 1.87 \text{ m}\). from the top (area shown hatched in Fig. 15).

† In this present case, where the mechanical percentage \(\sigma = (F_r/bh_1R)\) is very small \([(845/1.75 \times 2.70 \times 2.000) = 0.089, \lambda = 1]\; the simplified calculation is all the more adequate since the number of cables required is not determined by the value of \(F_r\). In those cases where \(\sigma\) is appreciably greater, it might be necessary to use the detailed methods of calculation given in Chapter IX (Section 13, Fig. 34).
The distance of $G_s$ from the top fibre is equal to:

$$\frac{1.75 \times 0.20 \times 0.10 + 0.18 \times 1.67 \times (0.20 + 0.83)}{1.75 \times 0.20 + 0.18 \times 1.67} = \frac{0.347}{0.652} = 0.53 \text{ m}$$

Lever arm $E_o G_s = 2.80 - (0.53 + 0.10) = 2.17 \text{ m}.$

Required force at the time of loss of compression $(F_f)$:

$$F_f = \frac{\gamma_s M}{E_o G_s}$$

$M$ (moment under maximum loading) = 650 + 60 + 840 = 1 550 tm;

$\gamma_s = 1.$

Hence:

$$F_f = \frac{1 550}{2.17} = 715 \text{ t}$$

The required cross-sectional area for the cables is $A_c = F_f/(T + \Delta T)$, where $\Delta T$ is the excess tension due to the loss of compression.

Denoting by $R_1$ the permissible conventional stress ($R_1 = R_{bm \text{ cube}}/3$), and as stated in Section 3 (Fig. 10), it can be assumed that:

$$\Delta T = m R_1 \left(1 - \frac{d'}{h}\right)$$

with $m = 5$.

In this case:

$$R_1 = \frac{450}{3} = 150 \text{ kg/cm}^2$$

$$\Delta T = 5 \times 150 \left(1 - \frac{0.10}{2.80}\right) \approx 700 \text{ kg/cm}^2 \text{ (T kg/mm}^2\text{)}$$

With $T = 93 \text{ kg/mm}^2$, $T + \Delta T \approx 100 \text{ kg/mm}^2$.

Therefore: $A_c = 715 \text{ 000}/100 = 7 150 \text{ mm}^2$.

Therefore $7 150/461 = 15.5$ cables are required.

Hence 16 cables are chosen.

(iii) Limit state of failure of the bottom flange under minimum loading (design of the bottom flange)

It has been found that $F_f = 715 \text{ t}$. 
The permanent force† is \( F = 715 \times \frac{T}{(T + \Delta T)} = 715 \times \frac{93}{100} = 665 \text{ t} \).

The moment in this particular state is:

\[
M_p + M_q = 650 + 60 = 710 \text{ tm}
\]

The chosen safety factors are \( \gamma_p = 0.9, \gamma_o = 1.3 \).

Design force: \( F^* = 665 \times 1.3 = 865 \text{ t} \).

Design moment: \( M^* = 710 \times 0.9 = 639 \text{ tm} \).

The design strength of the concrete is \( R^* = 2000 \text{ t/m}^2 \).

At the limit state of failure, the lower portion of the section is in uniform compression over a depth \( y \).

The depth must be such that:

(a) the area corresponding to \( y \) is equal to

\[
S_b = \frac{F^*}{R^*} = \frac{865}{2000} = 0.432 \text{ m}^2
\]

(b) the centroid of \( S_b \) coincides with the design centre of compression \( E^*_p \), which is obtained from \( E_o \) by the displacement

\[
E_oE^*_p = \frac{M^*}{F^*} = \frac{639}{865} = 0.74 \text{ m}
\]

The holes for the cables, and, if applicable, the complementary steel, must be taken into consideration in the calculation of areas (and of moments of areas). (The area of the complementary steel is multiplied by a suitable modular ratio \( m \).)

There are 16 cable holes, each 15.7 cm\(^2\) in cross-sectional area. The total area is therefore 250 cm\(^2\) (0.025 m\(^2\)) at 0.10 m from the bottom fibre.

It is not necessary to take into account the area of the complementary steel.

A preliminary design, similar to that in Fig. 2, enables the minimum dimensions of the bottom flange in order to accommodate the 16 cables to be determined. It is found that the flange must have a depth of 0.26 m.

With this depth, let the area of the bottom flange excluding the web be \( \Delta S \) (shown cross-hatched in Fig. 16).

† It is necessary to calculate as shown, and not to use the actual number of cables, which may be excessive. The design of the bottom flange is very sensitive to the value of \( F \), and it could be unduly increased by overestimating this force.
Taking moments about the centroid of $\Delta S$, situated at 0.13 m above the bottom fibre, for the centroid of $S_b$ to lie 0.74 m above the cable it is necessary to have:

$$0.18y \left( \frac{y}{2} - 0.13 \right) + [(-0.025) \times (-0.03)] = (0.74 - 0.03) \times S_b$$

Or, neglecting the effect of the holes:

$$0.09y^2 - 0.023\,4y = 0.71 \times 0.432 = 0.306$$

$$y^2 - 0.26y - 3.4 = 0$$

**Fig. 16.** Limit state of failure of the bottom flange under minimum loading (Class I).

Hence:

$$y = 1.97 \text{ m}$$

Then:

$$0.18y + \Delta S - 0.025 = S_b = 0.432 \text{ m}^2$$

Therefore:

$$\Delta S = 0.432 + 0.025 - 0.18y$$

$$= 0.457 - 0.18 \times 1.97 = 0.103 \text{ m}^2$$

or:

$$0.26 \text{ m} \times 0.40 \text{ m}$$
The breadth of the bottom flange must therefore be equal to 0.18 + 0.40 = 0.58 m.

(iv) Limit state of cracking of the top flange under minimum loading
The check on safety with regard to this limit state is tied to the position of the centre of compression relative to the lower boundary of the central core $G_t$.

$G_t$ is very nearly coincident with the centroid of that area which is limited to the bottom two-thirds of the depth. With the breadth of the bottom flange equal to 0.58 m, the position of $G_t$ results in the following calculation:

\[
\begin{align*}
0.58 \times 0.26 &= 0.151 \text{ m}^2 \\
0.18 (1.87 - 0.26) &= 0.290 \text{ m}^2 \\
\text{less holes} &= -0.025 \text{ m}^2 \\
\hline
0.416 \text{ m}^2
\end{align*}
\]

<table>
<thead>
<tr>
<th>Moment about the bottom fibre</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>at 0.13 m from the bottom</td>
<td>0.019 6 m$^3$</td>
</tr>
<tr>
<td>at 1.06 m from the bottom</td>
<td>0.308 0 m$^3$</td>
</tr>
<tr>
<td>at 0.10 m from the bottom</td>
<td>0.002 5 m$^3$</td>
</tr>
</tbody>
</table>

The position of $G_t$ is therefore at 0.325 $\frac{1}{0.416} = 0.78$ m above the bottom fibre, or 0.68 m above the cable. The centre of compression is $\gamma_p M / \gamma_o F$ above the cable, or, with $\gamma_p = \gamma_o = 1$, $(710 \text{ t/m})/665 \text{ t} = 1.07$ m above the cable.

Since the centre of compression is above the lower limit of the central core, the section is wholly in compression. The limit state under consideration does not therefore impose any conditions on the design.

(v) Limit state of failure of the bottom flange under initial prestress
The corresponding moment is $M_p = 650$ tm.

This limit enables the number of cables which can be initially tensioned during prestressing to be determined.

The chosen safety factors are $\gamma_p = 0.9$, $\gamma_{ol} = 1.15$.

The design strength is $R^*_b = 1330 \text{ t/m}^2$.

Let $F_t$ be the force which is exerted by the cables which are tensioned (in general, fewer than the total number of cables).

The area of concrete in compression which is required to resist this force is $S_l = F_t/R^*_b$.

Hence, when $F_t$ is known, the depth $y_l$ which would be in uniform compression if failure occurred at this load is obtained. Also, if failure occurs,
the centroid of the concrete in compression is coincident with the centre of compression $E^*_i$, which is defined by its distance from the cable:

$$E_o E^*_i = \frac{M^*_p}{F^*_i} = \frac{0.9 M_p}{F_i}$$

Taking $E^*_i$ as its centroid, the area $S'_i$ can be considered separately from the remainder of the section and calculated.

If $S'_i > S_i$, failure does not occur. Also, by calculating the depth $y'$ of the area $S'_i$ in uniform compression, failure does not occur if $y' > y_i$.

However, the calculation of $y'$ requires the solution of a second-order equation. The following calculation is easier.

With the force $F_i$ producing failure, then by definition $E^*_i$ is coincident with the centroid of $S_i$.

The problem is solved by trial and error: various values of $F_i$ are assumed, each one corresponding to a certain number of cables. Hence values of $E_o E^*_i$ are obtained.

To each value of $F_i$ would correspond, if failure occurred, a value of $S_i$, and therefore of $y_i$; $y_i$ is calculated and the distance $g_i$ of the centroid of $S_i$ above the bottom fibre is deduced.

Failure will occur when $y_i - d' = E_o E^*_i$ ($d'$ = distance of the cable from the bottom fibre).

By tabulating the values of $E_o E^*_i$ as a function of the number of cables, the value of $F_i$ for which the condition $g_i - d' + E_o E^*_i$ for failure occurs is found by interpolation.

It must be noted that, in the tabulation, $y_i$ is not the true depth of the zone in compression; this true depth would be $y'$ (not calculated for the tabulation), such that $E^*_i$ is the centroid of the area limited to $y_i$.

Denoting the area of the bottom flange outside the web by $\Delta S$ (cross-hatched in Fig. 16), then $\Delta S = 0.103$ m² [see (iii)].

If $S_i$ does not include any part of the top flange (that is, if $y_i < 2.60$ m), the value of the depth $y_i$ corresponding to $S_i$ is given by:

$$S_i = 0.103 - 0.025 + 0.18 y_i = 0.078 + 0.18 y_i$$ (allowance made for holes)

The following would apply if $y_i$ were greater than 2.60 m:

$$S_i = 0.103 - 0.025 + 0.18 \times 2.60 + (y_i - 2.60) \times 1.75$$

$$= 0.078 + 0.468 + 1.75 y_i = 4.004$$

The distance $g_i$ of the centroid of $S_i$ from the bottom fibre is easily calculated; the calculation must take the holes into account and, if applicable, the complementary steel, with a suitable modular ratio $m$. 
In this case $d' = 0.10$ m. The condition for failure is, therefore, $g_i - 0.10 = E_o E_i^*$. Each cable exerts a force of $55.4$ t (initial stress of 120 kg/mm$^2$ in the section considered).†

The corresponding moment is $M_p = 650$ tm, and $\gamma_p M_p = 0.9 \times 650 = 585$ tm.

The following table shows the results which are obtained, in terms of the number of cables tensioned.

<table>
<thead>
<tr>
<th>No. of tensioned cables</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_i$</td>
<td>498</td>
<td>554</td>
<td>609</td>
</tr>
<tr>
<td>$1.15 F_i$</td>
<td>572</td>
<td>640</td>
<td>702</td>
</tr>
<tr>
<td>$E_o E_i^* = \frac{585}{1.15} F_i$</td>
<td>1.02</td>
<td>0.91</td>
<td>0.83</td>
</tr>
<tr>
<td>$S_i = \frac{1.15}{1.330} F_i$</td>
<td>0.430</td>
<td>0.482</td>
<td>0.527</td>
</tr>
<tr>
<td>$\gamma_i$</td>
<td>1.96</td>
<td>2.24</td>
<td>2.49</td>
</tr>
<tr>
<td>$g_i$</td>
<td>0.82</td>
<td>0.96</td>
<td>1.08</td>
</tr>
<tr>
<td>$g_i - d' = g_i - 0.10$</td>
<td>0.72</td>
<td>0.86</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Failure occurs at a force which is between 554 t (10 cables) and 609 t (11 cables), since the difference $E_o E_i^* - (g_i - 0.10)$ changes sign in going from 10 to 11 cables.

Therefore 10 cables can be tensioned.

**Comment**

It was found that 12 cables could be tensioned, using the method based on elastic behaviour (Section 2, Class I). This difference is due to the fact that, in the calculation using limit states, and with the safety factors used, the size of the bottom flange (iii) is less than in the elastic case.

(vi) **Limit state of cracking of the top flange under initial prestress**

The factors $\gamma_p$ and $\gamma_{oi}$ are taken as being equal to unity. The centre of compression must lie above the lower limit, $G_b$, of the central core. $G_i$ is approximately 0.78 m above the bottom fibre [see (iv)].

† Not the initial stress at the anchors, which is higher because of the friction which must be overcome between the anchors and the section under consideration.
If 10 cables are tensioned, \( F_i = 554 \text{ t} \); therefore \( E_o E^*_{i} = 650/554 = 1.17 \text{ m} \).

The point \( E^*_{i} \) is \( 1.17 + 0.10 = 1.27 \text{ m} \) above the bottom fibre; the criterion for safety is therefore satisfied.

**Comment**

Because the number of cables which are tensioned must be a whole number, the initial prestressing force is here less than the force \( F_i \), which would cause failure in limit state \((v)\).

But it could happen that the two values are equal. The stress diagram in the compressive zone in limit state \((v)\) is then that diagram corresponding to the maximum plastic deformations in compression. But between limit states \((v)\) and \((vi)\), the difference lies in the different values of the safety factors adopted:

\[
\gamma_p = 0.9 \text{ in } (v), \quad 1 \text{ in } (vi)
\]

\[
\gamma_{ot} = 1.15 \text{ in } (v), \quad 1 \text{ in } (vi)
\]

The overall multiplication factor for the loads is about 25% less in state \((vi)\) than in state \((v)\).

Can it be the fact that this is sufficient to cause the stress distribution to pass from the form it takes in state \((v)\) to an elastic distribution in \((vi)\); that is, without any plastic deformation of the concrete? The criterion of safety, based on the position of the centre of compression relative to the central core as given above, rests on this hypothesis.

In fact, the assumption of a linear diagram for the limit state \((vi)\) can only be an approximation, but it is sufficient to guarantee safety against the risk (usually not very serious) which is involved.

The force \( F_i \), which would cause failure in limit state \((v)\), by interpolation of the values in the table, would equal about 572 t (corresponding to the condition \( g_i - d'' = E_o E^*_{i} \) at failure).

Then, in state \((vi)\), \( E_o E_i \) would be equal to \( M_p/F_i = 650/572 = 1.13 \text{ m} \); the point \( E^*_{i} \) would therefore be 1.23 m above the bottom fibre.

If the stress is calculated on the basis of elastic behaviour, the following results are obtained:

Properties of the section (allowance made for holes):

\[
S = 0.897 \text{ m}^2 \quad v = 1.06 \text{ m} \quad v' = 1.74 \text{ m} \quad I = 0.879 \text{ m}^4 \quad I/v = 0.83 \text{ m}^3
\]

\[
I/v' = 0.504 \text{ m}^3 \quad r^2/v = 0.925 \text{ m} \quad r^2/v' = 0.560 \text{ m}
\]

The eccentricity of the centre of compression \( E^*_{i} \) is

\[
1.23 - 1.74 = -0.51 \text{ m}
\]
The stress at the bottom fibre is:

$$\sigma' = \frac{572}{0.897} \left(1 + \frac{0.51}{0.56}\right) = 1220 \text{ t/m}^2$$

There is partial plasticity if the effective strength reduces to

$$\frac{R_{mk}}{\gamma_b} = 1330 \text{ t/m}^2$$

but this is still quite a long way from total plasticity. In other words, if the true stress diagrams are considered (and not the approximate diagrams such as the rectangular diagram), which correspond to the various stages of plasticity (Fig. 17), then diagram (a) corresponds to the final stage; diagram (b), in which the limit stress is only just reached at the bottom fibre, corresponds to a less advanced stage; and diagram (c) corresponds to a stage which is still less advanced.\(^\dagger\)

![Fig. 17. Various stages of plasticity.](image)

The present case corresponds roughly to diagram (b). In this case, and in the case of a rectangular section, the point of application of the resultant stress is distant 0.375x from the bottom, instead of 0.33x in the case of a diagram of triangular shape. In order to take possible plasticity into account, it would be sufficient in practice to take as the limit point, above which the centre of compression \(E_l\) must lie, a point \(\overline{G_l}\) which is slightly above the lower limit of the central core, at a distance about 10% greater

\(^\dagger\) See *Prestressed Concrete*, Volume II, Chapter XXXVI. Diagram (c) corresponds to the stage where the deformation of the bottom fibre is half of that corresponding to diagram (b).
from the bottom fibre. This correction is not absolutely necessary, but it can be made if judged useful.

