GREEK MATHEMATICS

II
SELECTIONS
ILLUSTRATING THE HISTORY OF
GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY
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IN TWO VOLUMES
II
FROM ARISTARCHUS TO PAPPUS

LONDON
WILLIAM HEINEMANN LTD
CAMBRIDGE, MASSACHUSETTS
HARVARD UNIVERSITY PRESS
MCMLI
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XVI. ARISTARCHUS OF SAMOS
XVI. ARISTARCHUS OF SAMOS

(a) General

Aët. i. 15. 5; Doxographi Graeci, ed. Diels 313. 16-18

'Αρισταρχος Σάμιος μαθηματικὸς ἀκούστης
Στράτωνος φῶς εἶναι τὸ χρῶμα τοῖς ύποκειμένοις
ἐπιπίπτον.

Archim. Aren. 1, Archim. ed. Heiberg ii. 218. 7-18

'Αρισταρχος δὲ ὁ Σάμιος ὑποθεσίων τινῶν ἐξ-
ἐδωκεν γραφᾶς, ἐν αἷς ἐκ τῶν ὑποκειμένων συμ-
βαίνει τὸν κόσμον πολλαπλάσιον εἴμεν τοῦ νῦν
εἰρημένου. ὑποτίθεται γὰρ τὰ μὲν ἀπλανεὰ τῶν
ἀστρων καὶ τὸν ἀλιον μὲνεν ἀκινητον, τὰν δὲ γὰν
περιφέρεσθαι περὶ τὸν ἀλιον κατὰ κύκλου περι-
φέρειαν, ὡς ἔστω ἐν μέσῳ τῶν δρόμων κείμενος,
τὰν δὲ τῶν ἀπλανέων ἀστρων σφαίραν περὶ τὸ

---

a Strato of Lampsacus was head of the Lyceum from 288/287 to 270/269 B.C. The next extract shows that Aristarchus formulated his heliocentric hypothesis before Archimedes wrote the Sand-Reckoner, which can be shown to have been written before 216 B.C. From Ptolemy, Syntaxis
iii. 2, Aristarchus is known to have made an observation of
the summer solstice in 281/280 B.C. He is ranked by
Vitruvius, De Architectura i. 1. 17 among those rare men,
such as Philolaus, Archytas, Apollonius, Eratosthenes,
XVI. ARISTARCHUS OF SAMOS

(a) General

Aëtius i. 15. 5; Doxographi Graeci, ed. Diels 313. 16-18

Aristarchus of Samos, a mathematician and pupil of Strato, held that colour was light impinging on a substratum.

Archimedes, Sand-Reckoner 1, Archim. ed. Heiberg ii. 218. 7-18

Aristarchus of Samos produced a book based on certain hypotheses, in which it follows from the premises that the universe is many times greater than the universe now so called. His hypotheses are that the fixed stars and the sun remain motionless, that the earth revolves in the circumference of a circle about the sun, which lies in the middle of the orbit, and that the sphere of the fixed stars, situated

Archimedes and Scopinas of Syracuse, who were equally proficient in all branches of science. Vitruvius, loc. cit. ix. 8. 1, is also our authority for believing that he invented a sun-dial with a hemispherical bowl. His greatest achievement, of course, was the hypothesis that the earth moves round the sun, but as that belongs to astronomy it can be mentioned only casually here. A full and admirable discussion will be found in Heath, Aristarchus of Samos: The Ancient Copernicus, together with a critical text of Aristarchus's only extant work.
GREEK MATHEMATICS

αὐτὸ κέντρον τῶν ἀλλίως κειμένων τῷ μεγέθει ταλικαύταν εἶμεν, ὡστε τὸν κύκλον, καθ' ὅν τὰν γᾶν ὑποτίθεται περιφέρεσθαι, τοιαύταν ἔχειν ἀναλογίαιν ποτὶ τὰν τῶν ἀπλανῶν ἀποστάσιαν, οὗαν ἔχει τὸ κέντρον τὰς σφαῖρας ποτὶ τὰν ἐπιφάνειαν.

Plut. De facie in orbe lunae 6, 922 f–923 a

Καὶ ὁ Λεύκιος γελάσας, "Μόνον," εἶπεν, "ὅ ταν, μὴ κρίσαι ἡμῖν ἁσβεῖας ἐπαγγειλῆς, ὡσπερ Ἄρισταρχὼν ᾖτο δεῖν Κλεάνθης τὸν Σάμιον ἁσβεῖας προσκαλεῖσθαι τοὺς "Ελλήνας, ὡς κινοῦντα τοῦ κόσμου τὴν ἐστίαν, ὅτι τὰ φαινόμενα σώζειν ἀνὴρ ἐπειράτο, μένειν τὸν οὐρανὸν ὑποτιθέμενος, ἐξελίττεσθαι δὲ κατὰ λοξοῦ κύκλου τὴν γῆν, ἀμα καὶ περὶ τὸν αὐτῆς ᾧδονα διωμένην."

(b) DISTANCES OF THE SUN AND MOON


<(Ὑποθέσεις)>

α'. Τὴν σελήνην παρὰ τοῦ ἠλίου τὸ φῶς λαμβάνειν.

β'. Τὴν γῆν σημεῖον τε καὶ κέντρον λόγον ἔχειν πρὸς τὴν τῆς σελήνης σφαίραν.

γ'. Ὑπαν η σελήνη διχότομος ἡμῶν φαίνεται,

1 ὑποθέσεις add. Heath.

* Aristarchus’s last hypothesis, if taken literally, would mean that the sphere of the fixed stars is infinite. All that he implies, however, is that in relation to the distance of the
ARISTARCHUS OF SAMOS

about the same centre as the sun, is so great that
the circle in which he supposes the earth to revolve
has such a proportion to the distance of the fixed
stars as the centre of the sphere bears to its surface.

Plutarch, On the Face in the Moon 6, 922 f–923 a.

Lucius thereupon laughed and said: "Do not, my
good fellow, bring an action against me for impiety
after the manner of Cleanthes, who held that the
Greeks ought to indict Aristarchus of Samos on a
charge of impiety because he set in motion the hearth
of the universe; for he tried to save the phenomena
by supposing the heaven to remain at rest, and the
earth to revolve in an inclined circle, while rotating
at the same time about its own axis."

(b) DISTANCES OF THE SUN AND MOON

Aristarchus of Samos, On the Sizes and Distances of the Sun
and Moon, ed. Heath (Aristarchus of Samos: The
Ancient Copernicus) 352. 1–354. 6

HYPOTHESES

1. The moon receives its light from the sun.
2. The earth has the relation of a point and centre
to the sphere in which the moon moves.
3. When the moon appears to us halved, the great
fixed stars the diameter of the earth’s orbit may be neglected.
The phrase appears to be traditional (v. Aristarchus’s second
hypothesis, infra).

Heraclides of Pontus (along with Ecphantus, a Pythagorean)
had preceded Aristarchus in making the earth
revolve on its own axis, but he did not give the earth a motion
of translation as well.

Lit. “sphere of the moon.”
GREEK MATHEMATICS

νεύειν εἰς τὴν ἡμετέραν ὡφιν τὸν διορίζοντα τὸ τε σκιερὸν καὶ τὸ λαμπρὸν τῆς σελήνης μέγιστον κύκλον.

δ'. "Οταν ἡ σελήνη διχότομος ἦμαι φαίνεται, τότε αυτήν ἀπέχει τοῦ ἡλίου ἑλασθεν τεταρτη-

μορίου τῶ τοῦ τεταρτημορίου τριακοστῶ.

ε'. Τὸ τῆς σκιᾶς πλάτος σελήνων εἶναι δύο.

ε''. Τὴν σελήνην ὑποτείνειν ὕπὸ πεντεκαίδεκατον μέρος ξυδίου.

Ἐπιλογίζεται οὖν τὸ τοῦ ἡλίου ἀπόστημα ἀπὸ τῆς γῆς τοῦ τῆς σελήνης ἀποστήματος μείζον μὲν ἡ ὀκτωκαίδεκαπλάσιον, ἑλάσθον δὲ ἡ εἰκοσα-

πλάσιον, διὰ τῆς περὶ τὴν διχοτομίαν ὑποθέσεως τοῦ αὐτοῦ δὲ λόγου ἔχειν τὴν τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς σελήνης διάμετρον. τὴν δὲ τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς γῆς διάμετρον μείζονα μὲν λόγου ἔχειν ἡ ὧν τὰ ἑθ πρὸς γ', ἑλάσθονα δὲ ἡ ὥν μὴ πρὸς ε', διὰ τοῦ εὐρεθέντος περὶ τὰ ἀποστήματα λόγου, τῆς <τε'> περὶ τὴν σκιᾶν ὑποθέσεως, καὶ τοῦ τῆς σελήνην ὕπὸ πεντεκαι-

δέκατον μέρος ξυδίου ὑποτείνειν.

Ibid., Prop. 7, ed. Heath 376. 1–380. 28

Τὸ ἀπόστημα ὧ ἀπέχει ὁ ἡλίος ἀπὸ τῆς γῆς τοῦ

1 τε add. Heath.

—a Lit. "verges towards our eye." For "verging," v. vol. i. p. 244 n. a. Aristarchus means that the observer's eye lies in the plane of the great circle in question. A great circle is a circle described on the surface of the sphere and having the same centre as the sphere; as the Greek implies, a great circle is the "greatest circle" that can be described on the sphere.
circle dividing the dark and the bright portions of the moon is in the direction of our eye.\textsuperscript{a}

4. When the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant.\textsuperscript{b}

5. The breadth of the earth’s shadow is that of two moons.\textsuperscript{c}

6. The moon subtends one-fifteenth part of a sign of the zodiac.\textsuperscript{d}

It may now be proved that the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon—this follows from the hypothesis about the halved moon; that the diameter of the sun has the aforesaid ratio to the diameter of the moon; and that the diameter of the sun has to the diameter of the earth a ratio which is greater than 19 : 3 but less than 43 : 6—this follows from the ratio discovered about the distances, the hypothesis about the shadow, and the hypothesis that the moon subtends one-fifteenth part of a sign of the zodiac.

\textit{Ibid.}, Prop. 7, ed. Heath 376. 1-380. 28

The distance of the sun from the earth is greater than

\textsuperscript{a} i.e., is less than 90° by 3°, and so is 87°. The true value is 89° 50'.

\textsuperscript{b} i.e., the breadth of the earth’s shadow where the moon traverses it during an eclipse. The figure is presumably based on records of eclipses. Hipparchus made the figure 2\frac{1}{2} for the time when the moon is at its mean distance, and Ptolemy a little less than 2\frac{1}{2} for the time when the moon is at its greatest distance.

\textsuperscript{c} i.e., the angular diameter of the moon is one-fifteenth of 30°, or 2°. The true value is about 1\frac{1}{2}°, and in the \textit{Sand-Reckoner} (Archim. ed. Heiberg ii. 222. 6-8) Archimedes says that Aristarchus “discovered that the sun appeared to be about \frac{1}{250}th part of the circle of the Zodiac”; as he believed
ἀποστήματος οὖ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς μεῖζον μὲν ἕστω ἡ ὀκτωκαιδεκαπλάσιον, ἑλασθον δὲ ἡ εἰκοσαπλάσιον.

"Εστώ γὰρ ἡλίου μὲν κέντρον τὸ Α, γῆς δὲ τὸ Β, καὶ ἐπίζευγθείσα ἡ ΑΒ ἐκβεβλήσθω, σελήνης δὲ κέντρον διχοτόμου οὐσῆς τὸ Γ, καὶ ἐκβεβλήσθω διὰ τῆς ΑΒ καὶ τοῦ Γ ἐπίπεδον, καὶ ποιεῖτω τομὴν ἐν τῇ σφαίρᾳ, καθ' ἡς φέρεται τὸ κέντρον τοῦ ἡλίου, μέγιστον κύκλον τὸν ΑΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΓΒ, καὶ ἐκβεβλήσθω ἡ ΒΓ ἐπὶ τὸ Δ.

"Εσται δὴ, διὰ τὸ τὸ Γ σημεῖον κέντρον εἶναι τῆς σελήνης διχοτόμου οὐσῆς, ὀρθὴ ἡ ὑπὸ τῶν

that the sun and moon had the same angular diameter he must, therefore, have found the approximately correct angular diameter of ½° after writing his treatise On the Sizes and Distances of the Sun and Moon.
ARISTARCHUS OF SAMOS

eighteen times, but less than twenty times, the distance of the moon from the earth.

For let A be the centre of the sun, B that of the earth; let AB be joined and produced; let Γ be the centre of the moon when halved; let a plane be drawn through AB and Γ, and let the section made by it in the sphere on which the centre of the sun moves be the great circle AΔE, let AΓ, ΓΒ be joined, and let BΓ be produced to Δ.

Then, because the point Γ is the centre of the moon when halved, the angle AΓΒ will be right.
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1 ἐστὶν add. Nizze.
2 ἔχει add. Wallis.

* Lit. "circumference," as in several other places in this proposition.
ARISTARCHUS OF SAMOS

From B let BE be drawn at right angles to BA. Then the arc \( E\Delta \) will be one-thirtieth of the arc \( E\Delta A \); for, by hypothesis, when the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant [Hypothesis 4]. Therefore the angle \( EBI \) is also one-thirtieth of a right angle. Let the parallelogram \( AE \) be completed, and let \( BZ \) be joined. Then the angle \( ZBE \) will be one-half of a right angle. Let the angle \( ZBE \) be bisected by the straight line \( BH \); then the angle \( HBE \) is one-fourth part of a right angle. But the angle \( \Delta BE \) is one-thirtieth part of a right angle; therefore angle \( HBE : \text{angle } \Delta BE = 15 : 2 \); for, of those parts of which a right angle contains 60, the angle \( HBE \) contains 15 and the angle \( \Delta BE \) contains 2.

Now since

\[
\text{HE} : \text{EO} > \text{angle HBE} : \text{angle } \Delta BE,\]

therefore \( \text{HE} : \text{EO} > 15 : 2 \).

And since \( BE = EZ \), and the angle at \( E \) is right, therefore

\[
ZB^2 = 2BE^2.
\]

But

\[
ZB^2 : BE^2 = ZH^2 : HE^2.
\]

Therefore

\[
ZH^2 = 2HE^2.
\]

Now

\[
49 < 2 \cdot 25,
\]

so that

\[
\]

Therefore

\[
ZH : HE > 7 : 5.
\]

* Aristarchus’s assumption is equivalent to the theorem

\[
\tan a > a
\]

\[
\tan \beta > \beta
\]

where \( \beta < a \leq \frac{1}{2} \pi \). Euclid’s proof in Optics 8 is given in vol. i. pp. 502-505.
ἐχει ἡ (ὁν') τὰ ζ πρὸς τὰ ἓ· καὶ συνθέντι ἡ ΖΕ ἄρα πρὸς τὴν ΕΗ μείζονα λόγον ἐχει ἡ ὁν τὰ ὑβ πρὸς τὰ ἓ· τουτέστων, ἡ ὁν (τὰ*) λς πρὸς τὰ ἓτ. ἐδείξθη δὲ καὶ ἡ ΗΕ πρὸς τὴν ΕΘ μείζονα λόγον ἐχουσα ἡ ὁν τὰ ἓτ πρὸς τὰ δυο· δὲ ἢσον ἄρα ἡ ΖΕ πρὸς τὴν ΕΘ μείζονα λόγον ἐχει ἡ ὁν τὰ λς πρὸς τὰ δυο, τουτέστων, ἡ ὁν τὰ ἓτ πρὸς α· ἡ ἄρα ΖΕ τῆς ΕΘ μείζονν ἐστίν ἡ ἐη. ἡ δὲ ΖΕ ἵση ἐστιν τῇ ΒΕ· καὶ ἡ ΒΕ ἄρα τῆς ΕΘ μείζονν ἐστίν ἡ ἐη· πολλῷ ἄρα ἡ ΒΗ τῆς ΘΕ μείζονν ἐστίν ἡ ἐη. ἀλλ' ὡς ἡ ΒΘ πρὸς τὴν ΘΕ, οὕτως ἐστίν ἡ ἂΒ πρὸς τὴν ΒΓ, διὰ τὴν ὁμοιότητα τῶν τριγώνων· καὶ ἡ ἂΒ ἄρα τῆς ΒΓ μείζονν ἐστίν ἡ ἐη· καὶ ἐστίν ἡ μὲν ἂΒ τὸ ἀπὸστημα δ' ἀπέχει ὁ ἡλιος ἀπὸ τῆς γῆς, ἡ δὲ ΓΒ τὸ ἀπὸστημα δ' ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἄρα ἀπὸστημα δ' ἀπέχει ὁ ἡλιος ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὗ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, μείζονν ἐστίν ἡ ἐη.

Λέγω δὴ ὅτι καὶ ἔλασσον ἡ ἐ. ἡχθω γὰρ διὰ τοῦ Δ τῇ ΕΒ παράλληλος ἡ ΔΚ, καὶ περὶ τὸ ΔΚΒ τρίγωνον κύκλος γεγράφθη ὁ ΔΚΒ· ἐστιν δὴ αὐτοῦ διάμετρος ἡ ΔΒ, διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Κ γωνίαν. καὶ ἐνημοῦσθω ἡ ΒΑ ἔξαγωνον. καὶ ἐπεὶ ἡ ὑπὸ τῶν ΔΒΕ γωνία Λ' ἐστιν ὀρθῆς, καὶ ἡ ὑπὸ τῶν ΔΚΒ ἄρα Λ' ἐστιν ὀρθῆς· ἡ ἄρα ΒΚ περιφέρεια ξ' ἐστίν τοῦ ὅλου κύκλου. ἐστιν δὲ καὶ ἡ ΒΑ ἑκτὸν μέρος τοῦ ὅλου κύκλου· ἡ ἄρα ΒΑ περιφέρεια τῆς ΒΚ περιφέρειας ἐστίν. καὶ ἐχει ἡ ΒΑ περιφέρεια πρὸς τὴν ΒΚ περιφέρειαν μείζονα λόγον ἢπερ ἡ ΒΑ

---

1 ὁν add. Wallis.  
2 τὰ add. Wallis.
Therefore, *componendo*, \( ZE : EH > 12 : 5 \),
that is, \( ZE : EH > 36 : 15 \).

But it was also proved that
\( HE : EO > 15 : 2 \).

Therefore, *ex aequali*, \( ZE : EO > 36 : 2 \),
that is, \( ZE : EO > 18 : 1 \).

Therefore \( ZE \) is greater than eighteen times \( EO \). And \( ZE \) is equal to \( BE \). Therefore \( BE \) is also greater than eighteen times \( EO \). Therefore \( BH \) is much greater than eighteen times \( OE \).

But \( BO : OE = AB : BG \),
by similarity of triangles. Therefore \( AB \) is also greater than eighteen times \( BG \). And \( AB \) is the distance of the sun from the earth, while \( BG \) is the distance of the moon from the earth; therefore the distance of the sun from the earth is greater than eighteen times the distance of the moon from the earth.

I say now that it is less than twenty times. For through \( \Delta \) let \( \Delta K \) be drawn parallel to \( EB \), and about the triangle \( \Delta KB \) let the circle \( \Delta KB \) be drawn; its diameter will be \( AB \), by reason of the angle at \( K \) being right. Let \( BA \), the side of a hexagon, be fitted into the circle. Then, since the angle \( \Delta BE \) is one-thirtieth of a right angle, therefore the angle \( B \Delta K \) is also one-thirtieth of a right angle. Therefore the arc \( BK \) is one-sixtieth of the whole circle. But \( BA \) is one-sixth part of the whole circle.

Therefore \( \text{arc } BA = 10 \cdot \text{arc } BK \).

And the arc \( BA \) has to the arc \( BK \) a ratio greater

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eὐθεία πρὸς τὴν ΒΚ εὐθείαν· ἡ ἀρα ΒΔ εὐθεία τῆς ΒΚ εὐθείας ἐλάσσων ἐστὶν ἡ ἢ. καὶ ἔστω ἀντὶς διπλῆ ἡ ΒΔ· ἡ ἀρα ΒΔ τῆς ΒΚ ἐλάσσων ἐστὶν ἡ κ. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΒΚ, ἡ ΑΒ πρὸς τὴν ΒΓ, ὥστε καὶ ἡ ΑΒ τῆς ΒΓ ἐλάσσων ἐστὶν ἡ κ. καὶ ἔστω ἡ μὲν ΑΒ τὸ ἀπόστημα δ ἀπέχει ὁ Ἧλιος ἀπὸ τῆς γῆς, ἡ δὲ ΒΓ τὸ ἀπόστημα δ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἀρα ἀπόστημα δ ἀπέχει ὁ Ἧλιος ἀπὸ τῆς γῆς τοῦ ἀπόστημα, οὐ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, ἐλάσσον ἐστὶν ἡ κ. ἐδείχθη δὲ καὶ μείζον ἡ ἢ.

(c) Continued Fractions (?)

Ibid., Prop. 13, ed. Heath 396. 1-2

"Ἐχεὶ δὲ καὶ τὰ ἥλιον πρὸς ἅν μείζονα λόγων ἦπερ τὰ πῆ πρὸς μὲ.

Ibid., Prop. 15, ed. Heath 406. 23-24

"Ἐχεὶ δὲ καὶ τὸ Μ. ἐσοε πρὸς ὅλα ἐφ μείζονα λόγων ὅ ὅν τὰ μὴ πρὸς λῆ.

1 τὴν add. Wallis.

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*a* This is proved in Ptolemy's *Syntaxis* i. 10, v. infra, pp. 435-439.

*b* If \( \frac{7921}{4050} \) is developed as a continued fraction, we obtain the approximation \( 1 + \frac{1}{1 + \frac{1}{21 + \frac{1}{2}} \) which is \( \frac{88}{153} \). Similarly, if \( \frac{71755875}{61735500} \) or \( \frac{21261}{18292} \) is developed as a continued fraction, we
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than that which the straight line $BA$ has to the straight line $BK$.

Therefore $BA < 10 \cdot BK$.
And $BA = 2 BA$.
Therefore $BA < 20 \cdot BK$.
But $BA : BK = AB : BG$.
Therefore $AB < 20 \cdot BG$.

And $AB$ is the distance of the sun from the earth, while $BG$ is the distance of the moon from the earth; therefore the distance of the sun from the earth is less than twenty times the distance of the moon from the earth. And it was proved to be greater than eighteen times.

(c) Continued Fractions (?)

_Ibid._, Prop. 13, ed. Heath 396. 1-2

But 7921 has to 4050 a ratio greater than that which 88 has to 45.

_Ibid._, Prop. 15, ed. Heath 406. 23-24

But 71755575 has to 61735500 a ratio greater than that which 48 has to 37.

obtain the approximation $1 + \frac{1}{6} + \frac{1}{6}$ or $\frac{48}{37}$. The latter result was first noticed in 1823 by the Comte de Fortia D'Urban (Traité d'Aristarque de Samos, p. 186 n. 1), who added: "Ainsi les Grecs, malgré l'imperfection de leur numération, avaient des méthodes semblables aux nôtres." Though these relations are hardly sufficient to enable us to say that the Greeks knew how to develop continued fractions, they lend some support to the theory developed by D'Arcy W. Thompson in _Mind_, xxxviii. pp. 43-55, 1929.
XVII. ARCHIMEDES
XVII. ARCHIMEDES

(a) General

Tzetzes, Chil. ii. 103-144

'O Ἀρχιμήδης ὁ σοφός, μηχανητής ἑκεῖνος,
Τῷ γένει Συρακούσιος ἂν, γέρων γεωμέτρης,
Χρόνος τε ἐβδομήκοντα καὶ πέντε παρελαύνω.
"Οστις εἰργάσαστο πολλὰς μηχανικάς δυνάμεις,
Καὶ τῇ τρισπάστῳ μηχανῇ χειρὶ λαῖρ καὶ μόνῃ
Πεντεμιριμέδιμον καθειλκυσεν ὀλκάδα
Καὶ τοῦ Μαρκέλλου στρατηγοῦ ποτε δὲ τῶν
Ῥωμαίων
Τῇ Συρακούσῃ κατὰ γῆν προσβάλλοντος καὶ
πόντον,
Τιμᾶς μὲν πρῶτον μηχαναίς ἀνείλκυσεν ὀλκάδας
Καὶ πρὸς τὸ Συρακούσιον τείχος μετεωρίσας
Ἀυτάνδρους πάλιν τῷ βυθῷ κατεπέμπειν ἀθρῶς,
Μαρκέλλου δὲ ἀποστήσαντος μικρὸν τι τὰς ὀλκάδας
Ο γέρων πάλιν ἀπαντᾷ ποιεὶ Συρακούσιος

* A life of Archimedes was written by a certain Heraclides—perhaps the Heraclides who is mentioned by Archimedes himself in the preface to his book On Spirals (Archim. ed. Heiberg ii. 2. 3) as having taken his books to Dositheus. We know this from two references by Eutocius (Archim. ed. Heiberg iii. 298. 20, Apollon. ed. Heiberg ii. 168. 3, where, however, the name is given as Ἡράκλειος), but it has not survived. The surviving writings of Archimedes, together with the commentaries of Eutocius of Ascalon (f. a.d. 520), have been edited by J. L. Heiberg in three volumes of the Teubner series (references in this volume are to the 2nd ed., Leipzig, 1910–1915). They have been put into mathematical notation by T. L. Heath, The Works of Archimedes (Cam-
XVII. ARCHIMEDES

(a) General

Tzetzes, *Book of Histories* ii. 103-144

Archimedes the wise, the famous maker of engines, was a Syracusan by race, and worked at geometry till old age, surviving five-and-seventy-years; he reduced to his service many mechanical powers, and with his triple-pulley device, using only his left hand, he drew a vessel of fifty thousand medimni burden. Once, when Marcellus, the Roman general, was assaulting Syracuse by land and sea, first by his engines he drew up some merchant-vessels, lifted them up against the wall of Syracuse, and sent them in a heap again to the bottom, crews and all. When Marcellus had withdrawn his ships a little distance, the old man gave all the Syracusans power to lift bridge, 1897), supplemented by *The Method of Archimedes* (Cambridge, 1922), and have been translated into French by Paul Ver Eecke, *Les Œuvres complètes d'Archimède* (Brussels, 1921).

The lines which follow are an example of the "political" (*πολιτικός*, popular) verse which prevailed in Byzantine times. The name is given to verse composed by accent instead of quantity, with an accent on the last syllable but one, especially an iambic verse of fifteen syllables. The twelfth-century Byzantine poet, John Tzetzes, preserved in his *Book of Histories* a great treasure of literary, historical, theological and scientific detail, but it needs to be used with caution. The work is often called the *Chiliades* from its arbitrary division by its first editor (N. Gerbel, 1546) into books of 1000 lines each—it actually contains 12,674 lines.

As he perished in the sack of Syracuse in 212 B.C., he was therefore born about 287 B.C.

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Μετεωρίζειν δύνασθαι λίθους ἀμαξιαίους
Καὶ τὸν καθένα πέμποντας τι βοθίζειν τὰς ὀλκάδας.
'Ος Μάρκελλος δ' ἀπέστησε βολήν ἐκείνας τόξου,
'Εξάγωνον τι κάτοπτρον ἑτέκτηνεν ὁ γέρων,
'Απὸ δὲ διαστήματος συμμέτρου τοῦ κατόπτρου
Μικρὰ τοιαύτα κάτοπτρα θείς τετραπλά γυναῖς
Κυνούμενα λεπίσα τε καὶ τισ γυγγυμίους,
Μέσον ἐκεῖνο τέθεικεν ἀκτίνων τῶν ἠλίου
Μεσημβρίνης καὶ θερμής καὶ χειμερωτάτης.
'Ανακλωμένων δὲ λουπὸν εἰς τούτο τῶν ἀκτίνων
'Εξαμῆς ἦρθη φοβερὰ πυρώδης ταῖς ὀλκάσι,
Καὶ ταύτας ἀπέτεφσεν ἐκ μήκους τοξοβόλου.
Οὕτω νικὰ τὸν Μάρκελλον ταῖς μυχαναῖς ὁ γέρων.
'Ελεγε δὲ καὶ δωριστὶ, φωνῇ Συρακούσια:
"Πά βῶ, καὶ χαριστίνω τὰν γᾶν κυνήσω πᾶσαν."

1 πέμποντας Cary, πέμποντα codd.

* Unfortunately, the earliest authority for this story is Lucian, Hipp. 2: τὸν δὲ (sc. Ἀρχιμήδην) τὰς τῶν πολεμῶν τρεῖρισ καταβλέαντα τῇ τέχνῃ. It is also found in Galen, Peri kras. iii. 2, and Zonaras xiv. 3 relates it on the authority of Dion Cassius, but makes Proclus the hero of it.

* Further evidence is given by Tzetzes, Chil. xii. 995 and Eutocius (Archim. ed. Heiberg iii. 132. 5-6) that Archimedes wrote in the Doric dialect, but the extant text of his best-known works, On the Sphere and Cylinder and the Measurement of a Circle, retains only one genuine trace of its original Doric—the form τῶν. Partial losses have occurred in other books, the Sand-Rectoner having suffered least. The subject is fully treated by Heiberg, Quaestiones Archimedae, pp. 69-94, and in a preface to the second volume of his edition of Archimedes he indicates the words which he has restored to their Doric form despite the manuscripts; his text is adopted in this selection.

The loss of the original Doric is not the only defect in the
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stones large enough to load a waggon and, hurling them one after the other, to sink the ships. When Marcellus withdrew them a bow-shot, the old man constructed a kind of hexagonal mirror, and at an interval proportionate to the size of the mirror he set similar small mirrors with four edges, moved by links and by a form of hinge, and made it the centre of the sun's beams—its noon-tide beam, whether in summer or in mid-winter. Afterwards, when the beams were reflected in the mirror, a fearful kindling of fire was raised in the ships, and at the distance of a bow-shot he turned them into ashes. In this way did the old man prevail over Marcellus with his weapons. In his Doric dialect, and in its Syracusan variant, he declared: "If I have somewhere to stand, I will move the whole earth with my charistion." text. The hand of an interpolator—often not particularly skilful—can be repeatedly detected, and there are many loose expressions which Archimedes would not have used, and occasional omissions of an essential step in his argument. Sometimes the original text can be inferred from the commentaries written by Eutocius, but these extend only to the books On the Sphere and Cylinder, the Measurement of a Circle, and On Plane Equilibrium. A partial loss of Doric forms had already occurred by the time of Eutocius, and it is believed that the works most widely read were completely recast a little later in the school of Isidorus of Miletus to make them more easily intelligible to pupils.

"The instrument is otherwise mentioned by Simplicius (in Aristot. Phys., ed. Diels 1110. 2-5) and it is implied that it was used for weighing: ταύτη δὲ τὴν ἀναλογία τοῦ κινούμενος καὶ τοῦ κινούμενον καὶ τοῦ διαστήματος τοῦ σταθμοτικὸν ὄργανον τῶν καλούμενων χαριστώνα συντήσας ὁ Ἀρχιμήδης ὁς μέχρι παντὸς τῆς ἀναλογίας προχωρούσης ἐκόμψατο ἐκεῖνο τὸ "πά βῶ καὶ κινῶ τὰν γάν." As Tzetzes in another place (Chil. iii. 61: ὁ γὰρ ἀνεστὼν μηχανὴ τῇ τρισπάστῳ βοῶν "ὅτα βῶ καὶ σαλεύσω τὴν χόνα") writes of a triple-pulley device in the same connexion, it may be presumed to have been of this nature.

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Οὖτος, κατὰ Διόδωρον, τῆς Συρακούσης ταύτης Προδότου πρὸς τὸν Μάρκκελλον ἀδρόως γενομένης, Εἰτε, κατὰ τὸν Δίωνα, Ὅρωμαίος πορθηθεῖσας, Ἀρτέμιδι τῶν πολιτῶν τότε παννυχίζοντων, Τοιουτοτρόπως τέθυκεν ὑπὸ τῶν Ὅρωμαίος. Ἡν κεκυφῶς, διάγραμμα μηχανικόν τι γράφων, Τις δὲ Ὅρωμαίος ἐπιστὰς εἶλκεν αἰχμαλωτίζων. Ὅ δὲ τοῦ διαγράμματος ὅλος ὑπάρχων τότε, Τις ὁ καθελκὸς οὐκ εἰδὼς, ἐλεγε πρὸς ἑκείνον. "Ἀπόστηθι, ὁ ἀνθρώπος, τοῦ διαγράμματός μου." Ὅς δὲ εἶλκε τούτων συντραφεῖς καὶ γνών Ὅρωμαίοι εἶναι, Ἐβόα, "τι μηχάνημα τις τῶν ἐμῶν μοι δότων." Ὅ δὲ Ὅρωμαίος πτοσθεῖσ εὐθὺς ἑκείνον κτείνει, "Ανδρα σαθρὸν καὶ γέροντα, δαμόνον τοῖς ἔργοις.

Plut. Marcellus xiv. 7–xvii. 7.

Καὶ μέντοι καὶ Ἀρχιμηνίδης, Ἴερων ποῦ βασιλεῖς συγχρησάμενος ὁν καὶ φίλος, ἐγράφεσαν ὡς τῇ δοθείᾳ δυνάμει τὸ δοθὲν βάρος κινῆσαι δυνατὸν ἔστι· καὶ νεανισσάμενος, ῥας φασί, ρώμη τῇς ἀποδείξεως εἰπεν ὡς, εἰ γῆν εἰχὲν ἔτεραν, εἰκάσθης αὖ ταύτην μεταβὰς εἰς ἑκεῖνην. θαυμάζαντος δὲ τοῦ Ἴερωνος, καὶ δεηθέντος εἰς ἐργὸν ἐμφάνισθη τὸ πρόβλημα καὶ δεῖξαν τί τῶν μεγάλων κινοῦμεν ὕπο σμικρὰς δυνάμεως, ὁλκάδα τριάρμενον τῶν βασιλικῶν πόνω μεγάλω καὶ χειρὶ πολλῇ νεωλυκηθείσων, ἐμβαλὼν ἀνθρώπους τε πολλοὺς καὶ τὸν συνήθη φόρτος, αὐτὸς ἀπώθεν καθήμενος, οὐ μετὰ σπουδῆς, ἀλλὰ

* The account of Dion Cassius has not survived.
* Zonaras ix. 5 adds that when he heard the enemy were
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Whether, as Diodorus asserts, Syracuse was betrayed and the citizens went in a body to Marcellus, or, as Dion tells, it was plundered by the Romans, while the citizens were keeping a night festival to Artemis, he died in this fashion at the hands of one of the Romans. He was stooping down, drawing some diagram in mechanics, when a Roman came up and began to drag him away to take him prisoner. But he, being wholly intent at the time on the diagram, and not perceiving who was tugging at him, said to the man: "Stand away, fellow, from my diagram." As the man continued pulling, he turned round and, realizing that he was a Roman, he cried, "Somebody give me one of my engines." But the Roman, scared, straightway slew him, a feeble old man but wonderful in his works.

Plutarch, Marcellus xiv. 7–xvii. 7

Archimedes, who was a kinsman and friend of King Hiero, wrote to him that with a given force it was possible to move any given weight; and emboldened, as it is said, by the strength of the proof, he averred that, if there were another world and he could go to it, he would move this one. Hiero was amazed and besought him to give a practical demonstration of the problem and show some great object moved by a small force; he thereupon chose a three-masted merchantman among the king's ships which had been hauled ashore with great labour by a large band of men, and after putting on board many men and the usual cargo, sitting some distance away and without any special effort, he pulled gently with his hand at coming "πᾶρο κεφαλάν" ἐπὶ "καὶ μὴ πορᾷ γραμμάν"—"Let them come at my head," he said, "but not at my line."
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ηρέμα τῇ χειρὶ σείων ἀρχὴν τινα πολυσπάστου προσηγάγετο λείως καὶ ἀπταίστως καὶ ὦσπερ διὰ θαλάττης ἐπιθέουσαν. ἐκπλαγεὶς οὖν ὁ βασιλεὺς καὶ συννόησας τῆς τέχνης τὴν δύναμιν, ἐπεισε τὸν Ἀρχιμήδην ὅπως αὐτῷ τὰ μὲν ἀμυνομένω, τὰ δὲ ἐπιχειροῦντι μηχανήματα κατασκευάσῃ πρὸς πᾶσαν ἱδέαν πολυρκίας, οἷς αὐτὸς μὲν ὦκ ἔχρησατο, τὸν βίου τὸ πλείστον ἀπόλειμον καὶ παντηγυρικὸν βιώσας, τότε δ' ὑπῆρχε τοῖς Συρακοσίοις εἰς δέον ἡ παρασκευὴ καὶ μετὰ τῆς παρασκευῆς ὁ δημιουργός.

Ὡς οὖν προσέβαλον οἱ Ῥωμαίοι διchengεν, ἐκπληθεὶς ἥν τῶν Συρακοσίων καὶ σιγῇ διὰ δεός, μηδέν ἂν ἄνθεξεν πρὸς βιαν καὶ δύναμιν οἰομένων τοσατήν. σγάσαντος δὲ τὰς μηχανὰς τοῦ Ἀρχιμήδου ἀμα τοῖς μὲν πεζίοις ἀπήντα τοξεύματα τε παντοδαπὰ καὶ λίθων ὑπέρονχα μεγέθη, ῥοῖζω καὶ τάχει καταφερμένων ἄπιστω, καὶ μηδενὸς ὅλως τὸ βρίθος στέγοντος ἀθρόου ἀνατρέποντων τοὺς ὑποπίπτοντας καὶ τὰς τάξεις συγχεόντων, ταῖς δὲ ναυσὶν ἀπὸ τῶν τειχῶν ἄφυν υπεραιωροῦμεναι κεραίας τὰς μὲν ὑπὸ βρίθους στήριζοντος ἀνωθὲν ὑδοίσαι κατέδυν έις βυθόν, τὰς δὲ χερι σίδηραις η πολύμας εἰκασμένους γεράνων ἀναπώσαι πρώραθεν ὅρθᾶς ἐπὶ πρύμναιν ἐβάπτιζον.