It should also be noted, with respect to the position of \( G_i \), the lower limit of the central core, determined above, that the rule which states that this point coincides with the centroid of the bottom two-thirds is only an approximation.

Once the size of the section is known; that is, when stage (iii) has been calculated, it is not difficult to obtain the position of \( G_i \) with greater accuracy.

In this case, it is \( v' - (r^2/v) = 1.74 - 0.925 = 0.815 \) m from the bottom (instead of 0.78 m). The difference of 3.5 cm does not affect any of the dimensions.

6. Calculation for Class II (limited prestress)

(i) Limit state of failure under maximum loading (same calculation as in Section 2).

The distance of the cables from the bottom fibre is taken as being equal to \( d' = 0.12 \) m. The effective depth is \( h_1 = 2.80 - 0.12 = 2.68 \) m.

The lever arm at failure is taken as \( z_r = 0.95h_1 = 2.54 \) m (this value is eventually checked in the final calculation).

Moment under maximum loading:

\[
M_p + M_q + M_z = 615 + 60 + 840 = 1515 \text{ tm}
\]

It is therefore necessary to have:

\[
M_r \geq \gamma_s \times 1515 = 1.4 \times 1515 = 2120 \text{ tm}
\]

Therefore:

\[
F_r = \frac{M_r}{z_r} = \frac{2120}{2.54} = 835 \text{ t (13 cables)}
\]

To resist this force, the area of concrete in compression, using a design strength of 2000 t/m², is:

\[
\frac{835}{2000} = 0.417 \text{ m}^2
\]

If it is required that the neutral fibre should not lie outside the top flange, the thickness of the flange must be taken as 0.24 m

\[
(1.75 \times 0.24 = 0.417 \text{ m}^2)
\]
It has been seen (Section 2) that this condition is not essential, and that the flange thickness could be reduced to 0·20 m by accepting a slight reduction in the value of the lever arm. (This can require a correction of \( F_r \).)

The flange cross-sectional area is then: \( 1·75 \times 0·20 = 0·350 \text{ m}^2 \).

This is deficient by an amount equal to \( 0·067 \text{ m}^2 \) compared with the required area in compression (\( 0·417 \text{ m}^2 \)).

Therefore (see Fig. 14) a portion of the web to a depth of 0·37 m must be used in compression (\( 0·18 \times 0·37 = 0·067 \text{ m}^2 \)).

The centroid of the area (\( 0·417 \text{ m}^2 \)) is at the following distance from the top fibre:

\[
\frac{0·350 \times 0·10 + 0·067(0·20 + 0·185)}{0·417} = 0·145 \text{ m}
\]

The lever arm \( z_r \) becomes \( h_1 - 0·145 \text{ m} = 2·68 - 0·145 = 2·535 \text{ m} \), approximately equal to the estimated value. It is therefore not necessary to correct \( F_r \).

(ii) Limit state of cracking at the bottom fibre under maximum loading (approximate design for the required prestressing force)

The safety factors are taken as \( \gamma_p = \gamma_s = \gamma_o = 1 \).

The force which would be necessary in Class I prestress must first be calculated.

\( G_s \) being the centroid of the top two-thirds (which is considered as coincident with the top boundary of the central core), then approximately:

\[
F_{f1} = \frac{M_{\text{max}}^*}{E_o G_s} = \frac{M_{\text{max}}}{E_o G_s}
\]

(\( F_{f1} \) is the force in the limit state under consideration, allowing for the excess tension \( \Delta T \) in the cables).

\( G_s \) is 0·53 m distant from the top fibre [Section 5(ii)] and:

\[
E_o G_s = 2·15 \text{ m}
\]

\( M_{\text{max}} = 1515 \text{ t/m}. \) Therefore: \( F_{f1} = 1515/2·15 = 705 \text{ t} \).

Formula (2) of Section 3 (limit state b (ii)b) is used to obtain an approximate value of the force \( F_{f11} \) which is required in Class II:

\[
\frac{F_{f11}}{F_{f1}} = 1 - k \frac{R_{b}^* \Delta M}{R_1 M_{\text{max}}}
\]

where \( R_{b}^* \) is the design tensile strength, \( R_1 \) is the permissible compressive
stress \(R_1 = R_b \text{cube}/3\), and \(k\) is the ratio \(R''_b/R'\), where \(R''_b\) is the apparent tensile strength.

If tables of specific strength moduli are available (\(\overline{W}\) and \(\overline{W}'\)) (Chapter IX, Section 7), \(k = \overline{W}'/(\text{I}/\text{v}')\).

For the range of sections considered, \(k\) is of the order of 1.6.

\[
R' = 30/1.2 = 25 \text{ kg/cm}^2
\]

\(R_1\) is taken as being equal to 150 kg/cm².

\[
\Delta M \text{ (moment variation)} = M_{\text{max}} - M_{\text{min}} = M_s = 840 \text{ tm}
\]

Hence:

\[
\frac{F_{f\text{II}}}{F_{f\text{I}}} = 1 - 1.6 \times \frac{25}{150} \times \frac{840}{1510} = 0.85 \text{ approx.}
\]

Therefore:

\[
F_{f\text{II}} = 0.85 \times 705 = 600 \text{ t}
\]

This force is the force which is exerted at the moment causing cracking, with a stress equal to \(T + \Delta T\).

To evaluate \(\Delta T\), formula (4) of Section 3 is used:

\[
\Delta T_{\text{II}} = m \left[ R_1 \left(1 - \frac{d''}{h} \right) + 2R'_b \right]
\]

or, with \(m = 5\),

\[
\Delta T = 5 \left[ 150 \left(1 - \frac{0.12}{2.80} \right) + 2 \times 25 \right] = 975 \text{ kg/cm}^2
\]

or 10 kg/mm² approx. \(T\) is taken as being equal to 93 kg/mm² (permanent stress).

The permanent prestressing force is then given by:

\[
F = F_f \times \frac{T}{T + \Delta T} = 600 \times \frac{93}{103} = 540 \text{ t (13 cables)}
\]

All the approximations which are made can be adjusted later (position of \(G_s, k, \Delta T\ldots\)). Only an approximate solution is required from this first trial design, and it is generally reasonably accurate.

(iii) Limit state of failure of the bottom flange under minimum loading
The chosen safety factors are \(\gamma_s = 1.3, \gamma_p = 0.9\).
The value of $F$ obtained for the previous limit state is used; that is, $F = 540$ t.

Moment under minimum loading $= M_p + M_q = 615 + 60 = 675$ tm.
Design force $F^* = 1.3 \times 540 = 702$ t.
Design moment $M^* = 0.9 \times 675 = 609$ tm.

It is required to design the bottom flange; that is, that portion of the flange which excludes the thickness of the web, and which is cross-hatched in Fig. 18.

Fig. 18. Limit state of rupture at the bottom flange under minimum loading (Class II).

The centre of compression goes to $E^*_{p}$, defined by:

$$E_oE^*_{p} = \frac{M^*}{F^*} = \frac{609}{702} = 0.87 \text{ m}$$

At the moment causing failure, the area in uniform compression must have:

(a) an area $S_b = \frac{F^*}{R^*} = \frac{702}{2000} = 0.351 \text{ m}^2$,

(b) a centroid which is coincident with $E^*_{p}$, at a distance of 0.99 m from the bottom.

These conditions define the depth $y$ of the zone in compression. The thickness of the bottom flange is taken as being equal to 0.26 m (based on a preliminary arrangement to determine the space required for the cables);
the area of the bottom flange, excluding the thickness equivalent to the web, is denoted by $\Delta S$.

The holes for the cables, as well as the complementary steel, must be taken into consideration. There are 13 holes, each 15.7 cm$^2$ in cross-sectional area, giving a total area of about 200 cm$^2$ (0.020 m$^2$). It is estimated that the complementary steel required is equivalent to six 12 mm diameter bars (to be adjusted later), with an area of approximately 7 cm$^2$. Since a state of failure in which $E_b$ is low is concerned, a high value of $m$ can be chosen. This is taken as $m = 15$; therefore $mA_a = 15 \times 7 = 105$ cm$^2$ (0.010 m$^2$) approximately, at 0.10 m from the bottom.

Taking moments with respect to the centroid of $\Delta S$, at 0.13 m from the base (Fig. 18):

$$0.18y \left(\frac{y}{2} - 0.13\right) + [-0.02 \times (-0.01)] + 0.01 \times (-0.03)$$

$$= 0.99 - 0.13 \times S_b = 0.86 \times 0.351$$

Neglecting the holes for the cables and the complementary steel:

$$0.09y^2 - 0.023 \times 4y = 0.302$$

or

$$y^2 - 0.26y - 3.34 = 0$$

Hence $y = 1.95$ m.

Also:

$$0.18y + \Delta S - 0.020 + 0.010 = S_b = 0.351$$

Hence:

$$\Delta S = 0.351 + 0.010 - 0.18 \times 1.95$$

$$= 0.351 + 0.010 - 0.350 = 0.011 \text{ m}^2$$

or 0.26 $\times$ 0.04 m.

From strength considerations alone, a bottom flange measuring 0.26 $\times$ 0.22 would be sufficient. This is not sufficient to accommodate the cables. The flange required is 0.26 $\times$ 0.42 m.

In other words, the flange is in this case determined by considerations of minimum size and not by the limit state of cracking.

(iv) Correction to the design obtained from the limit state of cracking; that is, to the calculation of the prestressing force obtained approximately in (ii)

It is recalled that, very approximately (Chapter IX, Section 7):

Moment at cracking (Class II) = moment due to loss of compression (Class I) + specific moment at cracking
The specific moment at cracking is the moment at cracking which is calculated for the section without prestress, taking into consideration the limit cracking strain \( \epsilon_f = 2.5(R_{b}^*/E_b) \) and the shape of the stress diagram; the evaluation of the specific moment is explained below.

Consequently, if \( F_f \) is the required prestressing force (at the stress \( T + \Delta T \) corresponding to this particular limit state), and if \( G_s \) is the top boundary of the central core, then:

\[
M_{fII} = E_o G_s \times F_f + \text{specific moment at cracking}
\]

Since it is required that \( M_{fII} \) should be greater than or equal to the design moment \( M^* \), that limit value of the prestressing force is given by:

\[
E_o G_s \times F_f = M^* - \text{specific moment at cracking}
\]

(a) Calculation of the specific moment at cracking. The stress diagram (in the section without prestress) is shown in Fig. 19. The stress in the tensile zone is uniform throughout the total depth of this zone and it is equal to \( R_{b}^* \) (Chapter IX, Section 7). The shape of the diagram is triangular in the zone under compression, and the maximum stress at the top fibre is equal to \( 2.5R_{b}^*(y/y') \)† where \( y' = \text{depth of the zone in tension} \), and \( y = \text{depth of the zone in compression} \).

† In the case of I-beams, \( y' \) is taken as being equal to \( x' \) (\( x' \) is the height of the neutral axis from the bottom). In the case of a beam of rectangular cross-section, \( y' \) should be taken as being equal to \( \frac{1}{2}x' \), and the equations modified accordingly (see Chapter IX, Fig. 14).
Since the section is not prestressed, the resultant of the compressive forces is equal to the resultant of the tensile forces. Taking the cable holes (0.020 m²) and the complementary steel into consideration

\[(A_a = 7 \text{ cm}^2, \ m = 5, \ mA_a = 35 \text{ cm}^2 = 0.0035 \text{ m}^2)\]

then:

resultant tensile force = \[0.26 \times 0.24 + 0.184y' - 0.20 + 0.0035 R'_b^*\]

\[= [0.046 + 0.18y'] R'_b^*\]

resultant compressive forces

\[= \left[\frac{1}{2} \times 1.75y - \frac{1}{2} \times 1.57(y - 0.20) \times \frac{y - 0.20}{y}\right] \times 2.5R'_b^* \frac{y}{y'}\]

\[= [0.875y^2 - 0.785(y - 0.20)^2] \times \frac{2.5}{y'} R'_b^*\]

Equating the two values:

\[0.046y' + 0.18y'^2 = 2.5 [0.875y^2 - 0.785(y - 0.20)^2]\]

Since \(y' = (2.80 - y)\), the following equation is obtained:

\[0.045y^2 + 1.809y - 1.6185 = 0\]

Thus \(y = 0.866 \text{ m}\) is obtained, hence \(y' = 1.934 \text{ m}\).

The value of the resultant tensile force is, therefore:

\[0.046 + 0.18 \times 1.934] R'_b^* = 0.395R'_b^*\]

The value of the resultant compressive force is:

\[0.875 \times 0.866^2 - 0.785 \times 0.666^2 \times \frac{2.5}{1.934} R'_b^* = 0.398R'_b^*\]

The check on the equality of the two values is acceptable.

The line of action of the resultant tensile force is through the centroid of the tensile zone (allowing for holes and complementary steel). It is found that the centroid is 0.86 m above the bottom fibre, or

\[1.934 - 0.86 = 1.074 \text{ m}\]

from the neutral axis.
The line of action of the resultant compressive force is at a distance from the neutral axis which is equal to:

\[
\frac{1.75y \times (2y/3) - 1.57[(y - 0.2)^2/y] \times 3(y - 0.2)}{1.75y - 1.57[(y - 0.2)^2/y]} = \frac{2}{3} \frac{1.75y^3 - 1.57(y - 0.2)^3}{1.75y^2 - 1.57(y - 0.2)^2}
\]

With \( y = 0.866 \) m, this distance is equal to 0.728 m.
The lever arm (distance between the resultants) is:

\[1.074 + 0.728 = 1.802 \text{ m}\]

Therefore:

specific moment at cracking = \(0.395R'_b\times 1.802\)

\[= 0.712R'_b\]

**COMMENT**
The specific cracking modulus is \(\overline{W'} = 0.719 \) m³. The properties of this section can now be calculated. They are as follows (allowing for holes):

\[S = 0.861 \text{ m}^2 \quad v = 0.99 \text{ m} \quad v' = 0.81 \text{ m} \quad I = 0.782 \text{ m}^4\]

\[\frac{I}{v} = 0.79 \text{ m}^3 \quad \frac{I}{v'} = 0.431 \text{ m}^3 \quad \frac{r^2}{v} = 0.92 \text{ m} \quad \frac{r^2}{v'} = 0.49 \text{ m}\]

Also, \( k = [\overline{W'}/(I/v')] = (0.712/0.431) = 1.65. \)

If tables of \(\overline{W'}\) (or \(k\)) are available, the calculation conducted in equation (a) above is not necessary. The specific moment at cracking is obtained immediately from \(\overline{W'R'_b}\) or \(k(I/v'R'_b)\).

But it is not necessary to know \(k\) in order to calculate the moment; it can be calculated directly as shown above (it is in fact this calculation which gives the values of \(k\), from which tables are compiled).