* πολυσπάστος. Galen, in Hipp. De Artic. iv. 47 uses the same word. Tzetzes (loc. cit.) speaks of a triple-pulley device (τῇ τρισπάστῳ μηχανῇ) in the same connexion, and Oribasius, Coll. med. xlix. 22 mentions the τρισπάστος as an invention of Archimedes; he says that it was so called because it had three ropes, but Vitruvius says it was thus named because it had three wheels. Athenaeus v. 207 a-b says that a helix was used. Heath, The Works of Archimedes, 24
the end of a compound pulley and drew the vessel smoothly and evenly towards himself as though she were running along the surface of the water. Astonished at this, and understanding the power of his art, the king persuaded Archimedes to construct for him engines to be used in every type of siege warfare, some defensive and some offensive; he had not himself used these engines because he spent the greater part of his life remote from war and amid the rites of peace, but now his apparatus proved of great advantage to the Syracusans, and with the apparatus its inventor.

Accordingly, when the Romans attacked them from two elements, the Syracusans were struck dumb with fear, thinking that nothing would avail against such violence and power. But Archimedes began to work his engines and hurled against the land forces all sorts of missiles and huge masses of stones, which came down with incredible noise and speed; nothing at all could ward off their weight, but they knocked down in heaps those who stood in the way and threw the ranks into disorder. Furthermore, beams were suddenly thrown over the ships from the walls, and some of the ships were sent to the bottom by means of weights fixed to the beams and plunging down from above; others were drawn up by iron claws, or crane-like beaks, attached to the prow and were

p. xx, suggests that the vessel, once started, was kept in motion by the system of pulleys, but the first impulse was given by a machine similar to the κορνᾶς described by Pappus viii. ed. Hultsch 1066, 1108 ff., in which a cog-wheel with oblique teeth moves on a cylindrical helix turned by a handle.

Similar stories of Archimedes' part in the defence are told by Polybius viii. 5. 3-5 and Livy xxiv. 34.
ἡ δι' ἀντιτόνων ἐνδον ἐπιστρεφόμεναι καὶ περιαγό-
μεναι τοῖς ὑπὸ τὸ τείχος πεφυκόσι κρημνοῖς καὶ
σκοπέλοις προσήρασσον, ἀμα φθόρω πολλῷ τῶν
ἐπιβατῶν συντριβομένων. πολλάκις δὲ μετέωροις
ἐξαρθείσαι ναῦς ἀπὸ τῆς θαλάσσης δεύρῳ κακεῖσε
περιδυνομένη καὶ κρεμαμένη θέαμα φρικώδες ἤν,
μέχρι οὗ τῶν ἀνδρῶν ἀπορριφέντων καὶ διασφεν-
δονυθέντων κενὴ προσπέσου τοῖς τείχεσιν ἡ περι-
ολίσθοι τῆς λαβῆσι ἀνείσης. ἦν δὲ ὁ Μάρκελλος
ἀπὸ τοῦ ζεύγματος ἐπίγει μηχανήν, σαμβύκη μὲν
ἐκαλεῖτο δι' ὀμοιότητα τίνα σχῆματος πρὸς τὸ
μουσικὸν ὄργανον, ἔτι δὲ ἀπωθεὶν αὐτῆς προσφερο-
μένης πρὸς τὸ τείχος ἐξήλατο λίθος δεκατάλαντος
ἐλκήν, εἶτα ἑτερος ἐπὶ τοῦτῳ καὶ τρῖτος, ὡς ἐν
μεν αὐτῇ ἐμπεσόντες μεγάλῳ κτύπῳ καὶ κλύδωνι
τῆς μηχανῆς τῇ τὲ βάσιν συνηλόσαν καὶ τὸ
γόμφωμα διέσεισαν καὶ διέσπασαν τοῦ ζεύγματος,
ὡς τὸν Μάρκελλον ἀπορούμενον αὐτὸν τὲ ταῖς
ναυσὶν ἀποπλεῖν κατὰ τάχος καὶ τοῖς πεζοῖς ἀνα-
χώρησιν παρεγγυησάι.

Βουλευομένους δὲ ἐδοξεῖν αὐτοῖς ἐτὶ νυκτὸς, ἀν
dύνωντα, προσμίζα τοῖς τείχεσιν τοὺς γὰρ τόνος,
ois χρήσθαι τὸν Ἀρχιμήδην, ρύμην ἔχοντας
ὑπερπετείς ποιήσειν τὰς τῶν βελῶν ἀφέσεις,
ἐγγύθεν δὲ καὶ τελέως ἀπράκτους εἶναι διάστημα
τῆς πληγῆς οὐκ ἐχοῦσης. ὦ δ' ἦν, ὡς οἰκεῖν, ἐπὶ
tαῦτα πάλαι παρασκευαζόμενοι ὄργανοι τὲ συμ-
μέτρουν πρὸς πάν διάστημα κυνήσεις καὶ βέλη
βραχέα, καὶ διὰ (τὸ τείχος 2) οὐ μεγάλων, πολλῶν

1 αὐτῇ Coraës, αὐτῆς codd.
2 τὸ τείχος add. Sintenis ex Polyb.
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plunged down on their sterns, or were twisted round and turned about by means of ropes within the city, and dashed against the cliffs set by Nature under the wall and against the rocks, with great destruction of the crews, who were crushed to pieces. Often there was the fearful sight of a ship lifted out of the sea into mid-air and whirled about as it hung there, until the men had been thrown out and shot in all directions, when it would fall empty upon the walls or slip from the grip that had held it. As for the engine which Marcellus was bringing up from the platform of ships, and which was called *sambuca* from some resemblance in its shape to the musical instrument, a while it was still some distance away as it was being carried to the wall a stone ten talents in weight was discharged at it, and after this a second and a third; some of these, falling upon it with a great crash and sending up a wave, crushed the base of the engine, shook the framework and dislodged it from the barrier, so that Marcellus in perplexity sailed away in his ships and passed the word to his land forces to retire.

In a council of war it was decided to approach the walls, if they could, while it was still night; for they thought that the ropes used by Archimedes, since they gave a powerful impetus, would send the missiles over their heads and would fail in their object at close quarters since there was no space for the cast. But Archimedes, it seems, had long ago prepared for such a contingency engines adapted to all distances and missiles of short range, and through openings in the

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*a* The sambuca was a triangular musical instrument with four strings. Polybius (viii. 6) states that Marcellus had eight quinqueremes in pairs locked together, and on each pair a sambuca had been erected; it served as a penthouse for raising soldiers on to the battlements.
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dē kai sunexē̂n trη̂matων ἀντων1, oi skɔrpiōi
βραχύτων μὲν, ἐγγύθεν δὲ πλῆξαι παρεστήκασαν
ἀόρατοι τοῖς πολεμίωσ.

Ὅς οὖν προσέμυζαν οἶομενοι λανθάνειν, αὕτης αὖ
βέλεσι πολλοὶς ἐντυγχάνοντες καὶ πληγαῖς, πετρῶν
μὲν ἐκ κεφαλῆς ἐπ' αὐτοῦς φερομένων ὥσπερ πρὸς
κάθετον, τοῦ δὲ τείχους τοξεύματα πανταχόθεν
ἀναπέμποντος, ἀνεχώρουν ὅπισώ. κάνταυθα πάλι
αὐτῶν εἰς μῆκος ἐκτεταγμένων, βελῶν ἐκθεόντων
καὶ καταλαμβανόντων ἀπίόντας ἐγίνετο πολὺς μὲν
αὐτῶν φθόρος, πολὺς δὲ τῶν νεῶν συγκρούσμοι,
οὖν ἀντιδέουσι τοὺς πολεμίους δυναμένων. τὰ
γὰρ πλείστα τῶν ὄργανων ὑπὸ τὸ τείχος ἐσκευο-
πούτη τῷ Ἀρχιμήδε, καὶ θεομαχοῦσιν ἑώκεσαν
οἱ Ρωμαῖοι, μυρίων αὐτοῖς κακῶν ἐξ ἀφανοῦς
ἐπιχειρομένων.

Οὐ μὴν ἀλλ' ὁ Μάρκελλος ἀπέφυγε τε καὶ τοὺς
σὺν ἐαυτῷ σκώπτων τεχνίτας καὶ μηχανοποιοὺς
ἐλέγεν: "οὐ πανοσμεθα πρὸς τὸν γεωμετρικὸν
τοῦτον Βριάρεων πολεμοῦντες, ὃς ταῖς μὲν ναυῶν
ἡμῶν κυαθίζει ἐκ τῆς θαλάσσης, τὴν δὲ σαμβύκην
ραπίζων μετ' αἰσχύνης ἐκβεβλήκε, τοὺς δὲ μυθικοὺς
ἐκατόγχειρας ὑπεραιρεῖ τοσαῦτα βάλλων ἀμα βέλη
καθ' ἡμῶν;" τῷ γὰρ ὄντι πάντες οἱ λοιποὶ Συρα-
κούσιοι σώμα τῆς Ἀρχιμήδους παρασκευῆς ἦσαν,
ἡ δὲ κυνοῦσα πάντα καὶ στρέφουσα ψυχὴ μία, τῶν
μὲν ἄλλων ὁπλῶν ἀτρέμα κειμένων, μόνοι δὲ τοῖς
ἐκείνου τότε τῆς πόλεως χρωμένης καὶ πρὸς
ἀμυναν καὶ πρὸς ἀσφάλειαν. τέλος δὲ τοὺς
Ῥωμαίους ὄντω περιφόρους γεγονότας ἄρῳν ὁ
Μάρκελλος ὁστ', εἰ καλοῦσιν ἢ ἐξόλων ὑπὲρ τοῦ

1 ἀντων add. Sintenis ex Polyb.
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wall, small in size but many and continuous, short-ranged engines called scorpions could be trained on objects close at hand without being seen by the enemy.

When, therefore, the Romans approached the walls, thinking to escape notice, once again they were met by the impact of many missiles; stones fell down on them almost perpendicularly, the wall shot out arrows at them from all points, and they withdrew to the rear. Here again, when they were drawn up some distance away, missiles flew forth and caught them as they were retiring, and caused much destruction among them; many of the ships, also, were dashed together and they could not retaliate upon the enemy. For Archimedes had made the greater part of his engines under the wall, and the Romans seemed to be fighting against the gods, inasmuch as countless evils were poured upon them from an unseen source.

Nevertheless Marcellus escaped, and, twitting his artificers and craftsmen, he said: "Shall we not cease fighting against this geometrical Briareus, who uses our ships like cups to ladle water from the sea, who has whipped our sambuca and driven it off in disgrace, and who outdoes all the hundred-handed monsters of fable in hurling so many missiles against us all at once?" For in reality all the other Syracusans were only a body for Archimedes' apparatus, and his the one soul moving and turning everything: all other weapons lay idle, and the city then used his alone, both for offence and for defence. In the end the Romans became so filled with fear that, if they saw a little piece of rope or of wood projecting over

2 ταῖς μὲν ναυσίν ... ἡμῶν καθίζων πρὸς τὴν βάλασσαν παῖζων codd.
τείχους μικρὸν ὀφθείς προτεινόμενον, τούτο ἐκείνο, μηχανήν τινα κινεῖν ἐπ' αὐτοὺς Ἀρχιμήδη βοῶντας ἀποτρέπεσθαι καὶ φεύγειν, ἀπέσχετο μάχης ἀπάσης καὶ προσβολῆς, τὸ λοιπὸν ἐπὶ τῷ χρόνῳ τὴν πολιορκίαν θέμενος.

Τηλικοῦτον μέντοι φρόνημα καὶ βάθος ψυχῆς καὶ τοσοῦτον ἐκέκτητο θεωρημάτων πλοῦτον Ἀρχιμήδης ὡστε, ἐφ' ὦς ὄνομα καὶ δόξαν οὐκ ἀνθρωπίνης, ἀλλὰ δαιμονίου τινὸς ἔσχε συνέσεως, μηθέν ἔθελεν ζύγημα φεύρειν καὶ πάσαιν ὁλοῖς τέχνην χρείας ἄφαρτομένην ἀγενήν καὶ βάναυσον ἠγγομένος, εἰς ἐκείνα καταθέσθαι μόνα τὴν αὐτοῦ φιλοτιμίαν ὦς τὸ καλὸν καὶ περίπτων ἁμιγές τοῦ ἀναγκαίου πρόσεστιν, ἀσύγκριτα μὲν ὅντα τοῖς ἄλλοις, ἐριν δὲ παρέχοντα πρὸς τὴν ὑλήν τῇ ἀποδείξει, τῆς μὲν τὸ μέγεθος καὶ τὸ κάλλος, τῆς δὲ τὴν ἀκρίβειαν καὶ τὴν δύναμιν ὑπερφυῆ παρεχομένης· οὐ γὰρ ἐστὶν ἐν γεωμετρίᾳ χαλεπωτέρας καὶ βαρυτέρας ὑποθέσεως ἐν ἀπλούστεροις λαβεῖν καὶ καθαρωτέροις στοιχείοις γραφομένας. καὶ τούθ' οἱ μὲν εὐφυία τοῦ ἄνδρος προσάπτουσιν, οἱ δὲ ὑπερβολὴ τινὶ πόνου νομίζουσιν ἀπόνως πεποιημένω καὶ ῥαδίως ἐκαστὸν ἐοικὸς γεγονέναι. Ξητῶν μὲν γὰρ οὐκ ἂν τις εὕροι δὴ αὐτοῦ τὴν ἀπόδειξιν, ἀμὴ δὲ τῇ μαθήσει παρίσταται δόξα τοῦ κἂν αὐτὸν εὑρεῖν ὅτως εἰ μὲν ὅδον ἀγεὶ καὶ ταχείαν ἐπὶ τὸ δεικνύμενον. οὐκοῦν οὐδὲ ἀπιστήσας τοῖς περὶ αὐτοῦ λεγομένοις ἐστὶν, ὥσ ὑπ' οἰκείας δή τινος καὶ συνοικοῦ θελομένοις αἰεί σειρῆνος ἐλέηστο καὶ σιτοῦ καὶ θεραπείας σώματος ἐξέλειπε, βία δὲ πολλάκις ἐλκόμενος ἐπ' άλειμμα καὶ
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the wall, they cried, "There it is, Archimedes is training some engine upon us," and fled; seeing this Marcellus abandoned all fighting and assault, and for the future relied on a long siege.

Yet Archimedes possessed so lofty a spirit, so profound a soul, and such a wealth of scientific inquiry, that although he had acquired through his inventions a name and reputation for divine rather than human intelligence, he would not deign to leave behind a single writing on such subjects. Regarding the business of mechanics and every utilitarian art as ignoble and vulgar, he gave his zealous devotion only to those subjects whose elegance and subtlety are untramelled by the necessities of life; these subjects, he held, cannot be compared with any others; in them the subject-matter vies with the demonstration, the former possessing strength and beauty, the latter precision and surpassing power; for it is not possible to find in geometry more difficult and weighty questions treated in simpler and purer terms. Some attribute this to the natural endowments of the man, others think it was the result of exceeding labour that everything done by him appeared to have been done without labour and with ease. For although by his own efforts no one could discover the proof, yet as soon as he learns it, he takes credit that he could have discovered it: so smooth and rapid is the path by which he leads to the conclusion. For these reasons there is no need to disbelieve the stories told about him—how, continually bewitched by some familiar siren dwelling with him, he forgot his food and neglected the care of his body; and how, when he was dragged by main force, as often happened, to the

1 ἄνα Bryan, ἄνω codd.
λοντρόν, εν ταῖς ἐσχάραις ἔγραφε σχῆματα τῶν
gεωμετρικῶν, καὶ τοῦ σώματος ἀληθιμένου διήγε
τῷ δακτύλῳ γραμμᾶς, ὑπὸ ἡδονῆς μεγάλης κάτοχος
ὡν καὶ μουσόληστος ἀληθῶς. πολλῶν δὲ καὶ
καλῶν εὐρετῆς γεγονός λέγεται τῶν φίλων δεη-
θηναι καὶ τῶν συγγενῶν ὅπως αὐτοῦ μετὰ τῆς
τελευτην ἐπιστήσωσι τῷ τάφῳ τῶν περιλαμβάνοντα
τὴν σφαῖραν ἐντὸς κυλινδροῦ, ἐπιγράφαντες τὸν
λόγον τῆς ὑπεροχῆς τοῦ περιέχοντος στερεοῦ πρὸς
τὸ περιεχόμενον.

Ibid. xix. 4-6

Μάλιστα δὲ τὸ Ἁρχιμήδους πάθος ἡνίασε Μάρ-
κελλον. ἔτυχε μὲν γὰρ αὐτὸς τι καθ᾽ ἕαυτὸν
ἀνασκοπῶν ἐπὶ διαγράμματος καὶ τῇ θεωρίᾳ
δεδωκὼς ἁμα τὴν τε διάνοιαν καὶ τὴν πρόσοψιν
οὐ προῆσθεν τὴν καταδρομὴν τῶν Ῥωμαίων
οὔδὲ τὴν ἀλώσιν τῆς πόλεως, ἀφ' ὅς δὲ ἐπιστάντος
αὐτῶ στρατιώτου καὶ κελεύοντος ἀκολουθεῖν πρὸς
Μάρκελλον οὐκ ἔβουλετο πρὶν ἡ τελέσαι τὸ πρό-
βλημα καὶ καταστήσαι πρὸς τὴν ἀπόδειξιν. ὁ δὲ
ὁργισθεὶς καὶ ὕπατος ἐν τῷ ξίφος ἀνέλευ αὐτοῦ.
ἐτεροὶ μὲν οὖν λέγουσιν ἐπιστήναι μὲν εὗθς ὡς
ἀποκτενοῦτα ἔφηρη τὸν Ῥωμαίον, ἐκεῖνον δ᾽
Ἰδόντα δεῖσαι καὶ ἀντιβολεῖν ἀναμεῖν βραχὺν
χρόνον, ὡς μὴ καταλίπῃ τὸ ζητούμενον ἀτελές
καὶ ἀθέωρητο, τὸν δὲ οὐ φροντίσαντα διαχρή-
σασθαι. καὶ τρῖτος ἔστι λόγος, ὡς κομίζοντι
πρὸς Μάρκελλον αὐτῷ τῶν μαθηματικῶν ὁργάνων
σκιόθηρα καὶ σφαῖρας καὶ γωνίας, αἰς ἑναρμόττει

* Cicero, when quaestor in Sicily, found this tomb over-
place for bathing and anointing, he would draw geometrical figures in the hearths, and draw lines with his finger in the oil with which his body was anointed, being overcome by great pleasure and in truth inspired of the Muses. And though he made many elegant discoveries, he is said to have besought his friends and kinsmen to place on his grave after his death a cylinder enclosing a sphere, with an inscription giving the proportion by which the including solid exceeds the included.\(^\text{a}\)

*Ibid. xix. 4-6*

But what specially grieved Marcellus was the death of Archimedes. For it chanced that he was alone, examining a diagram closely; and having fixed both his mind and his eyes on the object of his inquiry, he perceived neither the inroad of the Romans nor the taking of the city. Suddenly a soldier came up to him and bade him follow to Marcellus, but he would not go until he had finished the problem and worked it out to the demonstration. Thereupon the soldier became enraged, drew his sword and dispatched him. Others, however, say that the Roman came upon him with drawn sword intending to kill him at once, and that Archimedes, on seeing him, besought and entreated him to wait a little while so that he might not leave the question unfinished and only partly investigated; but the soldier did not understand and slew him. There is also a third story, that as he was carrying to Marcellus some of his mathematical instruments, such as sundials, spheres and grown with vegetation, but still bearing the cylinder with the sphere, and he restored it (*Tusc. Disp.* v. 64-66). The theorem proving the proportion is given *infra*, pp. 124-127.
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tο τοῦ ἡλίου μέγεθος πρὸς τὴν ὄψιν, στρατιῶται περιτυχόντες καὶ χρυσοῦν ἐν τῷ τεύχει δόξαντες φέρειν ἀπέκτειναν. ὅτι μὲντοι Μάρκελλος ἠληγεὶς καὶ τὸν αὐτόχειρα τοῦ ἀνδρός ἀπεστράφη καθάπερ ἐναγῇ, τοὺς δὲ οἰκείους ἀνευρῶν ἐτύμησεν, ὁμολογεῖται.

Papp. Coll. viii. 11. 19, ed. Hultsch 1060. 1-4

Τῆς αὐτῆς δὲ ἐστὶν θεωρίας τὸ δοθὲν βάρος τῇ δοθεißῃ δυνάμει κινήσαι τοῦτο γὰρ Ἀρχιμήδους μὲν εὐρήμα [λέγεται] μηχανικον, ἐφ' ὃ λέγεται εἰρηκάνα· "δός μοι (φησι) ποῦ στῶ καὶ κινῶ τὴν γῆν."

Diod. Sic. i. 34. 2

Ποταμόχωστος γὰρ οὖσα καὶ κατάρρυτος πολλοὺς καὶ πανταδαποὺς ἐκφέρει καρποὺς, τοῦ μὲν ποταμοῦ διὰ τὴν κατ' ἐτος ἀνάβασιν νεαρὰν ἤλθεν ἀεὶ καταχέοντος, τῶν δ' ἀνθρώπων ῥάφιος ἀπασάν ἀρδευόντων διὰ τῶν μηχανής, ἢν ἐπενόησε μὲν Ἀρχιμήδης ὁ Συρακόσιος, δονομάζεται δὲ ἀπὸ τοῦ σχῆματος κοχλίας.

Ibid. v. 37. 3

Τὸ πάντων παράδοξοτὸν, ἀπαρύτουσι τὰς ῥύσεις τῶν ὑδάτων τοὺς Αἰγυπτιακοῖς λεγομένους κοχλίας, οὐς Ἀρχιμήδης ὁ Συρακόσιος εὑρεῖν, ὅτε παρέβαλεν εἰς Αἰγυπτον.

1 λέγεται om. Hultsch.

a Diodorus is writing of the island in the delta of the Nile.

b It may be inferred that he studied with the successors of Euclid at Alexandria, and it was there perhaps that he made the acquaintance of Conon of Samos, to whom, as
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angles adjusted to the apparent size of the sun, some soldiers fell in with him and, under the impression that he carried treasure in the box, killed him. What is, however, agreed is that Marcellus was distressed, and turned away from the slayer as from a polluted person, and sought out the relatives of Archimedes to do them honour.

Pappus, *Collection* viii. 11. 19, ed. Hultsch 1060. 1-4

To the same type of inquiry belongs the problem: *To move a given weight by a given force.* This is one of Archimedes’ discoveries in mechanics, whereupon he is said to have exclaimed: “Give me somewhere to stand and I will move the earth.”

Diodorus Siculus i. 34. 2

As it is made of silt watered by the river after being deposited, it bears an abundance of fruits of all kinds; for in the annual rising the river continually pours over it fresh alluvium, and men easily irrigate the whole of it by means of a certain instrument conceived by Archimedes of Syracuse, and which gets its name because it has the form of a spiral or screw.

*Ibid.* v. 37. 3

Most remarkable of all, they draw off streams of water by the so-called Egyptian screws, which Archimedes of Syracuse invented when he went by ship to Egypt.

the preface to his books *On the Sphere and Cylinder* shows, he used to communicate his discoveries before publication, and Eratosthenes of Cyrene, to whom he sent the *Method* and probably the *Cattle Problem.*
Archimedis vero cum multa miranda inventa et varia fuerint, ex omnibus etiam infinita sollertia id quod exponam videtur esse expressum. Nimium Hiero Syracusis auctus regia potestate, rebus bene gestis cum auream coronam votivam diis immortalibus in quodam fano constituisset ponendam, manupretio locavit faciendam et aurum ad sacoma adpendit redemptori. Is ad tempus opus manu factum subtiliter regi adprobavit et ad sacoma pondus coronae visus est praestitisse. Posteaquam indicium est factum dempto auro tantundem argenti in id coronarium opus admixtum esse, indignatus Hiero se contemptum esse neque inveniens qua ratione id furtum deprehenderet, rogavit Archimeden uti insumeret sibi de eo cognitionem. Tunc is cum haberet eius rei curam, casu venit in balineum ibique cum in solium descenderet, animadvertit quantum corporis sui in eo insideret tantum aquae extra solium effluere. Idque cum eius rei rationem explicationis ostendisset, non est moratus sed exsiluit gudio motus de solio et nudus vadens domum versus significabat clara voce invenisse quod quaereret. Nam currens identidem graece clamabat εὐρήκα εὐρήκα.

Tum vero ex eo inventionis ingressu duas fecisse dicitur massas aequo pondere quo etiam fuerat corona, unam ex auro et alteram ex argento. Cum ita fecisset, vas amplum ad summa labra implevit

* "I have found, I have found."
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Vitruvius, On Architecture ix., Preface 9-1

Archimedes made many wonderful discoveries of different kinds, but of all these that which I shall now explain seems to exhibit a boundless ingenuity. When Hiero was greatly exalted in the royal power at Syracuse, in return for the success of his policy he determined to set up in a certain shrine a golden crown as a votive offering to the immortal gods. He let out the work for a stipulated payment, and weighed out the exact amount of gold for the contractor. At the appointed time the contractor brought his work skilfully executed for the king's approval, and he seemed to have fulfilled exactly the requirement about the weight of the crown. Later information was given that gold had been removed and an equal weight of silver added in the making of the crown. Hiero was indignant at this disrespect for himself, and, being unable to discover any means by which he might unmask the fraud, he asked Archimedes to give it his attention. While Archimedes was turning the problem over, he chanced to come to the place of bathing, and there, as he was sitting down in the tub, he noticed that the amount of water which flowed over the tub was equal to the amount by which his body was immersed. This indicated to him a means of solving the problem, and he did not delay, but in his joy leapt out of the tub and, rushing naked towards his home, he cried out with a loud voice that he had found what he sought. For as he ran he repeatedly shouted in Greek, *heureka, heureka.*

Then, following up his discovery, he is said to have made two masses of the same weight as the crown, the one of gold and the other of silver. When he had so done, he filled a large vessel right up to the brim
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aqua, in quo demisit argenteam massam. Cuius quanta magnitudo in vase depressa est, tantum aquae effluxit. Ita exempta massa quanto minus factum fuerat refudit sextario mensus, ut eodem modo quo prius fuerat ad labra aequaretur. Ita ex eo invenit quantum pondus argenti ad certam aquae mensuram responderet.

Cum id expertus esset, tum auream massam similiter pleno vase demisit et ea exempta eadem ratione mensura addita invenit deesse aquae non tantum sed minus, quanto minus magno corpore eodem pondere auri massa esset quam argenti. Postea vero repleto vase in eadem aqua ipsa corona demissa invenit plus aquae defluxisse in coronam quam in auream eodem pondere massam, et ita ex eo quod defuerit plus aquae in corona quam in massa, ratiocinatus deprehendit argenti in auro mixtionem et manifestum furtum redemptoris.

* The method may be thus expressed analytically.
Let \( w \) be the weight of the crown, and let it be made up of a weight \( w_1 \) of gold and a weight \( w_2 \) of silver, so that \( w = w_1 + w_2 \).
Let the crown displace a volume \( v \) of water.
Let the weight \( w \) of gold displace a volume \( v_1 \) of water; then a weight \( w_1 \) of gold displaces a volume \( \frac{w_1}{w} \cdot v \) of water.
Let the weight \( w \) of silver displace a volume \( v_2 \) of water;
with water, into which he dropped the silver mass. The amount by which it was immersed in the vessel was the amount of water which overflowed. Taking out the mass, he poured back the amount by which the water had been depleted, measuring it with a pint pot, so that as before the water was made level with the brim. In this way he found what weight of silver answered to a certain measure of water.

When he had made this test, in like manner he dropped the golden mass into the full vessel. Taking it out again, for the same reason he added a measured quantity of water, and found that the deficiency of water was not the same, but less; and the amount by which it was less corresponded with the excess of a mass of silver, having the same weight, over a mass of gold. After filling the vessel again, he then dropped the crown itself into the water, and found that more water overflowed in the case of the crown than in the case of the golden mass of identical weight; and so, from the fact that more water was needed to make up the deficiency in the case of the crown than in the case of the mass, he calculated and detected the mixture of silver with the gold and the contractor's fraud stood revealed.\(^a\)

then a weight \(w_2\) of silver displaces a volume \(\frac{v}{w_2} \cdot v_2\) of water.

It follows that

\[
v = \frac{w_1}{w} \cdot v_1 + \frac{w_2}{w} \cdot v_2
\]

\[
= \frac{w_1 v_1 + w_2 v_2}{w_1 + w_2}
\]

so that

\[
\frac{w_1}{w_2} = \frac{v_2 - v}{v - v_1}
\]

For an alternative method of solving the problem, \(v\). \textit{infra}, pp. 248-251.
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(b) Surface and Volume of the Cylinder and Sphere

Archim. De Sphaera et Cyl. i., Archim. ed. Heiberg
i. 2–132. 3.

"Ἀρχιμήδης Δοσιθέω χαίρειν

Πρῶτερον μὲν ἀπεσταλκά σοι τῶν ὑφ’ ἡμῶν τεθεωρημένων γράφας μετὰ ἀποδείξεως, ὅτι πᾶν τμῆμα τὸ περιεχόμενον ὑπὸ τε εὐθείας καὶ ὀρθογωνίου κῶνου τομῆς ἐπίτροπον ἐστὶ τριγώνου τοῦ βάσιν τὴν αὐτὴν ἔχοντος τῷ τμῆματι καὶ ὦσον ὦσιον ὑπερτερον δὲ ἡμῶν ὑποπεσόντων θεωρημάτων ἀξίων λόγον¹ πεπραγματεύμεθα περὶ τὰς ἀποδείξεις αὐτῶν. ἔστιν δὲ τάδε: πρῶτον μὲν, ὅτι πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶν τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ ἐπείτα δὲ, ὅτι πάντος τμῆματος σφαίρας τῇ ἐπιφάνεια ἴσος ἐστὶ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἐστὶ τῇ εὐθείᾳ τῇ ἀπὸ τῆς κορυφῆς τοῦ τμῆματος ἀγομένη ἐπὶ τὴν περιφερειαν τοῦ κύκλου, ὃς ἐστὶ βάσις τοῦ τμῆματος.

¹ ἀξίων λόγον cod., ἀνελεγκτῶν coni. Heath.

* The chief results of this book are described in the prefatory letter to Dositheus. In this selection as much as possible is given of what is essential to finding the proportions between the surface and volume of the sphere and the surface and volume of the enclosing cylinder, which Archimedes regarded as his crowning achievement (supra, p. 32). In the case of the surface, the whole series of propositions is reproduced so that the reader may follow in detail the majestic chain of reasoning by which Archimedes, starting from seemingly remote premises, reaches the desired conclusion; in the case of the volume only the final proposition (34) can be given, for reasons of space, but the reader will be able to prove the omitted theorems for himself. Pari passu with 40
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(b) SURFACE AND VOLUME OF THE CYLINDER AND SPHERE

Archimedes. On the Sphere and Cylinder i., Archim. ed. Heiberg i. 2-132. 3

Archimedes to Dositheus greeting

On a previous occasion I sent you, together with the proof, so much of my investigations as I had set down in writing, namely, that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base as the segment and equal height. Subsequently certain theorems deserving notice occurred to me, and I have worked out the proofs. They are these: first, that the surface of any sphere is four times the greatest of the circles in it; then, that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment; and, this demonstration, Archimedes finds the surface and volume of any segment of a sphere. The method in each case is to inscribe in the sphere or segment of a sphere, and to circumscribe about it, figures composed of cones and frusta of cones. The sphere or segment of a sphere is intermediate in surface and volume between the inscribed and circumscribed figures, and in the limit, when the number of sides in the inscribed and circumscribed figures is indefinitely increased, it would become identical with them. It will readily be appreciated that Archimedes' method is fundamentally the same as integration, and on p. 116 n. b this will be shown trigonometrically.

1 This is proved in Props. 17 and 24 of the Quadrature of the Parabola, sent to Dositheus of Pelusium with a prefatory letter, e. pp. 228-243, infra.
2 De Sphaera et Cyl. i. 30. "The greatest of the circles," here and elsewhere, is equivalent to "a great circle."
3 Ibid. i. 42, 43.
πρὸς δὲ τούτοις, ὅτι πάσης σφαῖρας ὁ κύλινδρος ὁ βάσις μὲν ἔχων ᾗν τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαῖρα, ὕψος δὲ ἴσων τῇ διαμέτρῳ τῆς σφαῖρας ἀυτὸς τε ἦμιολος ἔστιν τῆς σφαῖρας, καὶ ἡ ἐπιφάνεια αὐτοῦ τῆς ἐπιφανείας τῆς σφαῖρας. ταῦτα δὲ τὰ συμπτώματα τῇ φύσει προσημεῖα περὶ τὰ εἰρημένα σχῆματα, ἥγοιετο δὲ ύπὸ τῶν πρὸ ἡμῶν περὶ γεωμετρίαν ἀνεστραμμένων οὐδενὸς αὐτῶν ἐπιφανοκότος, ὅτι τούτων τῶν σχημάτων ἐστὶν ςυμμετρία. . . . ἔξεσται δὲ περὶ τούτων ἐπισκέψασθαι τοῖς δυνησομένοις. ὥφειλε μὲν οὖν Κόνωνος ἐπὶ ζῶντος ἐκδίδοσθαι ταῦτα τῆν γὰρ υπολαμβάνομεν που μᾶλλον ἂν δύνασθαι κατανοῆσαι ταῦτα καὶ τὴν ἀρμόζουσαν ὑπὲρ αὐτῶν ἀπόφασιν ποιήσασθαι δοκιμάζοντες δὲ καλῶς ἔχειν μεταδιδόναι τοῖς οἰκείοις τῶν μαθημάτων ἀποστέλλομεν σοι τὰς ἀποδείξεις ἀναγράφας, ὑπὲρ ὧν ἔξεσται τοῖς περὶ τὰ μαθήματα ἀναστρεφομένως ἐπισκέψασθαι. ἔρωμενως.

Γράφονται πρῶτον τὰ τὰ ἀξιώματα καὶ τὰ λαμβανόμενα εἰς τὰς ἀποδείξεις αὐτῶν.

'Αξιώματα

α'. Εἰσὶ τινὲς ἐν ἐπιπέδῳ καμπυλαὶ γραμμαὶ πεπερασμέναι, αἱ τῶν τὰ πέρατα ἐπιζευγνυσάνων αὐτῶν εὐθείων ἢτοι ὅλαι ἐπὶ τὰ αὐτὰ εἰσών ἢ οὐδὲν ἔχουσιν ἐπὶ τὰ ἐτερα.

β'. Ἐπὶ τὰ αὐτὰ δὴ κοίλην καλῶ τὴν τοιαύτην γραμμὴν, ἐν ἦ ἔαν δύο σημείων λαμβανόμενων

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*De Sphaera et Cyl. i. 34 coroll. The surface of the cylinder here includes the bases.*
further, that, in the case of any sphere, the cylinder having its base equal to the greatest of the circles in the sphere, and height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface is one-and-a-half times the surface of the sphere.}\(^a\) Now these properties were inherent in the nature of the figures mentioned, but they were unknown to all who studied geometry before me, nor did any of them suspect such a relationship in these figures.\(^b\) ... But now it will be possible for those who have the capacity to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I opine that he more than others would have been able to grasp them and pronounce a fitting verdict upon them; but, holding it well to communicate them to students of mathematics, I send you the proofs that I have written out, which proofs will now be open to those who are conversant with mathematics. Farewell.

In the first place, the axioms and the assumptions used for the proofs of these theorems are here set out.

**Axioms**\(^c\)

1. There are in a plane certain finite bent lines which either lie wholly on the same side of the straight lines joining their extremities or have no part on the other side.

2. I call *concave in the same direction* a line such that, if any two points whatsoever are taken on it, either

\(^a\) In the omitted passage which follows, Archimedes compares his discoveries with those of Eudoxus; it has already been given, vol. i. pp. 408-411.

\(^b\) These so-called axioms are more in the nature of definitions.
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όπουνον αἱ μεταξὺ τῶν σημείων εὐθεῖαι ήτοι πάσαι ἐπὶ τὰ αὐτά πέπτουσι τῆς γραμμῆς, ἣ τινὲς μὲν ἐπὶ τὰ αὐτά, τινὲς δὲ κατ' αὐτῆς, ἐπὶ τὰ ἐτερὰ δὲ μηδεμία.

γ’. Ὄμοιως δὴ καὶ ἐπιφάνειαί τινὲς εἰσιν πε-
περασμέναι, αὐταὶ μὲν οὐκ ἐν ἐπιπέδῳ, τὰ δὲ
πέρατα ἔχουσιν ἐν ἐπιπέδῳ, αἱ τοῦ ἐπιπέδου, ἐν
哼 τὰ πέρατα ἔχουσιν, ἦτοι ὅλαι ἐπὶ τὰ αὐτά
ἔσονται ἡ οὐδὲν ἕχουσιν ἐπὶ τὰ ἐτερὰ.