(b) *Calculation of the prestressing force* \(F_f\). The force results from equation (a) above:

\[E_oG_s \times F_f = M^* \text{ – specific moment at cracking}\]

The values of \(M_p\), estimated at 615 tm, can now be corrected.
\(S = 0.861 \text{ m}^2\). Therefore \(p = 0.861 \times 2.4 = 2.07 \text{ t/m}\), \(M_p = 2.07 \times (48^2/8) = 600 \text{ tm}\).
The value of $E_sG_s$ can also be corrected. $G_s$ is the top boundary of the central core, and its distance from the top fibre is equal to:

$$v - \frac{r^2}{v'} = 0.99 - 0.49 = 0.50 \text{ m (in place of the estimated 0.53 m)}$$

$$E_sG_s = 2.80 - (0.50 + 0.12) = 2.18 \text{ m (in place of 2.15 m)}$$

$$M^* = M_p + M_q + M_s = 600 + 60 + 840 = 1500 \text{ tm}$$

With $R'_{p*} = 250 \text{ t/m}^2$, the specific moment of cracking is equal to:

$$0.719 \times 250 = 180 \text{ tm approx.}$$

Therefore:

$$2.18F_f = 1500 - 180 = 1320 \text{ tm}$$

and

$$F_f = \frac{1320}{2.18} = 605 \text{ t}$$

[in place of 600 t obtained by the approximate calculation in (ii)]

$$T + \Delta T = 103 \text{ kg/mm}^2$$

$$T = 93 \text{ kg/mm}^2$$

Permanent prestressing force:

$$F = 605 \times \frac{93}{103} = 544 \text{ t}$$

Therefore $(544/43) = 12.7$ cables are required, say 13.

**Comment**

It can be shown that the approximate method which has been followed, consisting of the calculation of the specific moment at cracking (in the section without prestress) and solving equation (a), gives very nearly the same results as the 'exact' method. With this method, the following must be written:

(a) that the resultant compressive force is equal to the resultant tensile force $+ F_f$;

(b) that the moment of the stresses taken about the cable is equal to the design value of the moment; therefore two equations are available to determine the two unknowns $y$ and $F_f$. 
Let \( B \) and \( B' \) be the absolute values of the resultant compressive and tensile forces. Denoting the distances between these resultants and the cable \( z_c \) and \( z'_c \), the two equations are:

\[
B = B' + F_f \tag{a}
\]

\[
Bz_c - B'z'_c = M^* \tag{\beta}
\]

Now:

\[
B = \frac{1}{2} \left[ 1.75y^2 - 1.57(y - 0.20)^2 \right] \frac{2.5}{y'} R'_b^* \]

\[
z_c = y' - d' + \frac{2}{3} \frac{1.75y^3 - 1.57(y - 0.20)^3}{1.75y^2 - 1.57(y - 0.20)^2}
\]

\[
B' = (0.046 + 0.18y')R'_b^*
\]

\[
z'_c = \frac{0.006 + 0.18(y'^2/2)}{0.046 + 0.18y'} - 0.12
\]

---

**Fig. 20.** Stress diagrams (in the prestressed section).

Substituting in equation (\( \beta \)) and noting that \( y' = 2.80 - y \), an equation in \( y \) is obtained. Hence \( y \) is obtained, and \( F_f \) follows from equation (\( \alpha \)).

Equation (\( \beta \)) for \( y \) is very tedious to solve and it is preferable to solve it by trial and error by assuming several values of \( y \) until the equation is satisfied (\( M^* = 1500 \text{ tm} \)).
With $R'_{s^*} = 250$ t/m², the following results are obtained:

<table>
<thead>
<tr>
<th>$y$</th>
<th>1.90</th>
<th>1.92</th>
<th>1.94</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y'$</td>
<td>0.90</td>
<td>0.88</td>
<td>0.86</td>
<td>m</td>
</tr>
<tr>
<td>B</td>
<td>619</td>
<td>646</td>
<td>669</td>
<td>t</td>
</tr>
<tr>
<td>B'</td>
<td>52</td>
<td>51</td>
<td>50</td>
<td>t</td>
</tr>
<tr>
<td>$z_c$</td>
<td>2.39</td>
<td>2.38</td>
<td>2.37</td>
<td>m</td>
</tr>
<tr>
<td>$z'_c$</td>
<td>0.26</td>
<td>0.25</td>
<td>0.24</td>
<td>m</td>
</tr>
<tr>
<td>$Bz_c$</td>
<td>1475</td>
<td>1530</td>
<td>1580</td>
<td>tm</td>
</tr>
<tr>
<td>$B'z'_c$</td>
<td>13</td>
<td>13</td>
<td>12</td>
<td>tm</td>
</tr>
<tr>
<td>$Bz_c - B'z'_c$</td>
<td>1462</td>
<td>1517</td>
<td>1568</td>
<td>tm</td>
</tr>
<tr>
<td>$F_f = B - B'$</td>
<td>567</td>
<td>595</td>
<td>619</td>
<td>t</td>
</tr>
</tbody>
</table>

Equation ($\beta$), $Bz_c - B'z'_c = 1500$ tm, is therefore satisfied, by interpolation, when $y = 1.91$ m approximately. Then $y' = 0.89$ m, and $F_f = 586$ t.

The permanent prestressing force $F$ is therefore equal to $586 \times 93/103 = 526$ t in place of 544 t. The error is therefore 3.5%.

(The stress diagram in Fig. 20 is, of course, the 'true' diagram, the diagram in Fig. 19 being for a section without prestress.)

The approximate method which has been used is much simpler. It provides the solution immediately if tables of the specific strength moduli $\overline{W}$ and $\overline{W}'$ are available.

The 3.5% error is insignificant compared to the other causes of uncertainty. It is therefore the approximate method which is recommended.

(v) Limit state of cracking of the top flange under minimum loading

$$
\gamma_o = \gamma_p = 1
$$

$$
M = 600 + 60 = 660 \text{ tm}
$$

$$
F = 544 \text{ t}
$$

The centre of compression is $(660/544) = 1.21$ m above the cable, and therefore at 1.33 m from the bottom. The distance of the lower boundary of the central core from the bottom fibre is equal to

$$
\nu' = \frac{r^2}{v} = 1.81 - 0.92 = 0.89 \text{ m}
$$

[Refer to the section properties in (iv).] The centre of compression is therefore above this lower boundary, and the safety criterion is amply satisfied.
Limit state of failure under initial prestress and resultant loading

The following are used:

\[ \gamma_p = 0.9 \]
\[ \gamma_{oi} = 1.15 \]
\[ M_p = 600 \text{ tm} \]

As in Section 5 (Class I), various values of \( F_i \) are tried. The value of the area \( S_i \) in compression, sufficient to resist \( F_i \) is: \( S_i = (1.15F_i/R^*_{b}) \).

\( R^*_{b} = 1 \text{ 330 t/m}^2 \) at the time of prestress.

Hence the depth \( y'_i \) corresponding to \( S_i \), and the position \( G_i \) of the centroid of \( S_i \) (distance of the centroid from the bottom fibre = \( g_i \)).

Failure occurs if the centre of compression \( E_i \), defined by the displacement \( E_oE^*_{v_i} = (\gamma_pM_p/\gamma_{oi}F_i) = [(0.9 \times 600)/1.15F_i] \), is below \( g_i \).

The condition for failure is \( g'_i - d' = E_oE^*_{v_i} \) with \( d' = 0.12 \).

The following results are obtained (force in each cable 55.4 t):

<table>
<thead>
<tr>
<th>No. of cables tensioned</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_i )</td>
<td>444</td>
<td>498</td>
<td>554</td>
</tr>
<tr>
<td>1.15 ( F_i )</td>
<td>511</td>
<td>572</td>
<td>640</td>
</tr>
<tr>
<td>( E_oE^*_{v_i} )</td>
<td>( \frac{540}{1.15F_i} )</td>
<td>1.05</td>
<td>0.94</td>
</tr>
<tr>
<td>( S_i ) = ( \frac{1.15F_i}{1.330} )</td>
<td>( 0.384 )</td>
<td>( 0.430 )</td>
<td>( 0.480 )</td>
</tr>
<tr>
<td>( y'_i )</td>
<td>1.90</td>
<td>2.14</td>
<td>2.44</td>
</tr>
<tr>
<td>( g_i )</td>
<td>0.86</td>
<td>0.99</td>
<td>1.24</td>
</tr>
<tr>
<td>( g_i - d' = g_i - 0.12 )</td>
<td>0.74</td>
<td>0.87</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Therefore 9 cables can be tensioned.

Failure occurs \((g_i - 0.12 = E_oE^*_{v_i})\) with \( F_i = 510 \text{ t} \), found by interpolation from the table. It is for this value only that \( y'_i \) is equal to the depth in compression (see in Section 5 the remark on this same limit state).

This value of \( y'_i \) is equal to 2.20 m approximately, or 0.78\( h \). As a result, the neutral fibre would be at a height from the bottom equal to

\[ x = \frac{1}{2}y = 1.04h \]

Consequently, the section would be entirely in compression.

There is therefore no need to check the limit state of cracking of the top fibre under total prestress, since the conditions under which this check
would be made are even more favourable than those taken above

\[(\gamma_p = \gamma_{oi} = 1)\]

**Comment**

Since the strength values at the time of prestress are in this case

\[R_{bk} = 200 \text{ kg/cm}^2, \quad R'_{bk} = 20 \text{ kg/cm}^2\]

(two-thirds of the 28-day strengths), and since the modulus \([21 \, 000 (R_{bk})^4]\) is \(E_b = 300 \, 000 \text{ kg/cm}^2\), the limit tensile strain at cracking would be, at the time of prestress:

\[\varepsilon'_{f} = 2.5 \frac{R'_{bk}}{E_{b}y'_{b}} = \frac{2.5 \times 20}{300 \, 000 \times 1.2} = \frac{0.14}{1 \, 000}\]

in order to reach simultaneously the limit state of cracking of the top fibre and the limit state of failure of the bottom fibre, where the compressive strain is \(\varepsilon_r = 3.5/1 \, 000\), it would be necessary to have:

\[\frac{x'}{\varepsilon_r} = \frac{h - x'}{\varepsilon'_{f}} = \frac{h}{\varepsilon_r + \varepsilon'_{f}}\]

or:

\[\frac{x'}{h} = \frac{3.5}{3.5 + 0.14} = 0.96\]

Therefore, if the depth \(y'\) which is in compression at the time of initial prestress at the limit state of failure (found by interpolation of the tabulated values as indicated above) is greater than \(\frac{3}{5} \times 0.96h\), or \(y' > 0.72h\), it is not necessary to check the limit state of cracking of the top flange under the initial prestress: the check on the limit state of failure of the bottom flange is sufficient.

Where it is necessary to check the top flange in the condition of initial prestress it is done as for Class I, and the calculations give rise to the same comments as those given in Section 5.

**(vii) Complementary steel**

(a) In accordance with the ASP, these bars must resist, when stressed to 60% of the design value of the elastic limit \(\sigma'_{e*}\), half of the tensile resultant calculated for an homogeneous section. In this case,

\[\sigma'_{e*} = 43.6 \text{ kg/mm}^2; \quad 0.6\sigma'_{e*} = 26 \text{ kg/mm}^2\]
The tensile force resultant can be estimated in the following manner, under maximum loading; using the section properties given in (iv):

\[
F_f = 605 \text{ t}
\]

\[
\sigma_g = \frac{F}{S} = \frac{605}{0.861} = 700 \text{ t/m}^2 \quad \text{(prestress at the centroid)}
\]

The apparent cracking stress is:

\[
R'' = kR'_b = 1.66 \times 250 = 415 \text{ t/m}^2
\]

The depth \(y''\) in tension is such that:

\[
\frac{y''}{415} = \frac{v'}{700 + 415}
\]

Therefore \(y'' = 0.37v' = 0.67 \text{ m}\)

![Diagram](image)

**Fig. 21.** Design of complementary steel.

The tensile force resultant, with the fictitious stress diagram of Fig. 21, is equal to:

\[
0.42 \times 0.67 \times \frac{415}{2} - 0.41 \times 0.24 \times \frac{415}{2} \times \frac{0.41}{0.67} - 0.020 \times 350 = 38.4 \text{ t}
\]

The cross-sectional area of the steel must be therefore equal to:

\[
A_s = \frac{1}{2} \frac{38400}{2600} = 7.3 \text{ cm}^2, \quad \text{or seven 12 mm diameter bars}
\]
This calculation is obviously not very satisfactory, since it introduces elastic behaviour considerations which are contradictory to limit state hypotheses.

It would be theoretically possible to consider the tensile force resultant corresponding to the hypothesis of a rectangular stress diagram in the tensile zone. It was seen (Fig. 20 and corresponding table) that the resultant was equal to 51 t. It could then be stipulated that the steel should resist a third, instead of a half, of this resultant. But the calculation corresponding to Fig. 20 is tedious and it seems more simple, even though illogical, to calculate the quantity of steel by the method shown (Fig. 21).

(b) The quantity of complementary reinforcement must also be greater than the maximum defined in Chapter IX, Section 5 (threshold of effectiveness).

It is necessary to check that the moment of resistance, for the reinforced concrete with no prestress in the section, is greater than \( \frac{1}{4} (U/v')R'' \), and therefore greater than \( \frac{1}{2} \overline{W' R'_b}^* = \frac{1}{2} \times \text{specific moment at cracking} \).
Let $z_{BA}$ be the lever arm for the reinforced concrete:

$$z_{BA}A_u\sigma'_e^* \geq \frac{1}{2} \times \text{specific moment causing cracking}$$

But:

$$z_{BA} = 2.50 \text{ m;}$$

$$\sigma'_e^* = 43.6 \text{ kg/mm}^2; \text{ specific moment causing cracking} = 180 \text{ tm;}$$

$$A_u \geq \frac{1}{2} \frac{180000}{2.50 \times 4360} = 8.25 \text{ cm}^2 (7.2 \text{ 12 mm diameter bars})$$

![Diagram showing section for Class II.](image)

FIG. 23. Section for Class II.

The reinforcement calculated in (a) above is therefore considered to be adequate.

Figures 22 and 23 show the sections designed for Class I and Class II according to the calculations of Sections 5 and 6.
7. Variation in the size of the bottom flange as a function of the safety factors \( \gamma_p \) and \( \gamma_o \)

The following approximate reasoning may be followed; it is valid where the cables are close to the bottom face of the section (beams with spans greater than the critical span).

The area in compression \( S_b \) in the limit state of failure of the bottom flange under minimum loading is equal to \( S_b = \gamma_o F/R_b^* \). This defines the depth \( y \) in compression (Sections 5(iii) and 6(iii)).

![Diagram](image)

\( \text{Fig. 24.} \)

The centroid of \( S_b \) is at a distance from the centroid of the area (cross-hatched in Fig. 24) which is equal to:

\[
\frac{b'y[(y/2) - (h'/2)]}{S_b} = \frac{b'y}{2S_b} \cdot y(y - h')
\]

where \( h' \) is the thickness of the bottom flange, taken as being rectangular.

But the following is approximately true:

\[
\left( y - \frac{h'}{2} \right)^2 = y(y - h')
\]

Also, the design centre of compression \( E_p^* \) is at a distance \( E_o E_p^* \) from the cables which is equal to \( \gamma_p M/\gamma_o F \), \( M \) being the moment under minimum loading.
The section is at the limit state when \( E^*_d \) coincides with the centroid of \( S_b \); that is, denoting the distance of the cable from the bottom fibre by \( d' \), when:

\[
\frac{\gamma_p M}{\gamma_o F} + d' = b' \left( y - \frac{h'}{2} \right)^2 + \frac{h'}{2}
\]

\( d' \) is usually nearly equal to \( h'/2 \). The following approximate equations therefore apply:

\[
\frac{\gamma_p M}{\gamma_o F} = \frac{b'}{2S_b} \left( y - \frac{h'}{2} \right)^2
\]

or

\[
\frac{\gamma_p M}{\gamma_o F} = \frac{b'R^*_b}{2\gamma_o F} \left( y - \frac{h'}{2} \right)^2
\]

The force \( \gamma_o F \) is eliminated, so that \( y \) is practically independent of the prestressing force and of the factor \( \gamma_o \)

\[
y = \frac{h'}{2} + \left( \frac{2\gamma_p M}{b'R^*_b} \right)^{\frac{1}{2}}
\]

Also:

\[
b'y + \Delta S = S_b = \gamma_o \frac{F}{R^*_b}
\]

Therefore:

\[
\Delta S = \frac{\gamma_o F}{R^*_b} - \frac{b'h'}{2} - \left( \frac{2\gamma_p b'M}{R^*_b} \right)^{\frac{1}{2}}
\]

The design is therefore very sensitive to \( \gamma_o \). Any variation \( \Delta\gamma_o \) of this factor causes a variation of \( (F/R^*_b)\Delta\gamma_o \) in the cross-sectional area of the bottom flange.