δ’. Ἐπὶ τὰ αὐτὰ δὴ κοῖλας καλλῷ τὰς τοιαύτας
ἐπιφανείας, ἐν αἷς ἃν δύο σημεῖων λαμβανομένων
αἱ μεταξὺ τῶν σημείων εὐθεῖαι ήτοι πάσαι ἐπὶ τὰ
αὐτὰ πέπτουσι τῆς ἐπιφανείας, ἡ τινὲς μὲν ἐπὶ
tὰ αὐτά, τινὲς δὲ κατ' αὐτῆς, ἐπὶ τὰ ἐτερὰ δὲ
μηδεμία.

e’. Τομέα δὲ στερεοῦ καλῶ, ἐπειδὰν σφαῖραν
κώνος τέμνῃ κορυφὴν ἔχων πρὸς τῷ κέντρῳ
tῆς σφαίρας, τὸ ἐμπεριεχόμενον σχῆμα ὑπὸ τὲ
tῆς ἐπιφανείας τοῦ κώνου καὶ τῆς ἐπιφανείας τῆς
σφαίρας ἐντὸς τοῦ κώνου.

ζ’. Ῥόμβου δὲ καλῶ στερεοῦ, ἐπειδὰν δύο κώνοι
tὴν αὐτὴν βάσιν ἔχοντες τὰς κορυφὰς ἔχωσιν ἐφ’
ἐκάτερα τοῦ ἐπιπέδου τῆς βάσεως, ὅπως οἱ ἄξονες
αὐτῶν ἐπ’ εὐθείας ὅσοι κεῖμενοι, τὸ ἐξ ἀμφότερον
tοῦ κώνου συγκείμενον στερεοῦν σχῆμα.

Λαμβανόμενα

Λαμβάνω δὲ ταῦτα:

α’. Τῶν τὰ αὐτὰ πέρατα ἔχουσιν γραμμῶν
ἐλαχίστην εἶναι τῆν εὐθείαν.
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all the straight lines joining the points fall on the same side of the line, or some fall on one and the same side while others fall along the line itself, but none fall on the other side.

3. Similarly also there are certain finite surfaces, not in a plane themselves but having their extremities in a plane, and such that they will either lie wholly on the same side of the plane containing their extremities or will have no part on the other side.

4. I call concave in the same direction surfaces such that, if any two points on them are taken, either the straight lines between the points all fall upon the same side of the surface, or some fall on one and the same side while others fall along the surface itself, but none falls on the other side.

5. When a cone cuts a sphere, and has its vertex at the centre of the sphere, I call the figure comprehended by the surface of the cone and the surface of the sphere within the cone a solid sector.

6. When two cones have the same base have their vertices on opposite sides of the plane of the base in such a way that their axes lie in a straight line, I call the solid figure formed by the two cones a solid rhombus.

POSTULATES

I make these postulates:

1. Of all lines which have the same extremities the straight line is the least.a

a Proclus (in Eucl., ed. Friedlein 110. 10-14) saw in this statement a connexion with Euclid’s definition of a straight line as lying evenly with the points on itself: ὁ δὲ αὐτῷ Ἀρχιμήδης τῆν εὐθείαν ὁρίσατο γραμμὴν εὐαχίστην τῶν τὰ αὐτὰ πέρατα ἐχουσῶν. διότι γὰρ, ὡς ὁ Ἐὐκλείδης λόγος φησὶν, ἕξ ἴσων κεῖται τοῖς ἑπτά ἑαυτῆς σημείοις, διὰ τούτο ἐλαχίστη ἐστὶν τῶν τὰ αὐτὰ πέρατα ἐχουσῶν.
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β'. Τῶν δὲ ἄλλων γραμμῶν, ἐὰν ἐν ἑπιπέδῳ
οὐσὶν τὰ αὐτὰ πέρατα ἔχωσιν, ἀνίσουσι εἶναι τάς
τοιαύτας, ἐπειδὰν ὤσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ
κοίλαι, καὶ ἦτοι ὅλη περιλαμβάνηται ἡ ἔτερα
αὐτῶν ὑπὸ τῆς ἔτερας καὶ τῆς εὐθείας τῆς τὰ
αὐτὰ πέρατα ἔχουσις αὐτῇ, ἡ τινὰ μὲν περιλαμ-
βάνηται, τινὰ δὲ κοινὰ ἔχον, καὶ ἑλάσσονα εἶναι
τὴν περιλαμβανομένην.

γ'. Ὁμοίως δὲ καὶ τῶν ἐπιφανείων τῶν τὰ
αὐτὰ πέρατα ἔχουσῶν, ἐὰν ἐν ἑπιπέδῳ τὰ πέρατα
ἔχωσιν, ἑλάσσονα εἶναι τὴν ἑπιπέδον.

δ'. Τῶν δὲ ἄλλων ἐπιφανείων καὶ τὰ αὐτὰ πέ-
ρατα ἔχουσῶν, ἐὰν ἐν ἑπιπέδῳ τὰ πέρατα ἢ,
ἀνίσουσι εἶναι τάς τοιαύτας, ἐπειδὰν ὄσιν ἀμφότεραι
ἐπὶ τὰ αὐτὰ κοίλαι, καὶ ἦτοι ὅλη περιλαμβάνηται
ὑπὸ τῆς ἔτερας ἡ ἔτερα ἐπιφάνεια καὶ τῆς ἑπιπέδου
tῆς τὰ αὐτὰ πέρατα ἔχουσις αὐτῇ, ἡ τινὰ μὲν
περιλαμβάνηται, τινὰ δὲ κοινὰ ἔχον, καὶ ἑλάσσονα
εἶναι τὴν περιλαμβανομένην.

e'. Ἐπὶ δὲ τῶν ἀνίσων γραμμῶν καὶ τῶν ἀνίσων
ἐπιφανείων καὶ τῶν ἀνίσων στερεῶν τὸ μείζον τοῦ
ἐλάσσονος ὑπερέχειν τοιούτῳ, ὃ συντιθέμενον αὐτὸ
ἐαυτῷ δυνατὸν ἔστιν ὑπερέχειν παντὸς τοῦ προ-
tεθέντος τῶν πρὸς ἅλληλα λεγομένων.

Τούτων δὲ ὑποκειμένων, ἐὰν εἰς κύκλου πολύγω-
νον ἐγγραφῇ, φανερὸν, ὅτι ἡ περίμετρος τοῦ
ἐγγραφέοντος πολυγώνου ἑλάσσων ἐστὶν τῆς τοῦ
κύκλου περιφερείας ἡ ἐκάστη γὰρ τῶν τοῦ πολυ-
γώνου πλευρῶν ἑλάσσων ἐστὶ τῆς τοῦ κύκλου
περιφερείας τῆς ὑπὸ τῆς αὐτῆς ἀποτεμνομένης.

* This famous "Axiom of Archimedes" is, in fact, generally used by him in the alternative form in which it is proved
2. Of other lines lying in a plane and having the same extremities, [any two] such are unequal when both are concave in the same direction and one is either wholly included between the other and the straight line having the same extremities with it, or is partly included by and partly common with the other; and the included line is the lesser.

3. Similarly, of surfaces which have the same extremities, if those extremities be in a plane, the plane is the least.

4. Of other surfaces having the same extremities, if the extremities be in a plane, [any two] such are unequal when both are concave in the same direction, and one surface is either wholly included between the other and the plane having the same extremities with it, or is partly included by and partly common with the other; and the included surface is the lesser.

5. Further, of unequal lines and unequal surfaces and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those comparable with one another.

With these premises, if a polygon be inscribed in a circle, it is clear that the perimeter of the inscribed polygon is less than the circumference of the circle; for each of the sides of the polygon is less than the arc of the circle cut off by it.

in Euclid x. 1, for which v. vol. i. pp. 452-455. The axiom can be shown to be equivalent to Dedekind's principle, that a section of the rational points in which they are divided into two classes is made by a single point. Applied to straight lines, it is equivalent to saying that there is a complete correspondence between the aggregate of real numbers and the aggregate of points in a straight line; v. E. W. Hobson, The Theory of Functions of a Real Variable, 2nd ed., vol. i. p. 55.
Εὰν περὶ κύκλου πολύγωνον περιγραφῇ, ἡ τοῦ περιγραφέντος πολυγώνου περίμετρος μείζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Περὶ γὰρ κύκλου πολύγωνον περιγεγράφθω τὸ ὑποκείμενον. λέγω, ὅτι ἡ περίμετρος τοῦ πολυγώνου μείζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Ἔπει γὰρ συναμφότερος ἡ ΒΛΑ μείζων ἐστὶ τῆς ΒΛ περιφερείας διὰ τὸ τὰ αὐτὰ πέρατα ἔχουσαν περιλαμβάνειν τὴν περιφερείαν, ὁμοίως δὲ καὶ συναμφότερος μὲν ἡ ΔΓ, ΓΒ τῆς ΔΒ, συναμφότερος δὲ ἡ ΔΚ, ΚΘ τῆς ΛΘ, συναμφότερος δὲ ἡ ΖΘ τῆς ΖΘ, ἐτὶ δὲ συναμφότερος ἡ ΔΕ, EZ τῆς ΔΖ, ὅλη ἀρα ἡ περίμετρος τοῦ πολυγώνου μείζων ἐστὶ τῆς περιφερείας τοῦ κύκλου.

* It is here indicated, as in Prop. 3, that Archimedes added a figure to his own demonstration.
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Prop. 1

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the circumference of the circle.

For let the polygon be circumscribed about the circle as below. I say that the perimeter of the polygon is greater than the circumference of the circle.

For since \( BA + A\lambda > \text{arc} \, BA \),

owing to the fact that they have the same extremities as the arc and include it, and similarly

\[ \Delta \Gamma + \Gamma B > \text{[arc]} \, \Delta B, \]
\[ \Delta K + K\Theta > \text{[arc]} \, \Delta \Theta, \]
\[ ZH + H\Theta > \text{[arc]} \, Z\Theta, \]

and further \( \Delta E + EZ > \text{[arc]} \, \Delta Z, \)

therefore the whole perimeter of the polygon is greater than the circumference of the circle.
Δύο μεγεθῶν ἀνίσων δοθέντων δυνατὸν ἐστὶν εὑρεῖν δύο εὐθείας ἀνίσους, ὅστε τὴν μείζονα εὐθείαν πρὸς τὴν ἑλάσσονα λόγον ἔχειν ἑλάσσονα ἢ τὸ μείζον μέγεθος πρὸς τὸ ἑλάσσον.

"Εστω δύο μεγέθη ἀνίσα τὰ AB, Δ, καὶ ἐστω μείζον τὸ AB. λέγω, ὅτι δυνατὸν ἐστὶ δύο εὐθείας ἀνίσους εὑρεῖν τὸ εἰρημένον ἐπίταγμα ποιούσας.

Κείσθω διὰ τὸ β' τοῦ α' τῶν Εὐκλείδου τῷ Δ ᾗ συν τὸ ΒΓ, καὶ κείσθω τις εὐθεία γραμμή ἡ ZΗ. τὸ δὴ ΓΑ ἕαυτῷ ἐπισυντθέμενον ὑπερέξει τοῦ Δ. πεπολλαπλασιάσθω οὖν, καὶ ἐστω τὸ ΑΘ, καὶ ὀσαπλασίων ἐστὶ τὸ ΑΘ τοῦ ΑΓ, τοσαυταπλάσιος ἐστω τῇ ZΗ τῆς ΗΕ. ἐστὸν ἄρα, ὡς τὸ ΘΑ πρὸς ΑΓ, οὔτως τῇ ZΗ πρὸς ΗΕ. καὶ ἀνάπαλιν ἐστὶ, ὡς ἡ ΕΗ πρὸς ΗΖ, οὔτως τὸ ΑΓ πρὸς ΑΘ. καὶ ἐπεὶ μειζόν ἐστὶ τὸ ΑΘ τοῦ Δ, τουτέστι τοῦ ΓΒ, τὸ ἄρα ΓΑ πρὸς τὸ ΑΘ λόγον ἑλάσσονα ἔχει ἦπερ τὸ ΓΑ πρὸς ΓΒ. ἀλλ' ὡς τὸ ΓΑ πρὸς ΑΘ, οὔτως ἡ ΕΗ πρὸς ΗΖ. ἡ ΕΗ ἀρα πρὸς ΗΖ ἑλάσσονα λόγον ἔχει ἦπερ τὸ ΓΑ πρὸς ΓΒ. καὶ συνθέντι ἡ EZ [ἀρά] πρὸς ZΗ ἑλάσσονα λόγον ἔχει ἦπερ τὸ AB πρὸς ΒΓ [διὰ λήμμα] 2 ἢ συν δὲ τὸ ΓΒ τῷ Δ ἡ EZ ἀρα πρὸς ZΗ ἑλάσσονα λόγον ἔχει ἦπερ τὸ AB πρὸς τὸ Δ.
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Prop. 2

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let AB, Δ be two unequal magnitudes, and let AB be the greater. I say that it is possible to find two unequal straight lines satisfying the aforesaid requirement.

By the second proposition in the first book of Euclid let BG be placed equal to Δ, and let ZH be any straight line; then GA, if added to itself, will exceed Δ. [Post. 5.] Let it be multiplied, therefore, and let the result be AΘ, and as AΘ is to AG, so let ZH be to HE; therefore

ΘA : AΓ = ZH : HE \[cf. Eucl. v. 15\]

and conversely, EH : HZ = AΓ : AΘ. \[Eucl. v. 7, coroll.\]

And since AΘ > Δ

ΓA : AΘ < ΓA : ΓB. \[Eucl. v. 8\]

But

ΓA : AΘ = EH : HZ;

therefore

EH : HZ < ΓA : ΓB;

componendo,

EZ : ZH < AB : BG.\[a\]

Now

BG = Δ;

therefore

EZ : ZH < AB : Δ.

\[a\] This and related propositions are proved by Eutocius [Archim. ed. Heiberg iii. 16. 11–18. 22] and by Pappus, Coll. ed. Hultsch 684. 20 ff. It may be simply proved thus. If \(a : b < c : d\), it is required to prove that \(a + b : b < c + d : d\). Let \(e\) be taken so that \(a : b = e : d\). Then \(e : d < c : d\). Therefore \(e < c\), and \(e + d : d < c + d : d\). But \(e : d = a + b : b\) (ex hypothesis, componendo). Therefore \(a + b : b < c + d : d\).

1 δωο om. Heiberg. 2 δια λήμμα om. Heiberg.
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Εὑρημέναι εἰςὶν ἄρα δύο εὐθεῖαι ἀνίσοι ποιοῦσαι τὸ εἰρημένον ἐπίταγμα [τούτεστιν τὴν μεῖζον πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλασσόν].

γ′

Δύο μεγεθῶν ἀνίσων δοθέντων καὶ κύκλου δυνατόν ἔστω εἰς τὸν κύκλον πολύγωνον ἐγγράφαι καὶ ἄλλο περιγράφαι, ὅπως ἢ τοῦ περιγραφομένου πολυγώνου πλευρὰ πρὸς τὴν τοῦ ἐγγραφομένου πολυγώνου πλευρὰν ἐλάσσονα λόγον ἔχῃ ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλασσόν.

'Εστω τὰ δοθέντα δύο μεγέθη τὰ A, B, ὁ δὲ δοθεὶς κύκλος ὁ ὑποκείμενος: λέγω οὖν, ὅτι δυνατόν ἔστω ποιεῖν τὸ ἐπίταγμα.

Εὐρήσθωσαν γὰρ δύο εὐθείας 

αἱ Θ, ΚΛ, ὧν μεῖζον ἔστω ἢ Θ, ὅστε τὴν Θ πρὸς τὴν ΚΛ

[Diagram and figures]

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Accordingly there have been discovered two unequal straight lines fulfilling the aforesaid requirement.

Prop. 3

Given two unequal magnitudes and a circle, it is possible to inscribe a [regular] polygon in the circle and to circumscribe another, in such a manner that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which the greater magnitude has to the less.

Let A, B be the two given magnitudes, and let the given circle be that set out below. I say then that it is possible to do what is required.

For let there be found two straight lines Θ, ΚΛ, of which Θ is the greater, such that Θ has to ΚΛ a ratio

1 τούτον ... ἔλασσον verba subditiva esse suspicatur Heiberg.
ελάσσονα λόγον ἔχειν ἡ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλαττον, καὶ ἦχθω ἀπὸ τοῦ Α τῇ ΔΚ πρὸς ὁρθᾶς ἡ ΔΜ, καὶ ἀπὸ τοῦ Κ τῇ Θ ὅσα κατήχου ἡ ΚΜ [δυνατον γάρ τοῦ], καὶ ἦχθωσαν τοῦ κύκλου ὁρθοὺς διάμετροι πρὸς ὁρθὰς ἀλλήλαις αἱ ΓΕ, ΔΖ. τέμνοντες οὖν τὴν ὑπὸ τῶν ΔΗΓ γωνίαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ αἰεὶ τοῦτο ποιοῦντες λειψομένων τῶν γωνίας ἐλάσσονα ἡ διπλασία τῆς ὑπὸ ΛΚΜ. λειψάθω καὶ ἐστὶν ἡ ὑπὸ ΝΗΓ, καὶ ἐπεξεύγνω ἡ ΝΓ. ἡ ἡ ἄρα ΝΓ πολυγώνων ἐστὶ πλευρὰ ἰσοπλεύρου [ἐπειπέρ ἡ ὑπὸ ΝΗΓ γωνία μετρεῖ τὴν ὑπὸ ΔΗΓ ὀρθῶν ὁδόν, καὶ ἡ ΝΓ ἄρα περιφέρεια μετρεῖ τὴν ΓΔ τέταρτον ὁδόν κύκλου· ὅστε καὶ τὸν κύκλον μετρεῖ. πολυγώνων ἄρα ἐστὶ πλευρὰ ἰσοπλεύρου· φανερὸν γάρ ἐστὶ τοῦτο]. καὶ τετμήσω ἡ ὑπὸ ΓΗΝ γωνία δίχα τῇ ΗΞ εὐθεία, καὶ ἀπὸ τοῦ Ε ἔφαπτεσθω τοῦ κύκλου ἡ ΟΞΠ, καὶ ἐκβεβλήσωσαν αἱ ΗΝΠ, ΗΓΟ· ὅστε καὶ ἡ ΠΟ πολυγώνων ἐστὶ πλευρά τοῦ περιγραφομένου περὶ τὸν κύκλον καὶ ἰσοπλεύρου [φανερὸν, ὅτι καὶ ὁμοίου τῶν ἐγγραφομένων, οὐ πλευρά ἡ ΝΓ]. ἑπεὶ δὲ ἐλάσσων ἐστὶν ἡ διπλασία ἡ ὑπὸ ΝΗΓ τῆς ὑπὸ ΛΚΜ, διπλασία δὲ τῆς ὑπὸ ΤΗΓ, ἐλάσσων ἄρα ἡ ὑπὸ ΤΗΓ τῆς ὑπὸ ΛΚΜ. καὶ εἰσιν ὁρθαὶ αἱ πρὸς τοῖς Α, Τ· ἡ ἄρα ΜΚ πρὸς ΛΚ μεῖζονα λόγον ἔχει ἡπερ ἡ ΓΗ πρὸς ΗΤ. ἵστη δὲ ἡ ΓΗ τῇ ΗΞ· ὅστε ἡ ΗΞ πρὸς ΗΤ ἐλάσσονα λόγον ἔχει, τούτεστι ἡ ΠΟ πρὸς ΝΓ, ἡπερ ἡ ΜΚ πρὸς ΚΛ· ἑτεὶ δὲ ἡ ΜΚ πρὸς ΚΛ ἐλάσσονα λόγον ἔχει ἡπερ τὸ Α πρὸς τὸ Β. καὶ ἐστὶν ἡ μὲν ΠΟ πλευρά
less than that which the greater magnitude has to the less [Prop. 2], and from $\Lambda$ let $\Lambda\overline{M}$ be drawn at right angles to $\Lambda\overline{K}$, and from $K$ let $K\overline{M}$ be drawn equal to $\Theta$, and let there be drawn two diameters of the circle, $\Gamma\overline{E}$, $\Delta\overline{Z}$, at right angles one to another. If we bisect the angle $\Delta\overline{H}\Gamma$ and then bisect the half and so on continually we shall leave a certain angle less than double the angle $\Lambda\overline{KM}$. Let it be left and let it be the angle $\Theta\overline{H}\Gamma$, and let $\Theta\overline{N}\Gamma$ be joined; then $\Theta\overline{N}\Gamma$ is the side of an equilateral polygon. Let the angle $\Theta\overline{HN}$ be bisected by the straight line $\Theta\overline{E}$, and through $\overline{E}$ let the tangent $\Theta\overline{E}\Pi$ be drawn, and let $\Theta\Pi\Pi$, $\Theta\Pi\Theta$ be produced; then $\Pi\Theta$ is a side of an equilateral polygon circumscribed about the circle. Since the angle $\Theta\overline{H}\Gamma$ is less than double the angle $\Lambda\overline{KM}$ and is double the angle $\Theta\overline{T}\Gamma$, therefore the angle $\Theta\overline{T}\Gamma$ is less than the angle $\Lambda\overline{KM}$. And the angles at $\Lambda$, $\Theta$ are right; therefore

$$MK : \Lambda K > \Gamma H : HT.$$  

But

$$\Gamma H = \Theta E.$$  

Therefore

$$\Theta E : HT < MK : KA,$$

that is,

$$\Pi O : \Theta N < MK : KA.$$  

Further,

$$MK : KA < A : B.$$  

Then

$$\Pi O : \Theta N < A : B.$$  

This is proved by Eutocius and is equivalent to the assertion that if $a < \beta \leq \frac{\pi}{2}$, cosec $\beta >$ cosec $a$.

* For $\Theta E : HT = \Pi O : \Theta N$, since $\Theta E : HT = \Theta O ; \Gamma T = 2 \Theta E : 2 \Gamma T = \Pi O : \Theta N$.

* For by hypothesis $\Theta : KA < A : B$, and $\Theta = MK$.

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1. διναρέν
2. επείπερ
3. πανερον
5. τοῦτο om. Heiberg.
6. η $\Theta N$ om. Heiberg.
τοῦ περιγραφομένου πολυγώνου, ἢ δὲ ΓΝ τοῦ ἐγγραφομένου ὑπὲρ προέκειτο εὑρεῖν.

ε'

Κύκλου δοθέντος καὶ δύο μεγεθῶν ἀνίσων περιγράφοι περὶ τὸν κύκλον πολύγωνον καὶ ἄλλο ἐγγράφαι, ὡστε τὸ περιγραφέν πρὸς τὸ ἐγγραφέν ἐλάσσονα λόγον ἔχειν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλάσσον.

Ἐκκείσθω κύκλος ὁ Α καὶ δύο μεγέθη ἀνίσα

\[ \triangle \gamma \triangle \delta \triangle \eta \]

τὰ Ε, Ζ καὶ μεῖζον τὸ Ε· δεῖ οὖν πολύγωνον ἐγγράφαι εἰς τὸν κύκλον καὶ ἄλλο περιγράφαι, ἵνα γένηται τὸ ἐπίταχθεν.

Λαμβάνω γὰρ δύο εὐθείας ἀνίσους τὰς Γ, Δ, ὃν μεῖζον ἔστω ἢ Γ, ὡστε τὴν Γ πρὸς τὴν Δ

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And ΠΟ is a side of the circumscribed polygon, ΓΝ of the inscribed; which was to be found.

Prop. 5

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another, so that the circumscribed polygon has to the inscribed polygon a ratio less than the greater magnitude has to the less.

Let there be set out the circle A and the two unequal magnitudes E, Z, and let E be the greater; it is therefore required to inscribe a polygon in the circle and to circumscribe another, so that what is required may be done.

For I take two unequal straight lines Γ, Δ, of which let Γ be the greater, so that Γ has to Δ a ratio
Ελάσσονα λόγον ἐχειν ἣ τὴν Ε πρὸς τὴν Ζ· καὶ τῶν Γ, Δ μέσης ἀνάλογον ληφθείσης τῆς Η μείζων ἀρα καὶ ἢ Γ τῆς Η. περιγεγράφθη ὅτι περὶ κύκλου πολύγωνον καὶ ἀλλο ἐγγεγράφθω, ὡστε τὴν τοῦ περιγραφέντος πολυγώνου πλευράν πρὸς τὴν τοῦ ἐγγραφέντος ἐλάσσονα λόγον ἐχειν ἢ τὴν Γ πρὸς τὴν Η [καθὼς ἐμάθομεν]· διὰ τούτο ὅτι καὶ ὁ διπλάσιος λόγος τοῦ διπλασίου ἐλάσσων ἐστι· καὶ τοῦ μὲν τῆς πλευρᾶς πρὸς τὴν πλευρᾶν διπλάσιος ἐστι ὁ τοῦ πολυγώνου πρὸς τὸν πολυγώνου [ὅμως γάρ], τῆς δὲ Γ πρὸς τὴν Η ὁ τῆς Γ πρὸς τὴν Δ· καὶ τὸ περιγραφέν ἀρα πολύγωνον πρὸς τὸ ἐγγραφέν ἐλάσσονα λόγον ἐχει ἦπερ ἢ Γ πρὸς τὴν Δ· πολλῶ ἀρα τὸ περιγραφέν πρὸς τὸ ἐγγραφέν ἐλάσσονα λόγον ἐχει ἦπερ το Ε πρὸς τὸ Ζ.

η'

'Εαν περὶ κώνων ἰσοσκελῆ πυραμίδος περιγραφῇ, ἡ επιφάνεια τῆς πυραμίδος χωρίς τῆς βάσεως ἢ ἐστὶν τριγώνου βάσιν μὲν ἔχοντι τὴν ἵπην τῇ περιμέτρῳ τῆς βάσεως, ύψος δὲ τῆς πλευρᾶς τοῦ κώνου.

θ'

'Εαν κώνων τινὸς ἰσοσκελοὺς εἰς τὸν κύκλον, ὡς ἐστὶ βάσις τοῦ κώνου, εὐθεία γραμμὴ ἐμπέσῃ, ἀπὸ δὲ τῶν περάτων αὐτῆς εὐθείαι γραμμαί ἀχθώσιν ἐπὶ τὴν κορυφήν τοῦ κώνου, τὸ περιληφθὲν τρίγωνον ὑπὸ τε τῆς ἐμπεσοῦσης καὶ τῶν ἐπίζευξεισῶν ἐπὶ τὴν κορυφὴν ἐλάσσον ἐσται τῆς.
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less than that which E has to Z [Prop. 2]; if a mean proportional H be taken between \( \Gamma', \Delta \), then \( \Gamma' \) will be greater than H [Eucl. vi. 13]. Let a polygon be circumscribed about the circle and another inscribed, so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which \( \Gamma' \) has to H [Prop. 3]; it follows that the duplicate ratio is less than the duplicate ratio. Now the duplicate ratio of the sides is the ratio of the polygons [Eucl. vi. 20], and the duplicate ratio of \( \Gamma' \) to H is the ratio of \( \Gamma' \) to \( \Delta \) [Eucl. v. Def. 9]; therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which \( \Gamma' \) has to \( \Delta \); by much more therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which E has to Z.

Prop. 8

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid without the base is equal to a triangle having its base equal to the perimeter of the base [of the pyramid] and its height equal to the side of the cone. . . .

Prop. 9

If in an isosceles cone a straight line [chord] fall in the circle which is the base of the cone, and from its extremities straight lines be drawn to the vertex of the cone, the triangle formed by the chord and the lines joining it to

1 The "side of the cone" is a generator. The proof is obvious.

1 καθώς εὐάθροιν om. Heiberg.
2 δυοια γάρ om. Heiberg.
επιφανείας τοῦ κώνου τῆς μεταξὺ τῶν ἐπὶ τὴν κορυφὴν ἐπιζευγθεὶσῶν.

*Εστι κώνου ἰσοσκελοῦς βάσις ὁ ΑΒΓ κύκλος, κορυφὴ δὲ τὸ Δ, καὶ διήκῳ τις εἰς αὐτὸν εὐθεῖα ἡ ΛΓ, καὶ ἀπὸ τῆς κορυφῆς ἐπὶ τὰ Α, Γ ἐπεζεύχθωσαν αἱ ΛΔ, ΛΓ· λέγω, ὅτι τὸ ΛΔΓ τρίγωνον

ἐλασθὸν ἔστιν τῆς επιφανείας τῆς κωνικῆς τῆς μεταξὺ τῶν ΛΔΓ.

Τετμῆσθω ἡ ΑΒΓ περιφέρεια δίχα κατὰ τὸ B, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΓΒ, ΔΒ· ἔσται δὴ τὰ ΑΒΔ, ΒΓΔ τρίγωνα μείζονα τοῦ ΛΔΓ τριγώνου. ἦ δὴ ὑπερέχει τὰ εἰρημένα τρίγωνα τοῦ ΛΔΓ τριγώνου, ἔστω τὸ Θ. τὸ δὴ Θ ἦτοι τῶν ΑΒ, ΒΓ τμημάτων ἐλασθὸν ἔστιν ἢ οὐ.
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the vertex will be less than the surface of the cone between the lines drawn to the vertex.

Let the circle $AB\Gamma$ be the base of an isosceles cone, let $\Delta$ be its vertex, let the straight line $A\Gamma$ be drawn in it, and let $\Delta A$, $\Delta \Gamma$ be drawn from the vertex to $A$, $\Gamma$; I say that the triangle $\Delta A\Gamma$ is less than the surface of the cone between $\Delta A$, $\Delta \Gamma$.

Let the arc $AB\Gamma$ be bisected at $B$, and let $AB$, $\Gamma B$, $\Delta B$ be joined; then the triangles $AB\Delta$, $B\Gamma\Delta$ will be greater than the triangle $\Delta A\Gamma$\(^a\). Let $\Theta$ be the excess by which the aforesaid triangles exceed the triangle $\Delta A\Gamma$. Now $\Theta$ is either less than the sum of the segments $AB$, $B\Gamma$ or not less.

\(^a\) For if $h$ be the length of a generator of the isosceles cone, triangle $AB\Delta = \frac{1}{2}h \cdot AB$, triangle $B\Gamma\Delta = \frac{1}{2}h \cdot B\Gamma$, triangle $\Delta A\Gamma = \frac{1}{2}h \cdot A\Gamma$, and $AB + B\Gamma > A\Gamma$.

\(^1\) έσται ... τριγώνου; ex Eutociο videtur Archimodem scripsisse: μείζονα ἀρα έστι τὰ $AB\Delta$, $B\Delta \Gamma$ τρίγωνα τοῦ $\Delta A\Gamma$ τριγώνου.
Εστώ μη ἔλασσον πρότερον. ἔπει ὅν δύο εἰσιν ἐπιφάνειαι τής κοινής μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τριήματος καὶ τῆς τοῦ ΑΔΒ τριγώνου τὸ αὐτὸ πέρας ἔχουσα τὴν περίμετρον τοῦ τριγώνου τοῦ ΑΔΒ, μεῖζον ἐστὶν ἡ περιλαμβάνοντα τῆς περιλαμβανομένης μεῖζον ἀρα ἐστὶν ἡ κοινικὴ ἐπιφάνεια μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμήματος τοῦ ΑΒΔ τριγώνου. ὅμοιως δὲ καὶ τὴ μεταξὺ τῶν ΒΔΓ μετὰ τοῦ ΓΖΒ τμήματος μεῖζον ἐστὶν τοῦ ΒΔΓ τριγώνου· ὅλη ἀρα ἡ κοινικὴ ἐπιφάνεια μετὰ τοῦ Θ χωρίου μεῖζον ἐστὶ τῶν εἰρημένων τριγώνων. τὰ δὲ εἰρήμενα τρίγωνα ἢ ἐστὶν τὸ τοῦ ΑΔΓ τριγώνῳ καὶ τὸ τοῦ Θ χωρίῳ. κοινὸν ἀφηρήσθω τὸ Θ χωρίῳ· λοιπὴ ἀρα ἡ κοινικὴ ἐπιφάνεια μεταξὺ τῶν ΑΔΓ μεῖζον ἐστὶν τοῦ ΑΔΓ τριγώνου.

Εστὼ δὴ τοῦ Θ ἔλασσον τῶν ΑΒ, ΒΓ τμημάτων. τέμνοντες δὴ τὰς ΑΒ, ΒΓ περιφερείας δίχα καὶ τὰς ἡμισείας αὐτῶν δίχα λείφομεν τμήματα ἔλασσον δυντα τοῦ Θ χωρίου. λελείφθω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ εὐθείων, καὶ ἐπεξεύχθωσαν αἱ ΔΕ, ΔΖ. πάλιν τοῖνυν κατὰ τὰ αὐτὰ ἡ μὲν ἐπιφάνεια τοῦ κώνου μεταξὺ τῶν ΑΔΕ μετὰ τοῦ ἐπὶ τῆς ΑΕ τμήματος μεῖζον ἐστὶν τοῦ ΑΔΕ τριγώνου, ἢ δὲ μεταξὺ τῶν ΕΔΒ μετὰ τοῦ ἐπὶ τῆς ΕΒ τμήματος μεῖζον ἐστὶν τοῦ ΕΔΒ τριγώνου· ἢ ἀρα ἐπιφάνεια μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἐπὶ τῶν ΑΕ, ΕΒ τμημάτων μεῖζον ἐστὶν τῶν ΑΔΕ, ΕΒΔ τριγώνων. ἔπει δὲ τὰ ΑΕΔ, ΔΕΒ τρίγωνα μεῖζονά ἐστιν τοῦ ΑΒΔ τριγώνου, καθὼς δέδεικται, πολλῷ ἀρα ἡ ἐπιφάνεια τοῦ κώνου μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἐπὶ τῶν ΑΕ,
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Firstly, let it be not less. Then since there are two surfaces, the surface of the cone between \( \Delta \Delta \), \( \Delta \)B together with the segment \( \Delta \)E\( \Delta \)B and the triangle \( \Delta \Delta B \), having the same extremity, that is, the perimeter of the triangle \( \Delta \Delta B \), the surface which includes the other is greater than the included surface [Post. 3]; therefore the surface of the cone between the straight lines \( \Delta \Delta \), \( \Delta \)B together with the segment \( \Delta \)E\( \Delta \)B is greater than the triangle \( \Delta \)B\( \Delta \). Similarly the [surface of the cone] between \( B \Delta \), \( \Delta \)\( \Delta \) together with the segment \( \Gamma \)Z\( \Delta \)B is greater than the triangle \( B \Delta \)\( \Delta \); therefore the whole surface of the cone together with the area \( \Theta \) is greater than the aforesaid triangles. Now the aforesaid triangles are equal to the triangle \( \Delta \Delta \Gamma \) and the area \( \Theta \). Let the common area \( \Theta \) be taken away; therefore the remainder, the surface of the cone between \( \Delta \Delta \), \( \Delta \)\( \Delta \) is greater than the triangle \( \Delta \Delta \Gamma \).

Now let \( \Theta \) be less than the segments \( \Delta \)B, \( \Delta \)\( \Gamma \). Bisecting the arcs \( \Delta \)B, \( \Delta \)\( \Gamma \) and then bisecting their halves, we shall leave segments less than the area \( \Theta \) [Eucl. xii. 2]. Let the segments so left be those on the straight lines \( \Delta \)E, \( B \)B, \( \Delta \)Z, \( \Delta \)\( \Gamma \), and let \( \Delta \)E, \( \Delta \)Z be joined. Then once more by the same reasoning the surface of the cone between \( \Delta \Delta \), \( \Delta \)E together with the segment \( \Delta \)E is greater than the triangle \( \Delta \Delta \)E, while that between \( \Delta \)\( \Delta \), \( \Delta \)B together with the segment \( \Delta \)B is greater than the triangle \( \Delta \)B; therefore the surface between \( \Delta \Delta \), \( \Delta \)B together with the segments \( \Delta \)E, \( \Delta \)B is greater than the triangles \( \Delta \Delta \)E, \( \Delta \)B\( \Delta \). Now since the triangles \( \Delta \)E\( \Delta \), \( \Delta \)B\( \Delta \) are greater than the triangle \( \Delta \)B\( \Delta \), as was proved, by much more therefore the surface of the cone between \( \Delta \), \( \Delta \)B together with the segments \( \Delta \)E, \( \Delta \)B is
ΕΒ τιμημάτων μείζων ἐστὶ τοῦ ΑΔΒ τριγώνου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΒΔΓ μετὰ τῶν ἐπὶ τῶν ΒΖ, ΖΓ τιμημάτων μείζων ἐστὶν τοῦ ΒΔΓ τριγώνου. ὅλη ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μετὰ τῶν εἰρημένων τιμημάτων μείζων ἐστὶ τῶν ΑΒΔ, ΔΒΓ τριγώνων. ταῦτα δὲ ἐστιν ἵσα τοῦ ΑΔΓ τριγώνου καὶ τῷ Θ χωρίῳ ἃν τὰ εἰρημένα τιμήματα ἐλάσσονα τοῦ Θ χωρίου λοιπὴ ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μείζων ἐστὶν τοῦ ΑΔΓ τριγώνου.

Εἰς ἐπιφανεύομαι ἀχθώσων τοῦ κύκλου, ὃς ἐστὶ βάσις τοῦ κώνου, ἐν τῷ αὐτῷ ἐπιπέδῳ οὕτως τῷ κύκλῳ καὶ συμπίπτουσαι ἄλληλαις, ἀπὸ ἰε τῶν ἁφῶν καὶ τῆς συμπτώσεως ἐπὶ τὴν κορυφὴν τοῦ κώνου εὐθεῖα ἀχθώσων, τὰ περιεχόμενα τρίγωνα ὑπὸ τῶν ἐπιφανούσων καὶ τῶν ἐπὶ τὴν κορυφὴν τοῦ κώνου ἐπιζευγθέσεων εὐθείῶν μείζονα ἐστὶν τῆς τοῦ κώνου ἐπιφανείας τῆς ἀπολαμβανομένης ὑπὸ αὐτῶν. . . .