For example, in the case of Class I (Section 5), \( F = 665 \text{ t}, R^*_b = 2 \, 000 \text{ t/m}^2 \), and \((665/2 \, 000) = 0.332 \, \text{m}^2\).

If \( \gamma_o \) is changed from 1.3 (the value previously adopted) to 1.25, the area of the bottom flange decreases by \(0.05 \times 0.332 = 0.016 \, \text{m}^2\), the breadth of the flange decreases by \((0.016 \, \text{m}^2/0.26 \, \text{m}) = 0.061 \, \text{m} (6 \, \text{cm})\).

8. General comment

All the preceding designs have been conducted on the basis of an assumed web thickness.

This thickness can itself be designed on the basis of limit states. An entirely self-consistent design of the section is then obtained (see Volume 2, Chapter II).
Chapter XI

REINFORCED PRESTRESSED CONCRETE (CLASS III)

Reinforced prestressed concrete structures are defined as Class III by the FIP-CEB Committee.

The three accepted classes of prestressed concrete are characterised by the degree of safety specified against cracking, which may itself be defined by the value of the permissible tensile strain at the tensile face.

For Class I, total prestress, \( \varepsilon' = 0 \) is taken.

For Class II, limited prestress, \( \varepsilon' < \varepsilon'_f \) is taken, where \( \varepsilon'_f \) is the strain corresponding to the limit state of cracking.

For Class III, reinforced prestressed concrete, \( \varepsilon' \) is taken as not exceeding \( \varepsilon'_f \). This is a lower bound; the upper limit is discussed later.

The boundary \( \varepsilon'_f \) between Classes II and III is related to the definition of the limit state of cracking. This has been defined by the FIP-CEB Committee which has recognised, based on practical experience, that, provided suitably arranged complementary reinforcement is provided, cracking is preceded by phenomena which are plastic in nature (Chapter IX, Sections 6 and 7). Wherever the boundary actually lies, the distinction between the three classes of prestress is in fact bound up with the methods of design.

For Classes I and II the section is considered as being wholly effective, including the concrete which is in tension, and the strain (or the tensile stress) must not exceed a given limit.

For Class III it is recognised that cracking occurs under normal working conditions and the concrete under tension in the zones of cracking is neglected; the design is therefore in accordance with the methods appropriate to reinforced concrete in combined bending and compression.

The three classes may be distinguished in a third way according to the relative numbers of cables which are required to ensure safety against failure and safety against cracking.

For Class I the required cross-sectional area of the cables is greater to prevent cracking than to ensure safety against failure. The cable
Reinforced Prestressed Concrete (Class III)

cross-sectional area has to be increased, therefore, relative to that area which is sufficient to avoid failure.

For Class II the numbers of cables which are required to provide safety against either failure or cracking are about equal; the complementary reinforcement plays a secondary role in providing safety against rupture.

For Class III the number of cables which is considered adequate for the acknowledged reduced factor of safety against cracking is not sufficient to provide the necessary factor of safety against failure. The moment of resistance which is required to prevent failure can be obtained only by the introduction of additional reinforcement, which therefore performs an essential function.

The quantity of complementary reinforcement is thus notably greater than for the other classes, and the consequences resulting from the increase in the stiffness of the tensile flange cannot be ignored.

These consequences are:

- on the one hand, the tensile stresses introduced in the concrete, due to the reinforcement of shrinkage which is caused by the reinforcement;
- on the other hand, the reduction in the effective prestress $\sigma'_o$, since the prestressing force has to apply compression to both the reinforcement and the concrete.

For a given structure, its behaviour as reinforced prestressed concrete is generally limited to certain areas, local to the most highly loaded zones; outside these areas the structure is in a state of total prestress for the greater part of its length; zones of limited prestress provide intermediate areas between the zones of total prestress and the zones acting as reinforced prestressed concrete.

Where statically-determinate structures are concerned, this division into zones of higher or lower prestress does not affect the behaviour of the separate zones. This is not so for indeterminate structures, where the distribution of the zones can produce some considerable rearrangement of the distribution of the moments.

Also, where behaviour as Class II and Class III is likely to occur, it only does so above a certain level of loading. For all loading below this level the whole of the structure behaves as if it were totally prestressed.

Again, a distinction must be made between statically-determinate and statically-indeterminate structures. For the former each new loading cycle is independent of the previous cycle.† For the latter a load which causes

† Apart from the phenomena mentioned in Chapter V, Section 1 (redistribution of the stresses in a given section); there is, however, no redistribution of the moments.
the structure to behave as reinforced prestressed concrete in certain zones can definitely change the distribution of the moments.

In the following only statically-determinate structures are considered.

1. Prestressing indexes of a section in reinforced prestressed concrete

It is recommended that the memoires of ChaiKes should be read (in particular: Rome Congress 1962, Thesis III, Communication 8). His work is used in part in the following pages.

The following symbols are used:

- \( F \) for the permanent prestressing force; that is, as before, the force which is exerted by the cables under permanent conditions, all losses being deducted;
- \( F_u \) for the ultimate (breaking) force in the cables;
- \( F_a \) for the force exerted by the complementary reinforcement under service conditions, under maximum loading;
- \( F_{ae} \) for the force exerted by the same complementary reinforcement when it is subjected to a stress which is equal to its elastic limit (corresponding to the field point of the steel where this is applicable, and otherwise to a conventional 0.2% proof stress).

ChaiKes defines a prestressing index \( i \), which is equal to the ratio of the prestressing force \( F \) to the total force exerted by the cables and the reinforcement at a given section, or: \( i = F/(F + F_a) \).

An index \( i_r \) could also be defined, equal to the ratio of the forces exerted, at failure, by the cables to the forces exerted by the cables plus the reinforcement. Or, assuming that the breaking strength \( F_r \) of the cables and the elastic limit (force \( F_{ae} \)) of the reinforcement are reached:

\[
i_r = \frac{F_r}{F_r + F_{ae}}
\]

Whichever way it is defined, the index is simply a quantity which characterises the ‘degree’ of prestress. It is not an indispensable concept and the subject can be treated without necessarily referring to it.

2. Degree of prestress in a reinforced prestressed concrete section

In the foregoing definitions of the classes of prestress, the prestress ceases to be a ‘limited prestress’ (Class II) when the strain in the tensile fibre is greater than the limit cracking strain \( \varepsilon'_f \), or, which is equivalent, when the tensile stress under maximum loading is greater than \( R''_{28} \), the apparent 28-day tensile strength.
This ‘lower bound’ corresponds to a certain degree (or index) of prestress; that is, to a certain level of compression under permanent loading, which is a function of the variations of stress.

Below this limit, the class is that of reinforced prestressed concrete. The definition thus obtained would cover the whole range of concrete types between concrete with limited prestress and reinforced concrete. But when the degree of prestress becomes too low, it is difficult to consider the concrete as prestressed.

![Diagram of cable](image)

**Fig. 1.**

The boundary between the two classes, from this aspect, is necessarily arbitrary. It nevertheless seems rational to suggest that there should be total compression under permanent loading, due account being taken of the tensile stresses due to shrinkage,† or, at least, that the possible crack (the depth of which is evaluated by assuming zero tensile strength) does not go beyond the level of the cable nearest to the tensile face (Fig. 1) under the same condition of loading (permanent load plus shrinkage).

3. **Limits on the excess tension in the cables**

Certain regulations impose a limit to the stress variation $\Delta T$ in the cables under working conditions.

The ASP, for example, requires that the total stress $(T + \Delta T)$ must not exceed the initial stress $T_i$.

In itself, this is reasonable. Indeed, there are in the section two types of steel (cables and ordinary reinforcement) with different properties. Since

† The FIP-CEB Committee has nevertheless agreed that this rule can be departed from and that tensile stresses under permanent loading conditions can be accepted (in the case, for example, of claddings for buildings where the live loads are very small compared to self-weight), provided that the prestressing steel is not prone to corrosion and is efficiently protected.
the prestressing steel behaves more or less elastically until it reaches its initial tension, the requirement that $T + \Delta T \leq T_Y$ is equivalent to saying that the two types of steel should remain elastic under working conditions.

The FIP-CEB Committee does not impose any limitation on the grounds that the limitation is inherent in the design of the two types of steel on the basis of limit states, provided the safety factors are adequate.

Anticipating the later discussion in this chapter, the absence of a specification can be justified in the following manner.

Let:

- $A_c$ denote the cross-sectional area of the cables, $T$ their permanent stress, $T'$ the stress acquired at the time of loss of compression of the bottom fibre, $\Delta T$ the excess tension sustained, from this loss of compression, under the nominal moment $M$, and $T_r$ the rupture stress;
- $A_a$ denote the cross-sectional area of the complementary reinforcement, $\sigma'_a$ the stress in the reinforcement under working conditions, and $\sigma'_{e}$ the yield stress of the reinforcement;

$$
\text{Fig. 2.}
$$

$E_o$ denote the centroid of the total steel (cables and reinforcement), $z$ the lever arm under working conditions (distant $E_o$ from the resultant compressive force in the concrete), and $z_r$ the lever arm at time of failure. Then:

$$
M_r \geq \gamma_a M
$$
$$
M = z[A_a\sigma'_a + A_c(T' + \Delta T)]
$$
$$
M_r = z_r \frac{A_a\sigma'_a + A_c T_r}{\gamma_a}
$$
Now, in Class III, \( z = z_r \) (this will be illustrated in the examples dealt with later; in any case, if \( R'' \) is the apparent cracking strength, the eccentricity of the centre of compression \( > (r^2/\nu') [1 + (R''/\sigma_g)] \); or, under the usual condition for Class III, where \( \sigma_g \) is low, the eccentricity of the centre of compression is greater than \( 1.6(r^2/\nu') \); since \( r^2/\nu' \) is of the order of \( 0.5 \nu \), the distance of the centre of compression from the top fibre is less than \( 0.2 \nu \), and it can be assumed that \( z = z_r \).

In practice, it can therefore be written that:

\[
A_a\sigma'_a + A_c(T' + \Delta T) \leq \frac{A_a\sigma'_e + A_cT_r}{\gamma_a\gamma_s}
\]

The FIP–CEB Committee uses \( \gamma_a = 1.15 \) and \( \gamma_s = 1.4 \) under usual conditions; \( 1/\gamma_a\gamma_s = 0.62 \).

Therefore,

\[
A_a\sigma'_a + A_c(T' + \Delta T) \leq 0.62(A_a\sigma'_e + A_cT_r) \quad (a)
\]

(i) Assume that the cables and the reinforcement are at the same level; then \( \sigma'_a = \Delta T \); therefore:

\[
(A_a + A_c) \Delta T \leq 0.62A_a\sigma'_e + A_c(0.62T_r - T')
\]

\( T \) is of the order of \( 0.5 \) to \( 0.6 \) \( T_r; \ T' = T + 6 \) kg/mm\(^2\).

The calculations below are made with \( T_r = 160 \) kg/mm\(^2\). It can be easily seen that the conclusions remain valid for other values of \( T_r \).

Regarding the value of \( \Delta T \), the most favourable conditions correspond to the case where the value of \( \sigma'_e \) is high; assume \( \sigma'_e = 50 \) kg/mm\(^2\) (high strength steel with good bond characteristics).

(a) With \( T = 0.5T_r = 80 \) kg/mm\(^2\), \( T' = 86 \) kg/mm\(^2\), \( 0.62T_r - T' = 14 \) kg/mm\(^2\)

\[
(A_a + A_c) \Delta T \leq 0.62 \times 50A_a + 14A_c
\]

\[
(A_a + A_c) \Delta T \leq 31A_a + 14A_c
\]

or, approximately:

\[
\Delta T \leq 31 \frac{A_a + A_c - 0.5A_c}{A_a + A_c}
\]

or

\[
\Delta T \leq 31 \left( 1 - \frac{1}{2} \frac{A_c}{A_a + A_c} \right) \text{(kg/mm}^2\text{)}
\]

But \( A_c/(A_a + A_c) \) is in practice greater than \( 0.33 \).

Therefore \( \Delta T \leq 31 \times 0.84 = 25 \) kg/mm\(^2\) approximately.
(b) With $T = 0.6 T_r = 96 \text{ kg/mm}^2$, $T' = 102 \text{ kg/mm}^2$, $0.62 T_r - T' \approx 0$, therefore:

$$(A_a + A_c) \Delta T \leq 0.62 A_a \sigma'_e$$

$$\Delta T \leq 0.62 \sigma'_e \frac{A_a}{A_a + A_c}$$

In practice, $A_a/(A_a + A_c)$ is greater than 0.66.
Therefore $\Delta T \leq 0.62 \times 0.66 \times 50 \text{ kg/mm}^2$ and $\Delta T = 20 \text{ kg/mm}^2$.
The stresses ($T' + \Delta T$) are therefore less than $80 + 25 = 105 \text{ kg/mm}^2$ when $T = 0.5 T_r$, and less than $96 + 20 = 116 \text{ kg/mm}^2$ when $T = 0.6 T_r$.
These stresses are not dangerous.

(ii) If the cables and the reinforcement are not at the same level:
(a) Usually, the reinforcement is nearer to the tensile face than the cables. If $\sigma'_{am}$ is the mean stress in the total steel (cables and reinforcement) following the loss of compression at the bottom fibre, then $\sigma'_{am}$ has the values calculated in (i); $\Delta T$ is less than $\sigma'_{am}$ and the condition is more favourable than in (i).
(b) In very exceptional cases, the cables could be nearer to the tensile face than the reinforcement. Then $\sigma'_a$ would be less than $\Delta T$ and inequality (a) would imply that:

$$A_a \Delta T + A_c (T' + \Delta T) \leq 0.62 (A_a \sigma'_e + A_a T_r)$$

Therefore, as in (i):

$$(A_a + A_c) \Delta T \leq 0.62 A_a \sigma'_e + A_c (0.62 T_r - T')$$

which leads to the same consequences.

In principle, therefore, it is considered in the following that it is not necessary to specify rules for limiting the excess tension. Nevertheless, it is necessary to calculate the stress $\sigma'_a$ in the reinforcement, in order to apply the rules governing the maximum widths of cracks, and a method for this calculation is indicated in Section 6. The values of $\Delta T$ can be deduced immediately, and all the main properties are obtained by means of tabulations such as those given for the example in Section 9.

4. Limitation on the widths of cracks

The generally accepted limits for the widths of cracks are 0.1 mm in an aggressive environment and 0.3 mm for covered structures in a normal environment, according to the CEB.
Whether these limits are applicable to reinforced prestressed concrete can be questioned; two contradictory considerations could supervene:

it could be considered that the requirements can be made less stringent than in the case of reinforcement concrete, since the cracks are only temporary;

on the contrary, it could be considered that the requirements should be more stringent, since a higher standard is required.

This is a question of judgement which can be influenced by the frequency of application of the loads which provoke or can provoke cracking.

If it is accepted that the limits relative to reinforced concrete apply, the diameters of the complementary reinforcement must not exceed those fixed by the customary regulations for reinforced concrete.

![Diagram](image)

**Fig. 3.** Imaginary section for determining the maximum diameter of the bars.

With the BA 60 Code, for example, the regulations relative to these diameters are based on the following definitions:

$B'$: area $b' \times 2d''$ of an imaginary section whose centroid coincides with the centroid of the steel,

$A_a$: cross-sectional area of the steel in tension,

$\omega'_{f}$: ‘effective percentage’ of the steel in the section, defined by:

$$\omega'_{f} = 100 \frac{A_a}{B'}$$

$\sigma'_a$: tensile stress in the steel, kg/cm²,

$k$: a factor which is dependent on the consequences of cracking.