. . . Τούτων δὴ δεδειγμένων φανερῶν ἐπὶ μὲν τῶν προειρημένων, ὅτι, εἰς κάτων ἰσοσκελῆ πυραμίδος ἐγγραφή, ἡ ἐπιφάνεια τῆς πυραμίδος χωρίς τῆς βάσεως ἐλάσσον ἐστὶ τῆς κοινῆς ἐπιφανείας ἐκαστον γὰρ τῶν περιεχόντων τῆς πυραμίδα τριγώνων ἐλασσόν ἐστιν τῆς κοινῆς ἐπιφανείας τῆς μεταξὺ τῶν τοῦ τριγώνου πλευρῶν ἀστε καὶ ὅλη ἡ ἐπιφάνεια τῆς πυραμίδος χωρίς τῆς.
greater than the triangle $\triangle A\Delta B$. By the same reasoning the surface between $B\Delta, \Delta\Gamma$ together with the segments $BZ, Z\Gamma$ is greater than the triangle $B\Delta\Gamma$; therefore the whole surface between $\Lambda\Delta, \Delta\Gamma$ together with the aforesaid segments is greater than the triangles $\Lambda\Delta\Delta, \Delta\Delta\Gamma$. Now these are equal to the triangle $\Lambda\Delta\Gamma$ and the area $\Theta$; and the aforesaid segments are less than the area $\Theta$; therefore the remainder, the surface between $\Lambda\Delta, \Delta\Gamma$ is greater than the triangle $\Lambda\Delta\Gamma$.

Prop. 10

If tangents be drawn to the circle which is the base of an [isosceles] cone, being in the same plane as the circle and meeting one another, and from the points of contact and the point of meeting straight lines be drawn to the vertex of the cone, the triangles formed by the tangents and the lines drawn to the vertex of the cone are together greater than the portion of the surface of the cone included by them. . . .

Prop. 12

. . . From what has been proved it is clear that, if a pyramid is inscribed in an isosceles cone, the surface of the pyramid without the base is less than the surface of the cone [Prop. 9], and that, if a pyramid

\footnote{\textit{αιτι}\ldots\textit{προκειμένων} om. Heiberg.}
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βάσεως ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου χωρίς τῆς βάσεως], καὶ ὅτι, ἕαν περὶ κώνου ἰσοσκελῆ πυραμίδος περιγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως μείζων ἐστὶν τῆς ἐπιφανείας τοῦ κώνου χωρὶς τῆς βάσεως [κατὰ τὸ συνεχὲς ἐκεῖνῳ].

Φανερὸν δὲ ἐκ τῶν ἀποδεδειγμένων, ὅτι τε, ἕαν εἰς κυλίνδρου ὅρθον πρίσμα ἐγγραφῇ, ἡ ἐπιφάνεια τοῦ πρίσματος ἢ ἐκ τῶν παραλληλογράμμων συγκειμένη ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρὶς τῆς βάσεως [ἐλάσσων γὰρ ἔκαστον παραλληλογράμμον τοῦ πρίσματος ἐστὶ τῆς καθ' αὐτό τοῦ κυλίνδρου ἐπιφανείας], καὶ ὅτι, ἕαν περὶ κυλίνδρου ὅρθον πρίσμα περιγραφῇ, ἡ ἐπιφάνεια τοῦ πρίσματος ἢ ἐκ τῶν παραλληλογράμμων συγκειμένη μείζων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρὶς τῆς βάσεως.

Γ'...

Πάντως κυλίνδρου ὅρθοῦ ἡ ἐπιφάνεια χωρὶς τῆς βάσεως ἵση ἐστὶ κύκλω, οὗ ἡ ἐκ τοῦ κέντρου μέσων λόγου ἔχει τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς διάμετρον τῆς βάσεως τοῦ κυλίνδρου.

Ἐστὶ δὲ κυλίνδρου τῶν ὅρθων βάσις ὁ Κύκλος, καὶ ἐστὶ τῇ μὲν διάμετρῳ τοῦ A κύκλου ἵση ἡ ΓΔ, τῇ δὲ πλευρᾷ τοῦ κυλίνδρου ἡ EZ, ἔχετω δὲ μέσων λόγων τῶν ΔΓ, EZ ἡ H, καὶ κείσθω κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἱσθα ἐστὶ τῇ H, ὁ Β'. δεικτέον, ὅτι ὁ Β κύκλος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κυλίνδρου χωρὶς τῆς βάσεως.

Εἰ γάρ μὴ ἐστὶν ἴσος, ἦτοι μείζων ἐστὶ η

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is circumscribed about an isosceles cone, the surface of the pyramid without the base is greater than the surface of the cone without the base [Prop. 10].

From what has been demonstrated it is also clear that, if a right prism be inscribed in a cylinder, the surface of the prism composed of the parallelograms is less than the surface of the cylinder excluding the bases [Prop. 11], and if a right prism be circumscribed about a cylinder, the surface of the prism composed of the parallelograms is greater than the surface of the cylinder excluding the bases.

Prop. 13

The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base of the cylinder.

Let the circle A be the base of a right cylinder, let $\Gamma \Delta$ be equal to the diameter of the circle A, let $EZ$ be equal to the side of the cylinder, let $H$ be a mean proportional between $\Delta \Gamma$, $EZ$, and let there be set out a circle, B, whose radius is equal to $H$; it is required to prove that the circle B is equal to the surface of the cylinder excluding the bases.

For if it is not equal, it is either greater or less.

¹ ἐκαστὸν . . . βάσεως. Heiberg suspects that this demonstration is interpolated. Why give a proof of what is φανερὸν?
² κατὰ . . . ἐκείνο om. Heiberg.
³ ἔλασθον . . . ἐπιφανείας. Heiberg suspects that this proof is interpolated.
ёλάσσων. ἦστω πρῶτερον, εἰ δυνατόν, ἐλάσσων. δύο δὴ μεγεθῶν οὐτων ἀνίσων τῆς τε ἐπιφάνειας τοῦ κυλίνδρου καὶ τοῦ Β κύκλου δυνατόν ἦστω εἰς τὸν Β κύκλον ἴσόπλευρον πολύγωνον ἐγγράφαι καὶ ἄλλο περιγράφατε, ὡστε τὸ περιγραφέν πρὸς τὸ ἐγγραφὲν ἐλάσσων λόγου ἔχειν τοῦ, ὅν ἔχει ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν Β κύκλον. νοεῖσθαι δὴ περιγεγραμμένον καὶ ἐγγεγραμμένον, καὶ περὶ τὸν Α κύκλον περιγεγράφῳ εὐθύγραμμον ὁμοιον τῷ περὶ τὸν Β περιγεγραμμένῳ, καὶ ἀναγεγράφῳ ἀπὸ τοῦ εὐθυγράμμου πρίσμα· ἦσται δὴ περὶ τὸν κυλίνδρον περιγεγραμμένον. ἦστω δὲ καὶ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ περὶ

* One ms. has the marginal note, “equalis altitudinis chylindro,” on which Heiberg comments: “nec hoc omiserat Archimedes.” Heiberg notes several places in which the text is clearly not that written by Archimedes.

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Let it first be, if possible, less. Now there are two unequal magnitudes, the surface of the cylinder and the circle B, and it is possible to inscribe in the circle B an equilateral polygon, and to circumscribe another, so that the circumscribed has to the inscribed a ratio

less than that which the surface of the cylinder has to the circle B [Prop. 5]. Let the circumscribed and inscribed polygons be imagined, and about the circle A let there be circumscribed a rectilineal figure similar to that circumscribed about B, and on the rectilineal figure let a prism be erected; it will be circumscribed about the cylinder. Let KΔ be equal

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τον Α κύκλον ἰση ἡ ΚΔ καὶ τῇ ΚΔ ἵση ἡ ΔΖ, τῆς
de ΓΔ ἡμίσεια ἐστὼ ἡ ΓΤ· ἐσται δὴ τὸ ΚΔΓ
tρίγωνον ἵσον τῷ περιγεγραμμένῳ εὐθυγράμμῳ
περὶ τὸν Α κύκλον [ἐπείδη βάσιν μὲν ἔχει τῇ
peri-
μέτρῳ ἵσῃ, ὕψος δὲ ἵσον τῇ ἐκ τοῦ κέντρου τοῦ
Α κύκλου], ἡ δὲ ΕΔ παραλληλόγραμμον τῇ
ἐπιφανεία τοῦ πρίσματος τοῦ περὶ τὸν κύλινδρον
περιγεγραμμένον [ἐπείδη περιέχεται ὑπὸ τῆς
πλευρᾶς τοῦ κύλινδρον καὶ τῆς ἴσης τῇ περιμέτρῳ
tῆς βάσεως τοῦ πρίσματος].

κείσθω δὴ τῇ ΕΖ
ἰση ἡ ΕΡ· ἵσον ἁρὰ ἐστὶν τὸ ΖΡΔ τρίγωνον τῷ
ΕΔ παραλληλόγραμμῳ, ὡστε καὶ τῇ ἐπιφανείᾳ
tοῦ πρίσματος. καὶ ἐπεὶ ὁμοία ἐστὶν τὰ εὐθύ-
γραμμα τὰ περὶ τοὺς Α, Β κύκλους περιγεγραμ-
μένα, τὸν αὐτὸν ἔξει λόγον [τὰ εὐθύγραμμα].

ὅπερ αἱ ἐκ τῶν κέντρων δυνάμει ἔξει ἁρὰ τὸ ΚΤΔ
τρίγωνον πρὸς τὸ περὶ τὸν Β κύκλον εὐθύγραμμον
λόγον, ὅν ἡ ΤΔ πρὸς Η δυνάμει [αἱ γὰρ ΤΔ, Η
ἰσαι εἰσὶν ταῖς ἐκ τῶν κέντρων]. ἀλλ' ὅπως ἔχει
λόγον ἡ ΤΔ πρὸς Η δυνάμει, τοῦτον ἔχει τὸν λόγον
ἡ ΤΔ πρὸς ΡΖ μῆκει [ἡ γὰρ Η τῶν ΤΔ, ΡΖ μέσῃ
ἐστὶ ἀνάλογον διὰ τὸ καὶ τῶν ΓΔ, ΕΖ· πῶς ἔ
τοτε; ἐπεὶ γὰρ ἴσῃ ἐστὶν ἡ μὲν ΔΤ τῇ ΤΓ, ἡ
de PE τῇ EZ, διπλασία ἁρὰ ἐστὶν ἡ ΓΔ τῆς ΤΔ,
καὶ ἡ ΡΖ τῆς PE· ἐστὶν ἁρὰ, ὡς ἡ ΔΓ πρὸς ΔΤ,
οὕτως ἡ ΡΖ πρὸς ΖΕ. τὸ ἁρὰ ὑπὸ τῶν ΓΔ, ΕΖ
ἰσον ἐστὶν τῷ ὑπὸ τῶν ΤΔ, ΡΖ. τῷ ὑπὸ τῶν
ΓΔ, ΕΖ ἵσον ἐστὶν τὸ ἀπὸ Η. καὶ τῷ ὑπὸ τῶν
ΤΔ, ΡΖ ἁρὰ ἵσον ἐστὶ τὸ ἀπὸ τῆς Η. ἐστὶν ἁρὰ,
to the perimeter of the rectilineal figure about the circle A, let \( \Delta Z \) be equal to \( K\Delta \), and let \( \Gamma T \) be half of \( \Gamma \Delta \); then the triangle \( K\Delta T \) will be equal to the rectilineal figure circumscribed about the circle A, while the parallelogram \( EA \) will be equal to the surface of the prism circumscribed about the cylinder.

Let \( EP \) be set out equal to \( EZ \); then the triangle \( ZPA \) is equal to the parallelogram \( EA \) [Eucl. i. 41], and so to the surface of the prism. And since the rectilineal figures circumscribed about the circles A, B are similar, they will stand in the same ratio as the squares on the radii; therefore the triangle \( KT\Delta \) will have to the rectilineal figure circumscribed about the circle B the ratio \( T\Delta^2 : H^2 \).

But \[ T\Delta^2 : H^2 = T\Delta : PZ. \]

\(^4\) Because the base \( K\Delta \) is equal to the perimeter of the polygon, and the altitude \( \Delta T \) is equal to the radius of the circle A, \textit{i.e.}, to the perpendiculars drawn from the centre of A to the sides of the polygon.

\(^5\) Because the base \( \Delta Z \) is made equal to \( \Delta K \) and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude \( EZ \) is equal to the side of the cylinder and therefore to the height of the prism.

\(^6\) Eutocius supplies a proof based on Eucl. xii. 1, which proves a similar theorem for inscribed figures.

\(^7\) For, by hypothesis, \[ H^2 = \Delta T \cdot EZ \]
\[ = 2T\Delta \cdot \frac{1}{2}PZ \]
\[ = T\Delta \cdot PZ \]

Heiberg would delete the demonstration in the text on the ground of excessive verbosity, as Nizze had already perceived to be necessary.
για τη Δ προς Η, ούτως η Η προς ΡΖ· εστιν αρα, 
για τη Δ προς ΡΖ, το απο της Δ προς το απο 
της Η· ειν γαρ τρεις ευθειαι αναλογουν ωσιν, εστιν, 
για πρωτη προς την τριτην, το απο της πρωτης 
eιδος προς το απο της δευτερας ειδος το ομοιον και 
ομοιος αναγεγραμμενον[1]. ον δε λογον έχει η Δ 
προς ΡΖ μηκει, τουτον έχει το ΚΔ τριγωνον 
προς το ΡΑΖ [επειδηπερ ίσαι εισιν αι ΚΔ, ΑΖ]2· 
tον αυτον αρα λογον έχει το ΚΔ τριγωνον προς 
tο ευθυγραμμιον το περι τον Β κυκλον περιγεγρα 
μενον, οπερ το ΤΚΔ τριγωνον προς το ΡΖΑ 
τριγωνον. ίσον αρα εστιν το ΖΔΡ τριγωνον τω 
περι τον Β κυκλον περιγεγραμμενο ευθυγραμμι 
ωστε και η επιφανεια του πρισματος του περι 
tου Α κυλινδρον περιγεγραμμενου τω ευθυγραμμ 
περι τον Β κυκλον ίση εστιν. και επει δια 
αθοσονα λογον έχει το ευθυγραμμιον το περι τον 
Β κυκλον προς το εγγεγραμμενον εν τω κυκλω 
tου, ον έχει η επιφανεια του Α κυλινδρον προς 
tον Β κυκλον, δια 
αθοσονα λογον έχει και η επιφανεια του πρισ 
ματος του περι του κυλινδρον περιγεγραμ 
μενον προς το ευθυγραμμιον το εν τω κυκλω 
tου Β εγγεγραμμενου οπερ η επιφανεια του 
κυλινδρον προς τον Β κυκλον 
και εναλλας· οπερ αδινατον [η μεν γαρ επιφα 
νεια του πρισματος του περιγεγ 
ραμενου περι του 
κυλινδρον μειξων ουσα δεδεκται της 
επιφανειας του κυλι 
νδρου, το δε εγγεγ 
ραμενον ευθυ 
γραμμου 
eν τω Β κ 
κυκλον ε 
λασον ε 
στιν του Β κυ 
κλου].3 
ουκ αρα εσ 
tιν ο Β κυ 
κλος ελα 
σσων της 
επιφανειας 
tου κυ 
λινδρου.

1 η γαρ ... άμοιος αναγεγραμμένων om. Heiberg.
2 επειδήπερ ... ΚΔ, ΑΖ om. Heiberg.

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And $\Delta : PZ = \triangle K\Delta : \triangle P\Delta Z$. Therefore the ratio which the triangle $K\Delta$ has to the rectilineal figure circumscribed about the circle $B$ is the same as the ratio of the triangle $TK\Delta$ to the triangle $PZ\Delta$. Therefore the triangle $TK\Delta$ is equal to the rectilineal figure circumscribed about the circle $B$ [Eucl. v. 9]; and so the surface of the prism circumscribed about the cylinder $A$ is equal to the rectilineal figure about $B$. And since the rectilineal figure about the circle $B$ has to the inscribed figure in the circle a ratio less than that which the surface of the cylinder $A$ has to the circle $B$ [ex hypothesis], the surface of the prism circumscribed about the cylinder will have to the rectilineal figure inscribed in the circle $B$ a ratio less than that which the surface of the cylinder has to the circle $B$; and, permutando, [the prism will have to the cylinder a ratio less than that which the rectilineal figure inscribed in the circle $B$ has to the circle $B$] $a$; which is absurd. Therefore the circle $B$ is not less than the surface of the cylinder.

$^a$ By Eucl. vi. 1, since $AZ = KA$.

$^b$ From Eutocius's comment it appears that Archimedes wrote, in place of καὶ ἐναλλάξ ὀπερ ἀδύνατον in our text: ἐναλλάξ ὃ ἡμας ἀλλον εἶναι τὸ πρίσμα πρὸς τὸν κύλινδρον ἤπερ τὸ ἑγγεγραμμὲνον eis τὸν B κύκλων πολύγωνον πρὸς τὸν B κύκλων ὀπερ ἀτοπον. This is what I translate.

$^c$ For the surface of the prism is greater than the surface of the cylinder [Prop. 12], but the inscribed figure is less than the circle $B$; the explanation in our text to this effect is shown to be an interpolation by the fact that Eutocius supplies a proof in his own words.

$^a$ ἦ μὲν ... τοῦ B κύκλου om. Heiberg ex Eutocio.
"Εστω δή, εἰ δυνατόν, μείζων. πάλιν δὴ νοείσθω εἰς τὸν Β κύκλον εὐθύγραμμον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὡστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἑλάσσονα λόγον ἔχειν ἢ τὸν Β κύκλον πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἐγγεγράφη εἰς τὸν Α κύκλον πολύγωνον ὰμοιον τῷ εἰς τὸν Β κύκλον ἐγγεγραμμένων, καὶ πρίσμα ἀναγεγράφη ἀπὸ τοῦ ἐν τῷ κύκλῳ ἐγγεγραμμένου πολυγώνου· καὶ πάλιν ἢ KD ἢς ἐστώ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου, καὶ ἢ ZΔ ἢς αὐτῇ ἐστώ. ἔσται δὴ τὸ μὲν ΚΤΔ τρίγωνον μείζον τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου [διότι βάσιν μὲν ἔχει τὴν περιμέτρον αὐτοῦ, ὦφθω δὲ μείζον τῆς ἀπὸ τοῦ κέντρου ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου ἀγομένης καθέτου], τὸ δὲ ΕΔ παραλληλόγραμμον ἢσον τῇ ἐπιφανείᾳ τοῦ πρίσματος τῇ ἐκ τῶν παραλληλόγραμμων συγκεκριμένῃ [διότι περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἢσος τῆς περιμέτρου τοῦ εὐθυγράμμου, ὡστε καὶ τὸ ΡΔΖ τρίγωνον ἢσον ἐστὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὄμοια ἢστι τὰ εὐθύγραμμα τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἔχει λόγον πρὸς ἄλληλα, ὅπερ αἱ ἐκ τῶν κέντρων αὐτῶν δύναμιν. ἔχει δὲ καὶ τὰ ΚΤΔ, ΖΠΑ τρίγωνα πρὸς ἄλληλα λόγον, ὅπερ αἱ ἐκ τῶν κέντρων τῶν κύκλων δύναμιν· τῶν αὐτῶν ἀρα λόγον ἔχει

1 διότι ... καθέτου om. Heiberg.

* For the base KD is equal to the perimeter of the polygon and the altitude ΔT, which is equal to the radius of the
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Now let it be, if possible, greater. Again, let there be imagined a rectilineal figure inscribed in the circle B, and another circumscribed, so that the circumscribed figure has to the inscribed a ratio less than that which the circle B has to the surface of the cylinder [Prop. 5], and let there be inscribed in the circle A a polygon similar to the figure inscribed in the circle B, and let a prism be erected on the polygon inscribed in the circle [A]; and again let $K\Delta$ be equal to the perimeter of the rectilineal figure inscribed in the circle A, and let $Z\Delta$ be equal to it. Then the triangle $KT\Delta$ will be greater than the rectilineal figure inscribed in the circle A, and the parallelogram $E\Lambda$ will be equal to the surface of the prism composed of the parallelograms; and so the triangle $PAZ$ is equal to the surface of the prism. And since the rectilineal figures inscribed in the circles A, B are similar, they have the same ratio one to the other as the squares of their radii [Eucl. xii. 1]. But the triangles $KT\Delta$, $Z\Pi\Lambda$ have one to the other the same ratio as the squares of the radii; therefore the rectilineal figure inscribed in circle A, is greater than the perpendiculars drawn from the centre of the circle to the sides of the polygon; but Heiberg regards the explanation to this effect in the text as an interpolation.

Because the base $Z\Delta$ is made equal to $K\Delta$, and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude $EZ$ is equal to the side of the cylinder and therefore to the height of the prism.

For triangle $KT\Delta$: triangle $Z\Pi\Lambda = T\Delta : ZP$

\[ T\Delta^2 : H^2 \]

[cf. p. 71 n. d.]

But $T\Delta$ is equal to the radius of the circle A, and $H$ to the radius of the circle B.
GREEK MATHEMATICS

to eubugrammon to en tw A kúklw eγγεγραμμένον
pros to eubugrammon to en tw B eγγεγραμμένον
kai to KTD trígonon pros to LZR trígonon.
ēlasson de èsti to eubugrammon to en tw A kúklw
eγγεγραμμένον tou KTD trígonon.
ēlasson āra kai to eubugrammon to en tw B kúklw eγγεγραμ-
ménou tou ZRA trígonou. woste kai thei eπιφανειας
toui πρίσματος tou en tw kulíndrw eγγεγραμμένου:
òper ādúnaton [èpei gár ēlássoνa lógon èxei to
perigegramménou eubugrammon peri tou B kúklou
pros to eγγεγραμμένον èi B kúklou pros tw
eπιφανειαν tou kulíndrou, kai ënalláξ, mei̇zou de
èsti to perigegramménou peri tou B kúklou tou B
kúklou, mei̇zou āra èstín to eγγεγραμμένον en
tw B kúklw thei eπιφανειας tou kulíndrou. woste
kai thei eπιφανειας tou πρίσματος].
ûk āra mei̇zou èstín ò B kúklou thei eπιφανειας tou
kulíndrou. èdeîkhpè de, òti ou̇de ēlássoνi ìsos
āra èstín.

6"

PANTOS KÔWNO IΣOΣKELOUΣ CHWRIΣ TΗΣ BÁSEWS
h eπιφανεια ësti kúklw, òh ëk tou kéntron
mèson lógon èxei ñ πλευrās tou kûwnou kai ñ
ëk tou kéntron tou kúklou, òs èstín básis tou
kûwnou.

"ÈSTW KÔWNO IΣOΣKELHΣ, òH BÁSIS Ö A KÛKLW,
h ëk tou kéntron èstw ëi G, ñ ñ πλευrā tòu

1 èpei ... πrím̓ȧm̓atos om. Heiberg.

a For since the figure circumscribed about the circle B has
to the inscribed figure a ratio less than that which the circle B
has to the surface of the cylinder [ex hypothesi], and the
circle B is less than the circumscribed figure, therefore the
the circle A has to the rectilineal figure inscribed in the circle B the same ratio as the triangle $KT\Delta$ has to the triangle $AZP$. But the rectilineal figure inscribed in the circle A is less than the triangle $KT\Delta$; therefore the rectilineal figure inscribed in the circle B is less than the triangle $ZPA$; and so it is less than the surface of the prism inscribed in the cylinder; which is impossible. Therefore the circle B is not greater than the surface of the cylinder. But it was proved not to be less. Therefore it is equal.

Prop. 14

The surface of any cone without the base is equal to a circle, whose radius is a mean proportional between the side of the cone and the radius of the circle which is the base of the cone.

Let there be an isosceles cone, whose base is the circle A, and let its radius be $\Gamma$, and let $\Delta$ be equal inscribed figure is greater than the surface of the cylinder, and a fortiori is greater than the surface of the prism [Prop. 12]. An explanation on these lines is found in our text, but as the corresponding proof in the first half of the proposition was unknown to Eutocius, this also must be presumed an interpolation.
κώνου ἔστω ἤση ἡ Δ, τῶν δὲ Γ, Δ μέση ἀνάλογον ἡ Ε, ὁ δὲ Β κύκλος ἔχετω τὴν ἐκ τοῦ κέντρου τῆς Ε ἤσην λέγω, ὅτι ὁ Β κύκλος ἐστὶν ἴσος τῆς ἐπιφάνεια τοῦ κώνου χωρίς τῆς βάσεως.

Εἰ γὰρ μὴ ἔστω ἴσος, ἦτοι μεῖζων ἐστὶν ἡ ἐλάσσων. ἔστω πρότερον ἐλάσσων. ἔστι δὴ δύο μεγέθη ἀνίσα ἢ τε ἐπιφάνεια τοῦ κώνου καὶ ὁ Β κύκλος, καὶ μεῖζων ἢ ἐπιφάνεια τοῦ κώνου. δυνατὸν ἀρα εἰς τὸν Β κύκλον πολύγωνον ἴσοπλευρον ἐγγράψαι καὶ ἄλλο περιγράψαι ὁμοιον τῷ ἐγγεγραμμένῳ, ὡστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν τοῦ, ὅπερ ἔχει ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλον. νοεῖσθω δὴ καὶ περὶ τὸν Α κύκλον πολύγωνον περιγεγραμμένον ὁμοιον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ, καὶ ἀπὸ τοῦ περὶ τὸν Α κύκλον περιγεγραμμένου πολυγώνου πυραμίδος ἀνεστάτω ἀναγεγραμμένη τὴν αὐτὴν κορυφὴν ἐχουσα τῷ κώνῳ. ἐπεὶ οὖν ὁμοιά ἐστιν τὰ πολύγωνα τὰ περὶ

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to the side of the cone, and let $E$ be a mean proportional between $\Gamma$, $\Delta$, and let the circle $B$ have its

radius equal to $E$; I say that the circle $B$ is equal to the surface of the cone without the base.

For if it is not equal, it is either greater or less. First let it be less. Then there are two unequal magnitudes, the surface of the cone and the circle $B$, and the surface of the cone is the greater; it is therefore possible to inscribe an equilateral polygon in the circle $B$ and to circumscribe another similar to the inscribed polygon, so that the circumscribed polygon has to the inscribed polygon a ratio less than that which the surface of the cone has to the circle $B$ [Prop. 5]. Let this be imagined, and about the circle $A$ let a polygon be circumscribed similar to the polygon circumscribed about the circle $B$, and on the polygon circumscribed about the circle $A$ let a pyramid be raised having the same vertex as the cone. Now since the polygons circumscribed about
τούς Α, Β κύκλους περιγεγραμμένα, τῶν αὐτῶν ἔχει λόγον πρὸς ἄλληλα, ὅν αἱ ἐκ τοῦ κέντρου δυνάμει πρὸς ἄλληλας, τούτεστιν ὅν ἔχει η Ἰ Γ πρὸς Ε δύναμει, τούτεστιν η Ἰ Γ πρὸς Δ μήκει. ὅν δὲ λόγον ἔχει η Ἰ Γ πρὸς Δ μήκει, τούτον ἔχει τὸ περιγεγραμμένον πολύγωνον περὶ τὸν Α κύκλον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κώνον [ἡ μὲν γὰρ Ι ἵστη ἐστὶ τῇ ἀπὸ τοῦ κέντρου καθέτω ἐπὶ μίαν πλευράν τοῦ πολυγώνου, ἡ δὲ Δ τῇ πλευρᾷ τοῦ κώνου κοινών δὲ ύψος ἡ περίμετρος τοῦ πολυγώνου πρὸς τὰ ἡμίσει τῶν ἐπιφανειῶν]· τὸν αὐτὸν ἀρα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Α κύκλον πρὸς τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον καὶ αὐτὸ τὸ εὐθύγραμμον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κώνον· ὡστε ίση ἐστὶν ἡ ἐπιφάνεια τῆς πυραμίδος τοῦ εὐθύγραμμο τῷ περὶ τὸν Β κύκλον περιγεγραμμένω. ἐπεὶ οὖν ἔλασσον λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἦπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλου, ἔλασσον λόγον ἔχει ἡ ἐπιφάνειᾳ τῇ: πυραμίδος τῆς περὶ τὸν κώνον περιγεγραμμένης πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ Β κύκλῳ ἐγγεγραμμένον ἦπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν Β κύκλου· ὅπερ άδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τῆς πυραμίδος μείζων οὐσά δεδεικται τῆς ἐπιφανείας τοῦ κώνου, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ Β κύκλῳ ἔλασσον ἐσται τοῦ Β κύκλου].· οὐκ ἀρα ὁ Β κύκλος ἔλασσον ἐσται τῆς ἐπιφανείας τοῦ κώνου.

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the circles A, B are similar, they have the same ratio one toward the other as the square of the radii have one toward the other, that is \( \Gamma^2 : E^2 \), or \( \Gamma : \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma : \Delta \) is the same ratio as that of the polygon circumscribed about the circle A to the surface of the pyramid circumscribed about the cone \(^a\); therefore the rectilineal figure about the circle A has to the rectilineal figure about the circle B the same ratio as this rectilineal figure [about A] has to the surface of the pyramid circumscribed about the cone; therefore the surface of the pyramid is equal to the rectilineal figure circumscribed about the circle B. Since the rectilineal figure circumscribed about the circle B has towards the inscribed [rectilineal figure] a ratio less than that which the surface of the cone has to the circle B, therefore the surface of the pyramid circumscribed about the cone will have to the rectilineal figure inscribed in the circle B a ratio less than that which the surface of the cone has to the circle B; which is impossible.\(^b\) Therefore the circle B will not be less than the surface of the cone.

\(^a\) For the circumscribed polygon is equal to a triangle, whose base is equal to the perimeter of the polygon and whose height is equal to \( \Gamma \), while the surface of the pyramid is equal to a triangle having the same base and height \( \Delta \) [Prop. 8]. There is an explanation to this effect in the Greek, but so obscurely worded that Heiberg attributes it to an interpolator.

\(^b\) For the surface of the pyramid is greater than the surface of the cone [Prop. 12], while the inscribed polygon is less than the circle B.

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1 \( \tilde{\eta} \text{ m\'e\'v} \ldots \text{epifanei}\text{\'\'\i\'w} \) om. Heiberg.
2 \( \tilde{\eta} \text{ m\'e\'v} \ldots \text{tou\'} \text{B k\'uk\'lou} \) om. Heiberg.
Λέγω δή, ὅτι οὐδεὶς μείζων. εἰ γὰρ δυνατὸν ἔστιν, ἐστώ μείζων. πάλιν δὴ νοεῖσθω εἰς τὸν Β κύκλον πολύγωνον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν τοῦ, ὥστε εἰς τὸν Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου, καὶ εἰς τὸν Α κύκλον νοεῖσθω ἐγγεγραμμένον πολύγωνον ὁμοίον τῷ εἰς τὸν Β κύκλον ἐγγεγραμμένῳ, καὶ ἀναγεγράφθω ἀπ’ αὐτοῦ πυραμίς τὴν αὐτὴν κορυφήν ἔχουσα τῷ κώνῳ. ἐπεὶ οὖν ὁμοία ἔστι τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἔξει λόγον πρὸς ἀλλήλα, ὅτι αἱ ἐκ τῶν κέντρων δυνάμει πρὸς ἀλλήλας· τὸν αὐτὸν ἁρά λόγον ἔχει τὸ πολύγωνον πρὸς τὸ πολύγωνον καὶ ἡ Γ πρὸς τὴν Δ μήκει. ἢ δὲ Γ πρὸς τὴν Δ μείζωνα λόγων ἔχει ἢ τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐγγεγραμμένης εἰς τὸν κώνον [ἡ γὰρ ἐκ τοῦ κέντρου τοῦ Α κύκλου πρὸς τὴν πλευρὰν τοῦ κώνου μείζωνα λόγων ἔχει ἦπερ ἢ απὸ τοῦ κέντρου ἀγομένη κάθετος ἐπὶ μιᾶν πλευρὰν τοῦ πολυγώνου πρὸς τὴν ἐπὶ τὴν πλευρὰν τοῦ πολυγώνου κάθετον ἀγομένην ἀπὸ τῆς κορυφῆς τοῦ κώνου]1: μεί

1 ἡ γὰρ ... τοῦ κώνου om. Heiberg.

* Eutocius supplies a proof. ΖΘΚ is the polygon inscribed in the circle Α (of centre Α), AH is drawn perpendicular to
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I say now that neither will it be greater. For if it is possible, let it be greater. Then again let there be imagined a polygon inscribed in the circle B and another circumscribed, so that the circumscribed has to the inscribed a ratio less than that which the circle B has to the surface of the cone [Prop. 5], and in the circle A let there be imagined an inscribed polygon similar to that inscribed in the circle B, and on it let there be drawn a pyramid having the same vertex as the cone. Since the polygons inscribed in the circles A, B are similar, therefore they will have one toward the other the same ratio as the squares of the radii have one toward the other; therefore the one polygon has to the other polygon the same ratio as \( \Gamma \) to \( \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma \) has to \( \Delta \) a ratio greater than that which the polygon inscribed in the circle A has to the surface of the pyramid inscribed in the cone; therefore the polygon in-

\( K\Theta \) and meets the circle in \( M, A \) is the vertex of the isosceles cone (so that \( \Delta \)H is perpendicular to \( K\Theta \)), and \( HN \) is drawn parallel to \( MA \) to meet \( AA \) in \( N \). Then the area of the polygon inscribed in the circle = \( \frac{1}{2} \) perimeter of polygon \( \cdot \) \( AH \), and the area of the pyramid inscribed in the cone = \( \frac{1}{2} \) perimeter of polygon \( \cdot \) \( AH \), so that the area of the polygon has to the area of the pyramid the ratio \( \Delta H : \Delta H \). Now, by similar triangles, \( AM : MA = AH : HN \), and \( AH : HN > AH : HA \), for \( HA > HN \). Therefore \( AM : MA > AH : HA \); that is, \( \Gamma : \Delta \) exceeds the ratio of the polygon to the surface of the pyramid.
ξόνα ἀρα λόγον ἔχει τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὸ πολύγωνον τὸ ἐν τῷ Β ἐγγεγραμμένον ἢ αὐτὸ τὸ πολύγωνον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος· μείζων ἀρα ἔστιν ἢ ἐπιφάνεια τῆς πυραμίδος τοῦ ἐν τῷ Β πολυγώνου ἐγγεγραμμένου. ἔλασσον δὲ λόγον ἔχει τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· πολλῷ ἀρα τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἐγγεγραμμένης ἔλασσον λόγον ἔχει ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· ὅπερ ἀδύνατον [τὸ μὲν γὰρ περιγεγραμμένον πολύγωνον μείζον ἐστὶν τοῦ Β κύκλου, ἢ δὲ ἐπιφάνεια τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἐλάσσων ἐστὶ τῆς ἐπιφάνειας τοῦ κώνου].  

"Εάν κώνος ἰσοσκελής ἐπιπέδων τμηθῇ παράλληλω τῇ βάσει, τῇ μεταξὺ τῶν παραλλήλων ἐπιπέδων ἐπιφάνεια τοῦ κώνου ἰσος ἐστὶ κύκλος, οὗ ἢ ἐκ τοῦ κέντρου μέσον λόγου ἔχει τῆς τε πλευρᾶς τοῦ κώνου τῆς μεταξὺ τῶν παραλλήλων ἐπιπέδων καὶ τῆς ἴσης ἀμφοτέραις ταῖς ἐκ τῶν κέντρων τῶν κύκλων τῶν ἐν τοῖς παραλλήλοις ἐπιπέδοις.

"Εστω κώνος, οὗ τὸ διὰ τοῦ ἄξονος τρίγωνον ἢσον τῷ ΑΒΓ, καὶ τετμήσω ταῖς παράλληλω ἐπιπέδῳ τῇ βάσει, καὶ ποιεῖτο τομὴν τῆς ΔΕ, ἀξον ὑπὸ τοῦ κώνου ἐστω ὁ ΒΗ κύκλος δὲ τῆς ἐκκείσθω, οὗ ἢ
scribed in the circle A has to the polygon inscribed in
the circle B a ratio greater than that which the same
polygon [inscribed in the circle A] has to the surface
of the pyramid; therefore the surface of the pyramid
is greater than the polygon inscribed in B. Now
the polygon circumscribed about the circle B has to
the inscribed polygon a ratio less than that which
the circle B has to the surface of the cone; by much
more therefore the polygon circumscribed about the
circle B has to the surface of the pyramid inscribed
in the cone a ratio less than that which the circle B
has to the surface of the cone; which is impossible. a
Therefore the circle is not greater than the surface
of the cone. And it was proved not to be less;
therefore it is equal.

Prop. 16

If an isosceles cone be cut by a plane parallel to the
base, the portion of the surface of the cone between the
parallel planes is equal to a circle whose radius is a mean
proportional between the portion of the side of the cone
between the parallel planes and a straight line equal to
the sum of the radii of the circles in the parallel planes.

Let there be a cone, in which the triangle through
the axis is equal to $AB\Gamma$, and let it be cut by a plane
parallel to the base, and let [the cutting plane] make
the section $\Delta E$, and let $BH$ be the axis of the cone,

For the circumscribed polygon is greater than the
circle B, but the surface of the inscribed pyramid is less than
the surface of the cone [Prop. 12]; the explanation to this
effect in the text is attributed by Heiberg to an interpolator.

\(^1\) τὸ μὲν . . . τοῦ κῶνου om. Heiberg.

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ἐκ τοῦ κέντρου μέση ἀνάλογον ἔστι τῆς τῆς ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΗΛ, ἔστω δὲ κύκλος ὁ Θ.

λέγω, ὅτι ὁ Θ κύκλος ἴσος ἔστι τῇ ἐπιφανείᾳ τοῦ κόνου τῇ μεταξὺ τῶν ΔΕ, ΑΓ.

Ἐκκείσθωσαν γὰρ κύκλου οἱ Λ, Κ, καὶ τοῦ μὲν Κ κύκλου ἦ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΔΖ, τοῦ δὲ Λ ἦ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΑΗ. ὁ μὲν ἄρα Λ κύκλος ἴσος ἔστιν τῇ ἐπιφανείᾳ τοῦ ΑΒΓ κόνου, ὁ δὲ Κ κύκλος ἴσος ἔστι τῇ ἐπιφανείᾳ τοῦ ΔΕΒ. καὶ ἐπει τὸ ὑπὸ τῶν ΒΑ, ΑΗ ἴσου ἔστι τῷ τῇ ὑπὸ τῶν ΒΔ, ΔΖ καὶ τῷ ὑπὸ τῆς ΔΑ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ διὰ τὸ παράλληλον εἶναι τῇ ΔΖ τῇ ΑΗ, ἀλλὰ τὸ μὲν ὑπὸ ΑΒ, ΑΗ δύναται ἦ ἐκ τοῦ κέντρου τοῦ Δ κύκλου, τὸ δὲ ὑπὸ ΒΔ, ΔΖ δύναται ἦ ἐκ τοῦ κέντρου τοῦ Κ κύκλου, τὸ δὲ ὑπὸ τῆς ΔΑ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ δύναται ἦ ἐκ τοῦ κέντρου τοῦ Θ, τὸ ἄρα ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Δ κύκλου ἴσου ἔστι τοῖς ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Κ, Θ κύκλων. ὥστε καὶ ὁ Δ κύκλος ἴσος ἔστι τοῖς Κ, Θ κύκλως.
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and let there be set out a circle whose radius is a mean proportional between $\Delta \Delta$ and the sum of $\Delta Z$, $HA$, and let $\Theta$ be the circle; I say that the circle $\Theta$ is equal to the portion of the surface of the cone between $\Delta E$, $AG$.

For let the circles $\Lambda$, $K$ be set out, and let the square of the radius of $K$ be equal to the rectangle contained by $B\Delta$, $\Delta Z$, and let the square of the radius of $\Lambda$ be equal to the rectangle contained by $BA$, $AH$; therefore the circle $\Lambda$ is equal to the surface of the cone $ABG$, while the circle $K$ is equal to the surface of the cone $\Delta EB$ [Prop. 14]. And since

$$BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot (\Delta Z + AH)$$

because $\Delta Z$ is parallel to $AH$, while the square of the radius of $\Lambda$ is equal to $AB \cdot AH$, the square of the radius of $K$ is equal to $B\Delta \cdot \Delta Z$, and the square of the radius of $\Theta$ is equal to $\Delta \Lambda \cdot (\Delta Z + AH)$, therefore the square on the radius of the circle $\Lambda$ is equal to the sum of the squares on the radii of the circles $K$, $\Theta$; so that the circle $\Lambda$ is equal to the sum of the circles

* The proof is given by Eutocius as follows:

$$BA : AH = B\Delta : \Delta Z$$

$$\therefore BA \cdot \Delta Z = B\Delta \cdot AH.$$  \hspace{1cm} [Eucl. vi. 16]

But $BA \cdot \Delta Z = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$  \hspace{1cm} [Eucl. ii. 1]

$$\therefore B\Delta \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$$  

Let $\Delta \Delta \cdot AH$ be added to both sides.

Then $B\Delta \cdot AH + \Delta \Delta + AH$,

i.e. $BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z + A\Delta \cdot AH.$

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ἄλλ' ὁ μὲν Λ ἰσός ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΒΔΓ κώνου, ὁ δὲ Κ τῇ ἐπιφανείᾳ τοῦ ΔΒΕ κώνου· λοιπῇ ἣ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν παραλλήλων ἐπιπέδων τῶν ΔΕ, ΑΓ ἰση ἐστὶ τῷ Ὁ κύκλῳ.

κα'

'Εάν εἰς κύκλον πολύγωνον ἔγγραφη ἀρτιο-πλευρόν τε καὶ ἰσόπλευρον, καὶ διαχώσων εὐθεῖαι ἐπιζευγνώσαι τὰς πλευρὰς τοῦ πολυγώνου, ὡστε αὐτὰς παραλλήλους εἶναι μιὰ ὄποιαν τῶν ὑπὸ δύο πλευράς τοῦ πολυγώνου ὑποτεινοὺσαν, αἱ ἐπι-ζευγνώσαι πάσαι πρὸς τὴν τοῦ κύκλου διάμετρον τούτον ἔχουσι τῶν λόγων, δν ἔχει ἡ ὑποτεινοῦσα τὰς μιὰς ἐλάσσονας τῶν ἡμίσεων πρὸς τὴν πλευρὰν τοῦ πολυγώνου.

"Εστω κύκλος ο ΑΒΓΔ, καὶ ἐν αὐτῶ πολύγωνον ἔγγεγράφθω τὸ ΑΕΖΒΗΘΟΓΜΝΔΑΚ, καὶ ἐπι-ζευχθῶσαν αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ· δήλον δὴ, ὅτι παράλληλοι εἰσὶν τῇ ὑπὸ δύο πλευράς τοῦ πολυγώνου ὑποτεινοῦσῃ· λέγω σὺν, ὅτι αἱ εἰρημέναι πάσαι πρὸς τὴν τοῦ κύκλου διάμετρον τὴν ΑΓ τοῦ αὐτὸν λόγων ἔχουσι τῷ τῆς ΓΕ πρὸς ΕΑ.

'Επιζευχθῶσαν γὰρ αἱ ΖΚ, ΛΒ, ΗΔ, ΘΝ· παράλληλος ἄρα ἡ μὲν ΖΚ τῇ ΕΑ, ἡ δὲ ΒΛ τῇ ΖΚ, καὶ ἔτι ἡ μὲν ΔΗ τῇ ΒΛ, ἡ δὲ ΘΝ τῇ ΔΗ, καὶ ἡ ΓΜ τῇ ΘΝ [καὶ ἐπεὶ δύο παράλληλοι εἰσὶν αἱ ΕΑ, ΚΖ, καὶ δύο διηγοῦσαι εἰσὶν αἱ ΕΚ, ΑΟ]. ἔστιν ἄρα, ὡς ἡ ΕΞ πρὸς ΕΑ, ὁ ΚΕ πρὸς ΞΟ. ὡς δὴ ἡ ΚΕ πρὸς ΞΟ, ἡ ΖΠ πρὸς ΠΘ, ὡς δὲ

1 καὶ ἐπεὶ ... ΕΚ, ΑΟ om. Heiberg.
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K, Θ. But A is equal to the surface of the cone BAΓ, while K is equal to the surface of the cone ΔΒΕ; therefore the remainder, the portion of the surface of the cone between the parallel planes ΔΕ, ΑΓ, is equal to the circle Θ.

Prop. 21

If a regular polygon with an even number of sides be inscribed in a circle, and straight lines be drawn joining the angles of the polygon, in such a manner as to be parallel to any one whatsoever of the lines subtended by two sides of the polygon, the sum of these connecting lines bears to the diameter of the circle the same ratio as the straight line subtended by half the sides less one bears to the side of the polygon.

Let ΑΒΓΔ be a circle, and in it let the polygon ΑΕΖΒΗΘΓΜΝΔΚ be inscribed, and let ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ be joined; then it is clear that they are parallel to a straight line subtended by two sides of the polygon; I say therefore that the sum of the aforementioned straight lines bears to ΑΓ, the diameter of the circle, the same ratio as ΓΕ bears to ΕΑ.

For let ΖΚ, ΔΒ, ΗΔ, ΘΝ be joined; then ΖΚ is parallel to ΕΑ, BΔ to ΖΚ, also ΔΗ to BΔ, ΘΝ to ΔΗ and ΓΜ to ΘΝ ; therefore

\[ ΕΞ : ΕΑ = ΚΞ : ΞΟ. \]

But

\[ ΚΞ : ΞΟ = ΖΠ : ΠΟ, \]

[Eucl. vi. 4]

- “Sides” according to the text, but Heiberg thinks Archimedes probably wrote γωνίας where we have πλευράς.
- For, because the arcs KA, EZ are equal, \( \angle EKZ = \angle KZA \) [Eucl. iii. 27]; therefore EK is parallel to AZ; and so on.
- For, as the arcs AK, EZ are equal, \( \angle AEK = \angle EKZ \), and therefore AE is parallel to ZK; and so on.
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ἡ ΖΠ πρὸς ΠΩ, ἡ ΛΠ πρὸς ΠΡ, ὡς δὲ ἡ ΛΠ πρὸς ΠΡ, οὕτως ἡ ΒΣ πρὸς ΣΡ, καὶ ἐτί, ὡς ἡ μὲν ΒΣ πρὸς ΣΡ, ἡ ΔΣ πρὸς ΣΤ, ὡς δὲ ἡ ΔΣ πρὸς ΣΤ, ἡ ΗΥ πρὸς ΥΤ, καὶ ἐτί, ὡς ἡ μὲν ΗΥ πρὸς ΥΤ, ἡ ΝΥ πρὸς ΥΦ, ὡς δὲ ἡ ΝΥ πρὸς ΥΦ, ἡ ΘΧ πρὸς ΧΦ, καὶ ἐτί, ὡς μὲν ἡ ΘΧ πρὸς ΧΦ, ἡ ΜΧ πρὸς ΧΓ [καὶ πάντα ἀρα πρὸς πάντα ἐστίν, ὡς εἰς τῶν λόγων πρὸς ἕνα]. ὡς ἄρα ἡ ΕΞ πρὸς ΕΑ, οὕτως αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον. ὡς δὲ ἡ ΕΞ πρὸς ΕΑ, οὕτως ἡ ΓΕ πρὸς ΕΑ· ἐσται ἀρα καὶ, ὡς ἡ ΓΕ πρὸς ΕΑ, οὕτω πάσαι αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον.

1 καὶ ... ἕνα om. Heiberg.
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while $ZP : PO = \Delta P : IP$, [ibid.]
and $\Delta P : IP = BS : SP$, [ibid.]
Again, $BS : SP = \Delta S : ST$, [ibid.]
while $\Delta S : ST = HY : YT$, [ibid.]
Again, $HY : YT = NY : Y\Phi$, [ibid.]
while $NY : Y\Phi = \Theta X : X\Phi$, [ibid.]
Again, $\Theta X : X\Phi = MX : X\Gamma$, [ibid.]
therefore $E\Xi : \Xi A = EK + ZA + BA + HN + \Theta M : \Lambda \Gamma$. [Eucl. v. 12
But $E\Xi : \Xi A = \Gamma E : EA$; [Eucl. vi. 4
therefore $\Gamma E : EA = EK + ZA + BA + HN + \Theta M : \Lambda \Gamma$.

* By adding all the antecedents and consequents, for
$E\Xi : \Xi A = E\Xi + K\Xi + ZP + \Delta P + BS + \Delta S + HY + NY + \Theta X + MX : \Xi A + \Xi O + PO + IP + SP + ST + YT + Y\Phi + X\Phi + X\Gamma$

$= EK + ZA + BA + HN + \Theta M : \Lambda \Gamma$.

* If the polygon has $4n$ sides, then
$\angle \Lambda \Gamma K = \frac{\pi}{2n}$ and $EK : \Lambda \Gamma = \sin \frac{\pi}{2n}$,
$\angle Z \Gamma A = \frac{2\pi}{2n}$ and $ZA : \Lambda \Gamma = \sin \frac{2\pi}{2n}$,
$\angle \Theta \Gamma M = (2n - 1) \frac{\pi}{2n}$ and $\Theta M : \Lambda \Gamma = \sin (2n - 1) \frac{\pi}{2n}$.

Further, $\angle \Lambda \Gamma E = \frac{\pi}{4n}$ and $\Gamma E : EA = \cot \frac{\pi}{4n}$.

Therefore the proposition shows that
$\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n - 1) \frac{\pi}{2n} = \cot \frac{\pi}{4n}$.
κυ’

"Εστω ἐν σφαίρα μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐγγεγράφθω εἰς αὐτὸν πολύγωνον ἴσοπλευρὸν, τὸ δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρεῖσθω ὑπὸ τετράδος, αἱ δὲ ΑΓ, ΔΒ διάμετροι ἔστωσαν. ἐὰν δὴ μενούσης τῆς ΑΓ διαμέτρου περιενεχθῇ ὁ ΑΒΓΔ κύκλος ἔχων τὸ πολύγωνον, δὴ λοι, ὅτι ἡ μὲν περιφέρεια αὐτοῦ κατὰ τῆς ἐπιφανείας τῆς σφαίρας ἐνεχθῆσται, αἱ δὲ τοῦ πολυγώνου γωνίαι χωρίς τῶν πρὸς τὸν Α, Γ σημείοις κατὰ κύκλων περιφερείων ἐνεχθήσονται ἐν τῇ ἐπιφανείᾳ τῆς σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΑΒΓΔ κύκλον· διάμετροι δὲ αὐτῶν ἔσονται αἱ ἐπιζευγγυνοῦσαι τὰς γωνίας τοῦ πολυγώνου παρὰ τὴν ΒΔ ὀδοίᾳ. αἱ δὲ τοῦ πολυγώνου πλευραὶ κατὰ τῶν κώνων ἐνεχθήσονται, αἱ μὲν AZ, AN κατ’ ἐπιφανείας κώνου, οὐ βάσις μὲν ὁ κύκλος ὁ περὶ διαμετροῦ τὴν ΖΝ, κορυφὴ δὲ τὸ αὐτὸν σημεῖον, αἱ δὲ 92
Prop. 23

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and let there be inscribed in it an equilateral polygon, the number of whose sides is divisible by four, and let $A\Gamma, \Delta B$ be diameters. If the diameter $A\Gamma$ remain stationary and the circle $AB\Gamma\Delta$ containing the polygon be rotated, it is clear that the circumference of the circle will traverse the surface of the sphere, while the angles of the polygon, except those at the points $A, \Gamma$, will traverse the circumferences of circles described on the surface of the sphere at right angles to the circle $AB\Gamma\Delta$; their diameters will be the [straight lines] joining the angles of the polygon, being parallel to $B\Delta$. Now the sides of the polygon will traverse certain cones; $AZ, AN$ will traverse the surface of a cone whose base is the circle about the diameter $ZN$ and whose vertex is the point $A$; $ZH,$
ZH, MN κατὰ τινὸς κωνικῆς ἐπιφανείας οἰσθήσονται, ἂς βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΜΗ, κορυφὴ δὲ τὸ σημεῖον, καθ' ὁ συμβάλλομεν ἐκβαλλόμεναι αἱ ZH, MN ἀλλήλαις τε καὶ τῇ ΑΓ, αἱ δὲ BH, MD πλευραὶ κατὰ κωνικῆς ἐπιφανείας οἰσθήσονται, ἂς βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ διάμετρον τὴν BD ὁρθὸς πρὸς τὸν ABΓΔ κύκλον, κορυφὴ δὲ τὸ σημεῖον, καθ' ὁ συμβάλλομεν ἐκβαλλόμεναι αἱ BH, DM ἀλλήλαις τε καὶ τῇ ΓΑ· ὀμοίως δὲ καὶ αἱ ἐν τῷ ἔτερῳ ἡμικυκλίῳ πλευραὶ κατὰ κωνικῶν ἐπιφανειῶν οἰσθήσονται πάλιν ὀμοίων ταύταις. ἔσται δὴ τὰ σχήμα ἐγγεγραμμένον ἐν τῇ σφαῖρᾳ ὑπὸ κωνικῶν ἐπιφανειῶν περιεχόμενον τῶν προειρημένων, οὐ ἡ ἐπιφάνεια ἐλάσσων ἔσται τῆς ἐπιφανείας τῆς σφαῖρας.

Διαπεδεύοις γὰρ τῆς σφαῖρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν BD ὁρθοῦ πρὸς τὸν ABΓΔ κύκλον ἡ ἐπιφάνεια τοῦ ἔτερου ἡμισφαίριον καὶ ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν αὐτῷ ἐγγεγραμμένου τὰ αὐτὰ πέρατα ἔχουσιν ἐν ἐνὶ ἐπιπέδῳ ἀμφοτέρως, γὰρ τῶν ἐπιφανειῶν πέρας ἐστὶν τοῦ κύκλου ἡ περιφέρεια τοῦ περὶ διάμετρον τῆς BD ὁρθοῦ πρὸς τὸν ABΓΔ κύκλον καὶ εἰσὶν ἀμφοτέρως ἐπὶ τὰ αὐτὰ κολλᾶν, καὶ περιλαμβάνεται αὐτῶν ἡ ἔτερα ὑπὸ τῆς ἐτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἐχουσίας αὐτῆς. ὀμοίως δὲ καὶ τοῦ ἐν τῷ ἔτερῳ ἡμισφαίριῳ σχήματος ἡ ἐπιφάνεια ἐλάσσων ἐστὶ τῆς τοῦ ἡμισφαίριον ἐπιφανείας· καὶ ὅλη ὑπὸ τῆς ἐπιφάνειας τοῦ σχήματος τοῦ ἐν τῇ σφαῖρᾳ ἐλάσσων ἐστὶ τῆς ἐπιφανείας τῆς σφαῖρας.

* Archimedes would not have omitted to make the deduc-
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MN will traverse the surface of a certain cone whose base is the circle about the diameter MH and whose vertex is the point in which ZH, MN produced meet one another and with AG; the sides BH, MD will traverse the surface of a cone whose base is the circle about the diameter BA at right angles to the circle AβΓΔ and whose vertex is the point in which BH, DM produced meet one another and with GA; in the same way the sides in the other semicircle will traverse surfaces of cones similar to these. As a result there will be inscribed in the sphere and bounded by the aforesaid surfaces of cones a figure whose surface will be less than the surface of the sphere.

For, if the sphere be cut by the plane through BA at right angles to the circle AβΓΔ, the surface of one of the hemispheres and the surface of the figure inscribed in it have the same extremities in one plane; for the extremity of both surfaces is the circumference of the circle about the diameter BA at right angles to the circle AβΓΔ; and both are concave in the same direction, and one of them is included by the other surface and the plane having the same extremities with it. Similarly the surface of the figure inscribed in the other hemisphere is less than the surface of the hemisphere; and therefore the whole surface of the figure in the sphere is less than the surface of the sphere.

*translation, from Postulate 4, that the surface of the figure inscribed in the hemisphere is less than the surface of the hemisphere.*
'Η τοῦ ἐγγραφομένου σχῆματος εἰς τὴν σφαίραν ἐπιφάνεια ἢστι κύκλω, οὐ δὲ ἐκ τοῦ κέντρου δύναται τὸ περιεχόμενον ὑπὸ τε τῆς πλευρᾶς τοῦ σχῆματος καὶ τῆς ὑστή πάσαις ταῖς ἐπιζευγνωσίαις τὰς πλευρὰς τοῦ πολυγώνου παράλληλοις ὑσταὶ τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτεινοῦσθ᾽ εὐθεῖα.

'Εστὶν ἐν σφαίρα μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθω ἰσόπλευρον, οὐ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπὸ τοῦ πολυγώνου τοῦ ἐγγεγραμμένου νοείσθω τι εἰς τὴν σφαίραν ἐγγραφέν σχῆμα, καὶ ἐπεζευγνωσαν αἱ ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ παράλληλοι οὐσαὶ τῇ ὑπὸ δύο πλευρὰς ὑποτεινοῦσθ᾽ εὐθεία, κύκλος δὲ τῆς ἐκκείσθω ὁ Σ, οὐ δὲ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ τῆς ὑστή ταῖς ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· λέγω, ὅτι ὁ κύκλος οὕτως ἠπὸ ἐστὶ τῇ ἐπιφάνειᾳ τοῦ εἰς τὴν σφαίραν ἐγγραφομένου σχῆματος.

'Εκκείσθωσαν γὰρ κύκλοι οἱ Ω, Π, Ρ, Σ, Τ, Υ, καὶ τοῦ μὲν Ω δὲ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τῆς ΕΑ καὶ τῆς ἡμισειάς τῆς ΕΖ, ή δὲ ἐκ τοῦ κέντρου τοῦ Π δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΕΑ καὶ τῆς ἡμισειάς τῶν ΕΖ, ΗΘ, ΓΔ, ή δὲ ἐκ τοῦ κέντρου τοῦ Ρ δυνάσθω τὸ περιεχόμενον ὑπὸ τῆς ΕΑ καὶ τῆς ἡμισειάς τῶν ΗΘ, ΓΔ, ή δὲ ἐκ τοῦ κέντρου τοῦ Σ δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΕΑ καὶ τῆς ἡμισειάς τῶν ΓΔ, ΚΛ, ή δὲ ἐκ τοῦ κέντρου τοῦ Τ δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ τῆς ἡμισειάς
Prop. 24

The surface of the figure inscribed in the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by the side of the figure and a straight line equal to the sum of the straight lines joining the angles of the polygon, being parallel to the straight line subtended by two sides of the polygon.

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from the inscribed polygon, let there be imagined a figure inscribed in the sphere, and let $EZ$, $HO$, $\Gamma\Delta$, $KA$, $MN$ be joined, being parallel to the straight line subtended by two sides; now let there be set out a circle $\Xi$, the square of whose radius is equal to the rectangle contained by $AE$ and a straight line equal to the sum of $EZ$, $HO$, $\Gamma\Delta$, $KA$, $MN$; I say that this circle is equal to the surface of the figure inscribed in the sphere.

For let the circles $O$, $\Pi$, $P$, $\Sigma$, $T$, $Y$ be set out, and let the square of the radius of $O$ be equal to the rectangle contained by $EA$ and the half of $EZ$, let the square of the radius of $\Pi$ be equal to the rectangle contained by $EA$ and the half of $EZ + HO$, let the square of the radius of $P$ be equal to the rectangle contained by $EA$ and the half of $HO + \Gamma\Delta$, let the square of the radius of $\Sigma$ be equal to the rectangle contained by $EA$ and the half of $\Gamma\Delta + KA$, let the square of the radius of $T$ be equal to the rectangle...
τῶν ΚΑ, ΜΝ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Υ δυνάσθω τὸ περιεχόμενον ύπό τε τῆς ΑΕ καὶ τῆς ἡμισείας τῆς ΜΝ. διὰ δὴ ταύτα ὁ μὲν Ο κύκλος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΑΕΖ κώνου, ὁ δὲ Π τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΕΖ, ΗΘ, ὁ δὲ Ρ τῇ μεταξὺ τῶν ΗΘ, ΓΔ, ὁ δὲ Σ τῇ μεταξὺ τῶν

ΔΓ, ΚΑ, καὶ ἕττα ὁ μὲν Τ ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΚΑ, ΜΝ, ὁ δὲ Υ τῇ τοῦ ΜΒΝ κώνου ἐπιφανείᾳ ἵσος ἐστίν· οἱ πάντες ἀρα κύκλοι ἵσοι εἰσὶν τῇ τοῦ ἐγγεγραμμένου σχῆματος ἐπιφανείᾳ. καὶ φανερῶν, ὅτι αἱ ἕκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ύπό τε τῆς ΑΕ καὶ δις τῶν ἡμισεών τῆς ΕΖ, ΗΘ, ΓΔ, ΚΑ, ΜΝ, αἱ ὅλαι εἰσὶν
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contained by AE and the half of KA + MN, and let the square of the radius of Y be equal to the rectangle contained by AE and the half of MN. Now by these constructions the circle O is equal to the surface of the cone AEZ [Prop. 14], the circle Π is equal to the surface of the conical frustum between EZ and ΗΘ, the circle P is equal to the surface of the conical frustum between ΗΘ and ΓΔ, the circle Σ is equal to the surface of the conical frustum between ΔΓ and KA, the circle T is equal to the surface of the conical frustum between KA, MN [Prop. 16], and the circle Y is equal to the surface of the cone MNB [Prop. 14]; the sum of the circles is therefore equal to the surface of the inscribed figure. And it is manifest that the sum of the squares of the radii of the circles O, Π, P, Σ, T, Y is equal to the rectangle contained by AE and twice the sum of the halves of EZ, ΗΘ, ΓΔ, KA, MN, that is to say, the sum of EZ,
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ai EΩ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. αi ἄρα ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Γ κύκλων δύναται τὸ περιεχόμενον ὑπὸ τὸ τῆς AE καὶ πασῶν τῶν EZ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. ἀλλὰ καὶ ἢ ἐκ τοῦ κέντρου τοῦ Ξ κύκλου δύναται τὸ ὑπὸ τῆς AE καὶ τῆς συγκεκμένης ἐκ πασῶν τῶν EZ, ΗΘ, ΓΔ, ΚΛ, ΜΝ—ἤ ἄρα ἐκ τοῦ κέντρου τοῦ Ξ κύκλου δύναται τὰ ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων· καὶ ὁ κύκλος ἄρα ὁ Ξ ἰσος ἐστὶ τοῖς Ο, Π, Ρ, Σ, Τ, Υ κύκλωι. οἱ δὲ Ο, Π, Ρ, Σ, Τ, Υ κύκλοι ἀπεδείχθησαν ἰσοὶ τῇ εἰρημένῃ τοῦ σχήματος ἐπιφανείᾳ· καὶ ὁ Ξ ἄρα κύκλος ἰσος ἐσται τῇ ἐπιφανείᾳ τοῦ σχήματος.

κε'

Τοῦ ἐγγεγραμμένου σχήματος εἰς τὴν σφαῖραν ἢ ἐπιφάνεια ἢ περιεχόμενη ὑπὸ τῶν κωνικῶν ἐπιφανειῶν ἑλάσσων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαῖρᾳ.

'Εστω ἐν σφαῖρα μέγιστος κύκλος ο ἈΒΓΔ, καὶ ἐν αὐτῷ ἐγγεγράφη πολύγωνον [ἀρτιώμων] ἰσοπλευρον, οὐ αἱ πλευραί ὑπὸ τετράδος μετροῦνται, καὶ ἀπ' αὐτοῦ νοεῖσθω ἐπιφάνεια ἢ ὑπὸ τῶν

* If the radius of the sphere is a this proposition shows that
Surface of inscribed figure = circle Ξ

= \pi \cdot AE \cdot (EZ + \Theta + \Delta + E + MN).

Now AE = 2a \sin \frac{\pi}{4a}, and by p. 91 n. 6
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\[ \text{H} \Theta, \Gamma \Delta, \text{K} \Lambda, \text{M} \text{N} \]; therefore the sum of the squares of the radii of the circles \( O, \Pi, P, \Sigma, T, Y \) is equal to the rectangle contained by \( \text{A} \text{E} \) and the sum of \( \text{EZ}, \text{H} \Theta, \Gamma \Delta, \text{K} \Lambda, \text{M} \text{N} \). But the square of the radius of the circle \( \Xi \) is equal to the rectangle contained by \( \text{A} \text{E} \) and a straight line made up of \( \text{EZ}, \text{H} \Theta, \Gamma \Delta, \text{K} \Lambda, \text{M} \text{N} \) \([ex \ hypothesis]\); therefore the square of the radius of the circle \( \Xi \) is equal to the sum of the squares of the radii of the circles \( O, \Pi, P, \Sigma, T, Y \); and therefore the circle \( \Xi \) is equal to the sum of the circles \( O, \Pi, P, \Sigma, T, Y \). Now the sum of the circles \( O, \Pi, P, \Sigma, T, Y \) was shown to be equal to the surface of the aforesaid figure; and therefore the circle \( \Xi \) will be equal to the surface of the figure.

Prop. 25

The surface of the figure inscribed in the sphere and bounded by the surfaces of cones is less than four times the greatest of the circles in the sphere.

Let \( \text{ABI} \Gamma \Delta \) be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from it, let a surface bounded by surfaces of

\[ \text{EZ} + \text{H} \Theta + \Gamma \Delta + \text{K} \Lambda + \text{MN} = 2a \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \right. \]

\[ \left. \frac{(2n - 1) \pi}{2n} \right]. \]

\[ \therefore \text{Surface of inscribed figure} = 4\pi a^2 \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} \right. \]

\[ \left. + \ldots + \sin \left(2n - 1\right) \frac{\pi}{2n} \right] \]

\[ = 4\pi a^2 \cos \frac{\pi}{4n} \]

[by p. 91 n. b.]

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κωνικών ἐπιφανειῶν περιεχομένη· λέγω, ὅτι ἡ ἐπιφάνεια τοῦ ἐγγραφέντος ἔλασσων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

Ἐπεξεύχθωσαν γὰρ αἱ ὑπὸ δύο πλευρὰς υποτείνουσαι τοῦ πολυγώνου αἱ ΕΙ, ΘΩΜ καὶ ταύταις παράλληλαι αἱ ΖΚ, ΔΒ, ΗΛ, εἰκείσθω δὲ τις κύκλος οὐ Ρ, οὐ ἡ ἐκ τοῦ κέντρου δύναται τὸ ὑπὸ τῆς ΕΑ καὶ τῆς ἴσης πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΩΜ· διὰ δὴ τὸ προδειχθὲν ἵσος ἐστὶν ὁ κύκλος τῇ τοῦ εἰρημένου σχήματος ἐπιφανεία. καὶ ἐπεὶ ἐδείχθη, ὅτι ἐστὶν, ὡς ἡ ἴση πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΩΜ πρὸς τὴν διάμετρον τοῦ κύκλου τῆς ΑΓ, οὕτως ἡ ΓΕ πρὸς ΕΑ, τὸ ἄρα ὑπὸ τῆς ἴσης πάσαις ταῖς εἰρημέναις καὶ τῆς ΕΑ, τούτεστιν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ κύκλου, ἵσον ἐστὶν τῷ ὑπὸ τῶν ΑΓ, ΓΕ. ἀλλὰ καὶ τὸ ὑπὸ ΑΓ, ΓΕ ἐλάσσων ἔστι τοῦ ἀπὸ τῆς ΑΓ· ἐλάσσων ἄρα ἐστὶν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ τοῦ ἀπὸ τῆς ΑΓ· ἐλάσσων ἄρα ἐστὶν ἡ ἐκ τοῦ κέντρου τοῦ Ρ τῆς ΑΓ· ἄστε ἡ διάμετρος τοῦ Ρ κύκλου ἐλάσσων ἐστὶν ἡ διπλασία τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, καὶ δύο ἄρα τοῦ ΑΒΓΔ κύκλου διάμετροι μεῖζον εἰσὶ τῆς διαμέτρου τοῦ Ρ κύκλου, καὶ τὸ τετράκις ἀπὸ τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, τούτεστι τῆς ΑΓ, μεῖζον ἐστὶ τοῦ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου. ὡς δὲ τὸ τετράκις ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου, οὕτως τέσσαρες κύκλοι οἱ ΑΒΓΔ πρὸς τοῦ Ρ κύκλου· τέσσαρες ἄρα κύκλοι οἱ ΑΒΓΔ μεῖζον εἰσὶν τοῦ Ρ κύκλου. τοῦ ἐλάσσων ἐστὶν τῇ τετραπλάσιος τοῦ 1

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1 ἐλάσσων...κύκλου om. Heiberg.
cones be imagined; I say that the surface of the inscribed figure is less than four times the greatest of the circles inscribed in the sphere.

For let $EI, \Theta M$, subtended by two sides of the polygon, be joined, and let $ZK, \Delta B, H\Lambda$ be parallel to them, and let there be set out a circle $P$, the square of whose radius is equal to the rectangle contained by $EA$ and a straight line equal to the sum of $EI, ZK, B\Delta, H\Lambda, \Theta M$; by what has been proved above, the circle is equal to the surface of the aforesaid figure. And since it was proved that the ratio of the sum of $EI, ZK, B\Delta, H\Lambda, \Theta M$ to $A\Gamma$, the diameter of the circle, is equal to the ratio of $GE$ to $EA$ [Prop. 21], therefore

$$EA \cdot (EI + ZK + B\Delta + H\Lambda + \Theta M)$$

that is, the square on the radius of the circle $P$

$$= A\Gamma \cdot GE.$$  \[ \text{ex hyp.} \]

$$[\text{Eucl. vi. 16}]$$

But $$A\Gamma \cdot GE < A\Gamma^2.$$  \[ \text{Eucl. iii. 15} \]

Therefore the square on the radius of $P$ is less than the square on $A\Gamma$; therefore the circle $P$ is less
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μεγιστου κύκλου. ο δὲ Π κύκλος ἵσος ἐδείχθη τῇ εἰρημένη ἐπιφανεία τοῦ σχήματος· ἡ ἀρα ἐπιφάνεια τοῦ σχήματος ἕλάσσων ἐστὶν ἡ τετραπλασία τοῦ μεγιστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

κτ'

"Εστω ἐν σφαίρᾳ μέγιστος κύκλος ο ἈΒΓΔ, περὶ δὲ τὸν ΑΒΓΔ κύκλον περιγεγράφθω πολύγωνον ἰσόπλευρον τε καὶ ἰσογώνιον, τὸ δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρεῖσθω ὑπὸ τετράδος, τὸ δὲ περὶ τὸν κύκλον περιγεγραμμένον πολύγωνον κύκλος περιγεγραμμένος περιλαμβανόμενων περὶ τὸ αὐτὸ κέντρον γυνόμενος τῷ ΑΒΓΔ. μενούσης δὴ τῆς ΕΗ περιενεχθῆτω τὸ ΕΖΗΘ ἐπίπεδον, ἐν ᾧ τὸ τε πολύγωνον καὶ ὁ κύκλος· δήλου οὖν, ὅτι ἡ μὲν περιφέρεια τοῦ ΑΒΓΔ κύκλου κατὰ τῆς ἐπιφανείας τῆς σφαίρας οὐσθήσεται, ἡ δὲ περιφέρεια τοῦ ΕΖΗΘ κατ' ἄλλης ἐπιφανείας σφαίρας τὸ 104.
than four times the greatest circle. But the circle $P$ was proved equal to the aforesaid surface of the figure; therefore the surface of the figure is less than four times the greatest of the circles in the sphere.

Prop. 28

Let $A\Gamma\Delta$ be the greatest circle in a sphere, and about the circle $A\Gamma\Delta$ let there be circumscribed an equilateral and equiangular polygon, the number of whose sides is divisible by four, and let a circle be described about the polygon circumscribing the circle, having the same centre as $A\Gamma\Delta$. While $EH$ remains stationary, let the plane $EZ\Theta$, in which lie both the polygon and the circle, be rotated; it is clear that the circumference of the circle $A\Gamma\Delta$ will traverse the surface of the sphere, while the circumference of $EZ\Theta$ will traverse the surface of another
αὐτὸ κέντρον ἑκούσης τῇ ἐλάσσονι οἰσθῆσεται, αἱ δὲ ἀφαί, καθ' ἂς ἐπιφαίνοντον αἱ πλευραί, γράφοντοι κύκλον ὀρθὸν πρὸς τὸν ΑΒΓΔ κύκλον ἐν τῇ ἐλάσσονι σφαῖρα, αἱ δὲ γωνίαι τοῦ πολυγώνου χωρίς τῶν πρὸς τοῖς Ε, Ἡ σημείως κατὰ κύκλων περιφερεῖσθαι εἰς τῇ ἐπιφανείᾳ τῆς μεῖζονος σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΕΖΗΘ κύκλον, αἱ δὲ πλευραὶ τοῦ πολυγώνου κατὰ κωνικῶν ἐπιφανειῶν οἰσθῆσονται, καθάπερ ἐπὶ τῶν πρὸ τούτου ἔσται οὕν τὸ σχῆμα τὸ περιεχόμενον ὑπὸ τῶν ἐπιφανεῖων τῶν κωνικῶν περὶ μὲν τὴν ἐλάσσονα σφαῖραν περιγεγραμμένον, ἐν δὲ τῇ μείζον ἐγγεγραμμένον. ὅτι δὲ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος μεῖζων ἐστὶ τῆς ἐπιφανείας τῆς σφαίρας, οὕτως δειχθῇ.