One of the BA 60 rules which applies to generalised cracking (which is the case in the zone under consideration) is equivalent to taking for the
maximum diameter $d$ (mm) of the complementary reinforcement:

$$d = k \frac{\sigma'_f}{(10 + \sigma'_f)\sigma'_a}$$

for mild steel, and

$$d = \frac{1.6k\sigma'_f}{(10 + \sigma'_f)\sigma'_a}$$

for steel with good bond characteristics, where $k$ has the value 150 000, 100 000 or 50 000, depending on whether the effects of the crack are slight, undesirable or very serious.

For example, with $\sigma'_f = 4$, $k = 100 000$ and $\sigma'_a = 2 100$ kg/cm$^2$, $d = 25$ mm approximately for steels with good bond characteristics.

The application of these rules, or equivalent rules, is simple where ordinary non-tensioned reinforcement is concerned, as in the case of reinforced concrete.

In the case of reinforced prestressed concrete, cables are present as well as the reinforcement.

Some codes specify that the cables must be included in the area $A_a$, and consequently in the percentage $\sigma'_f$. Such requirements seem debatable because the ratio between the active area of the cable and the bonded perimeter of the cable sheath can be quite different to that corresponding to the ordinary reinforcement; the bonding strength is also badly defined because the bonding phenomena intervene twice—once inside the sheath (wire bundle in contact with the grout) and a second time between the sheath and the concrete, the degree of bond being dependent on the type of sheath employed.

In any event, the cables are generally on a different level to the complementary reinforcement.

It seems that the most simple solution is to ignore the cables when determining the maximum permissible diameter of the reinforcement. The imaginary section is then as shown in Fig. 3, and it is dependent on the position of the complementary reinforcement alone, and the area $A_a$ is the area of the reinforcement.

The maximum reinforcement diameter can be slightly underestimated by proceeding in this manner, but this does not in general cause any serious disadvantages. This reasoning is equivalent to saying that if the cables only were present, without the reinforcement, then the cables could not in any way control the cracking. This is perhaps a little severe, but it is not too far removed from practical observations.
This hypothesis is in any case not true with reinforced prestressed concrete, since the definition of the material itself implies that reinforcement is present. A minimum quantity of reinforcement is necessary, if only to control the cracking. It seems that the percentage \( \rho' \), defined above (relative to the imaginary section shown in Fig. 3), cannot be taken at less than 2%. The BA 60 Code gives a reasonable value for the maximum diameter\(^\dagger\) of the reinforcement.

All these empirical rules can be considered only as indicative, and the distribution of the reinforcement, both as regards position and diameter, is primarily a question of good judgement.

5. Resistance to prestressing compression due to the presence of complementary reinforcement

Due to shrinkage and to the presence of complementary reinforcement, the concrete is subjected to tensile stresses before prestress and the reinforcement is subjected to compressive stresses.

![Fig. 4. Shrinkage effect.](image)

These effects, of only slight importance for Classes I and II, cannot be ignored in the case of Class III.

Let \( A_a \) be the total cross-sectional area of the reinforcement (assumed local only to one of the faces of the beam), \( e_a \) the eccentricity of the reinforcement relative to the centroid, and \( \eta \) the shrinkage; the section

\(^\dagger\) There is no requirement for minimum diameters. Diameters which are too small should not, however, be used in an attempt to reduce the opening of cracks because of the risks of corrosion (low cover protection).
properties \((S, I, v, v', r^2/v, r^2/v', \ldots)\), are denoted by the same symbols as before. The concrete elastic modulus is denoted in general by \(E_b\), but the values of \(E_b\) depend on age and on the type of loading; the elastic modulus of steel is denoted by \(E_a\); the modular ratio \(E_a/E_b\) is denoted by \(m\) and, as with \(E_b\), its value depends on the particular conditions in the member.

If the reinforcement could slide freely in the concrete, a unit length of concrete would shorten by an amount \(\eta\) and the reinforcement, assumed fixed at one end, would project beyond the other end by an amount \(\eta\). Due to bonding, forces \(\pm F_\eta\) in equilibrium are created; the force \(-F_\eta\) is tensile in the concrete and the force \(+F_\eta\) is compressive in the reinforcement.

The stress in the concrete at the level of the steel is equal to:

\[
\sigma'_b = -\left(\frac{F_\eta}{S} + F_\eta \frac{e_a}{I} e_a\right) = -\frac{F_\eta}{S} \left(1 + \frac{e_a^2}{r^2}\right)
\]

If the steel is very close to the bottom fibre, \(e_a\) can be taken as equal to \(v'\), and:

\[
\sigma'_b = -\frac{F_\eta}{S} \left(1 + \frac{v'^2}{r^2}\right)
\]

Let \(E'_b\) be the elastic modulus of the concrete at the time of the deformation (green concrete, subjected to tensile stresses upon setting).

The strain \(\varepsilon'_b\) of the unit length is \(\sigma'_b/E'_b\) or:

\[
\varepsilon'_b = -\frac{F_\eta}{E'_b S} \left(1 + \frac{v'^2}{r^2}\right)
\]

The compressive strain in the steel is:

\[
\varepsilon'_a = \frac{F_\eta}{E_a A_a}
\]

But \(|\varepsilon'_b| + \varepsilon'_a = \eta\). Therefore:

\[
F_\eta \left[\frac{1}{E'_b S} \left(1 + \frac{v'^2}{r^2}\right) + \frac{1}{E_a A_a}\right] = \eta \quad \text{(a)}
\]

Let \(A_a/S = \lambda\) (Chaikes’ percentage reinforcement), and write

\(1 + v'^2/r^2 = K\) (shape coefficient).

Let \(m' = E_a/E'_b\).

Equation (a) can then be written:

\[
\frac{F_\eta}{E_a A_a} (1 + m'\lambda K) = \eta
\]
Hence the compressive stress in the reinforcement is:

\[ \sigma'_a = \frac{F\eta}{A_a} = \frac{E_a\eta}{1 + m'\lambda K} \]

The tensile stress in the concrete is:

\[ \sigma'_b = -K \frac{F\eta}{S} = -K \frac{F_e\lambda}{A_a} = -\frac{E_a\lambda K}{1 + m'\lambda K} \eta \] (b)

or

\[ \sigma'_b = -\frac{m'\lambda K}{1 + m'\lambda K} E'_b \eta \] (c)

The strain in the concrete is:

\[ \varepsilon'_b = \frac{\sigma'_b}{E'_b} = \frac{m'\lambda K}{1 + m'\lambda K} \eta \] (d)

As regards the concrete, it is this tensile strain which is of interest; the tensile stress \( \sigma'_b \) is temporary and it disappears after prestress. It is not usually very high, since \( m' \) is large (formula b) because of the low values of the modulus at the time of the deformation and because of the creep of green concrete in tension.†

It is nevertheless safer to avoid the probability of cracking by prestressing partially as soon as possible.

It is necessary to discuss the amount of shrinkage which has to be taken into consideration, as well as the values of \( m' \) to use.

(a) **Amount of shrinkage to be taken into account for prestressing losses**

Consider (Fig. 5) a length of concrete initially equal to unity. Assume that the reinforcement is free to slide and that it is fixed to the right-hand end of the concrete and that it projects from the left-hand end.

If \( \eta_1 \) is the proportion of shrinkage before prestress, the reinforcement projects by this amount before the cables are tensioned. When the cables are tensioned, the concrete is subjected instantaneously to a compressive strain \( \varepsilon_1 \) and in the long-term to a stress \( \varepsilon_\pi \). Shrinkage continues, and because of this a reduction in length occurs, equal to \( \eta_2 \).

† The modulus of elasticity increases with time from completion of setting, according to a law of the form \( E'_t = E'_\infty[1 - \exp (-kt)] \).

Shrinkage varies with time according to a law of similar form \( \eta_t = \eta_\infty[1 - \exp (-kt)] \), but stability is reached much more slowly than with the modulus. The stress \( \sigma'_t \) would be equal to \( \int E'd\eta \) if the behaviour were elastic, but relaxation occurs which significantly reduces the stress.
The total projection of the reinforcement, assumed to be free, is therefore \( \eta_1 + \varepsilon_0 + \eta_2 \). The reinforcement must be shortened and the concrete lengthened in order to re-establish the equality of the reinforcement with the concrete.

The portion \( \varepsilon_0 \) of the relative strain is considered separately in the distribution of the prestressing force \( F \) between the concrete and the reinforcement, and it can be considered independently. With regard to shrinkage, the relative strain which must be eliminated (by compression in the reinforcement and tension in the concrete) is equal to \( \eta_1 + \eta_2 \).

![Diagram](image)

**Fig. 5.**

Hence the total shrinkage must be taken into account, and not only that portion of it which occurs before prestress. This requirement is, however, perhaps a little too severe.

The consequences can in any case be alleviated by applying a fraction \( F_1 \) of the prestressing force as soon as possible, in order to compensate for the shrinkage tensile stresses. The compressive stress due to \( F_1 \) on the bottom fibre is:

\[
\frac{F_1}{S} \left(1 + \frac{|\varepsilon|}{r^2/v'} \right) \quad \text{or, approximately} \quad \frac{F_1}{S} K
\]

The resulting strain is \( F_1/E_b SK \). This must be equal to \( \varepsilon'_b \) (value (d) above).

Therefore \( F_1 = E_b(S/K)\varepsilon'_b \).
The value of $E_b$ to use is that which corresponds to the age of the concrete when it is prestressed to the value $F_1$ and to long-term compressive deformations.

Under normal conditions $E_b$ would be of the order of 150,000 kg/cm². By applying the prestress $F_1$ sufficiently early, this can be reduced to 120,000 or 100,000 kg/cm². So as not to have too many values of the modulus to consider, the value of 150,000 kg/cm² can be retained, but a lower value must be taken for $\eta$ in order to obtain the same value of the product $E_b\eta$. A value of 3/10 000 for $\eta$ can be used if the true value is 4/10 000.

(b) Values of $m'$

By definition $m' = E_d/E'_b$, where $E'_b$ is the mean elastic modulus during the deformation which occurs under the influence of the tensile stresses due to shrinkage.

![Fig. 6. Variations of shrinkage and of the tensile stress.](image)

If $\sigma'_b$ is the tensile stress which is reached in the concrete and if $\eta_b$ is the strain: $E'_b = \sigma'_b/\eta_b$.

Some values are available from tests. In these tests a ring with a metallic core is used. The ring is cast around the core. The stress in the metallic core is measured by means of vibrating gauges and the tensile force in the concrete ring is obtained, hence the stress $\sigma'_b$. On another ring which is free (that is, without the core), the free shrinkage is measured.

The principle of these tests is due to L’HERMITE, and examples are described in a paper by A. DE SOUZA CONTINHO (A fissurabilidade dos cimentos, argamassas e betoes por efeito da sua contracção, Lisbon 1954).

Diagrams showing the variations of $\sigma'_b$ and $\eta_b$ with time are of the shape shown in Fig. 6. Generally, $\sigma'_b$ goes through a maximum (between 5 and 7 weeks) and then decreases, due to relaxation, and it tends towards a final limit at the end of three to six months. The modulus $E'_b = \sigma'_b/\eta_b$ therefore
varies with time and it tends towards a limit. It is this limit which is of interest, since it is usually reached before the structure is commissioned. Some results are given below; values of tensile strength after 180 days are included in the table, determined from bending tests on bars.

\[ E'_b = \frac{\sigma'_b}{\eta} \]

<table>
<thead>
<tr>
<th>Mix (kg of cement per m³)</th>
<th>Diameter aggregate of (mm)</th>
<th>Water/cement ratio</th>
<th>Final shrinkage (\eta (10^{-4}))</th>
<th>Final tensile strength (\sigma'_b) (kg/cm²)</th>
<th>Modulus of elasticity (E'_b) (kg/cm²)</th>
<th>Tensile strength at 180 days (kg/cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>6.35</td>
<td>0.75</td>
<td>8</td>
<td>17</td>
<td>21 000</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>12.7</td>
<td>0.6</td>
<td>5</td>
<td>11</td>
<td>22 000</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>25.4</td>
<td>0.5</td>
<td>4</td>
<td>10</td>
<td>25 000</td>
<td>42</td>
</tr>
<tr>
<td>300</td>
<td>6.35</td>
<td>0.65</td>
<td>6.5</td>
<td>18</td>
<td>27 000</td>
<td>48</td>
</tr>
<tr>
<td>500</td>
<td>12.7</td>
<td>0.38</td>
<td>5</td>
<td>18</td>
<td>36 000</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>7.2</td>
<td></td>
<td>18</td>
<td>25 000</td>
<td>39</td>
</tr>
<tr>
<td>500</td>
<td>12.7</td>
<td>0.44</td>
<td>6</td>
<td>23</td>
<td>38 000</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>25.4</td>
<td>0.4</td>
<td>4</td>
<td>15</td>
<td>37 000</td>
<td>50</td>
</tr>
</tbody>
</table>

\(E'_b\) can therefore vary between large limits, between 25 000 and 40 000 kg/cm² in practice for the usual mixes and aggregate sizes.

The modulus \(m' = E_a/E_b\), where \(E_a = 2 000 000\) kg/cm², can therefore vary between 50 and 80. Shrinkage itself is variable but measures can be taken (principally by limiting the water content) so as not to exceed a shrinkage strain of 4/10 000.

Consider these results when applied to formula (d). Two cases are considered for the shape factor:

\[ K = 1 + \frac{v^2}{r^2} = 1 + \frac{v'}{r^2/v'} \]

If \(\rho\) is the efficiency of the section, \(K = 1 + (v'/\rho v')\) and if the section is symmetrical, \(K = 1 + (1/\rho)\).

(i) Rectangular section:

\[ \rho = \frac{1}{3} \quad K = 4 \]
<table>
<thead>
<tr>
<th>$\lambda = \frac{A_g}{S}$ (%)</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m' = 50$ &amp; $E_b'$ during shrinkage $40,000 \text{ kg/cm}^2$</td>
<td>Rect. section ($K = 4$)</td>
<td>0.85</td>
<td>1.33</td>
<td>1.63</td>
<td>1.84</td>
<td>2.00</td>
<td>2.11</td>
<td>2.20</td>
<td>2.28</td>
<td>2.34</td>
</tr>
<tr>
<td>$m' = 80$ &amp; $E_b'$ during shrinkage $25,000 \text{ kg/cm}^2$</td>
<td>Rect. section ($K = 4$)</td>
<td>1.16</td>
<td>1.67</td>
<td>1.97</td>
<td>2.16</td>
<td>2.28</td>
<td>2.36</td>
<td>2.46</td>
<td>2.52</td>
<td>2.55</td>
</tr>
<tr>
<td></td>
<td>Rect. section ($K = 3$)</td>
<td>0.97</td>
<td>1.47</td>
<td>1.77</td>
<td>1.97</td>
<td>2.11</td>
<td>2.22</td>
<td>2.31</td>
<td>2.37</td>
<td>2.43</td>
</tr>
</tbody>
</table>

The second decimal place is obviously meaningless, but it has been included in order to plot a consistent graph; only orders of magnitude are intended to be shown, obtained in any case from rather uncertain basic data.
(ii) I-beam (symmetrical) of normal efficiency:

\[ \rho = \frac{1}{2} \quad K = 3 \]

Table II and Fig. 7 give the values of the strain at the tensile face of the concrete as a function of the percentage \( A_w/S = \lambda \) of the reinforcement for the two types of section (i) and (ii), for the shrinkage value of 3/10 000 considered above in (a).

![Graph showing strains at the tensile face](image)

**Fig. 7.** Strains at the tensile face (in 1/10 000) as a function of the percentage \( A_w/S \) of complementary reinforcement (on the assumption that the shrinkage \( \eta = 3/10 000 \)).

Figure 7 shows the variation with the percentage of reinforcement (according to ChaiKes) of the strains which are possible before cracking.

6. Estimates of the excess stresses in the cables and of the stresses in the reinforcement at the planes of cracking (concrete in tension neglected)

This estimate is required to determine the diameter of the complementary reinforcement from empirical formulae such as those given in Section 4, and it is useful for an approximate calculation of the excess stresses in the cables.