"Εστω γὰρ ἡ ΚΔ διάμετρος κύκλου τινὸς τῶν ἐν τῇ ἐλάσσονι σφαίρα τῶν Κ, Δ σημείων ὄντων, καθ' ἂς ἀπτοῦται τὸν ΑΒΓΔ κύκλον αἱ πλευραὶ τοῦ περιγεγραμμένου πολυγώνου. διηρημένης δὴ τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν ΚΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον καὶ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν διαιρεθήσεται ὑπὸ τοῦ ἐπιπέδου. καὶ φανερῶν, ὅτι τὰ αὐτὰ πέρατα ἑξούσιν ἐν ἐπιπέδῳ ἀμφότεροι γὰρ τῶν ἐπιπέδων πέρας ἐστὶν ἡ τοῦ κύκλου περιφέρεια τοῦ περὶ διάμετρον τὴν ΚΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον καὶ εἰσὶν ἀμφότεροι ἐπὶ τὰ αὐτὰ κοίλαι, καὶ περιλαμβάνεται ἡ ἐτέρα αὐτῶν ὑπὸ τῆς ἑτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἑξούσις ἐλάσσουν οὖν ἐστὶν ἡ περιλαμβανομένη τοῦ τριήματος τῆς σφαίρας ἐπιφάνεια τῆς ἐπιφανείας τοῦ σχήματος τοῦ περι-
sphere, having the same centre as the lesser sphere; the points of contact in which the sides touch [the smaller circle] will describe circles on the lesser sphere at right angles to the circle \( \text{ABG} \Delta \), and the angles of the polygon, except those at the points \( \text{E, H} \) will traverse the circumferences of circles on the surface of the greater sphere at right angles to the circle \( \text{EZHO} \), while the sides of the polygon will traverse surfaces of cones, as in the former case; there will therefore be a figure, bounded by surfaces of cones, described about the lesser sphere and inscribed in the greater. That the surface of the circumscribed figure is greater than the surface of the sphere will be proved thus.

Let \( \text{K\Delta} \) be a diameter of one of the circles in the lesser sphere, \( \text{K, D} \) being points at which the sides of the circumscribed polygon touch the circle \( \text{ABG} \Delta \). Now, since the sphere is divided by the plane containing \( \text{K\Delta} \) at right angles to the circle \( \text{ABG} \Delta \), the surface of the figure circumscribed about the sphere will be divided by the same plane. And it is manifest that they \(^a\) have the same extremities in a plane; for the extremity of both surfaces \(^b\) is the circumference of the circle about the diameter \( \text{K\Delta} \) at right angles to the circle \( \text{ABG} \Delta \); and they are both concave in the same direction, and one of them is included by the other and the plane having the same extremities; therefore the included surface of the segment of the sphere is less than the surface of

\(^a\) \text{i.e., the surface formed by the revolution of the circular segment} \( \text{KA\Delta} \) \text{and the surface formed by the revolution of the portion} \( \text{K...E...D} \) \text{of the polygon.}

\(^b\) \text{In the text} \text{ἐπισέδω} \text{should obviously be} \text{ἐπισφανεῖων.}

\(^1\) \text{ai πλευρai Heiberg; om. codd.}
γεγραμμένου περὶ αὐτήν. ὡμοίως δὲ καὶ ἡ τοῦ
λοιποῦ τμῆματος τῆς σφαίρας ἐπιφάνεια ἐλάσσων
ἐστὶν τῆς ἐπιφανείας τοῦ σχήματος τοῦ περι-
γεγραμμένου περὶ αὐτήν: δὴ λοιπὸ ν οὖν, ὅτι καὶ ὅλη ἡ
ἐπιφάνεια τῆς σφαίρας ἐλάσσων ἐστὶ τῆς ἐπιφανείας
toῦ σχήματος τοῦ περιγεγραμμένου περὶ αὐτήν.

κυβέρνησις

Τῇ ἐπιφανείᾳ τοῦ περιγεγραμμένου σχήματος
περὶ τῆς σφαίρας ἴσος ἐστὶ κύκλος, οὐ ἡ ἐκ τοῦ
κέντρου ἴσον δύναται τῷ περιεχομένῳ ὑπὸ τε μᾶς
πλευράς τοῦ πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς
ἐπιζευγνυσίαις τὰς γωνίας τοῦ πολυγώνου οὖσαις
παρὰ τινὰ τῶν ὑπὸ δύο πλευρὰς τοῦ πολυγώνου
ὑποτεινοῦσών.

Τὸ γὰρ περιγεγραμμένον περὶ τῆς ἐλάσσωνा
σφαίραν ἐγγεγραπται εἰς τὴν μείζωνα σφαίραν τοῦ
δὲ ἐγγεγραμμένου ἐν τῇ σφαίρᾳ περιεχομένῳ ὑπὸ
tῶν ἐπιφανειῶν τῶν κωνικῶν δέδεικται ὅτι τῇ
ἐπιφανείᾳ ἴσος ἐστὶν ὁ κύκλος, οὐ ἡ ἐκ τοῦ κέντρου
δύναται τὸ περιεχόμενον ὑπὸ τε μᾶς πλευράς τοῦ
πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνυ-
νύσιαις τὰς γωνίας τοῦ πολυγώνου οὖσαις παρὰ
τινὰ τῶν ὑπὸ δύο πλευρὰς ὑποτεινοῦσών δῆλον
οὖν ἐστὶ τὸ προερημένον.

* If the radius of the inner sphere is \(a\) and that of the outer
spheres \(a'\), and the regular polygon has \(4n\) sides, then

\[ a' = a \sec \frac{\pi}{4n} \]

This proposition shows that

Area of figure circumscribed

Area of figure inscribed in
to circle of radius \(a\)

circle of radius \(a'\)

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the figure circumscribed about it [Post. 4]. Similarly
the surface of the remaining segment of the sphere
is less than the surface of the figure circumscribed
about it; it is clear therefore that the whole surface
of the sphere is less than the surface of the figure
circumscribed about it.

Prop. 29

The surface of the figure circumscribed about the sphere
is equal to a circle, the square of whose radius is equal
to the rectangle contained by one side of the polygon and
a straight line equal to the sum of all the straight lines
joining the angles of the polygon, being parallel to one of
the straight lines subtended by two sides of the polygon.

For the figure circumscribed about the lesser sphere
is inscribed in the greater sphere [Prop. 28]; and it
has been proved that the surface of the figure inscribed
in the sphere and formed by surfaces of cones is equal to a circle, the square of whose radius
is equal to the rectangle contained by one side of the
polygon and a straight line equal to the sum of all
the straight lines joining the angles of the polygon,
being parallel to one of the straight lines subtended
by two sides [Prop. 24]; what was aforesaid is there-
fore obvious.\textsuperscript{a}

\[
=4\pi a^2 \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{(2n-1)\pi}{2n} \right],
\]

or

\[
4\pi a^2 \cos \frac{\pi}{4n} \quad \text{[by p. 91 n. b]}
\]

\[
=4\pi a^2 \sec^2 \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{(2n-1)\pi}{2n} \right],
\]

or

\[
4\pi a^2 \sec \frac{\pi}{4n}.
\]

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Τοῦ σχήματος τοῦ περιγεγραμμένου περὶ τὴν σφαῖραν ἡ ἐπιφάνεια μείζων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαῖρᾳ.

"Εστώ γὰρ ἡ τε σφαῖρα καὶ ὁ κύκλος καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρῶτοι προκειμένους, καὶ ὁ Δ κύκλος ἴσος τῇ ἐπιφάνειᾳ ἐστώ τοῦ προκειμένου περιγεγραμμένου περὶ τῆν ἐλάσσονα σφαῖραν.

'Επεί οὖν ἐν τῷ ΕΖΗΘ ΚΥΚΛΩ πολύγωνον ἰσόπλευρον ἐγγέγραπται καὶ ἄρτιογώνιον, αἱ ἐπιζευγνύουσαι τὰς τοῦ πολυγώνου πλευρὰς παράλληλοι οὐσίᾳ τῇ ΖΘ πρὸς τὴν ΖΘ τῶν αὐτῶν λόγων ἔχουσιν, ὅτι ΘΚ Ἰσόν ἀρα ἐστὶν τὸ περιεχόμενον σχῆμα ὑπὸ τε μίας πλευρᾶς τοῦ πολυγώνου καὶ τῆς ἰσημε ἔλεγχου ταῖς ἐπιζευγνυούσαις τὰς γωνίας τοῦ πολυγώνου τῷ περιεχόμενῳ ὑπὸ τῶν ΖΘΚ· ὅστε ἡ ἐκ τοῦ κέντρου τοῦ Δ κύκλου ἰσον δύναται τῷ ὑπὸ ΖΘΚ· μείζων· ἀρὰ

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Prop. 30

The surface of the figure circumscribed about the sphere is greater than four times the greatest of the circles in the sphere.

For let there be both the sphere and the circle and the other things the same as were posited before, and let the circle $\Lambda$ be equal to the surface of the given figure circumscribed about the lesser sphere.

Therefore since in the circle $\text{EZHO}$ there has been inscribed an equilateral polygon with an even number of angles, the [sum of the straight lines] joining the sides of the polygon, being parallel to $Z\Theta$, have the same ratio to $Z\Theta$ as $\Theta K$ to $KZ$ [Prop. 21]; therefore the rectangle contained by one side of the polygon and the straight line equal to the sum of the straight lines joining the angles of the polygon is equal to the rectangle contained by $Z\Theta$, $\Theta K$ [Eucl. vi. 16]; so that the square of the radius of the circle $\Lambda$ is equal to the rectangle contained by $Z\Theta$, $\Theta K$.
ἐστιν ἡ ἐκ τοῦ κέντρου τοῦ Λ κύκλου τῆς ΘΚ. ἢ δὲ ΘΚ ἴση ἐστὶ τῇ διαμέτρῳ τοῦ ΑΒΓΔ κύκλου [διπλασία γάρ ἐστιν τῆς ΧΣ οὐσίας ἐκ τοῦ κέντρου τοῦ ΑΒΓΔ κύκλου].¹ δὴ λοιπὸν οὖν, ὅτι μεῖζων ἐστὶν ἡ τετραπλάσιος ὁ Λ κύκλος, τούτοισιν ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν ἐλάσσονα σφαῖραν, τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

λγ’

Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλάσια ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.

"Εστω γὰρ σφαίρα τις, καὶ ἐστω τετραπλάσιος τοῦ μεγίστου κύκλου ὁ Λ· λέγω, ὅτι ὁ Λ ἴσος ἐστὶν τῇ ἐπιφάνειᾳ τῆς σφαίρας.

Εἴ γὰρ μῆ, ἦτοι μεῖζων ἐστὶν ἡ ἐλάσσων. ἐστω πρότερον μεῖζων ἡ ἐπιφάνεια τῆς σφαίρας τοῦ κύκλου. ἔστι δὴ δύο μεγέθη ἄνισα ἡ τε ἐπιφάνεια τῆς σφαίρας καὶ ὁ Λ κύκλος· δυνατὸν ἀρα ἐστὶ λαβεῖν δύο ἑυθείας ἄνισους, ὡστε τὴν μείζωνα πρὸς τὴν ἐλάσσονα λόγον ἐχειν ἐλάσσονα τοῦ, ὃν ἔχει ἡ

¹ διπλασία . . . κύκλου om. Heiberg.

* Because ΖΘ > ΘΚ [Eucl. iii. 15].
[Prop. 29]. Therefore the radius of the circle A is greater than ΩK.\(^6\) Now ΩK is equal to the diameter of the circle ABΓΔ; it is therefore clear that the circle A, that is, the surface of the figure circumscribed about the lesser sphere, is greater than four times the greatest of the circles in the sphere.

**Prop. 33**

The surface of any sphere is four times the greatest of the circles in it.

For let there be a sphere, and let A be four times the greatest circle; I say that A is equal to the surface of the sphere.

For if not, either it is greater or less. First, let the surface of the sphere be greater than the circle.

Then there are two unequal magnitudes, the surface of the sphere and the circle A; it is therefore possible to take two unequal straight lines so that the greater bears to the less a ratio less than that which the sur-
ἐπιφάνεια τῆς σφαίρας πρὸς τὸν κύκλον. εἰλήφθωσαν αἱ Β, Γ, καὶ τῶν Β, Γ μέση ἀνάλογον ἔστω ἡ Δ, νοεῖσθω δὲ καὶ ἡ σφαῖρα ἐπιπέδω τετραμένη διὰ τοῦ κέντρου κατὰ τὸν ΕΖΗΘ κύκλον, νοεῖσθω δὲ καὶ εἰς τὸν κύκλον ἐγγεγραμμένον καὶ περιγεγραμμένον πολύγωνον, ὡστε ὁμοιον εἶναι τὸ περιγεγραμμένον τῷ ἐγγεγραμμένῳ πολύγωνῳ καὶ τὴν τοῦ περιγεγραμμένου πλευρὰν ἑλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ἡ Β πρὸς Δ [καὶ ὁ διπλάσιος ἀρά λόγος τοῦ διπλασίου λόγου ἐστὶν ἑλάσσων. καὶ τοῦ μὲν τῆς Β πρὸς Δ διπλασίος ἐστὶν ὁ τῆς Β πρὸς τὴν Γ, τῆς δὲ πλευρᾶς τοῦ περιγεγραμμένου πολύγωνου πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμμένου διπλάσιος ὁ τῆς ἐπιφάνειας τοῦ περιγεγραμμένου στερεοῦ πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου]¹. ἡ ἐπιφάνεια ἀρά τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου σχήματος ἑλάσσονα λόγον ἔχει ἤπερ ἡ ἐπιφάνεια τῆς σφαίρας πρὸς τὸν Α κύκλον· ὁπερ ἀτοποῦ· ἡ μὲν γὰρ ἐπιφάνεια τοῦ περιγεγραμμένου τῆς ἐπιφάνειας τῆς σφαίρας μεῖζων ἐστὶν, ἡ δὲ ἐπιφάνεια τοῦ ἐγγεγραμμένου σχήματος τοῦ Α κύκλου ἑλάσσων ἐστὶ [δεδεικται γὰρ ἡ ἐπιφάνεια τοῦ ἐγγεγραμμένου ἑλάσσων τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαῖρα τῇ τετραπλάσια, τοῦ δὲ μεγίστου κύκλου τετραπλάσιος ἐστιν ὁ Α κύκλος].² οὐκ ἀρα ἡ ἐπιφάνεια τῆς σφαίρας μεῖζων ἐστὶ τοῦ Α κύκλου.

¹ καὶ ... ἐγγεγραμμένου om. Heiberg.
² δεδεικται ... κύκλος “repetitionem inutilem Prop. 25,” om. Heiberg.

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face of the sphere bears to the circle [Prop. 2]. Let
B, Γ be so taken, and let Δ be a mean proportional
between B, Γ, and let the sphere be imagined as cut
through the centre along the [plane of the] circle
EZΗΘ, and let there be imagined a polygon inscribed
in the circle and another circumscribed about it in
such a manner that the circumscribed polygon is
similar to the inscribed polygon and the side of the
circumscribed polygon has [to the side of the inscribed
polygon] a ratio less than that which B has to Δ
[Prop. 3]. Therefore the surface of the figure cir-
cumscribed about the sphere has to the surface of
the inscribed figure a ratio less than that which the
surface of the sphere has to the circle A; which is
absurd; for the surface of the circumscribed figure
is greater than the surface of the sphere [Prop. 28],
while the surface of the inscribed figure is less than
the circle A [Prop. 25]. Therefore the surface of the
sphere is not greater than the circle A.

* Archimedes would not have omitted: πρὸς τὴν τοῦ ἐγγε-
γραμμένου.
Λέγω δὴ, ὅτι οὕτω ἐλάσσων· εἰ γὰρ δυνατόν, ἔστω· καὶ ὁμοίως εὑρήσθωσαν αἰ Β, Γ εὐθείαι, ὥστε τὴν Β πρὸς Γ ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαῖρας, καὶ τῶν Β, Γ μέση ἀνάλογον ἡ Δ, καὶ ἐγγεγράφθω καὶ περιγεγράφθω πάλιν, ὥστε τὴν τοῦ περιγεγραμμένου ἐλάσσονα λόγον ἔχειν τοῦ τῆς Β πρὸς Δ [καὶ τὰ διπλάσια ἀρα]. 1 ἡ ἐπιφανεία ἄρα τοῦ περιγεγραμμένου πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἔχει ἦπερ [ἡ Β πρὸς Γ. ἡ δὲ Β πρὸς Γ ἐλάσσονα λόγον ἔχει ἦπερ] 2 ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαῖρας· ὁπερ ἀτοπον ἡ μὲν γὰρ τοῦ περιγεγραμμένου ἐπιφάνεια μείζων ἐστὶ τοῦ Α κύκλου, ἡ δὲ τοῦ ἐγγεγραμμένου ἐλάσσων τῆς ἐπιφανείας τῆς σφαῖρας.

Οὕτω ἄρα οὕτω ἐλάσσων ἡ ἐπιφάνεια τῆς σφαῖρας τοῦ Α κύκλου. ἔδειξθη δὲ, ὅτι οὕτω μείζων ἡ ἄρα ἐπιφάνεια τῆς σφαῖρας ἢ ἐστὶ τῷ Α κύκλῳ, τούτῳ τῷ τετραπλασίῳ τοῦ μεγίστου κύκλου.

1 καὶ . . . ἄρα om. Heiberg.
2 ἡ Β . . . ἦπερ om. Heiberg.

* Archimedes would not have omitted these words.
* On p. 100 n. a it was proved that the area of the inscribed figure is

\[4\pi a^2 \sin \frac{\pi}{n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(2n - 1\right) \frac{\pi}{2n} \right],\]

or \(4\pi a^2 \cos \frac{\pi}{4n}\).

On p. 108 n. a it was proved that the area of the circumscribed figure is

\[4\pi a^2 \sec \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(2n - 1\right) \frac{\pi}{2n} \right],\]

or \(4\pi a^2 \sec \frac{\pi}{4n}\).
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I say now that neither is it less. For, if possible let it be; and let the straight lines B, Γ' be similarly found, so that B has to Γ a less ratio than that which the circle A has to the surface of the sphere, and let Δ be a mean proportional between B, Γ', and let [polygons] be again inscribed and circumscribed, so that the [side] of the circumscribed polygon has [to the side of the inscribed polygon] a less ratio than that of B to Δ; then the surface of the circumscribed polygon has to the surface of the inscribed polygon a ratio less than that which the circle A has to the surface of the sphere; which is absurd; for the surface of the circumscribed polygon is greater than the circle A, while that of the inscribed polygon is less than the surface of the sphere.

Therefore the surface of the sphere is not less than the circle A. And it was proved not to be greater; therefore the surface of the sphere is equal to the circle A, that is to four times the greatest circle.

When n is indefinitely increased, the inscribed and circumscribed figures become identical with one another and with the circle, and, since \( \cos \frac{\pi}{4n} \) and \( \sec \frac{\pi}{4n} \) both become unity, the above expressions both give the area of the circle as \( 4\pi a^2 \).

But the first expressions are, when n is indefinitely increased, precisely what is meant by the integral

\[
4\pi a^2 \cdot \int_0^\pi \sin \phi \, d\phi,
\]

which is familiar to every student of the calculus as the formula for the area of a sphere and has the value \( 4\pi a^2 \).

Thus Archimedes' procedure is equivalent to a genuine integration, but when it comes to the last stage, instead of saying, "Let the sides of the polygon be indefinitely
Πάσα σφαίρα τετραπλασιά ἐστὶ κῶνον τοῦ βάσιν μὲν ἑχοντος ἵσην τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὡς ὅ ἐν τῇ ἐκ τοῦ κέντρου τῆς σφαίρας.

"Εστώ γὰρ σφαίρα τίς καὶ ἐν αὐτῇ μέγιστος κύκλος ὁ ABΓΔ. ἐἰ οὖν μὴ ἐστὶν ἡ σφαίρα τε-

τραπλασιὰ τοῦ εἰρημένου κώνου, ἐστὼ, εἰ δυνατὸν, μεῖζων ἡ τετραπλασιὰ· ἐστὼ δὲ ὁ Σ κώνος βάσιν μὲν ἑχων τετραπλασιάν τοῦ ABΓΔ κύκλου, ὡς ὅ ἐσον τῇ ἐκ τοῦ κέντρου τῆς σφαίρας· μεῖζων οὖν ἐστὶν ἡ σφαίρα τοῦ Σ κώνου. ἔσται δὴ δύο μεγέθη ἀνίσα ἡ τε σφαίρα καὶ ὁ κώνος· δυνατὸν οὖν δύο εὐθείας λαβεῖν ἀνίσους, ὡστε ἐχεῖν τὴν

increased," he prefers to prove that the area of the sphere cannot be either greater or less than $4\pi a^2$. By this double reductio ad absurdum he avoids the logical difficulties of dealing with indefinitely small quantities, difficulties that were not fully overcome until recent times.

The procedure by which in this same book Archimedes
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Prop. 34

Any sphere is four times as great as the cone having a base equal to the greatest of the circles in the sphere and height equal to the radius of the sphere.

For let there be a sphere in which $AB\Gamma\Delta$ is the greatest circle. If the sphere is not four times the

aforesaid cone, let it be, if possible, greater than four times; let $\Xi$ be a cone having a base four times the circle $AB\Gamma\Delta$ and height equal to the radius of the sphere; then the sphere is greater than the cone $\Xi$. Accordingly there will be two unequal magnitudes, the sphere and the cone; it is therefore possible to take two unequal straight lines so that finds the surface of the segment of a sphere is equivalent to the integration

$$\pi a^2 \int_{\alpha}^{\pi} 2 \sin \theta d\theta = 2\pi a^2 (1 - \cos \alpha).$$

Concurrently Archimedes finds the volumes of a sphere and segment of a sphere. He uses the same inscribed and circumscribed figures, and the procedure is equivalent to multiplying the above formulae by $\frac{1}{4} a$ throughout. Other "integrations" effected by Archimedes are the volume of a segment of a paraboloid of revolution, the volume of a segment of a hyperboloid of revolution, the volume of a segment of a spheroid, the area of a spiral and the area of a segment of a parabola. He also finds the area of an ellipse, but not by a method equivalent to integration. The subject is fully treated by Heath, The Works of Archimedes, pp. cxlii-cliv, to whom I am much indebted in writing this note.
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μείζονα πρὸς τὴν ἐλάσσονα ἐλάσσονα λόγον τοῦ, ὅν ἔχει ἡ σφαῖρα πρὸς τὸν Σ κῶνον. ἔστωσαν οὖν αἱ Κ, Η, αἱ δὲ Ι, Θ εἰλημμέναι, ὡστε τῷ ἀπὸ ἀλλήλων ὑπερέχειν τὴν Κ τῆς Ι καὶ τῆς Τ τῆς Θ καὶ τῆς Η, νοεῖσθω δὲ καὶ εἰς τὸν ΑΒΓΔ κύκλον ἐγγεγραμμένον πολύγωνον, οὐ τὸ πλήθος τῶν πλευρῶν μετρεῖσθω ὑπὸ τετράδος, καὶ ἀλλὸ περιγεγραμμένον ὁμοιὸν τῷ ἐγγεγραμμένῳ, καθάπερ ἐπὶ τῶν πρῶτων, ἢ δὲ τοῦ περιγεγγραμμένου πολυγώνου πλευρὰ πρὸς τὴν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἔχετω τοῦ, ὅν ἔχει ἡ Κ πρὸς Ι, καὶ ἔστωσαν αἱ ΑΓ, ΒΔ διάμετροι πρὸς ὅρθας ἀλλήλαις. εἰ δὲν μενοῦσας τῆς ΑΓ διαμέτρου περιενεχθεῖν τὸ ἐπίπεδον, ἐν ὧ τὰ πολύγωνα, ἔσται σχήματα τὸ μὲν ἐγγεγραμμένον ἐν τῇ σφαίρᾳ, τὸ δὲ περιγεγραμμένον, καὶ ἔξει τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον τριπλασίονα λόγον ἥπερ ἡ πλευρὰ τοῦ περιγεγγραμμένον πρὸς τὴν τοῦ ἐγγεγραμμένου εἰς τὸν ΑΒΓΔ κύκλον. ἡ δὲ πλευρὰ πρὸς τὴν πλευρὰν ἐλάσσονα λόγον ἔχει ἥπερ ἡ Κ πρὸς τὴν Ι. ὡστε τὸ σχῆμα τὸ περιγεγγραμμένον ἐλάσσονα λόγον ἔχει ἡ τριπλασίωνα τοῦ Κ πρὸς Ι.

1 σχήματα Heiberg, τὸ σχῆμα codd.

* Eutocius supplies a proof on these lines. Let the lengths of K, I, Θ, H be a, b, c, d. Then a - b = b - c = c - d, and it is required to prove that a : d > a² : b².

Take x such that

\[ a : b = b : x. \]

Then

\[ a - b : a = b : x : b. \]

and since \(a > b\),

\[ a - b > b - x. \]

But, by hypothesis,

\[ a - b = b - c. \]

Therefore

\[ b - c > b - x, \]

and so

\[ x > c. \]
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the greater will have to the less a less ratio than that which the sphere has to the cone Ξ. Therefore let the straight lines K, H, and the straight lines I, θ, be so taken that K exceeds I, and I exceeds θ and θ exceeds H by an equal quantity; let there be imagined inscribed in the circle ABΓΔ a polygon the number of whose sides is divisible by four; let another be circumscribed similar to that inscribed so that, as before, the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that K : I; and let ΑΓ, ΒΔ be diameters at right angles. Then if, while the diameter ΑΓ remains stationary, the surface in which the polygons lie be revolved, there will result two [solid] figures, one inscribed in the sphere and the other circumscribed, and the circumscribed figure will have to the inscribed the triplicate ratio of that which the side of the circumscribed figure has to the side of the figure inscribed in the circle ABΓΔ [Prop. 32]. But the ratio of the one side to the other is less than K : I [ex hypothesi]; and so the circumscribed figure has [to the inscribed] a ratio less than K³ : I³. But a K : H > K³ : I³; by much more there-

Again, take y such that \( b : z = x : y \).
Then, as before \( b - x > x - y \).
Therefore, a fortiori, \( b - e > x - y \).
But, by hypothesis, \( b - e = c - d \).
Therefore \( e - d > x - y \).
But \( x > c \),
and so \( y > d \).
But, by hypothesis, \( a : b = b : x = x : y \).
\( a : y = a^2 : b^3 \) [Eucl. v. Def. 10, also vol. i. p. 258 n. b.]

Therefore \( a : d > a^2 : b^3 \).

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ἔχει δὲ καὶ ἡ Κ πρὸς Η μεῖζον λόγον ἡ τριπλάσιον τοῦ, ὅπως ἐχει ἡ Κ πρὸς Ι [τούτω γὰρ φανερῶν διὰ λημμάτων]

πολλῷ ἀρα τὸ περιγραφέν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἐχει τοῦ, ὅπως ἐχει ἡ Κ πρὸς Η. ἡ δὲ Κ πρὸς Η ἐλάσσονα λόγον ἐχει ἦπερ ἡ σφαίρα πρὸς τὸν Σ κώνον: καὶ ἐναλλαξὶ ὡστε ἀδύνατον τὸ γὰρ σχῆμα τὸ περιγεγραμμένον μεῖζον ἐστὶ τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον ἐλάσσον τοῦ Σ κώνον [διὸς ὦ μὲν Σ κώνος τετραπλάσιός ἐστι τοῦ κώνοι τοῦ βάσιν μὲν ἐχοντος ἢσην τῷ ΑΒΓΔ κύκλῳ, ύψος δὲ ἤσον τῇ ἑκ τοῦ κέντρου τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον σχῆμα ἐλάσσον τοῦ εἰρημένου κώνον ἡ τετραπλάσιον].

οὐκ ἀρα μεῖζον ἡ τετραπλάσια ἡ σφαίρα τοῦ εἰρημένου.

"Εστώ, εἰ δύνατον, ἐλάσσον ἡ τετραπλάσια ὡστε ἐλάσσων ἔστιν ἡ σφαίρα τοῦ Σ κώνον. εἰ-ληψθοσαν δὴ αἱ Κ, Η εὐθεῖαι, ὡστε τὴν Κ μεῖζον εἶναι τῆς Η καὶ ἐλάσσονα λόγον ἔχειν πρὸς αὐτήν τοῦ, ὅπως ἐχει ὁ Σ κώνος πρὸς τὴν σφαίραν, καὶ αἱ Θ, Ι ἐκκεισθοσαν, καθὼς πρότερον, καὶ εἰς τὸν ΑΒΓΔ κύκλων νοείσθω πολύγωνον ἐγγεγραμμένον καὶ ἀλλο περιγεγραμμένον, ὡστε τὴν πλευρὰν τοῦ περιγεγραμμένου πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμ-μένου ἐλάσσονα λόγον ἔχειν ἦπερ ἡ Κ πρὸς Ι, καὶ τὰ ἀλλα κατεσκευασμένα τοῦ αὐτὸν τρόπον τοῖς πρότερον ἔχει ἀρα καὶ τὸ περιγεγραμμένον στερεών σχῆμα πρὸς τὸ ἐγγεγραμμένον τριπλᾶσιονα λόγον ἦπερ ἡ πλευρὰ τοῦ περιγεγραμμένου περὶ τὸν ΑΒΓΔ κύκλων πρὸς τὴν τοῦ ἐγγεγραμμένου. ἡ δὲ πλευρὰ πρὸς τὴν πλευρὰν ἐλάσσονα λόγον ἔχει
fore the circumscribed figure has to the inscribed a ratio less than $K : H$. But $K : H$ is a ratio less than that which the sphere has to the cone $\Xi$ [ex hypothesi]; [therefore the circumscribed figure has to the inscribed a ratio less than that which the sphere has to the cone $\Xi$]; and permutando, [the circumscribed figure has to the sphere a ratio less than that which the inscribed figure has to the cone] $^a$; which is impossible; for the circumscribed figure is greater than the sphere [Prop. 28], but the inscribed figure is less than the cone $\Xi$ [Prop. 27]. Therefore the sphere is not greater than four times the aforesaid cone.

Let it be, if possible, less than four times, so that the sphere is less than the cone $\Xi$. Let the straight lines $K$, $H$ be so taken that $K$ is greater than $H$ and $K : H$ is a ratio less than that which the cone $\Xi$ has to the sphere [Prop. 2]; let the straight lines $\Theta$, $I$ be placed as before; let there be imagined in the circle $\Delta \Gamma \Delta$ one polygon inscribed and another circumscribed, so that the side of the circumscribed figure has to the side of the inscribed a ratio less than $K : I$; and let the other details in the construction be done as before. Then the circumscribed solid figure will have to the inscribed the triplicate ratio of that which the side of the figure circumscribed about the circle $\Delta \Gamma \Delta$ has to the side of the inscribed figure [Prop. 32]. But the ratio of the sides

$^a$ A marginal note in one ms. gives these words, which Archimedes would not have omitted.

$^1$ τούτο . . . λημμάτων om. Heiberg.
$^2$ διότι . . . τετραπλάδου om. Heiberg.
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η'περ η' Κ προς Ι. έξει οὖν τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἡ τριπλάσιον τοῦ, οὖν ἔχει η' Κ προς τὴν Ι. ἡ δὲ Κ προς τὴν Η μεῖζον λόγον ἔχει η' τριπλάσιον τοῦ, οὖν ἔχει η' Κ προς τὴν Ι. ὡστε ἐλάσσονα λόγον ἔχει τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἡ' Κ προς τὴν Η. ἡ δὲ Κ προς τὴν ἐλάσσονα λόγον ἔχει η' ὁ Σ κώνος πρὸς τὴν σφαίραν ὀπερ ἀδύνατον τὸ μὲν γὰρ ἐγγεγραμμένον ἐλασσόν ἐστὶ τῆς σφαίρας, τὸ δὲ περιγεγραμμένον μεῖζον τοῦ Σ κώνου. οὐκ ἀρα οὐδὲ ἐλάσσων ἐστὶν ἡ τετραπλασία ἡ σφαίρα τοῦ κώνου τοῦ βάσιν μὲν ἐχοντος ἵσην τῷ ΑΒΓΔ κύκλῳ, ὡσος δὲ τὴν ἵσην τῇ ἐκ τοῦ κέντρου τῆς σφαίρας. ἐδείχθη δὲ, ὅτι οὐδὲ μεῖζων τετραπλασία ἀρα.

[Πόρισμα]¹

Προδεδευμένων δὲ τούτων φανερῶν, ὅτι πᾶς κύλινδρος βάσιν μὲν ἔχων τὸν μέγιστον κύκλον τῶν ἐν τῇ σφαίρᾳ, ὡσος δὲ ἵσην τῇ διαμέτρῳ τῆς σφαίρας, ἥμισυός ἐστι τῆς σφαίρας καὶ ἡ ἐπιφάνεια αὐτοῦ μετὰ τῶν βάσεων ἡμιολία τῆς ἐπιφάνειας τῆς σφαίρας.

'Ο μὲν γὰρ κύλινδρος ὁ προειρημένος ἐξαπλάσιος ἐστὶ τοῦ κώνου τοῦ βάσιν μὲν ἐχοντος τὴν αὐτήν, ὡσος δὲ ἵσην τῇ ἐκ τοῦ κέντρου, ἡ δὲ σφαίρα δεδεικταὶ τοῦ αὐτοῦ κώνου τετραπλασία οὐδαμῶς ἰσθον οὖν, ὅτι ὁ κύλινδρος ἡμιολίος ἐστι τῆς σφαίρας. πάλιν, ἐπεὶ ἡ ἐπιφάνεια τοῦ κυλίνδρου χωρὶς τῶν βάσεων ἵση δεδεικταὶ κύκλῳ, οὐ ἦκ

¹ πόρισμα. The title is not found in some mss.
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is less than $K : I$ [ex hypothesi]; therefore the circumscribed figure has to the inscribed a ratio less than $K^3 : I^3$. But $K : H > K^3 : I^3$; and so the circumscribed figure has to the inscribed a ratio less than $K : H$. But $K : H$ is a ratio less than that which the cone $\Xi$ has to the sphere [ex hypothesi]; [therefore the circumscribed figure has to the inscribed a ratio less than that which the cone $\Xi$ has to the sphere] $^a$; which is impossible; for the inscribed figure is less than the sphere [Prop. 28], but the circumscribed figure is greater than the cone $\Xi$ [Prop. 31, coroll.]. Therefore the sphere is not less than four times the cone having its base equal to the circle $AB\Gamma\Delta$, and height equal to the radius of the sphere. But it was proved that it cannot be greater; therefore it is four times as great.

[Corollary]

From what has been proved above it is clear that any cylinder having for its base the greatest of the circles in the sphere, and having its height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface including the bases is one-and-a-half times the surface of the sphere.

For the aforesaid cylinder is six times the cone having the same basis and height equal to the radius [from Eucl. xii. 10], while the sphere was proved to be four times the same cone [Prop. 34]. It is obvious therefore that the cylinder is one-and-a-half times the sphere. Again, since the surface of the cylinder excluding the bases has been proved equal to a circle

$^a$ These words, which Archimedes would not have omitted, are given in a marginal note to one ms.
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tou kentrou meosi analoignon esti tis tou kulindrou pleurais kai tis diametrou tis basewos, tou d' eirhmeinou kulindrou tou peri twn sphairon h pleura isis esti tis diametrou tis basewos [deioun, oti h meosi autwn analoignon isis ginetai tis diametrou tis basewos], 1 o d' kiklos o twn ek tou kentrou exwn isis tis diametrou tis basewos tetraplaosi esti tis basewos, toutestoi tou megiston kiklou twon en tis sphaira, estai ara kai h epifaneia tou kulindrou xoris twon basewon tetraplasia tou megiston kiklou. Oly ara metata twon basewon h epifaneia tou kulindrou exaplasia estai tou megiston kiklou. Estin d' kai h tis sphairas epifaneia tetraplasia tou megiston kiklou. Oly ara h epifaneia tou kulindrou hmiolia esti tis epifaneias tis sphairas.

(c) Solution of a Cubic Equation

Archim. De Sphaera et Cyl. ii., Prop. 4, Archim. ed. Heiberg i. 186. 15-192. 6

Tis dotheisan sphairon tenein, osoete ta trimata tis sphairas pros allhla logon exein ton auton tou dotheini.

1 deioun . . . basewos om. Heiberg.

As the geometrical form of proof is rather diffuse, and may conceal from the casual reader the underlying nature of the operation, it may be as well to state at the outset the various stages of the proof. The problem is to cut a given sphere by a plane so that the segments shall have a given ratio, and the stages are:

(a) Analysis of this main problem in which it is reduced to a particular case of the general problem, "so to cut a given straight line AZ at X that XZ bears to the given
whose radius is a mean proportional between the side of the cylinder and the diameter of the base [Prop. 13], and the side of the aforementioned cylinder circumscribing the sphere is equal to the diameter of the base, while the circle having its radius equal to the diameter of the base is four times the base [Eucl. xii. 2], that is to say, four times the greatest of the circles in the sphere, therefore the surface of the cylinder excluding the bases is four times the greatest circle; therefore the whole surface of the cylinder, including the bases, is six times the greatest circle. But the surface of the sphere is four times the greatest circle. Therefore the whole surface of the cylinder is one-and-a-half times the surface of the sphere.

(c) Solution of a Cubic Equation

Archimedes, On the Sphere and Cylinder ii., Prop. 4, Archim. ed. Heiberg i. 186. 15-192. 6

To cut a given sphere, so that the segments of the sphere shall have, one towards the other, a given ratio. a

straight line the same ratio as a given area bears to the square on \( \Delta X \); in algebraical notation, to solve the equation

\[
\frac{a - x}{b} = \frac{c^2}{x^2}, \text{ or } x^2(a - x) = bc^2.
\]

(b) Analysis of this general problem, in which it is shown that the required point can be found as the intersection of a parabola \([ax^2 = c^2y]\) and a hyperbola \([(a - x)y = ab]\). It is stated, for the time being without proof, that \(x^2(a - x)\) is greatest when \(x = \frac{a}{2}\); in other words, that for a real solution \(bc^2 > \frac{1}{2}a^2\).

(c) Synthesis of this general problem, according as \(bc^2\) is greater than, equal to, or less than \(\frac{1}{2}a^2\). If it be greater, there is no real solution; if equal, there is one real solution; if less, there are two real solutions.