Because of the approximate nature of the formulae, the stresses themselves can only be obtained approximately. The method which is used is due to Franco Levi.†

† Report of the work of the Joint FIP-CEB Committee, April 1964.
Let $M_d$ be the moment under which there is no compression in the bottom fibre and $M$ the maximum moment. Consider the corresponding stress diagrams (concrete in tension ignored). The distribution ($M$) is obtained from distribution ($M_d$) by subtracting a triangle ($\beta$) (cross-hatched) and by adding a triangle ($\gamma$) (diagonal hatching). The steel (cables and reinforcement) is in addition subjected to a mean stress $\sigma'_{am}$ during the transition from $M_d$ to $M$ (from which the values of $\Delta T$ and $\sigma_a$ are ultimately determined). Let $A$ be their total cross-sectional area ($A = A_c + A_a$) and let $\Delta F$ be the force $A\sigma'_{am}$ due to the excess stresses.

**Fig. 8.** Estimate of the excess cable stresses and of the complementary reinforcement stresses (FRANCO LEVI).

This force passes through the point $E'_o$, the centroid of the combined cables and reinforcement.

Let $\gamma$ be the resultant compressive force corresponding to triangle ($\gamma$) and $G_\gamma$ its point of application; let $\beta$ be the resultant of the compressive forces corresponding to the triangle ($\beta$) and $G_\beta$ its point of application. Let ($\alpha$) be the top quadrilateral of the distribution $M_d$ (portion which is separate from $\beta$) and $\alpha$ the corresponding resultant compressive force.

Denoting by $F$ the value of the prestressing force at the time of the loss of compression, then:

$$F = \alpha + \beta$$
$$F + \Delta F = \alpha + \gamma$$
Hence

$$\Delta F + \beta = \gamma$$

The moment $M - M_d$ is therefore balanced by the couple formed by the two equal forces $\gamma$ and $\Delta F + \beta$.

Therefore $M - M_d = E'_o G_\gamma \Delta F + G_\rho G_\gamma \times \beta$ or, by writing $E'_o G_\gamma = z$ and $G_\rho G_\gamma = z_\beta$,

$$M - M_d = z \Delta F + z_\beta \beta$$

Hence:

$$\Delta F = \frac{M - M_d}{z} - \beta \frac{z_\beta}{z}$$

(1)

Let $N$ be the intersection of diagrams $(M)$ and $(M_d)$ and let $P$ be the intersection of diagram $(M)$ with the ordinate at zero stress.

The depth in compression under combined bending and compression is at a distance $P$ from the top fibre; let this depth be $y$ and let $y' = h - y$ be the depth in tension (Fig. 8).

Also, let $y_n$ and $y'_n$ be the distances of $N$ from the top and bottom fibres respectively.

The approximation consists of assuming that the depth $y_n$ is equal to the depth in compression which would occur in simple bending (with a total reinforcement area equal to $A$). It can therefore be calculated by the normal methods of reinforced concrete design. Hence the position of $G_\gamma$ is obtained (point of action of the resultant compressive force in simple bending) and the value of $z = E'_o G_\gamma$; the position of $N$ is also obtained [on the known diagram $(M_d)$ at the distance $y_n$ from the top fibre].

If $\sigma'_{am}$ were known, diagram (M) would be known since it passes through $N$ and since the stress (of an equivalent area of concrete) would be $\sigma'_{am}/m$ at $E'_o$; hence $\beta$ is found.

$\beta$ and $z_\beta$ are therefore functions of $\sigma'_{am}$ and eqn. (1) is an equation in $\sigma'_{am}$. It can be solved approximately in the following manner.

The approximate value of $\sigma'_{am}$ obtained from eqn. (1) by neglecting the term $-\beta(z_\beta/z)$ is first obtained (this is the stress in the two types of steel if they were in simple bending). If $\sigma'_1$ is this approximate value:

$$\sigma'_1 = \frac{\Delta F}{A} = \frac{M - M_d}{Az}$$

$\beta$ and $z_\beta$ are calculated using this value of $\sigma'_1$. A corrected value of $\Delta F$ is then obtained from eqn. (1) and therefore a corrected value of $\sigma'_{am}$.

This corrected value is considered to be sufficiently accurate.
(If the correction to \( \sigma'_1 \) was too great to stop at this first approximation, the process could be repeated, but this is usually not necessary.)

Once \( \sigma'_{am} \), and consequently \( P \), are known, the stress \( \sigma'_a \) in the steel and the excess stress \( \Delta T \) in the cables are derived, relative to condition (M.a), in terms of the position of point P.

In order completely to define the method, \( \beta \) and \( z_\beta \) must be calculated once \( \sigma'_{am} \) is known.

(a) Let \( \sigma_n \) be the stress at point N (known, since N is known). Neglecting \( d'_1 \), the distance of \( E'_{o} \) from the bottom fibre, then:

\[
y' = y'_n \left( \frac{\sigma'_{am}/m}{(\sigma'_{am}/m) + \sigma_n} \right) = y'_n \frac{\sigma'_{am}}{\sigma'_{am} + m\sigma_n}
\]

Therefore, if \( b' \) is the web thickness:

\[
\beta = \frac{1}{2} b'y'_n \sigma_n = \frac{1}{2} b'y'_n \frac{\sigma'_{am}\sigma_n}{\sigma'_{am} + m\sigma_n}
\]

(b) \( G_\beta \) is the centroid of triangle \( \beta \). Its distance from the bottom fibre (and therefore approximately from \( E'_{o} \)), is equal to \( (y' + y'n)/3 \), or:

\[
y'_n \left( 1 + \frac{\sigma'_{am}}{\sigma'_{am} + m\sigma_n} \right) = \frac{y'_n}{3} \frac{2\sigma'_{am} + m\sigma_n}{\sigma'_{am} + m\sigma_n}
\]

Therefore:

\[
z_\beta = z - \frac{y'_n}{3} \frac{2\sigma'_{am} + m\sigma_n}{\sigma'_{am} + m\sigma_n}
\]

and

\[
\frac{z_\beta}{z} = 1 - \frac{y'_n}{3z} \frac{2\sigma'_{am} + m\sigma_n}{\sigma'_{am} + m\sigma_n}
\]

The terms of equation (1) are therefore determined. Consequently, having taken the approximate value \( \Delta F_1 = (M - M_a)/z \) for \( \Delta F \), and calling \( \sigma'_{a1} (=\Delta F_1/A) \) the approximate corresponding value of \( \sigma'_{am} \), equation (1) can be written:

\[
\Delta F = \Delta F_1 - \frac{1}{2} b'y'_n \frac{\sigma'_{a1}\sigma_n}{\sigma'_{a1} + m\sigma_n} \left( 1 - \frac{y'_n}{3z} \frac{2\sigma'_{a1} + m\sigma_n}{\sigma'_{a1} + m\sigma_n} \right)
\]

(2)

The quantity \( m\sigma_n \) is generally small compared to \( \sigma'_1 \) (\( \sigma'_1 \) is of the order of 1 500 kg/cm², \( \sigma_n \) of the order of 40 kg/cm², \( m\sigma_n \) of the order of 200 kg/cm²).
Therefore

\[ \frac{\sigma'_{a1}}{\sigma'_{a1} + m_0 \sigma_n} = 1 - \frac{m_0 \sigma_n}{\sigma'_{a1}} \]

approximately

and, by writing \( m_0 \sigma_n / \sigma'_{a1} = \phi \):

\[ \Delta F = \frac{M - M_d}{z} - \frac{b' y'_n \sigma_n}{2} (1 - \phi) \left[ 1 - \frac{2}{3} \frac{y'_n}{z} \left( 1 - \frac{\phi}{2} \right) \right] \] (3)

7. Design on the combined basis of failure and elastic behaviour

The limit state of failure under maximum loading provides, as in Chapter X, a relation between the forces \( F_r \) and \( F_{ae} \), and, consequently, a relation between the cross-sectional areas of the cables and the reinforcement. It also enables the flange in compression to be designed.

(i) It is proposed to treat the curve of limit strains as a second design basis

If this criterion is accepted it would enable the required prestressing force to be determined. The force should be such that the strain in the tensile fibre (resulting from the compressive strain due to prestress and the tensile strain due to loading), at the maximum loading, is less than the ‘available’ strain. This is the difference between the limit strain, corresponding to tensile failure (cracking) (top curve in Fig. 7), and the tensile strain due to shrinkage (intermediate curves in Fig. 7).

Since it is necessary in practice to bring in the elastic moduli when evaluating the strains under load and the compressive strains under prestress, this is the same as saying that, taking into account the tensile stress due to shrinkage, the calculated resultant stress at the bottom fibre (on the basis of a homogeneous section) must not exceed a certain permissible limit, obtained by multiplying the limit strain by a certain modulus of elasticity.

This permissible limit would be that strength which is called the apparent tensile strength \( R'' \) in Chapter IX. \( R'' \) would be a function of the percentage of reinforcement, \( A_a/S \), given in Fig. 7.

A third condition is provided by the compressive stress at the bottom fibre which must not exceed a limit \( R_1 \) under minimum loading.

Calculations based on elastic behaviour then provide the design information which is not given by the limit state of failure, under maximum loading.

In other words (the design of the top flange being determined by the limit state of failure under maximum loading), three unknowns must be determined: \( F, F_a \) and \( I/v' \).
The limit state of failure gives a relation between $F$ and $F_a$, which is written, for the usual case:

$$0.9 \text{ (or } 0.95) \left[h_1 F^*_r + h_a F^*_{ae}\right] \geq \gamma_s M_{\text{max}}$$  \hspace{1cm} (a)

where $\gamma_s$ is the safety factor for the loads.

Consideration of the available strain would provide a second relation:

$$\frac{F}{S} \left(1 + \frac{|e|}{r^2/v'}\right) - \frac{M}{I/v'} \geq -R''$$  \hspace{1cm} (b)

and $I/v'$ is defined by

$$I/v' \geq \frac{\Delta M}{R_1 + R''}$$  \hspace{1cm} (c)

Usually, with this class of prestressed concrete, the modulus $I/v'$ resulting from the least dimensions which are required to accommodate the cables and the reinforcement is greater than that which is required to satisfy exactly the relation (c).†

This modulus then becomes one of the parameters; only the two unknowns $F$ and $F_a$ remain, and these are determined using equations (a) and (b) if the available strains are taken as the criteria.

The most simple is a trial and error method. A value of $F^*_r$, is taken, corresponding to a whole number of cables. Then $F^*_{ae}$ is obtained [equation (a)] and then $A_a$.

$F$ is obtained from $F^*_r$, and a check is made on inequality (b), the apparent strength $R''$ being known when $A_a$ (and therefore $\lambda$) is known. The value of $F_r$ is varied until the inequality is satisfied in the most economical manner.

The practical application of this trial and error method is further explained in Section 9.

If it is not decided to use a ‘mean’ modulus to evaluate $R''$ (by means of Fig. 7), the strains induced by the different types of loading can be introduced into inequality (b): prestress, permanent and live loads, with different moduli, corresponding to the type of loading: $E_o$ for sustained loading, $E_i$ for instantaneous loading; that is:

$$\frac{M_p}{E_o(I/v')} + \frac{M_s}{E_i(I/v')} - \frac{F}{E_o S} \left(1 + \frac{|e|}{r^2/v'}\right) \leq \varepsilon' \text{ available}$$

† It can sometimes happen that conditions under initial prestress are more unfavourable than conditions for minimum size. The bottom flange design should then be changed and the calculation done again with the corrected value of $I/v'$.  

This relation still represents an elastic condition in an homogeneous section.

(ii) Proposed method
In the author's opinion the limit state criterion is not the one to use; it is a criterion for Class II (partial prestress) and not for Class III. The values of the limit strains are in any case debatable, and it is probable that this limit strain method far from guarantees absolute freedom from cracking. This is not too serious, since, by definition, this risk is recognised. But why pick a criterion which purports to avoid it?

The criteria which it seems should be chosen are those which correspond to the imposed conditions, that is:

1. That there should be no tensile stresses under permanent loading plus a proportion $\alpha$ of the live loading, that is, under the moment $M_p + \alpha M_s$ taking into account the tensile shrinkage stresses.

   This condition is expressed by the following inequality:

   \[
   \frac{F}{S} \left( 1 + \frac{|e|}{r^2/v'} \right) - E_b \varepsilon'_b - \frac{M_p + \alpha M_s}{I/v'} \geq 0
   \]  

   (b')

   Ultimately, the value of $\alpha$ can become zero.

2. That the tensile stress $\sigma'_a$ in the reinforcement should not exceed a certain permissible limit (account being taken of the compressive stresses to which the reinforcement is subjected through shrinkage and prestress).

   Or, by giving $\gamma_a$ a suitable value:

   \[ \sigma'_a < \frac{\sigma'^e}{\gamma_a} \]  

   (d)

   In any event, this condition is automatically satisfied by the design against failure from inequality (a), if sufficiently high safety factors are chosen (see Section 3). The stress $\sigma'_a$ must be calculated, however, as it is necessary to know the value of $\sigma'_a$ when applying the regulations relative to the width of cracks. The calculation is done on the basis of a section with cracks, ignoring the concrete in tension.

   Replacing the inequalities by equalities, or, in other words, by considering the limit for each condition, the two equations (a) and (b') [or ultimately (a) and (d)] are used to determine $F$ and $F_a$, it being assumed that $I/v'$ is known (minimum dimensions of the bottom flange to accommodate the cables).
Again, a trial and error method is the most simple. Having decided on a value of $F^*$, corresponding to a whole number of cables, $F_{ae}^*$ is obtained from equation (a), and $F^*$, than $A_a$ and $\lambda = (A_a/S)$ are determined.

The shrinkage strain $\varepsilon'_b$ is deduced \{\varepsilon'_b = [m'\lambda K/(1 + m'\lambda K)]\eta}\}.

By substituting the values of $F$ and $\varepsilon'_b$ in equation (b'), the fraction $\alpha$ of the live load for which the bottom flange ceases to be in compression is obtained.\dagger

Also, $\sigma'_a$ is determined by a calculation for combined bending and compression in which the axial force is equal to $F$ and the bending moment taken about the cable is $M_p + \alpha M_a$ (\(\alpha\) being given the value previously obtained). $F$ is varied until inequalities (b') and (d) are satisfied; that is, until $\alpha$ is equal at least to its fixed value and until $\sigma'_a$ is less than the permissible limit [condition (d) is in general automatically satisfied].

Concerning the calculations relating to (b'), the presence of reinforcement and of holes for the cables must be taken into consideration when estimating the stresses due to prestress and permanent loads; in addition, the cables must be taken into consideration when calculating the live load stress.

A calculation assuming elastic behaviour, or the approximate calculation in Section 6, can be made for condition (d). This also provides the value of the excess stress $\Delta T$.

With the section thus designed, a check is necessary to show that the compressive stress is less than the permissible limit under minimum loading. This condition is satisfied in general. If it is not, the design of the bottom flange is corrected and the calculations done again, as in the foregoing.

\textit{Formulae for the application of the method given in Section 6 (determination of $\sigma'_a$).} It is necessary to calculate the position of the neutral axis under the assumption of simple bending, defined by the depth $y_n$ shown in Fig. 8. It is calculated by means of the usual formula. If $h_1$ is the effective depth (distance of $E'$ from the top fibre, Fig. 8), $u$ the thickness of the top flange, and if $v = y_n/h_1$, this formula is written:

$$v^2 - \left(1 - \frac{b'}{b}\right)\left(v - \frac{u}{h_1}\right)^2 - 2m\frac{A_e + A_a}{bh_1} (1 - v) = 0$$

\dagger For the type of construction envisaged in the note in Section 2, it could be taken that the loss of compression occurs under a load which is less than the permanent load. In any event, equation (b') enables the loss of compression to be determined during the course of the trial and error calculations.
The distance of point $G_s$ (centre of compression under simple bending) from point N on Fig. 8 (neutral axis in simple bending) is given by:

\[
\frac{2}{3} \frac{b y_n^3}{y_n'} - \frac{(b - b')(y_n' - u)^3}{(b - b')(y_n' - u)^2}
\]

The lever arm $z$ in simple bending is equal to the distance of $G_s$ from the centroid $E_n$ of the cables and reinforcement.

Thus all the data are available for the calculation of the mean excess stress $\sigma'_{am}$ in the cables and the reinforcement (formulae 2 or 3 in Section 6); hence $\sigma'_u$ and $\Delta T$ are obtained.

**Values of the proportion of the prestressing force used in compensating for shrinkage effects.** This proportion $F_1$ of the total force $F$ is not available for prestress and it is useful to know its value.

If $|e|$ is the absolute value of the eccentricity, then:

\[
\frac{F_1}{S} \left(1 + \frac{|e|}{r^2/v'}\right) = E_b e' b = E_b m' \lambda K 1 + m' \lambda K \eta
\]

Now: $K = 1 + (v'^2/r^2)$ and $m'$ is chosen as equal to 50; $\lambda = A_n/S$.

But $1 + [|e|/(r^2/v')] \approx K$. Therefore $F_1/S = [m' \lambda/(1 + m' \lambda K)]E_b \eta$.

$K$ can be taken as equal to 4·4 for the usual type of I-beams; $E_b \eta$ can be taken as equal to 45 kg/cm² under usual conditions (150 000 × 3/10 000) [see Section 5, sub-section (a)]. The following approximate values of $F_1/S$ and hence $F_1$ are then obtained, as a function of $\lambda$:

<table>
<thead>
<tr>
<th>$\lambda$ (%)</th>
<th>$\frac{m' \lambda}{1 + m' \lambda K}$</th>
<th>$\frac{F_1}{S}$ = $\frac{m' \lambda}{1 + m' \lambda K}$ × 450 t/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>0·1</td>
<td>0·041</td>
<td>18 t/m² (1·8 kg/cm²)</td>
</tr>
<tr>
<td>0·3</td>
<td>0·09</td>
<td>40 t/m² (4·0 kg/cm²)</td>
</tr>
<tr>
<td>0·5</td>
<td>0·119</td>
<td>54 t/m² (5·4 kg/cm²)</td>
</tr>
<tr>
<td>0·8</td>
<td>0·145</td>
<td>66 t/m² (6·6 kg/cm²)</td>
</tr>
<tr>
<td>1</td>
<td>0·156</td>
<td>70 t/m² (7·0 kg/cm²)</td>
</tr>
</tbody>
</table>

8. Note on design on the basis of limit states. Similarity with design in previous section

Since there is no need to consider the limit state of tensile strain in the concrete with Class III standards, as opposed to Classes I and II, and since safety with respect to the limit state of failure of the bottom flange under
minimum loading is generally satisfied, it is not to be expected that methods of calculation different from those given in the previous section would be applicable.

Indeed, in view of the suitability of the bottom flange the limit states which are sufficient to satisfy the design are:

- the limit state of failure of the top flange under maximum loading;
- the limit state of loss of compression in the bottom fibre under a proportion of the loads \( M_p + \alpha M_s \);
- the limit state of tensile strain in the reinforcement under maximum loading.

The limit state of failure under maximum loading is the same as in the previous section as is the limit state of tensile strain in the reinforcement (method of Section 6).

Concerning the limit state of loss of compression under the effect of the moment \( M_p + \alpha M_s \), it could be evaluated as in the case of Class II (approximately) without previously knowing the section properties, if all the prestressing force \( F \) were usefully employed; that is, if a portion \( F_1 \) were not used to compensate the shrinkage effects. It would then be sufficient to write that under the moment \( M_p + \alpha M_s \) the centre of compression rises to the upper limit \( G_s \) of the central core, and that \( G_s \) is approximately coincident with the centroid of the top two-thirds, whatever the dimensions of the bottom flanges (as yet unknown).

But since the force \( F_1 \) must be added, it is necessary to write:

\[
F = F_1 + \frac{M_p + \alpha M_s}{E_o G_s}
\]

To calculate \( F_1 \), however, it is necessary to know the section, since \( F_1 \) is such that:

\[
\frac{F_1}{S} \left(1 + \frac{|e|}{r^2/\nu'} \right) = E_o \varepsilon'_b
\]

Since \( E_o G_s \) is equal to \( |e| + (r_2/\nu') \), the equation

\[
F = F_1 + \frac{M_p + \alpha M_s}{E_o G_s}
\]

is the same as equation (b') of Section 7.

In the following section, therefore, only the design in accordance with Section 7 is considered.
COMMENT
Should the initial conditions render it necessary to reinforce the bottom flange, the design could nevertheless be corrected on the basis of the limit state of failure under initial prestress and accompanying loads. After this, the design would be the same as in the previous case.

9. Application
The example previously taken for Classes I and II is considered. Span 48 m. Depth 2·80 m. Width at top 1·75 m. Permanent load in addition to self-weight: \( q = 0.208 \text{ t/m} \) (\( M_q = 60 \text{ tm} \)); Live load: \( s = 2.91 \text{ t/m} \) (\( M_s = 840 \text{ tm} \)). The FIP–CEB safety recommendations are adopted.

The strength of the concrete is taken to be the same as in Chapter X. 28-day cube strength = 450 kg/cm², giving a design strength (see Chapter X) of:

\[
R^*_b = 200 \text{ kg/cm}^2 (2000 \text{ t/m}^2)
\]

Cube strength at the times of prestress = \( R_{cube} = 300 \text{ kg/cm}^2 \) (design strength \( R^*_b = 1330 \text{ t/m}^2 \)).

The strengths of the cables and bars are also taken to be the same as in Chapter X:

- cables: design rupture stress 139 kg/mm²;
- reinforcement: design elastic limit 43.6 kg/mm² (4.36 t/cm²).

Cables: 7 mm diameter, 12-wire cables (461 mm²). Breaking force: 64 t per cable.

(For simplification, the asterisk in \( F^*_r \) and \( F^*_ae \) is omitted; it is understood that these forces are calculated on the basis of the design strength values, including the safety factor \( 1/\gamma_a \).)

Web thickness: \( b' = 0.18 \text{ m} \).

\( M_p = 600 \text{ tm} \) is taken as a first approximation.

It is assumed that the centroid of the cables is at 0·10 m and the reinforcement at 0·04 m from the bottom fibre. \( F \) denotes the force which is exerted by the cables under permanent conditions, \( F_r \) the breaking force in the cables, \( \Delta T \) the increase in tension in the cables at maximum loading from the time of loss of compression in the bottom fibre, \( F_a \) the force exerted by the reinforcement under maximum loading working conditions, and \( F_{ae} \) the force in the reinforcement at the 0·2% proof stress. It is assumed that this limit is reached at failure of the section (see Chapter IX, Section 11).
(1) **First stage of design: conditions at failure (design of the top flange and relation between the cross-sectional areas of the cables and the reinforcement)**

It is assumed (subject to a later correction) that the distance of the centre of compression from the centroid \( E_o \) of the cables is \( 0.95h_1 \) at failure, where \( h_1 \) is the effective depth \( (h_1 = 2.80 - 0.10 = 2.70 \text{ m}) \). Therefore \( 0.95h_1 = 0.95 \times 2.70 = 2.56 \text{ m} \).

The distance of the centre of compression from the reinforcement is then \( 2.56 + 0.06 = 2.62 \text{ m} \).

![Diagram](image)

**Fig. 9.** Limit state of failure under maximum loading.

The factor of safety for the load is taken as \( \gamma_s = 1.4 \):

\[
M_{\text{max}} = 600 + 60 + 840 = 1500
\]

Therefore

\[
M_r = 1.4 \times 1500 = 2100 \text{ tm}
\]

But

\[
M_r = 2.56F_r + 2.62F_{ae}
\]

Hence:

\[
2.56F_r + 2.62F_{ae} = 2100 \text{ tm}
\]

or:

\[
F_r + 1.02F_{ae} = 820 \text{ t}
\]  

(a)
The distribution of forces between the cables and the reinforcement is not yet known, but, since the lever arm values are very nearly equal, \( F_r + F_{ae} = 820 \text{ t} \) approximately.

The area of concrete in compression must therefore resist this force under a stress equal to \( R_{ub}^* \).

This area must therefore be equal to \( 820/2000 = 0.41 \text{ m}^2 \) \((1.75 \times 0.234)\). The thickness of the top flange is therefore \( 0.23 \text{ m} \) if it is required that the neutral axis should lie within the top flange. But, as in Chapter X, the thickness can be reduced to \( 0.20 \text{ m} \), accepting as a consequence a reduction in the lever arm, relative to the estimated values.

To obtain an area of \( 0.41 \text{ m}^2 \), part of the web must be added to the flange. This is \( 0.33 \text{ m} \) deep \((1.75 \times 0.20 + 0.18 \times 0.33 = 0.41 \text{ m}^2)\).

The centroid of the area is \( 0.14 \text{ m} \) from the top of the flange. The lever arms are then \( 2.70 - 0.14 = 2.56 \text{ m} \) and \( 2.56 + 0.06 = 2.62 \text{ m} \); that is, they agree with those which have been assumed.

The assumption made concerning the lever arm corresponding to the cables \((0.95h_1)\) does not therefore require correction.

The total mechanical percentage \( w + w_a = (F_r + F_{ae})/(bh_{1}\, R_{ub}^*) \) is approximately equal to \( 820/(1.75 \times 2.70 \times 2000) = 0.087 \). With this value the cables and the reinforcement effectively reach their rupture stress and proof stress respectively at failure. Equation (a) is therefore valid.

(2) Design of the bottom flange

From Table IV, given later, the apparent tensile stress at the bottom flange reaches at least \(-80 \text{ kg/cm}^2\) under maximum loading.\( \ddagger \) If it were required that under permanent loading \((p + q)\) the compressive stress should reach the permissible limit, of the order of \(140 \text{ kg/cm}^2\), the stress variation under the moment \( M_x \) should be \(220 \text{ kg/cm}^2 \) \((2200 \text{ t/m}^2\)). This would determine the value of \( I/v' \):

\[
\frac{I}{v'} = \frac{\Delta M}{\Delta R} = \frac{840}{2200} = 0.38 \text{ m}^3
\]

This would give a flange size which is definitely insufficient to accommodate the cables and the reinforcement. It is therefore necessary to size

\( \ddagger \) If the lever arms were very different, it would be necessary to make other assumptions relative to the distribution of \( F_r \) and \( F_{ae} \). Trial and error methods for both eqn. (a) and the following equations would be required.

\( \ddagger \) Although 'apparent' tensile stresses of this magnitude are excessive, they are considered to provide an indication of the order of size of the modulus \( I/v' \) which would be sufficient from strength considerations if the criterion of 'limit' strains were accepted.
the flange so that the cables and reinforcement are accommodated, and, after a few preliminary trials, a flange size of 0.42 m wide by 0.26 m thick is chosen (see Chapter X, Section 6).

The properties of this section are then:

\[
\begin{align*}
S &= 0.879 \text{ m}^2 \\
v &= 1.03 \text{ m} \\
v' &= 1.77 \text{ m} \\
I &= 0.838 \text{ m}^4 \\
I/v &= 0.812 \text{ m}^3 \\
I/v' &= 0.473 \text{ m}^3 \\
r^2/v &= 0.922 \text{ m} \\
r^2/v' &= 0.537 \text{ m}
\end{align*}
\]

Trial and error will be based on this section, by varying the distribution between the cables and the reinforcement.

The tables obtained from the trial and error calculations enable the prestressing force \( F \) which satisfies the criteria defined in Section 7 to be determined.

(3) **Trial and error**

Different values of \( F \) are tried, each corresponding to a whole number of cables; for each value of \( F \), a value of \( F_{ae} \) which satisfies equation (a) for failure is obtained.

(a) Corrections are made for the holes in the section, the reinforcement and the cables. The correction factors given in Chapter VII, Section 7, are applied.

Let \( \sigma'_o \) be the gross value of the prestress at the bottom fibre; that is, the value for a full section and without reinforcement; and let \( A_a \) be the cross-sectional area of the reinforcement and \( A_t \) that of the holes for the cables. The true prestress value is:

\[
\sigma'_o = \sigma'_o \left[ 1 - \frac{mA_a}{S} - \frac{A_t}{S} \left( 1 + \frac{e^2}{r^2} \right) \right]
\]

If \( \sigma'_p \) is the gross value of the stress at the bottom fibre due to the loads \( p \) acting before grouting, the true value is:

\[
\sigma'_p = \sigma'_p \left[ 1 - \frac{mA_a}{S} - \frac{A_t}{S} \left( \frac{e^2}{r^2} + \frac{|e|}{v'} \right) \right]
\]

If \( \sigma'_q \) and \( \sigma'_s \) are the gross values of the stresses at the bottom fibre due to the loads \( q \) and \( s \) which act after grouting, their true values are:

\[
\sigma'_q \text{ (or } \sigma'_s) = \sigma'_q \text{ (or } \sigma'_s) \left[ 1 - \frac{m(A_c + A_a) - A_t}{S} \left( \frac{e^2}{r^2} + \frac{|e|}{v'} \right) \right]
\]
(b) To calculate the tensile strain at the bottom fibre due to the shrinkage \( \eta \), formula (d) of Section 5 is applied, or:

\[
\varepsilon'_b = \frac{m'(A_a/S)K}{1 + m'(A_a/S)K} \eta, \quad \text{where} \quad K = 1 + \frac{v'^2}{t^2}
\]

This strain is compensated by a portion \( F_1 \) of the prestressing force, such that it would create in the bottom fibre a compression which is equal to \( E_b \varepsilon'_b \).

This is equivalent to saying that on the calculated prestress \( \sigma'_o \), a portion \( \sigma'_{o1} = E_b \varepsilon'_b \) is cancelled by shrinkage, so that the available prestress is \( \sigma'_o - E_b \varepsilon'_b \).

For the reasons given in Section 5, \( \eta = 3/10 \, 000 \) and \( E_b = 150 \, 000 \, \text{kg/cm}^2 \) are used.

Then:

\[
1 + \frac{e^2}{r^2} = 3.93
\]

\[
\frac{e^2}{r^2} + \frac{|e|}{v'} = 3.87
\]

\[
K = 1 + \frac{v'^2}{r^2} = 4.3
\]

The following are used in the formulae:

\( m = 15 \) for corrections relative to prestress and self-weight;
\( m = 5 \) for corrections relative to loading which is applied after grouting;
\( m' = 50 \) for the calculation of the tensile strain in the concrete under the effect of restrained shrinkage.

The methods which are indicated still apply if different values are chosen for \( m', m, \eta \) and \( E_b \).

The value of the prestress \( \overline{\sigma}'_o \) is \( \overline{\sigma}'_o = (F/S)\{1 + [\varepsilon/(r^2v')]\} = 4.11(F/S) \).

The calculations are done on the assumption that under permanent conditions each cable exerts a force of 43 t. Excess tension need not be considered, since the section is taken as homogeneous.