(d) Proof that \(x^2(a - x)\) is greatest when \(x = \frac{a}{2}\), deferred
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*Εστώ ἡ δοθείσα σφαῖρα ἡ ΑΒΓΔ· δεῖ δὴ αὐτὴν
tetmēain επιπέδῳ, ὥστε τὰ τμῆμα τῆς σφαίρας
pros ἀλληλα λόγον έχειν τὸν δοθέντα.

Τετμήσθω διὰ τῆς ΑΓ επιπέδῳ· λόγος ἀρα τοῦ
ΔΔΓ τμῆμας τῆς σφαίρας πρὸς τὸ ΑΒΓ τμῆμα
τῆς σφαίρας δοθείς. τετμήσθω δὲ ἡ σφαίρα διὰ
τοῦ κέντρου, καὶ ἔστω ἡ τομὴ μέγιστος κύκλος ὁ
ΑΒΓΔ, κέντρον δὲ τὸ Κ καὶ διάμετρος ἡ ΔΒ, καὶ
πεποιήσθω, ὡς μὲν συναμφότερος ἡ ΚΔΧ πρὸς
ΔΧ, οὕτως ἡ ΡΧ πρὸς ΧΒ, ὡς δὲ συναμφότερος
ἡ ΚΒΧ πρὸς ΒΧ, οὕτως ἡ ΛΧ πρὸς ΧΔ, καὶ
ἐπεξεύθυναν αἱ ΑΑ, ΛΓ, ΑΡ, ΡΓ· ἵσος ἄρα ἔστιν ὁ
μὲν ΔΔΓ κώνος τῶν ΔΔΓ τμῆμα τῆς
σφαίρας, ὁ δὲ ARG τῶν AΒΓ· λόγος ἀρα καὶ τοῦ
ΔΔΓ κώνου πρὸς τὸν AΡΓ κώνον δοθείς.
ográf 

οὗτος ἡ ΛΧ πρὸς ΧΡ [ἐπείπερ τὴν αὐτὴν βάσιν ἔχουσιν τὸν περὶ
diάμετρον τῆς AΓ κύκλον].· λόγος ἀρα καὶ τῆς
ΔΧ πρὸς ΧΡ δοθείς. καὶ διὰ ταύτα τοῖς πρό-

*ἐπείπερ . . . κύκλον om. Heiberg.

in (b). This is done in two parts, by showing that (1) if \( x \) has
any value less than \( \frac{4}{3} a \), (2) if \( x \) has any value greater than \( \frac{4}{3} a \),
then \( x^2(a - x) \) has a smaller value than when \( x = \frac{4}{3} a \).

(e) Proof that, if \( b o^2 < \frac{4}{3} a^3 \), there are always two real
solutions.

(f) Proof that, in the particular case of the general problem
to which Archimedes has reduced his original problem, there
is always a real solution.

(g) Synthesis of the original problem.

Of these stages, (a) and (g) alone are found in our texts of
Archimedes; but Eutocius found stages (b)-(d) in an old
book, which he took to be the work of Archimedes; and
he added stages (e) and (f) himself. When it is considered
that all these stages are traversed by rigorous geometrical

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Let $AB\Gamma\Delta$ be the given sphere; it is required so to cut it by a plane that the segments of the sphere shall have, one towards the other, the given ratio.

Let it be cut by the plane $\Lambda \Gamma'$; then the ratio of the segment $\Lambda\Delta\Gamma'$ of the sphere to the segment $AB\Gamma'$ of the sphere is given. Now let the sphere be cut through the centre [by a plane perpendicular to the plane through $\Lambda \Gamma'$], and let the section be the great circle $AB\Gamma\Delta$ of centre $K$ and diameter $\Delta B$, and let $[\Lambda, P$ be taken on $B\Delta$ produced in either direction so that]

$$KA + AX : AX = PX : XB,$$

$$KB + BX : BX = AX : XD,$$

and let $\Lambda \Delta, \Lambda \Gamma, \Lambda P, \Delta P$ be joined; then the cone $\Lambda \Delta \Gamma$ is equal to the segment $\Lambda \Delta \Gamma'$ of the sphere, and

the cone $\Lambda \Gamma \Pi$ to the segment $AB\Gamma$ [Prop. 2]; therefore the ratio of the cone $\Lambda \Delta \Gamma$ to the cone $\Lambda \Pi \Gamma'$ is given. But cone $\Lambda \Delta \Gamma :$ cone $\Lambda \Pi \Gamma' = AX : XP$.\(^{a}\)

Therefore the ratio $AX : XP$ is given. And in the methods, the solution must be admitted a veritable tour de force. It is strictly analogous to the modern method of solving a cubic equation, but the concept of a cubic equation did not, of course, come within the purview of the ancient mathematicians.

\(^a\) Since they have the same base.
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terov dia tis katakeunhs, ws h Lambda pros KD, h KB pros BP kai h Delta pros XB. kai epeiei estin, ws h PB pros BK, h KD pros Lambda, sunvedenti, ws h PK pros KB, toutestei pros KD, owtos h KL pros Lambda kai olh ara h RL pros olyn thn KL estin, ws h KL pros Lambda. ison ara to upo town RDA tw apo AK. ws ara h RL pros Lambda, to apo KL pros to apo Lambda. kai epeiei estin, ws h Lambda pros Delta, owtos h Delta pros XB, estai anapalin kai sunvedenti, ws h KL pros Lambda, owtos h Beta pros Delta [kai ws ara to apo KL pros to apo Lambda, owtos to apo Beta pros to apo Delta. palin, epeiei estin, ws h Delta pros Delta, sylvaghoteros h KB, BX pros BX, dieilonti, ws h Lambda pros Delta, owtos h KB pros BX].

1 kai keivos twi KB isthi h BZ oni gar ekotos tou P pneumatai, dhelon [kai estai, ws h Lambda pros Delta, owtos h Zeta pros BX. wste kai, ws h Lambda pros Lambda, h Beta pros Zeta].

2 epeiei de logos esti ths Lambda pros Delta dotheis, kai tis RL ara pros Delta logos esti

1 The words kai ... apo Delta are shown by Eutocius's comment to be an interpolation. The words palin ... pros BX and kai ... pros Zeta must also be interpolated, as, in order to prove that Delta : Lambda is given, Eutocius first proves that Beta : Zeta = Lambda : Lambda, which he would hardly have done if Archimedes had himself provided the proof.

2 This is proved by Eutocius thus:

Since

\[ KA + \Delta X : \Delta X = PX : XB, \]

dirimendo,

\[ KD : \Delta X = PB : BX, \]

and permutando,

\[ KD : BP = \Delta X : XB, \]

i.e.,

\[ KB : BP = \Delta X : XB, \]

Again, since

\[ KB + BX : XB = \Delta X : XD, \]
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same way as in a previous proposition [Prop. 2], by construction,

$$\Delta \Delta : KB : BP = AX : XB.$$ 

And since \[ PB : BK = K \Delta : \Delta, \quad [\text{Eucl. v. 7, coroll.} \]
componendo, \[ PK : KB = K \Delta : \Delta, \quad [\text{Eucl. v. 18} \]
i.e., \[ PK : K \Delta = K \Delta : \Delta \Delta. \]
\[ \therefore PA : K \Delta = K \Lambda : \Delta \Delta. \quad [\text{Eucl. v. 12} \]
\[ \therefore PA : \Delta \Delta = \Delta K^2. \quad [\text{Eucl. vi. 17} \]
\[ \therefore PA : \Delta \Delta = K \Lambda^2 : \Delta \Delta^2. \]
And since \[ \Delta \Delta : \Delta K = \Delta X : XB, \]
\[ \text{invertendo et componendo,} \quad KA : \Delta \Delta = \Delta X : XB, \quad [\text{Eucl. v. 7, coroll. and v. 18} \]
Let BZ be placed equal to KB. It is plain that [Z] will fall beyond P. Then, \[ \Delta X : XB = \Delta \Delta : \Delta K. \]
Now \[ \Delta X : XB = KB : BP. \]
\[ \therefore \Delta \Delta : \Delta K = \Delta X : XB = KB : BP. \]
\[ \therefore BZ > BP. \]
\[ \text{As Eutocius' note shows, what Archimedes wrote was:} \]
"Since the ratio \( \Delta \Lambda : \Delta X \) is given, and the ratio \( PA : \Delta X \),
therefore the ratio \( PA : \Delta \Delta \) is also given." Eutocius' proof is:
Since \[ KB + BX : BX = AX : XD, \]
\[ ZX : XB = \Delta X : XD; \]
\[ XZ : ZB = \Delta \Lambda : \Delta \Delta; \]
\[ BZ : ZX = \Delta \Delta : \Delta X. \]
But the ratio BZ : ZX is given because ZB is equal to the radius of the given sphere and BX is given. Therefore \( \Delta \Delta : \Delta X \) is given.

Again, since the ratio of the segments is given, the ratio of
δοθείς. ἐπεὶ οὖν ὁ τῆς ΡΑ πρὸς ΛΧ λόγος συν-
ήπται ἐκ τε τοῦ, οὖν ἔχει η ῬΑ πρὸς ΛΔ, καὶ ἡ
ΔΔ πρὸς ΛΧ, ἀλλ' ὅσ μὲν ἡ ῬΑ πρὸς ΛΔ, τὸ
ἀπὸ ΔΒ πρὸς τὸ ἀπὸ ΔΧ, ὡς δὲ ἡ ΔΔ πρὸς ΛΧ,
οὕτως ἡ ΒΖ πρὸς ΖΧ, ὁ ἀρα τῆς ΡΑ πρὸς ΛΧ
λόγος συνήπται ἐκ τε τοῦ, οὖν ἔχει τὸ ἀπὸ ΒΔ
πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ΖΧ. πεποίησθω
δὲ, ὡς ἡ ΡΑ πρὸς ΛΧ, ἡ ΒΖ πρὸς ΖΘ· λόγος δὲ
tῆς ΡΑ πρὸς ΛΧ δοθεῖς· λόγος ἀρα καὶ τῆς ΖΒ
πρὸς ΖΘ δοθεῖς. δοθείσα δὲ ἡ ΒΖ—ὑσ γάρ ἔστι
τῇ ἐκ τοῦ κέντρου· δοθείσα ἀρα καὶ ἡ ΖΘ. καὶ
ὁ τῆς ΒΖ ἀρα λόγος πρὸς ΖΘ συνήπται ἐκ τε τοῦ,
evin ἔχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ
πρὸς ΖΧ. ἀλλ' ὁ ΒΖ πρὸς ΖΘ λόγος συνήπται
ἐκ τε τοῦ τῆς ΒΖ πρὸς ΖΧ καὶ τοῦ τῆς ΖΧ πρὸς
ΖΘ [κοινὸς ἀφηρήσθω ὁ τῆς ΒΖ πρὸς ΖΧ].
λοιπὸν ἀρα ἑστὶν, ὡς τὸ ἀπὸ ΒΔ, τούτεστὶ δοθέν,
πρὸς τὸ ἀπὸ ΔΧ, οὕτως ἡ ΖΧ πρὸς ΖΘ, τούτεστι
πρὸς δοθέν. καὶ ἑστὶν δοθείσα ἡ ΖΔ εὐθεία·
eὐθείαν ἀρα δοθείσαν την ΔΖ τεμεῖν δεῖ κατὰ τὸ
Χ καὶ ποιεῖν, ὡς τῆν ΖΧ πρὸς δοθείσαν [τῆν ΖΘ],
οὕτως τὸ δοθέν [τὸ ἀπὸ ΒΔ] πρὸς τὸ ἀπὸ ΔΧ.
tοῦτο οὕτως ἀπλῶς μὲν λεγόμενον ἔχει διορισμόν,

1 κοινὸς ... πρὸς ΖΧ. Eutocius's comment shows that
these words are interpolated.

2 τῆν ΖΘ, τὸ ἀπὸ ΒΔ. Eutocius's comments show these
words to be glosses.

the cones ΛΛΓ, ΑΡΓ is also given, and therefore the ratio
ΔΧ:ΧΡ. Therefore the ratio ΡΑ:ΛΧ is given. Since
the ratios ΡΑ:ΛΧ and ΛΔ:ΛΧ are given, it follows that
the ratio ΡΑ:ΛΔ is given.

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since the ratio PA : AX is composed of the ratios PA : ΔΔ and ΔΔ : AX,
and since

\[ PA : \Delta \Delta = \Delta B^2 : \Delta X^2, \]

\[ \Delta \Delta : \Delta X = BZ : ZX, \]

therefore the ratio PA : AX is composed of the ratios BΔ^2 : ΔX^2 and BZ : ZX. Let [θ be chosen so that]

\[ PA : \Delta X = BZ : Z\theta. \]

Now the ratio PA : AX is given; therefore the ratio ZB : Zθ is given. Now BZ is given—for it is equal to the radius; therefore Zθ is also given. Therefore the ratio BZ : Zθ is composed of the ratios BΔ^2 : ΔX^2 and BZ : ZX. But the ratio BZ : Zθ is composed of the ratios BZ : ZX and ZX : Zθ. Therefore, the remainder BΔ^2 : ΔX^2 = XZ : Zθ, in which BΔ^2 and Zθ are given. And the straight line ΔZ is given; therefore it is required so to cut the given straight line ΔZ at X that XZ bears to a given straight line the same ratio as a given area bears to the square on ΔX.

When the problem is stated in this general form, it is necessary to investigate the limits of possibility,

\[ \text{For} \]

\[ PA : \Delta \Delta = \Delta K^2 : \Delta \Delta^2 \]

\[ = B\Delta^2 : \Delta X^2. \]

\[ \text{For } \]

\[ \frac{a - x}{b} = \frac{c^2}{a^2}, \]

or

\[ x^2(a - x) = bc^2. \]
προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε ύπαρχόντων [τούτοστι τοῦ τε διπλασίαν εἶναι τὴν ΔΒ τῆς BΖ καὶ τοῦ μείζων τῆς ZΘ τῆν ZΒ, ὡς κατὰ τὴν ἀνάλυσιν] οὐκ ἔχει διορισμόν· καὶ ἐσται τὸ πρόβλημα τουτοῦτον· δύο δοθεσῶν εὐθειῶν τῶν ΒΔ, ΒΖ καὶ διπλασίας οὐσῆς τῆς ΒΔ τῆς ΒΖ καὶ σημείου ἐπὶ τῆς ΒΖ τοῦ Θ τεμεῖν τὴν ΔΒ κατὰ τὸ Χ καὶ ποιεῖν, ὡς τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, τὴν ΧΖ πρὸς ZΘ· ἐκάτερα δὲ ταῦτα ἐπὶ τέλει ἀναλυθήσεται τε καὶ συντεθήσεται.


Ἐπὶ τέλει μὲν τὸ προρήθην ἐπηγγείλατο δεῖξαι, ἐν οὐδενὶ δὲ τῶν ἀντιγράφων εὐρεῖν ἔνεστι τὸ ἑπάγγελμα. οἶθεν καὶ Διονυσίδωρον μὲν εὕρισκομεν μὴ τῶν αὐτῶν ἑπιτυχόντα, ἀδυνατήσαντα δὲ ἐπιβαλεῖν τῷ καταλειφθέντι λήμματι, ἐφ’ ἔτεραν ὅδον τοῦ ὅλου προβλήματος ἔλθειν, ἦντων εξῆς γράψομεν: Διοκλῆς μέντοι καὶ αὐτὸς ἐν τῷ Περὶ πυρίων αὐτῷ συγγεγραμμένῳ βιβλίῳ ἐπηγγέλθαι νομίζων τὸν 'Ἀρχιμήδη, μὴ πεποιηκέναι δὲ τὸ ἑπάγγελμα, αὐτὸς ἀναπληροῦν ἐπεχείρησεν, καὶ τὸ ἐπιχείρημα εξῆς γράψομεν· ἐστὶν γὰρ καὶ αὐτὸ οὐδένα μὲν ἔχων πρὸς τὰ παραλειμμένα λόγουν, ὁμοίως δὲ τῷ Διονυσίδωρῳ δι’ ἔτερας ἀποδείξεως κατασκευάζον τὸ πρόβλημα. ἐν τινὶ μέντοι παλαιῷ

1 τούτεστι . . . ἀνάλυσιν. Eutocius’s notes make it seem likely that these words are interpolated.

* In the technical language of Greek mathematics, the
but under the conditions of the present case no such investigation is necessary. In the present case the problem will be of this nature: Given two straight lines $\Delta B, BZ$, in which $B\Delta = 2BZ$, and a point $\Theta$ upon $BZ$, so to cut $\Delta B$ at $X$ that

$$B\Delta^2 : \Delta X^2 = XZ : Z\Theta;$$

and the analysis and synthesis of both problems will be given at the end.

Eutocius, Commentary on Archimedes' Sphere and Cylinder ii., Archim. ed. Heiberg iii. 130. 17–150. 22

He promised that he would give at the end a proof of what is stated, but the fulfilment of the promise cannot be found in any of his extant writings. Dionysodorus also failed to light on it, and, being unable to tackle the omitted lemma, he approached the whole problem in an altogether different way, which I shall describe in due course. Diocles, indeed, in his work On Burning Mirrors maintained that Archimedes made the promise but had not fulfilled it, and he undertook to supply the omission himself, which attempt I shall also describe in its turn; it bears, however, no relation to the missing discussion, but, like that of Dionysodorus, it solves the problem by a construction reached by a different proof. But

general problem requires a diorismos, for which v. vol. i. p. 151 n. $h$ and p. 396 n. $a$. In algebraic notation, there must be limiting conditions if the equation

$$x^2(a - x) = bx^2$$

is to have a real root lying between 0 and $a$.

* Having made this promise, Archimedes proceeded to give the formal synthesis of the problem which he had thus reduced.
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βιβλίω—οὐδὲ γὰρ τῆς εἰς πολλὰ ζητήσεως ἀπέστη-μεν—ἐντετύχαμεν θεωρήμασι γεγραμμένοις οὔκ ὀλίγην μὲν τὴν ἐκ τῶν πταισμάτων ἐχουσιν ἀσάφειαν περὶ τε τὰς καταγραφὰς πολυτρόπως ἡμαρτημένοις, τῶν μὲντοι ζητουμένων εἰχον τὴν ὑπόστασιν, ἐν μὲρει δὲ τὴν Ἀρχιμήδει τῇ μὴν παραβολῆς ὁρθογωνίου κώνου τομῆς ὄνομαζο-μένης, τῆς δὲ ὑπερβολῆς ἀμβλυγωνίου κώνου τομῆς, ὡς ἐξ αὐτῶν διανοεῖσθαι, μὴ ἁρα καὶ αὐτά εἰς τὰ ἐν τῷ τέλει ἐπηγγελμένα γράφεσθαι. ὃθεν σπουδιαίτερον ἐντυγχάνοντες αὐτὸ μὲν τὸ ῥητὸν, ὡς γέγραπται, διὰ πλῆθος, ὡς εἴρηται, τῶν πται-σμάτων δυσχέρες εὐρύνετε τὰς ἑννοιας κατὰ μικρὸν ἀποσυλήσαντες κοινοτέρα καὶ σαφέστερα κατὰ τὸ δυνατὸν λέξει γράφομεν. καθόλου δὲ πρῶτον τὸ θεώρημα γραφήσεται, ἐνα τὸ λεγόμενον ὑπ' αὐτοῦ σαφήνηθη περὶ τῶν διορισμῶν εἰτα καὶ τοῖς ἀναλεγμένοις ἐν τῷ προβλήματι προσαρμοσθή-σεται.

"Εὐθείας δοθείσης τῆς AB καὶ ἑτέρας τῆς AG καὶ χωρίου τοῦ Δ προκειόμενον λαβεῖν ἐπὶ τῆς AB σημείων ὡς τὸ E, ὅστε εἶναι, ὡς τὴν AE πρὸς AG, οὕτω τὸ τ Δ χωρίου πρὸς τὸ ἀπὸ EB.

"Γεγονέτω, καὶ κείσθω ἡ AG πρὸς ὀρθάς τῇ AB, καὶ ἐπιζευγθείσα ἡ GE διήγηθω ἐπὶ τὸ Z, καὶ ἤχθω διὰ τοῦ Γ τῇ AB παράλληλος ἡ ΓΗ, διὰ δὲ τοῦ Β τῇ AG παράλληλος ἡ ZBH συμπίπτουσα. 136
in a certain ancient book—for I pursued the inquiry thoroughly—I came upon some theorems which, though far from clear owing to errors and to manifold faults in the diagrams, nevertheless gave the substance of what I sought, and furthermore preserved in part the Doric dialect beloved by Archimedes, while they kept the names favoured by ancient custom, the parabola being called a section of a right-angled cone and the hyperbola a section of an obtuse-angled cone; in short, I felt bound to consider whether these theorems might not be what he had promised to give at the end. For this reason I applied myself with closer attention, and, although it was difficult to get at the true text owing to the multitude of the mistakes already mentioned, gradually I routed out the meaning and now set it out, so far as I can, in more familiar and clearer language. In the first place the theorem will be treated generally, in order to make clear what he says about the limits of possibility; then will follow the special form it takes under the conditions of his analysis of the problem.

"Given a straight line AB and another straight line $\Gamma \Delta$ and an area $\Delta$, let it be required to find a point $E$ on $AB$ such that $AE : \Gamma \Delta = \Delta : EB^2$.

"Suppose it found, and let $\Gamma \Delta$ be at right angles to $AB$, and let $\Gamma E$ be joined and produced to $Z$, and through $\Gamma$ let $\Gamma H$ be drawn parallel to $AB$, and through $B$ let $ZBH$ be drawn parallel to $\Gamma \Delta$, meeting
ékatéra tôn GE, GH, kai svmpetplhrwsw tò
HΘ parallhlógrammon, kai dia tôn E opotéra
tòn GΘ, HZ parálhllos ήxhò ή KEL, kai tò
Δ ίson éstw to ùpò GHM.

"Επεi oúv éstw, ós h ΕA prós ΑΓ, outhws tò
Δ prós tò ápò EB, ós de h ΕA prós ΑΓ, outhws
h ΓΗ prós HZ, ós de h ΓΗ prós HZ, outhws
tò ápò ΓΗ prós tò ùpò ΓΗΖ, óws ára tò ápò
ΓΗ prós tò ùpò ΓΗΖ, outhws tò Δ prós tò ápò
EB, toutésti prós tò ápò ΚΖ· kai énalláξ, ós
tò ápò ΓΗ prós tò Δ, toutésti prós tò ùpò
ΓΗΜ, outhws tò ùpò ΓΗΖ prós tò ápò ΖΚ.
álλ' óws tò ápò ΓΗ prós tò ùpò ΓΗΜ, outhws
h ΓΗ prós HM· kai ós ára h ΓΗ prós HM,
outhws tò ùpò ΓΗΖ prós tò ápò ΖΚ. álλ' óws
h ΓΗ prós HM, τής HZ kououv υψους λαμβανο-
ménghs outhws tò ùpò ΓΗΖ prós tò ùpò MΗZ·
ós ára tò ùpò ΓΗΖ prós tò ùpò MΗZ, outhws
tò ùpò ΓΗΖ prós tò ápò ΖΚ· ίson ára tò ùpò
MΗZ tò ápò ΖΚ. éavn ára peri áξova tìn ZΗ
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both $\Gamma E$ and $\Gamma H$, and let the parallelogram $H\Theta$ be completed, and through $E$ let $KE\Delta$ be drawn parallel to either $\Gamma \Theta$ or $HZ$, and let $[M$ be taken so that $]$

$\Gamma H \cdot HM = \Delta.$

"Then, since $EA : AG = \Delta : EB^2$ [ex hyp.

and $EA : AG = \Gamma H : HZ,$

and $\Gamma H : HZ = \Gamma H^2 : \Gamma H \cdot HZ,$

$\therefore \Gamma H^2 : \Gamma H \cdot HZ = \Delta : EB^2$

$= \Delta : KZ^2;$

and, permutando, $\Gamma H^2 : \Delta \ [= \Gamma H \cdot HZ : ZK^2,]$

i.e., $\Gamma H^2 : \Gamma H \cdot HM = \Gamma H \cdot HZ : ZK^2.$

But $\Gamma H^2 : \Gamma H \cdot HM = \Gamma H : HM;

\therefore \Gamma H : HM = \Gamma H \cdot HZ : ZK^2.$

But, by taking a common altitude $HZ,$

$\Gamma H \cdot HM = \Gamma H \cdot HZ : MH \cdot HZ;$

$\therefore \Gamma H \cdot HZ : MH \cdot HZ = \Gamma H \cdot HZ : ZK^2;$

$\therefore MH \cdot HZ = ZK^2.$
ГРЕКСКИЕ МАТЕМАТИКИ

γραφὴ діά τοῦ Η παραβολή, ὥστε τὰς καταγο-
μένας δύνασθαι παρὰ τὴν ΗΜ, ἦξει διὰ τοῦ Κ, καὶ ἔσται δέσις δεδομένη διὰ τὸ δεδομένην εἶναι 
τὴν ΗΜ τῷ μεγέθη περιέχουσαν μετὰ τῆς HG 
δεδομένης δοθὲν τὸ Δ· τὸ ἄρα Κ ἀπτεται δέσις 
dεδομένης παραβολῆς. γεγράφθω ὁρ, ὥς εἴρηται, 
καὶ ἔστω ὡς ἡ HK.

"Πάλιν, ἐπειδὴ τὸ ΘΛ χωρίον ἰσον ἔστι τῷ 
ΓΒ, τούτεστι τὸ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΗ, ἕαν 
dιὰ τοῦ Β περὶ ἀσυμπτώτους τὰς ΘΓ, ΓΗ γραφὴ 
ὑπερβολή, ἦξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν 
τοῦ η’ θωρήματος τοῦ δευτέρου βιβλίου τῶν 
Ἀπολλωνίου Κωνικῶν στοιχείων, καὶ ἔσται δέσις 
dεδομένη διὰ τὸ καὶ ἐκατέραν τῶν ΘΓ, ΓΗ, ἕτι 
μὴν καὶ τὸ Β τῇ δέσις δεδόθαι. γεγράφθω, ὥς 
eἰρηται, καὶ ἔστω ὡς ἡ ΚΒ· τὸ ἄρα Κ ἀπτεται 
δέσις δεδομένης ὑπερβολῆς. ἦπερτο δέ καὶ δέσις 
dεδομένης παραβολῆς· δεδοται ἄρα τὸ Κ. καὶ 
ἔστων ἀπ’ αὐτοῦ κάθετος ἡ ΚΕ ἐπὶ δέσις δεδομένην 
τὴν ΑΒ· δεδοται ἄρα τὸ Ε. ἐπεὶ ὅν ἔστων, ὡς 
ἡ ΕΑ πρὸς τὴν δοθέαν τὴν ΑΓ, οὕτως δοθὲν τὸ 
Δ πρὸς τὸ ἀπὸ EB, δύο στερεών, ὥν βάσεως τὸ 
ἀπὸ EB καὶ τὸ Δ, ὑψῆ δὲ αἱ ΕΑ, ΑΓ, ἀντιπεπόν-

* Let $AB = a$, $AG = b$, and $\Delta = GH$. $HM = c^2$, so that $HM = \frac{c^2}{a}$.

Then if $HG'$ be taken as the axis of $x$ and $HZ$ as the axis of $y$, the equation of the parabola is

$$x^2 = \frac{c^2}{a} y,$$

and the equation of the hyperbola is

$$(a - x)y = ab.$$

Their points of intersection give solutions of the equation

$$x^2(a - x) = bc^2.$$
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If, therefore, a parabola be drawn through H about the axis ZH with the parameter HM, it will pass through K [Apoll. Con. i. 11, converse], and it will be given in position because HM is given in magnitude [Eucl. Data 57], comprehending with the given straight line HI the given area $\Delta$; therefore K lies on a parabola given in position. Let it then be drawn, as described, and let it be HK.

"Again, since the area $\Theta \Lambda = \Gamma B$ [Eucl. i. 43

\[\Theta K \cdot K \Lambda = AB \cdot BH,\]

i.e.,

if a hyperbola be drawn through B having $\Theta \Gamma$, $\Gamma H$ for asymptotes, it will pass through K by the converse to the eighth theorem of the second book of Apollonius’s Elements of Conics, and it will be given in position because both the straight lines $\Theta \Gamma$, $\Gamma H$, and also the point B, are given in position. Let it be drawn, as described, and let it be KB; therefore K lies on a hyperbola given in position. But it lies also on a parabola given in position; therefore K is given. And KE is the perpendicular drawn from it to the straight line AB given in position; therefore E is given. Now since the ratio of EA to the given straight line $\Lambda \Gamma$ is equal to the ratio of the given area $\Delta$ to the square on EB, we have two solids, whose bases are the square on EB and $\Delta$ and whose altitudes are EA, $\Lambda \Gamma$, and the bases are inversely pro-

to which, as already noted, Archimedes had reduced his problem. (N.B.—The axis of $x$ is for convenience taken in a direction contrary to that which is usual; with the usual conventions, we should get slightly different equations.)
θαυμαί βάσεις τοῖς ύψεσιν ὡστε ἵσα ἐστὶ τὰ στερεά· τὸ ἀρα ἀπὸ EB ἐπί τὴν EA ἵσον ἐστὶ τῷ δοθέντι τῷ Δ ἐπὶ δοθείσαι τὴν ΓΑ. ἀλλὰ τὸ ἀπὸ BE ἐπί τὴν EA μέγιστὸν ἐστὶ πάντων τῶν ὰμοίως λαμβανομένων ἐπὶ τῆς BA, ὅταν ἡ διπλασία ἡ BE τῆς EA, ὡς δειχθῆσεται· δεῖ ἀρα τὸ δοθὲν ἐπὶ τὴν δοθείσαι μὴ μείζον εἶναι τοῦ ἀπὸ τῆς BE ἐπὶ τὴν EA.

"Συντεθήσεται δὲ οὕτως· ἐστώ ἡ μὲν δοθείσα εὐθεία ἡ AB, ἀλλὰ δὲ τις δοθείσα ἡ AG, τὸ δὲ δοθὲν χωρίον τὸ Δ, καὶ δέον ἐστῷ τεμεῖν τὴν AB, ὡστε εἶναι, ὡς τὸ ἐν τῷ μή μα πρὸς τὴν δοθείσαι τὴν AG, οὕτως τὸ δοθὲν τὸ Δ πρὸς τὸ ἀπὸ τοῦ λοιποῦ τμῆματος.

"Εἰλήφθω τῆς AB τρίτον μέρος ἡ AE· τὸ ἀρα Δ ἐπὶ τὴν AG ἣτοι μείζον ἐστὶ τοῦ ἀπὸ τῆς BE ἐπὶ τὴν EA ἡ ἵσον ἡ ἔλασσον.

"Εἰ μὲν οὖν μείζον ἐστιν, οὐ συντεθῆσεται, ὡς ἐν τῇ ἀναλύσει δέδεκται· εἰ δὲ ἵσον ἐστὶ, τὸ Ε σημείον ποιήσει τὸ πρόβλημα. ἵσοιν γὰρ οὖν τῶν στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ύψεσιν, καὶ ἐστιν, ὡς ἡ EA πρὸς AG, οὕτως τὸ Δ πρὸς τὸ ἀπὸ BE.

"Εἰ δὲ ἔλασσόν ἐστι τὸ Δ ἐπὶ τὴν AG τοῦ ἀπὸ BE ἐπὶ τὴν EA, συντεθῆσεται οὕτως· κείσθω ἡ AG πρὸς ὑπὸ τῇ AB, καὶ διὰ τοῦ Γ τῇ AB παρ-

* In our algebraical notation, $x^2(a - x)$ is a maximum when $x = \frac{2a}{3}$. We can easily prove this by the calculus. For, by differentiating and equating to zero, we see that $x^2(a - x)$ has 142
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portional to the altitudes; therefore the solids are equal [Eucl. xi. 34]; therefore

\[ EB^2 \cdot EA = \Delta \cdot \Gamma A, \]

in which both \( \Delta \) and \( \Gamma A \) are given. But, of all the figures similarly taken upon \( BA, BE^2 \). \( BA \) is greatest when \( BE = 2EA \), as will be proved; it is therefore necessary that the product of the given area and the given straight line should not be greater than

\[ BE^2 \cdot EA. \]

"The synthesis is as follows: Let \( AB \) be the given straight line, let \( AI' \) be any other given straight line, let \( \Delta \) be the given area, and let it be required to cut \( AB \) so that the ratio of one segment to the given straight line \( AI' \) shall be equal to the ratio of the given area \( \Delta \) to the square on the remaining segment.

"Let \( AE \) be taken, the third part of \( AB \); then \( \Delta \cdot AI' \) is greater than, equal to or less than \( BE^2 \cdot EA \).

"If it is greater, no synthesis is possible, as was shown in the analysis; if it is equal, the point \( E \) satisfies the conditions of the problem. For in equal solids the bases are inversely proportional to the altitudes, and \( EA : AI' = \Delta : BE^2 \).

"If \( \Delta \cdot AI' \) is less than \( BE^2 \cdot EA \), the synthesis is thus accomplished: let \( AI' \) be placed at right angles to \( AB \), and through \( \Gamma \) let \( \Gamma Z \) be drawn parallel to a stationary value when \( 2ax - 3x^3 = 0 \), i.e., when \( x = 0 \) (which gives a minimum value) or \( x = \frac{x}{a} \) (which gives a maximum). No such easy course was open to Archimedes.

\* Sc. "not greater than \( BE^2 \cdot EA \) when \( BE = 2EA."

\* Figure on p. 146.
άλληλος ἡχθω ἦ ΓΖ, διὰ δὲ τοῦ Β τῇ ΑΓ παράλληλος ἡχθω ἦ ΒΖ καὶ συμπεπτέτω τῇ ΓΕ ἐκβληθεῖσα κατὰ τὸ Η, καὶ συμπεπληρώσθω τὸ ΖΘ παράλληλόγραμμον, καὶ διὰ τοῦ Ε τῇ ΖΗ παράλληλος ἡχθω ἦ ΚΕΛ. ἐπεὶ οὖν τὸ Δ ἐπὶ τὴν ΑΓ ἐλασσὸν ἐστὶ τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, ἔστω, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς ἐλασσὸν τι τοῦ ἀπὸ τῆς ΒΕ, τουτέστι τοῦ ἀπὸ τῆς HK. ἔστω οὖν, ὡς ἡ EA πρὸς ΑΓ, οὔτως τὸ Δ πρὸς τὸ ἀπὸ ΗΜ, καὶ τῷ Δ ἴσον ἔστω τὸ ὑπὸ ΓΖΝ. ἐπεὶ οὖν ἔστω, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ, τουτέστι τὸ ὑπὸ ΓΖΝ, πρὸς τὸ ἀπὸ ΗΜ, ἀλλ' ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως ἦ ΓΖ πρὸς ΖΗ, ὡς δὲ ἦ ΓΖ πρὸς ΖΗ, οὔτως τῷ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, καὶ ὡς ἄρα τῷ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, οὔτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ἀπὸ ΗΜ καὶ ἐναλλάξ, ὡς τῷ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, οὔτως τῇ ὑπὸ ΓΖΗ πρὸς τῷ ἀπὸ ΗΜ. ἀλλ' ὡς τῷ ἀπὸ ΓΖ πρὸς τῷ ὑπὸ ΓΖΝ, ἦ ΓΖ πρὸς ΖΝ, ὡς δὲ ἦ ΓΖ πρὸς ΖΝ, τῆς ΖΗ κοινοῦ ύφους λαμβανομένης οὔτως τῇ ὑπὸ ΓΖΗ πρὸς τῷ ὑπὸ ΝΖΗ καὶ ὡς ἄρα τῷ ὑπὸ ΓΖΗ πρὸς τῷ ὑπὸ ΝΖΗ, οὔτως τῷ ὑπὸ ΓΖΗ πρὸς τῷ ἀπὸ ΗΜ· ἴσον ἄρα ἔστι τῷ ἀπὸ ΗΜ τῷ ὑπὸ ΗΖΝ.