The breaking force in each cable is 64 t. Thus the force \( F_r \) corresponding to the chosen number of cables is obtained, and the force \( F_{ae} \) is given by eqn. (a):

\[
F_{ae} = \frac{820 \, t - F_r}{1.02}
\]
### Table III

Tensile strain at the bottom fibre due to shrinkage. Correction factors

<table>
<thead>
<tr>
<th>Number of cables</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_e$</td>
<td>768</td>
<td>704</td>
<td>640</td>
<td>576</td>
<td>518</td>
</tr>
<tr>
<td>$F_{ae}$</td>
<td>51</td>
<td>112</td>
<td>177</td>
<td>239</td>
<td>302</td>
</tr>
<tr>
<td>Cable area (4.61 cm² per cable): $A_c$</td>
<td>55.4</td>
<td>50.8</td>
<td>46.1</td>
<td>41.4</td>
<td>36.9</td>
</tr>
<tr>
<td>Reinforcement area ($F_{ae}/4.36$ t/cm²): $A_a$</td>
<td>11.7</td>
<td>25.7</td>
<td>40.6</td>
<td>54.7</td>
<td>69.1</td>
</tr>
<tr>
<td>Area of holes (15.7 mm² per cable): $A_t$</td>
<td>188</td>
<td>173</td>
<td>157</td>
<td>141</td>
<td>126</td>
</tr>
</tbody>
</table>

\[
m'(A_a/S)K = \frac{m'(A_a/S)K}{1 + m'(A_a/S)K} \times \frac{3}{10000}
\]

\[
\varepsilon' \approx \frac{m'(A_a/S)K}{1 + m'(A_a/S)K} \times \frac{3}{10000}
\]

\[
\begin{align*}
(mA_a - A_t)/S, & \text{ where } m = 15 \\
& \text{(for corrections relative to prestress and self-weight)} \\
& +0.002 +0.023 +0.049 +0.076 +0.101
\end{align*}
\]

\[
\begin{align*}
[m(A_c + A_a) - A_t]/S, & \text{ with } m = 5 \\
& \text{(for corrections relative to loads acting after grouting)} \\
& +0.0165 0.0235 0.0306 0.038 0.045
\end{align*}
\]

**Correction factors:**

- For $\sigma'_o$ and $\sigma'_p$
  \[ k_o = 1 - (mA_a - A_t)/S \times 3.93 \]
  \[ 1.008 0.910 0.807 0.702 0.604 \]

- For $\sigma'_q$ or $\sigma'_s$
  \[ k_s = 1 - [m(A_c + A_a) - A_t]/S \times 3.87 \]
  \[ 0.936 0.909 0.882 0.853 0.826 \]

The same value is taken for the correction factors for $\sigma'_o$ and $\sigma'_p$ (their difference is too small to be of any significance).
### Table IV

Successive resultant stresses at the bottom fibre. Proportion of live load causing loss of compression at the bottom fibre

<table>
<thead>
<tr>
<th>Number of cables</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prestressing force (43 t per cable): $F$</td>
<td>516</td>
<td>473</td>
<td>430</td>
<td>387</td>
<td>344</td>
</tr>
<tr>
<td>Mean stress $\sigma_e = \frac{F}{S}$</td>
<td>586</td>
<td>537</td>
<td>490</td>
<td>441</td>
<td>392</td>
</tr>
<tr>
<td>Gross prestress $\overline{\sigma'_o} = 4\cdot11 \times \frac{F}{S}$</td>
<td>2 410</td>
<td>2 210</td>
<td>2 020</td>
<td>1 815</td>
<td>1 610</td>
</tr>
<tr>
<td>Corrected prestress $\sigma'_o = k_o \sigma'_o$</td>
<td>2 429</td>
<td>2 010</td>
<td>1 630</td>
<td>1 275</td>
<td>974</td>
</tr>
<tr>
<td>Deduction for shrinkage $(\varepsilon'_b \text{ Table III } \times 1.5 \times 10^6)$</td>
<td>-101</td>
<td>-174</td>
<td>-224</td>
<td>-256</td>
<td>-284</td>
</tr>
<tr>
<td>Available prestress $\overline{\sigma'_{edisp}}$</td>
<td>2 328</td>
<td>1 836</td>
<td>1 406</td>
<td>1 019</td>
<td>690</td>
</tr>
<tr>
<td></td>
<td>(t/m²)</td>
<td>(t/m²)</td>
<td>(t/m²)</td>
<td>(t/m²)</td>
<td>(t/m²)</td>
</tr>
<tr>
<td>---------------------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>Self-weight stress</td>
<td>1048</td>
<td>677</td>
<td>381</td>
<td>127</td>
<td>79</td>
</tr>
<tr>
<td>(7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stress due to load</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma' = k_0 \sigma_p$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resultant stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>under self-weight</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resultant stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>under permanent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>loading $(p + q)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(10)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stress due to total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>live loads $\sigma' = k_0 \sigma_p$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(11)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proportion $\alpha$ of live loads for which the resultant stress is zero (line 10/line 11)</td>
<td>0.55</td>
<td>0.35</td>
<td>0.17</td>
<td>0.01</td>
<td>−0.125</td>
</tr>
</tbody>
</table>
The same values are also taken for \( k_q \) and \( k_s \). Theoretically, \( m = 15 \) should be used for \( q \), since this is a sustained load. But since \( M_q \) is small compared to \( M_d \) + \( M_s \), equal values of \( k_q \) and \( k_s \) simplify the calculations and the error is insignificant.

Now: \( \bar{\sigma}'_o = 4.11(F/S) \); therefore \( \bar{\sigma}'_o = k_o \times 4.11(F/S) \); the available prestress is \( \sigma'_o \text{ disp} = \sigma'_o - E_b \bar{e}'_b \) (\( E_b = 150000 \text{ kg/cm}^2 \) or \( 1500000 \text{ t/m}^2 \)). Also:

\[
\bar{\sigma}'_o = \frac{600}{0.473} = -1270 \text{ t/m}^2
\]

\[
\bar{\sigma}'_q = \frac{60}{0.473} = -127 \text{ t/m}^2
\]

\[
\bar{\sigma}'_s = \frac{840}{0.473} = -1780 \text{ t/m}^2
\]

### Table V

*Reinforcement stresses \( \sigma'_o \) and cable stress increases \( \Delta T \)

<table>
<thead>
<tr>
<th>Number of cables</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_c + A_e )</td>
<td>67.1</td>
<td>76.5</td>
<td>86.7</td>
<td>96.1</td>
<td>106</td>
</tr>
<tr>
<td>( y_n )</td>
<td>0.320</td>
<td>0.346</td>
<td>0.370</td>
<td>0.400</td>
<td>0.431</td>
</tr>
<tr>
<td>( y'_n )</td>
<td>2.40</td>
<td>2.37</td>
<td>2.35</td>
<td>2.32</td>
<td>2.29</td>
</tr>
<tr>
<td>( z )</td>
<td>2.63</td>
<td>2.63</td>
<td>2.63</td>
<td>2.63</td>
<td>2.63</td>
</tr>
<tr>
<td>( F' = A_c \times 9600 )</td>
<td>541</td>
<td>488</td>
<td>444</td>
<td>395</td>
<td>354</td>
</tr>
<tr>
<td>( \sigma'_o )</td>
<td>615</td>
<td>555</td>
<td>504</td>
<td>444</td>
<td>404</td>
</tr>
<tr>
<td>Loss of ( \sigma'_o )</td>
<td>( \frac{F}{S} )</td>
<td>21</td>
<td>46</td>
<td>67</td>
<td>89</td>
</tr>
<tr>
<td>( \sigma'_e )</td>
<td>804</td>
<td>680</td>
<td>581</td>
<td>466</td>
<td>374</td>
</tr>
<tr>
<td>( M_{\text{max}} - M_d )</td>
<td>378</td>
<td>546</td>
<td>697</td>
<td>823</td>
<td>905</td>
</tr>
<tr>
<td>( \Delta F_{11} )</td>
<td>144</td>
<td>207</td>
<td>265</td>
<td>312</td>
<td>358</td>
</tr>
<tr>
<td>( \sigma'_{a1} )</td>
<td>2140</td>
<td>2710</td>
<td>3050</td>
<td>3240</td>
<td>3390</td>
</tr>
<tr>
<td>( \varphi )</td>
<td>0.186</td>
<td>0.125</td>
<td>0.095</td>
<td>0.072</td>
<td>0.054</td>
</tr>
<tr>
<td>( \sigma'_o \approx \Delta T )</td>
<td>1210</td>
<td>1980</td>
<td>2500</td>
<td>2840</td>
<td>3070</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.55</td>
<td>0.35</td>
<td>0.17</td>
<td>0.02</td>
<td>-0.125</td>
</tr>
</tbody>
</table>
Hence the corrected values $-\sigma'_p - \sigma'_q - \sigma'_s$, and therefore the successive resultant stresses are derived. They are calculated in Table IV in terms of the number of cables.

**Reinforcement stresses $\sigma'_a$. Increasing stresses $\Delta T$ in the cables due to loss of compression at the bottom fibre**

(i) To get a clear idea of the method (Section 6), the complete calculation is given in the following for the case of eleven cables.

In this case, the moment after loss of compression is:

$$M_p + M_q + 0.35M_s = 660 + 294 = 954 \text{ tm}$$

The moment $M_{\text{max}} - M_d = 1500 - 954 = 546 \text{ tm}$.

Also:

$$A_c + A_a = 50.8 + 25.7 = 76.5 \text{ cm}^2$$

The combined centroid of the reinforcement and the cables is 0.08 m from the bottom face. Effective depth $h_1 = 2.72$ m.

With the same symbols as in Section 7, the equation giving the neutral axis (depth $y_n$) is obtained by writing $y_n = vh_1$:

$$v^2 - \left(1 - \frac{b'}{b}\right)\left(v - \frac{u}{h_1}\right)^2 - 2m \frac{A_c + A_a}{bh_1} (1 - v) = 0$$

$$1 - \frac{b'}{b} = 1 - \frac{0.18}{1.75} = 0.897$$

$$\frac{u}{h_1} = \frac{0.20}{2.72} = 0.073$$

$$\frac{2 \times 5 \times 76.5}{1.75 \times 2.72} = 0.0161$$

Therefore:

$$v^2 - 0.897(v - 0.073)^2 - 0.0161(1 - v) = 0$$

Hence:

$$v = 0.127, \quad y_n = 0.346 \text{ m}$$

$$y'_n = 2.72 - 0.346 = 2.37 \text{ m}$$

From the formula given in Section 7, it is found that the centre of compression is 0.09 m from the top fibre.

Then:

$$z = G_sE'_o = 2.72 - 0.09 = 2.63 \text{ m}$$
The prestressing force after loss of compression is:

\[ F' = 50.8 \times 9\,600 \text{ kg/cm}^2 = 488 \text{ t} \]

The corresponding mean stress is 488/0.879 = 555 t/m².

The portion \( F_1 \) of the prestressing force which is absorbed by the tensile stresses due to shrinkage (line 5 of Table IV) is such that:

\[
\frac{F_1}{S} \times 4.11 \times 0.91 = 174 \text{ t/m}^2
\]

Therefore \( F_1/S = 46 \text{ t/m}^2 \). This quantity is deducted from the effective compression, so that the remaining mean compression is \( \sigma_y = 509 \text{ t/m}^2 \).

The stress \( \sigma_n \) at the depth corresponding to point N (Fig. 8) (see Section 6) is \( \sigma_n = \sigma_y (y'/v') \), or:

\[
\sigma_n = 509 \times \frac{2.37}{1.77} = 680 \text{ t/m}^2 (68 \text{ kg/cm}^2)
\]

A first approximation of \( \Delta F \) is:

\[
\Delta F_1 = \frac{M_{\text{max}} - M_d}{z} = \frac{546}{2.63} = 207 \text{ t}
\]

The corresponding stress \( \sigma_{a1} \) in the steel is equal to 207 t/76.5 = 2.71 t/cm², or 2,710 kg/cm².

Then, with the above values of \( \sigma_n \) and \( \sigma_{a1} \):

\[
\varphi = \frac{m \sigma_n}{\sigma_{a1}} = \frac{5 \times 68}{2.710} = 0.125
\]

\[
\frac{b'y_n \sigma_n}{2} = \frac{0.18 \times 2.37 \times 680}{2} = 144 \text{ t}
\]

Hence, from formula (3) of Section 6:

\[
\Delta F = \Delta F_1 - 144 \text{ t} (1 - 0.125) \left[ 1 - \frac{2}{3} \times \frac{2.37}{2.63} \left( 1 - \frac{0.125}{2} \right) \right]
\]

\[
= 207 - 55 = 152 \text{ t}
\]

† The calculation is not done on an homogeneous section and the properties of the gross section are taken. The increase in stress between the permanent state and the state of loss of compression must then be taken into consideration, or \( m \times 56 \text{ kg/cm}^2 = 280 \text{ kg/cm}^2 (T = 9\,300 + 280 = 9\,600 \text{ kg/cm}^2) \).

‡ Including the correction factor \( k_a \) relative to prestress (Table III).
Then:
\[
\sigma'_a = \frac{152}{76.5} = 1.98 \text{ t/cm}^2 (1980 \text{ kg/cm}^2)
\]

And, approximately:

\[
\sigma'_a \approx \Delta T = 20 \text{ kg/mm}^2
\]

If it is required to improve on this approximation, this value of \( \sigma'_a \) is taken as the starting point; \( \varphi \) becomes \((5 \times 68)/1980 = 0.171\). Hence:

\[
\Delta F = \Delta F_1 - 144 t (1 - 0.171) \left[ 1 - \frac{2}{3} \times \frac{2.37}{2.63} \left( 1 - \frac{0.171}{2} \right) \right]
\]

\[
= 207 - 53 = 154 \text{ t}
\]

The correction is therefore not necessary in this case.

Similar calculations for other numbers of cables give the results shown in Table V, showing the effects of the various factors. In particular, it can be seen that the first approximation giving \( \sigma'_{a1} \) is better still for smaller values of prestress. The tables show the sequence of the design in relation to the proportion of the loading under which it is required that the bottom fibre should remain in compression; that is in relation to the factor \( \alpha \).

Assume that \( \alpha = \frac{1}{3} \) (loss of compression under \( \frac{1}{3} \) of the live loading) is fixed. Table IV shows that eleven cables are required and Table III shows that the required area of reinforcement is 25.7 cm². The check calculation is completed on this basis.

Final stress under maximum loading \( \Delta T = 1980 \text{ kg/cm}^2 \) (20 kg/mm²).

\( T' \) (due to loss of compression) = 96 kg/mm². \( T + \Delta T = 116 \text{ kg/mm}^2 \).

Stress at the bottom fibre under initial prestress is (120 kg/mm²).

The prestress at this fibre under permanent tension (93 kg/mm²) is calculated in Table IV.

With eleven cables, the gross prestress \( \overline{\sigma}'_g \) is 2,010 t/m².

Under initial tension \( \sigma'_o = 2010 \times 120/93 = 2590 \text{ t/m}^2 \)

Less shrinkage effect (Table IV) \( - 174 \)

Less self-weight stress \( - 1159 \)

Resultant stress \( 1257 \text{ t/m}^2 \)

This stress is less than 0.55\(R_b\text{cyl}\). (the calculation is done on the elastic basis so that the ASP strength criterion is adopted); hence it is acceptable and all the cables can be tensioned initially.
Diameter of reinforcement (see Section 4)

Area of the section B' whose centroid is coincident with that of the reinforcement:

\[ B' = 42 \times 8 = 336 \text{ cm}^2 \]
\[ A_a = 25.7 \text{ cm}^2 \]
\[ w'_f = 100 \times \frac{25.7}{336} = 7.6 \]

![Diagram](image)

**Fig. 10.** Section designed as Class III.

With steel of high bond strength, when cracking is to be limited, Section 4 gives the following formula:

\[ d_{max} = 100000 \times \frac{1.6w'_f}{(w'_f + 10)s'_a} \quad (d \text{ in mm}) \]
\[ s'_a = 1980 \]
Therefore:

\[ d_{\text{max}} = \frac{100000 \times 1.6 \times 7.6}{17.6 \times 1.980} = 35 \text{ mm} \quad \text{(say 32 mm)} \]

It is considered that this diameter is high and five 26 mm diameter bars are chosen (five 26 mm diameter = 26.5 > 25.7).

The section is shown in Fig. 10.

The comparison with Classes I and II is summarised in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Class I</th>
<th>Class II</th>
<th>Class III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(( \alpha = \frac{1}{2} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top flange (m)</td>
<td>1.75 \times 0.20</td>
<td>1.75 \times 0.20</td>
<td>1.75 \times 0.20</td>
</tr>
<tr>
<td>Bottom flange (m)</td>
<td>0.58 \times 0.26</td>
<td>0.42 \times 0.26</td>
<td>0.42 \times 0.26</td>
</tr>
<tr>
<td>Number of cables</td>
<td>16</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>Complementary</td>
<td>negligible</td>
<td>7.3 (seven 12</td>
<td>25.7 (five 26</td>
</tr>
<tr>
<td>reinforcement (cm²)</td>
<td></td>
<td>mm diameter)</td>
<td>mm diameter)</td>
</tr>
</tbody>
</table>
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