"Εάν ἄρα διὰ τοῦ Ζ περὶ ἄξονα τῆς ΖΗ γράφω-μεν παραβολήν, ὡστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΖΝ, ἦξει διὰ τοῦ Μ. γεγράφω, καὶ ἔστω ὡς ἦ ΜΕΖ. καὶ ἐπεὶ ἴσον ἔστι τῷ ΘΔ τῷ AZ, τουτέστι τῷ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΖ, εάν διὰ
AB, and through B let BZ be drawn parallel to AG, and let it meet GE produced at H, and let the parallelogram Zθ be completed, and through E let KEA be drawn parallel to ZH. Now since
\[ \Delta \cdot AG < BE^2 \cdot EA, \]
\[ \therefore \quad \frac{EA}{AG} = \Delta : (the \ square \ of \ some \ quantity \ less \ than \ BE) \]
\[ = \Delta : (the \ square \ of \ some \ quantity \ less \ than \ HK). \]

Let
\[ \frac{EA}{AG} = \Delta : HM^2, \]
and let
\[ \Delta = \frac{EZ}{ZN}. \]
Then
\[ \frac{EA}{AG} = \Delta : HM^2 \]
\[ = \frac{EZ}{ZN} : HM^2. \]

But
\[ \frac{EA}{AG} = \frac{EZ}{ZH}, \]
and
\[ \frac{EZ}{ZH} = \frac{EZ^2}{ZH^2} ; \]
\[ \therefore \quad \frac{EZ^2}{ZH} : \frac{EZ}{ZH} = \frac{EZ}{ZN} : HM^2; \]
and permutando, \[ \frac{EZ^2}{ZH} : \frac{EZ}{ZN} = \frac{EZ}{ZH} : HM^2. \]

But
\[ \frac{EZ^2}{ZH} : \frac{EZ}{ZN} = \frac{EZ}{ZN}, \]
and
\[ \frac{EZ}{ZN} = \frac{EZ}{ZH} : NZ \cdot ZH, \]
by taking a common altitude ZH;
and, \[ \frac{EZ}{ZH} : NZ \cdot ZH = \frac{EZ}{ZH} : HM^2; \]
\[ \therefore \quad HM^2 = HZ \cdot ZN. \]

"Therefore if we describe through Z a parabola about the axis ZH and with parameter ZN, it will pass through M. Let it be described, and let it be as MEZ. Then since
\[ \Theta A = AZ, \quad \text{[Eucl. i. 43]} \]
\[ i.e. \quad \Theta K \cdot KA = AB \cdot BZ, \]
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τοῦ Β περὶ ἀσυμμετρῶτον τὰς ΘΓ, ΓΖ γράψωμεν ὑπερβολήν, ἢξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν

τοῦ η' 'βεωρήματος τῶν Απολλωνίου Κωνικῶν στοιχείων. γεγράφθω, καὶ ἔστω ὡς η' ΒΚ τεύ- νουσα τὴν παραβολήν κατὰ τὸ Σ, καὶ ἀπὸ τοῦ Σ ἑπὶ τὴν ΑΒ κάθετος ἢχθω η' ΣΟΠ, καὶ διὰ τοῦ Σ τη' ΑΒ παράλληλος ἢχθω η' ΡΕΣ. ἐπεὶ οὖν ὑπερβολή ἑστιν ἡ ΒΣΚ, ἀσυμμετρωτοὶ δὲ αἱ ΘΓ, ΓΖ, καὶ παράλληλοι ἤγμεναι εἰσὶ αἱ ΡΕΠ τὰς ΑΒΖ, ἢσον ἑστὶ τὸ ὑπὸ ΡΕΠ τῶ ὑπὸ ΑΒΖ· ὥστε καὶ τὸ ΡΟ τῶ ΟΖ. ἔὰν ἀρα ἀπὸ τοῦ Γ ἑπὶ τὸ Σ ἐπιζευξθῇ εὐθεία, ἢξει διὰ τοῦ Ο. ἐρχέσθω, καὶ ἔστω ὡς η' ΓΟΣ. ἐπεὶ οὖν ἑστὶ, ὡς ἡ ΟΑ πρὸς ΑΓ, ὡστὶς ἡ ΟΒ πρὸς ΒΣ, τούτεστιν ἡ' ΓΖ πρὸς ΖΣ, ὡς δὲ ἡ ΓΖ πρὸς ΖΣ, τῆς ΖΝ κοινοῦ ὑφον λαμβανομένης ὡστὶς τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΣΖΝ, καὶ ὡς ἀρα ἡ ΟΑ πρὸς ΑΓ, ὡστὶς τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΣΖΝ. καὶ ἑστὶ τῶ μὲν ὑπὸ ΓΖΝ ἢσον τὸ Δ χωρίον, τῷ δὲ ὑπὸ ΣΖΝ ἢσον τὸ ἀπὸ ΣΞ, τούτεστι τὸ ἀπὸ ΒΟ, διὰ τὴν παραβολήν· ὡς ἀρα ἡ ΟΑ πρὸς ΑΓ, ὡστὶς τὸ Δ
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if we describe through $B$ a hyperbola in the asymptotes $\Theta \Gamma, \Gamma \Sigma$, it will pass through $K$ by the converse of the eighth theorem [of the second book] of Apollonius's *Elements of Conics*. Let it be described, and let it be as $BK$ cutting the parabola in $E$, and from $E$ let $EO\Pi$ be drawn perpendicular to $AB$, and through $E$ let $P\Xi\Sigma$ be drawn parallel to $AB$. Then since $BE\Pi$ is a hyperbola and $\Theta \Gamma, \Gamma \Sigma$ are its asymptotes, while $P\Xi, \Xi\Pi$ are parallel to $AB, BZ$,

$$P\Xi \cdot \Xi\Pi = AB \cdot BZ;$$

[Apoll. ii. 12

$$PO = OZ.$$ Therefore if a straight line be drawn from $\Gamma$ to $\Sigma$ it will pass through $O$ [Eucl. i. 43, converse]. Let it be drawn, and let it be as $\Gamma O\Sigma$. Then since

$$OA : AG = OB : B\Sigma$$

[Eucl. vi. 4

$$= \Gamma \Sigma : Z\Sigma,$$

and

$$\Gamma \Sigma : Z\Sigma = \Gamma \Sigma \cdot ZN : \Sigma \Sigma \cdot ZN,$$

by taking a common altitude $ZN$,

$$OA : AG = \Gamma \Sigma \cdot ZN : \Sigma \Sigma \cdot ZN.$$ And $\Gamma \Sigma \cdot ZN = \Delta, \Sigma \Sigma \cdot ZN = \Sigma \Xi^2 = BO^2$, by the property of the parabola [Apoll. i. 11].

$$OA : AG = \Delta : BO^2;$$

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χωρίον πρὸς τὸ ἀπὸ τῆς ΒΟ. εἶληται ἂρα τὸ Ο
σημεῖον ποιοῦν τὸ πρόβλημα.

""Οτι δὲ διπλασίας οὖσας τῆς ΒΕ τῆς ΕΑ τὸ
ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ μέγιστον ἐστὶ πάντων τῶν
ὀμοίων λαμβανομένων ἐπὶ τῆς ΒΑ, δειχθήσεται
οὕτως. ἐστω γάρ, ὡς ἐν τῇ ἀναλύσει, πάλιν
δοθεῖσα εὐθεία πρὸς ὅρθας τῇ ΑΒ ἡ ΑΓ, καὶ ἐπὶ-
ζευγθεῖσα ἡ ΓΕ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ
dιὰ τοῦ Β παράλληλων ἡμινὲν τῇ ΑΓ κατὰ τὸ Ζ,
καὶ διὰ τῶν Γ, Ζ παράλληλοι τῇ ΑΒ ἡχθωσιν αἱ
ΘΖ, ΓΗ, καὶ ἐκβεβλήσθω ἡ ΖΑ ἐπὶ τὸ Θ, καὶ
tαῦτη παράλληλος διὰ τοῦ Ε ἡξθω ἡ ΚΕΛ, καὶ
geγονέτω, ὡς ἡ ΕΑ πρὸς ΑΓ, οὕτως τὸ ὑπὸ ΓΗΜ
πρὸς τὸ ἀπὸ ΕΒ· τὸ ἄρα ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ ἴσον
ἐστὶ τῷ ὑπὸ ΓΗΜ ἐπὶ τὴν ΑΓ διὰ τὸ δύο στερεῶν
ἀντιπεπονθέναι τὰς βάσεις τοῖς ὑφεσιν. λέγω οὖν,
ὅτι τὸ ὑπὸ ΓΗΜ ἐπὶ τὴν ΑΓ μέγιστον ἐστὶ
πάντων τῶν ὁμοίων ἐπὶ τῆς ΒΑ λαμβανομένων.

"Γεγράφθω γάρ διὰ τοῦ Η περὶ ἀξόνα τῆς ΖΗ
παραβολῆ, ὡστε τὰς καταγομένας δύνασθαι παρὰ
tὴν HM· ἦξει δὴ διὰ τοῦ K, ὡς ἐν τῇ ἀναλύσει
deδεικται, καὶ συμπεσεῖται ἐκβαλλομένη τῇ
ΘΓ παράλληλω οὐσῃ τῇ διαμέτρῳ τῆς τομῆς
diὰ τὸ ἐβδομὸν καὶ ἐκκοστὸν θεώρημα τοῦ πρώτου βιβλίου
tῶν Ἀπολλωνίου Κανικῶν στοιχείων. ἐκβεβλήσθω
cαὶ συμπιπτέτω κατὰ τὸ N, καὶ διὰ τοῦ
Β περὶ ἀσυμπτώτους τὰς ΝΓΗ γεγράφθω ὑπερ-
βολῆ· ἦξει ἂρα διὰ τοῦ K, ὡς ἐν τῇ ἀναλύσει
εἰρηται. ἐρχέσθω οὖν ὡς ἡ ΒΚ, καὶ ἐκβληθείσῃ
tῇ ΖΗ ἴσῳ κεῖσθω ἡ ΗΞ, καὶ ἐπεζεύχθων ἡ ΞΚ
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therefore the point $O$ has been found satisfying the conditions of the problem.

"That $BE^2 \cdot EA$ is the greatest of all the figures similarly taken upon $BA$ when $BE = 2EA$ will be thus proved. Let there again be, as in the analysis, a given straight line $AG$ at right angles to $AB$, and let $GE$ be joined and let it, when produced, meet at $Z$ the line through $B$ drawn parallel to $AG$, and through $G$, $Z$ let $OZ$, $GH$ be drawn parallel to $AB$, and let $GA$ be produced to $O$, and through $E$ let $KEA$ be drawn parallel to it, and let

$$EA : AG = GH : HM : EB^2;$$

then

$$BE^2 \cdot EA = (GH \cdot HM) \cdot AG,$$

owing to the fact that in two [equal] solids the bases are inversely proportional to the altitudes. I assert, then, that $(GH \cdot HM) \cdot AG$ is the greatest of all the figures similarly taken upon $BA$.

"For let there be described through $H$ a parabola about the axis $ZH$ and with parameter $HM$; it will pass through $K$, as was proved in the analysis, and, if produced, it will meet $OG$, being parallel to the axis of the parabola, by the twenty-seventh theorem of the first book of Apollonius's *Elements of Conics*. Let it be produced, and let it meet at $N$, and through $B$ let a hyperbola be drawn in the asymptotes $NG$, $GH$; it will pass through $K$, as was shown in the analysis. Let it be described as $BK$, and let $ZH$ be produced to $E$ so that $ZH = HE$, and let $EK$ be joined

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* Figure on p. 151.

* Lit. "diameter," in accordance with Archimedes' usage.

* Apoll. i. 26 in our texts.
καὶ ἐκβεβλήσθω ἐπὶ τὸ Ο· φανερὸν ἀρα, ὅτι ἐφάπτεται τῆς παραβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τετάρτου καὶ τριακοστοῦ θεωρήματος τοῦ πρώτου βιβλίου τῶν Ἄπολλωνίου Κωνικῶν στοιχείων. ἐπεὶ οὖν διπλὴ ἐστιν ἡ ΒΕ τῆς ΕΑ—οὕτως γὰρ ὑπόκειται—τοιτέστων ἡ ΖΚ τῆς ΚΘ,

* Apoll. i. 33 in our texts.
and produced to \( O \); it is clear that it will touch the parabola by the converse of the thirty-fourth theorem of the first book of Apollonius's *Elements of Conics*. Then since \( BE = 2EA \)—for this hypothesis has been made—therefore \( ZK = 2K\Theta \), and the triangle
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καὶ ἐστὶν ὁμοιὸν τὸ ΩΘΚ τρίγωνον τῷ ΞΖΚ τριγώνῳ, διπλασία ἐστὶ καὶ ἡ ΞΚ τῆς ΚΟ. ἐστὶν δὲ καὶ ἡ ΞΚ τῆς ΚΠΠ διπλῆ διὰ τὸ καὶ τὴν ΞΖ τῆς ΞΗ καὶ παράλληλον εἰναι τὴν ΠΗ τῇ ΚΖ. ἢ ἀρα ἡ ΟΚ τῇ ΚΠΠ. ἢ ἀρα ΟΚΠ ψαύουσα τῆς ύπερβολῆς καὶ μεταξὺ ὦσα τῶν ἀσυμπτωτῶν δίχα τέμνεται. ἐφάπτεται ἢ ἀρα τῆς ύπερβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τρίτου θεωρήματος τοῦ δευτέρου βιβλίου τῶν 'Ἀπολλωνίου Κωνικῶν στοιχείων. ἐφήπτετο δὲ καὶ τῆς παραβολῆς κατὰ τὸ αὐτὸ Κ. ἢ ἀρα παραβολὴ τῆς ύπερβολῆς ἐφάπτεται κατὰ τὸ Κ.

"Νευόμασθω οὖν καὶ ἡ ύπερβολὴ προσεκβαλλομένη ὡς ἐπὶ τὸ Ρ, καὶ εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Σ, καὶ διὰ τοῦ Σ τῇ ΚΛ παράλληλος ἡχθὼ τῇ ΣΥΛ καὶ συμβαλλεῖ τῇ ύπερβολῇ κατὰ τὸ Τ, καὶ διὰ τοῦ Τ τῇ ΓΗ παράλληλος ἡχθὼ τῇ ΦΛΧ. ἐπει οὖν διὰ τὴν ύπερβολὴν καὶ τὰς ἀσυμπτωτὰς ἰσον ἐστὶ τὸ ΦΥ τῷ ΓΒ, κοινοὶ ἀφαίρεθαι τοῦ ΓΣ ἰσον γίνεται τὸ ΦΣ τῷ ΣΗ, καὶ διὰ τοῦ ἦ ἀπὸ τοῦ Γ ἐπὶ τὸ Χ ἐπίζευγμα μένῃ εὐθεία ἥξει διὰ τοῦ Σ. ἐργεῖσθω καὶ ἐστὶν ὡς ἡ ΓΣΧ. καὶ ἐπεὶ τὸ ἀπὸ ΨΧ ἰσον ἐστὶ τῷ ὑπὸ ΧΗΜ διὰ τὴν παραβολὴν, τὸ ἀπὸ ΤΧ ἔλασσον

* In the same notation as before, the condition $BE^2 = (\Gamma \Pi, HM)$. $\alpha \Gamma$ is $\frac{4}{27} a^3 = 6c^2$; and Archimedes has proved that, when this condition holds, the parabola $x^2 = \frac{c}{a} y$ touches the hyperbola $(a - x)y = ab$ at the point $\left(\frac{2}{3} a, 3b\right)$ because they both touch at this point the same straight line, that is the 152
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ΟΘΚ is similar to the triangle ΞΖΚ, so that ΞΚ = 2ΚΟ. But ΞΚ = 2ΚΠι because ΞΖ = 2ΞΗ and ΠΗ is parallel to ΚΖ; therefore ΟΚ = ΚΠι. Therefore ΟΚΠι, which meets the hyperbola and lies between the asymptotes, is bisected; therefore, by the converse of the third theorem of the second book of Apollonius's *Elements of Conics*, it is a tangent to the hyperbola. But it touches the parabola at the same point Κ. Therefore the parabola touches the hyperbola at Κ.

Let the hyperbola be therefore conceived as produced to Ρ, and upon ΑΒ let any point Σ be taken, and through Σ let ΣΥ be drawn parallel to ΚΑ and let it meet the hyperbola at Τ, and through Τ let ΦΤΧ be drawn parallel to ΓΗ. Now by virtue of the property of the hyperbola and its asymptotes, ΦΥ = ΓΒ, and, the common element ΓΣ being subtracted, ΦΣ = ΣΗ, and therefore the straight line drawn from Γ to Χ will pass through Σ [Eucl. i. 43, conv.]. Let it be drawn, and let it be as ΓΣΧ. Then since, in virtue of the property of the parabola,

\[ ΨΧ^2 = ΧΗ . ΗΜ, \]

line \[ 96x - ay - 3ab = 0, \] as may easily be shown. We may prove this fact in the following simple manner. Their points of intersection are given by the equation

\[ x^2(a - x) = bc^2, \]

which may be written

\[ x^2 - ax^2 + \frac{4}{27}a^3 = \frac{4}{27}a^3 - bc^2, \]

or

\[ (x - \frac{2}{3}a)^2(x + a) = \frac{4}{27}a^3 - bc^2. \]

Therefore, when \( bc^2 = \frac{4}{27}a^3 \) there are two coincident solutions, \( x = \frac{2}{3}a \), lying between 0 and \( a \), and a third solution \( x = -\frac{a}{3} \), outside that range.

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ἔστι τοῦ ὑπὸ ΧΗΜ. γεγονέτω ὁδὸν τῷ ἀπὸ ΤΧ ἴσον τὸ ὑπὸ ΧΗΩ. ἐπεὶ οὖν ἐστὶν, ὡς η ΣΑ πρὸς ΑΓ, οὔτως ἡ ΓΗ πρὸς ΗΧ, ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΧ, τῆς ΗΩ κοινῷ ύψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ ΧΗΩ καὶ πρὸς τὸ ἴσον αὐτῷ τὸ ἀπὸ ΧΤ, τούτεστι τὸ ἀπὸ ΒΣ, τὸ ἀρα ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἴσον ἐστὶ τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ. τὸ δὲ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ ἔλασσον ἐστὶ τοῦ ὑπὸ ΓΗΜ ἐπὶ τὴν ΓΑ· τὸ ἀρα ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἔλαστον ἐστὶ τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ. ὁμοίως δὴ δεικτὸς εἶται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ λαμβανομένων τῶν E, B.

"Ἀλλὰ δὴ εἰλῆθνος μεταξὺ τῶν E, A σημείων τὸ 5. λέγω, ὅτι καὶ οὔτως τὸ ἀπὸ τῆς BE ἐπὶ τὴν EA μείζον ἐστὶ τοῦ ἀπὸ BS ἐπὶ τὴν 5A.

"Τῶν γὰρ αὐτῶν κατασκευασμένων ἕκασθ' διὰ τοῦ 5 τῇ ΚΛ παράλληλος η ζε'Ρ καὶ συμβαλλέτω τῇ ύπερβολῇ κατὰ τὸ Ρ: συμβαλεῖ γὰρ αὐτῇ διὰ τὸ παράλληλος εἰναι τῇ ἀσυμπτώτῳ· καὶ διὰ τοῦ Ρ παράλληλος ἀχθεῖσα τῇ ΑΒ η Α'ΡΒ' συμβαλλέτω τῇ ΗΖ ἐκβαλλομένη κατὰ τὸ Β'. καὶ ἐπεὶ πάλιν διὰ τὴν ύπερβολὴν ἴσον ἐστὶ τὸ Γ'ζ τῷ ΑΗ, η ἀπὸ τοῦ Γ ἐπὶ τὸ Β' ἐπιζευγνυμένη εὐθεία ἦξει διὰ τοῦ 5· ἐρχέσθω καὶ ἐστὶν ὡς η Γ'ζΒ'. καὶ ἐπεὶ πάλιν διὰ τὴν παραβολήν ἴσον ἐστὶ τὸ ἀπὸ AΒ' τῷ ὑπὸ B'HM, τὸ ἀρα ἀπὸ PB' ἔλασσον ἐστὶ τοῦ ὑπὸ B'HM. γεγονέτω τὸ ἀπὸ PB' ἴσον τῷ ὑπὸ B'ΗΩ. ἐπεὶ οὖν ἐστιν, ὡς η 5A πρὸς ΑΓ, οὔτως ἡ ΓΗ πρὸς HB', ἀλλ' ὡς ἡ ΓΗ πρὸς HB', τῆς

* Figure on p. 156.
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∴ \[ TX^2 < XH \cdot HM. \]

Let \[ TX^2 = XH \cdot H\Omega. \]

Then since \[ \Sigma A : A\Gamma = \Gamma H : HX, \]

while \[ \Gamma H : HX = \Gamma H : H\Omega : XH \cdot H\Omega, \]

by taking a common altitude \( H\Omega, \)

\[ = \Gamma H \cdot H\Omega : XT^2 \]

\[ = \Gamma H \cdot H\Omega : \Sigma^2, \]

∴ \[ \Sigma^2 \cdot \Sigma A = (\Gamma H \cdot H\Omega) \cdot \Gamma A. \]

But \( (\Gamma H \cdot H\Omega) \cdot \Gamma A < (\Gamma H \cdot HM) \cdot \Gamma A; \)

∴ \[ \Sigma^2 \cdot \Sigma A < \Sigma B^2 \cdot EA. \]

This can be proved similarly for all points taken between \( E, B. \)

"Now let there be taken a point \( \sigma \) between \( E, A. \)

I assert that in this case also \( \Sigma B^2 \cdot EA > \Sigma \sigma A. \)

"With the same construction, let \( \zeta \sigma P \) be drawn parallel to \( KA \) and let it meet the hyperbola at \( P; \) it will meet the hyperbola because it is parallel to an asymptote [Apoll. ii. 13]; and through \( P \) let \( \Lambda'PB' \) be drawn parallel to \( AB \) and let it meet \( \Lambda Z \) produced in \( B'. \) Since, in virtue of the property of the hyperbola, \( \Gamma \zeta \sigma = \Lambda H, \) the straight line drawn from \( \Gamma \) to \( B' \) will pass through \( \sigma \) [Eucl. i. 43, conv.]. Let it be drawn and let it be as \( \Gamma \sigma B'. \) Again, since, in virtue of the property of the parabola,

\[ A'B'^2 = B'H \cdot HM, \]

∴ \[ PB'^2 < B'H \cdot HM. \]

Let \[ PB'^2 = B'H \cdot H\Omega. \]

Then since \( \sigma A : A\Gamma = \Gamma H : HB', \)

while \[ \Gamma H : HB' = \Gamma H \cdot H\Omega : B'H \cdot H\Omega, \]
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ΗΩ κοινού ύψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ ΒΗΩ, τούτεστι πρὸς τὸ ἀπὸ ΡΒ', τούτεστι πρὸς τὸ ἀπὸ Βζ, τὸ ἀρα ἀπὸ Βζ ἐπὶ τὴν εΑ ἵσον ἐστὶ τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ, καὶ μείζον τὸ ὑπὸ ΓΗΜ τοῦ ὑπὸ ΓΗΩ· μείζον ἀρα καὶ τὸ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ τοῦ ἀπὸ Βζ ἐπὶ 156
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by taking a common altitude $H\Omega$,

$$= \Gamma H \cdot H\Omega : PB'^2$$

$$= \Gamma H \cdot H\Omega : B^2,$$

$$\therefore \quad B^2 \cdot sA = (\Gamma H \cdot H\Omega) \cdot GA.$$

And

$$\Gamma H \cdot HM > \Gamma H \cdot H\Omega;$$

$$\therefore \quad BE^2 \cdot EA > B^2 \cdot sA.$$
Επιστήσαι δὲ χρὴ καὶ τοῖς ἀκολουθοῦσιν κατὰ τὴν εἰρημένην καταγραφήν. ἐπεὶ γὰρ δεδεικται τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ καὶ τὸ ἀπὸ Βς ἐπὶ τὴν ε eius ἔλασσον τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, δυνατὸν ἦσσι καὶ τοῦ δοθέντος χωρίου ἐπὶ τὴν δοθέον ἔλασσον οὖν τοῦ ἀπὸ τῆς BE ἐπὶ τῆς EA κατά δύο σημεία τὴν AB τεμνομένην ποιεῖν τὸ ἐξ ἀρχής πρόβλημα. τούτο δὲ γίνεται, εἰ νοήσαμεν περὶ διάμετρον τῆς ΧΘ γραφομένην παραβολὴν, ὡστε τὰς καταγωμένας δύνασθαι παρὰ τὴν ΗΩ· ἡ γὰρ τοιαύτη παραβολὴ πάντως ἔρχεται διὰ τοῦ Τ. καὶ ἐπειδὴ ἀνάγκη αὐτήν συμπίπτειν τῇ ΙΓΝ παραλληλήν οὐσίᾳ τῇ διαμέτρῳ, δῆλον, ὅτι τέμνει τὴν ύπερβολὴν καὶ κατ' ἄλλο σημείον ἀνωτέρω τοῦ Κ, ὡς ἐνταῦθα κατὰ τὸ Ρ, καὶ ἀπὸ τοῦ Ρ ἐπὶ τὴν AB κάθετος ἀγομένη, ὡς ἐνταῦθα ἡ Ρς, τέμνει τὴν AB κατὰ τὸ σ, ὡστε τὸ σ σημείον ποιεῖν τὸ πρό-

* There is some uncertainty where the quotation from Archimedes ends and Eutocius's comments are resumed. Heiberg, with some reason, makes Eutocius resume his comments at this point.

* In the mss. the figures on pp. 150 and 156 are com-
This can be proved similarly for all points taken between E, A. And it was proved for all points between E, B; therefore for all figures similarly taken upon AB, BE². EA is greatest when BE = 2EA."

The following consequences a should also be noticed in the aforementioned figure. b Inasmuch as it has been proved that

\[ BS^2 : SA < BE^2, \]

and

\[ BS^2 : SA < BE^2, EA, \]

if the product of the given space and the given straight line is less than BE². EA, it is possible to cut AB in two points satisfying the conditions of the original problem. c This comes about if we conceive a parabola described about the axis XH with parameter \( H\Omega \); for such a parabola will necessarily pass through "T. d And since it must necessarily meet \( P'N \), being parallel to a diameter [Apoll. Con. i. 26], it is clear that it cuts the hyperbola in another point above K, as at \( P \) in this case, and a perpendicular drawn from \( P \) to AB, as \( P' \) in this case, will cut AB in \( \varepsilon \), so that the point \( \varepsilon \) satisfies the conditions of the

bined; in this edition it is convenient, for the sake of clarity, to give separate figures.

With the same notation as before this may be stated: when \( bc^2 < \frac{4}{27}a^2 \), there are always two real solutions of the cubic equation \( x^2(a - x) = bc^2 \) lying between 0 and a. If the cubic has two real roots it must, of course, have a third real root as well, but the Greeks did not recognize negative solutions.

By Apoll. i. 11, since \( TX^2 = XH \cdot H\Omega \).
βλήμα, καὶ ἵσον γίνεσθαι τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ
tῷ ἀπὸ Βζ ἐπὶ τὴν ζΑ, ὡς ἐστὶ διὰ τῶν προειρη-
mένων ἀποδείξεων ἐμφανές. ὡστε δυνατοῦ ὅντος ἐπὶ τῆς ΒΑ δύο σημεῖα λαμβάνεις ποιοῦντα τὸ
ξητούμενον, ἐξεστιν, ὅποτέρον τις βούλοιτο, λαμ-
βάνειν ἢ τὸ μεταξὺ τῶν Ε, Β ἢ τὸ μεταξὺ τῶν Ε,
Α. εἰ μὲν γὰρ τὸ μεταξὺ τῶν Ε, Β, ὡς εἰρηται,
τῆς διὰ τῶν Η, Τ σημείων γραφομένης παραβολῆς
catὰ δύο σημεία τεμνοῦσας τὴν ὑπερβολὴν τὸ μὲν
ἐγγύτερον τοῦ Η, τουτέστι τοῦ ἀξονος τῆς παρα-
βολῆς, εὑρίσκει τὸ μεταξὺ τῶν Ε, Β, ὡς ἐνταῦθα
tὸ Τ εὑρίσκει τὸ Σ, τὸ δὲ ἀπωτέρω τὸ μεταξὺ
tῶν Ε, Α, ὡς ἐνταῦθα τὸ Ρ εὑρίσκει τὸ σ.

Καθόλου μὲν οὖν οὕτως ἀναλέωνται καὶ συντέ-
θεται τὸ πρόβλημα: ἵνα ἐδὲ καὶ τοῖς Ἀρχιμενήθείοις
ῥήμαισιν ἐφαρμοσθῇ, νενομήθω ὡς ἐν αὐτῇ τῇ τοῦ
ῥητοῦ καταγραφῆ διάμετρος μὲν τῆς σφαιρᾶς ἢ
ΔΒ, ἢ δὲ ἐκ τοῦ κέντρου ἡ ΒΖ, καὶ ἡ δεδομὲνη
ἡ ΖΘ. κατηντήσαμεν ἀρα, φησίν, εἰς τὸ "τὴν
ΔΖ τεμεῖν κατὰ τὸ Χ, ὡστε εἶναι, ὡς τὴν ΧΖ
πρὸς τὴν δοθείσαν, οὕτως τὸ δοθὲν πρὸς τὸ ἀπὸ
tῆς ΔΧ. τούτῳ δὲ ἀπλῶς μὲν λεγόμενον ἔχει
dιορισμὸν." εἰ γὰρ τὸ δοθὲν ἐπὶ τὴν δοθείσαν
μεῖζον ἐνώγχανε τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ,
ἀδύνατον ἦν τὸ πρόβλημα, ὡς δεδεικται, εἰ δὲ
ἰσον, τὸ Β σημεῖον ἔποιει τὸ πρόβλημα, καὶ οὕτως
δὲ οὐδὲν ἦν πρὸς τὴν ἐξ ἀρχῆς Ἀρχιμήθους πρό-
θεσιν: ἢ γὰρ σφαῖρα οὐκ ἐτέμνετο εἰς τὸν δοθέντα

a Archimedes' figure is re-drawn (v. page 162) so that
B, Z come on the left of the figure and Δ on the right;
instead of B, Z on the right and Δ on the left.
b v. supra, p. 133.
problem, and \( BS^2 \cdot \Sigma A = Bz^2 \cdot \xi A \), as is clear from the above proof. Inasmuch as it is possible to take on \( BA \) two points satisfying what is sought, it is permissible to take whichever one wills, either the point between \( E, B \) or that between \( E, A \). If we choose the point between \( E, B \), the parabola described through the points \( H, T \) will, as stated, cut the hyperbola in two points; of these the one nearer to \( H \), that is to the axis of the parabola, will determine the point between \( E, B \), as in this case \( T \) determines \( \Sigma \), while the point farther away will determine the point between \( E, A \), as in this case \( P \) determines \( \xi \).

The analysis and synthesis of the general problem have thus been completed; but in order that it may be harmonized with Archimedes' words, let there be conceived, as in Archimedes' own figure, a diameter \( DB \) of the sphere, with radius [equal to] \( BZ \), and a given straight line \( Z\Theta \). We are therefore faced with the problem, he says, "so to cut \( DZ \) at \( X \) that \( XZ \) bears to the given straight line the same ratio as the given area bears to the square on \( DX \). When the problem is stated in this general form, it is necessary to investigate the limits of possibility." If therefore the product of the given area and the given straight line chanced to be greater than \( DB^2 \cdot BZ \), the problem would not admit a solution, as was proved, and if it were equal the point \( B \) would satisfy the conditions of the problem, which also would be of no avail for the purpose Archimedes set himself at the outset; for the sphere would not be

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For \( DB = \frac{3}{4} DZ \) [ex hyp.], and so \( DB \) in the figure on p. 162 corresponds with \( BE \) in the figure on p. 146, while \( BZ \) in the figure on p. 162 corresponds with \( EA \) in the figure on p. 146.
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λόγον. ἀπλῶς ἂρα λεγόμενον εἶχεν προσδιορισμόν.
"προστιθεμένων δὲ τῶν προβλημάτων τῶν ἑνήδε

υπαρχόντων," τοιτέστι τοῦ τε διπλασίαν εἶναι τὴν
ΔΒ τῆς ΖΒ καὶ τοῦ μείζονα εἶναι τὴν βΖ τῆς ΖΘ,
"οὖν ἔχει διορισμόν." τὸ γάρ ἀπὸ ΔΒ τὸ δοθὲν
ἐπὶ τὴν ΖΘ τὴν δοθεῖσαν ἑλαττών ἔστι τοῦ ἀπὸ
tῆς ΔΒ ἐπὶ τὴν βΖ διὰ τὸ τὴν βΖ τῆς ΖΘ μείζονα
eἶναι, οὔτερ ὑπάρχοντος ἐδείξαμεν δυνατὸν, καὶ
ὅπως προβαίνει τὸ πρόβλημα.

* Eutocius proceeds to give solutions of the problem by Dionysodorus and Dioecles, by whose time, as he has explained, Archimedes' own solution had already disappeared. Dionysodorus solves the less general equation by means of the intersection of a parabola and a rectangular hyperbola; Dioecles solves the general problem by the intersection of an ellipse with a rectangular hyperbola, and his proof is both ingenious and intricate. Details may be consulted in Heath, H.G.M. ii. 46-49 and more fully in Heath, 162
cut in the given ratio. Therefore when the problem was stated generally, an investigation of the limits of possibility was necessary as well; "but under the conditions of the present case," that is, if $\Delta B = 2ZB$ and $BZ > Z\Theta$, "no such investigation is necessary." For the product of the given area $\Delta B^2$ into the given straight line $Z\Theta$ is less than the product of $\Delta B^2$ into $BZ$ by reason of the fact that $BZ$ is greater than $Z\Theta$, and we have shown that in this case there is a solution, and how it can be effected.  

The Works of Archimedes, pp. cxxiii-cxlii, which deals with the whole subject of cubic equations in Greek mathematical history. It is there pointed out that the problem of finding mean proportionals is equivalent to the solution of a pure cubic equation, $\frac{a^2}{x} = \frac{a}{b}$, and that Menaechmus's solution, by the intersection of two conic sections (v. vol. i. pp. 278-283), is the precursor of the method adopted by Archimedes, Dionysodorus and Diocles. The solution of cubic equations by means of conics was, no doubt, found easier than a solution by the manipulation of parallelepipeds, which would have been analogous to the solution of quadratic equations by the application of areas (v. vol. i. pp. 186-215). No other examples of the solution of cubic equations have survived, but in his preface to the book On Conoids and Spheroids Archimedes says the results there obtained can be used to solve other problems, including the following, "from a given spheroidal figure or conoid to cut off, by a plane drawn parallel to a given plane, a segment which shall be equal to a given cone or cylinder, or to a given sphere" (Archim. ed. Heiberg i. 258. 11-15); the case of the paraboloid of revolution does not lead to a cubic equation, but those of the spheroid and hyperboloid of revolution do lead to cubics, which Archimedes may be presumed to have solved. The conclusion reached by Heath is that Archimedes solved completely, so far as the real roots are concerned, a cubic equation in which the term in $x$ is absent: and as all cubic equations can be reduced to this form, he may be regarded as having solved geometrically the general cubic.
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(d) CONOIDS AND SPHEROIDS

(i.) Preface


'Αρχιμήδης Δοσιθέω εὖ πράττεω.
'Αποστέλλω τοι γράφας ἐν τῷ διὰ τῶν βιβλίων τῶν τε λοιπῶν θεωρημάτων τὰς ἀποδείξεις, ὅιον ous eixes εν τοῖς πρότερον ἀπεσταλμένοις, καὶ ἄλλων ὑστερον ποτεξευρημένων, ἀ πρότερον μὲν ἡδη πολλάκις ἐγχειρήσας ἐπισκέπτεσθαι δύσκολον ἐχειν τι φανείσας μοι τὰς εὐφέσιος αὐτῶν ἀπόρησα: διόπερ οὔδε συνεξεύθεντο τοῖς ἄλλοις αὐτά τὰ προ- βεβλημένα. ὑστερον δὲ ἐπιμελέστερον ποτ' αὐτοῖς γενόμενος ἐξεύρον τὰ ἀπορηθέντα. ἦν δὲ τὰ μὲν λοιπὰ τῶν προτέρων θεωρημάτων περὶ τοῦ ὀρθο- γωνίου κωνοειδέους προβεβλημένα, τὰ δὲ νῦν ἐντι ποτεξευρημένα περὶ τε ἀμβλυγωνίου κωνοειδέους καὶ περὶ σφαιροειδέων σχημάτων, ὅν τὰ μὲν παραμάκεα, τὰ δὲ ἐπιπλατέα καλέω.

(ii.) Two Lemmas

Ibid., Lemma ad Prop. 1, Archim. ed. Heiberg i. 260. 17-24

Εἰ κα ἐὼντι μεγέθεα ὀποσαοῦν τῷ ἴσῳ ἀλλάλων

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* In the books On the Sphere and Cylinder, On Spirals and on the Quadrature of a Parabola.
* i.e., the paraboloid of revolution.
* i.e., the hyperboloid of revolution.
* An oblong spheroid is formed by the revolution of an
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(d) CONOIDS AND SPHEROIDS

(i.) Preface

Archimedes, On Conoids and Spheroids, Preface,
Archim. ed. Heiberg i. 246. 1-14

Archimedes to Dositheus greeting.
I have written out and now send you in this book
the proofs of the remaining theorems, which you did
not have among those sent to you before, and also
of some others discovered later, which I had often
tried to investigate previously but their discovery
was attended with some difficulty and I was at a loss
over them; and for this reason not even the propo-
sitions themselves were forwarded with the rest. But
later, when I had studied them more carefully, I
discovered what had left me at a loss before. Now
the remainder of the earlier theorems were proposi-
tions about the right-angled conoid; but the dis-
coveries now added relate to an obtuse-angled conoid
and to spheroidal figures, of which I call some oblong
and others flat.

(ii.) Two Lemmas

Ibid., Lemma to Prop. 1, Archim. ed. Heiberg
i. 260. 17-24

If there be a series of magnitudes, as many as you
please, in which each term exceeds the previous term by an
ellipse about its major axis, a flat spheroid by the revolution
of an ellipse about its minor axis.

In the remainder of our preface Archimedes gives a number
of definitions connected with those solids. They are of
importance in studying the development of Greek mathe-
matical terminology.
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υπερέχοντα, ἢ δὲ ἢ υπεροχὰ ὑπα τῷ ἑλαχίστῳ, καὶ ἄλλα μεγέθεα τῷ μὲν πλήθει ὑπα τοῦτοι, τῷ δὲ μεγέθει ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντα τὰ μεγέθεα, ὅπερ ἦστιν ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντων μὲν τῶν τῷ ἵσῳ υπερεχόντων ἔλασσονα ἐσσοῦνται ἡ διπλάσια, τῶν δὲ λοιπῶν χωρὶς τοῦ μεγίστου μείζονα ἡ διπλάσια. ἢ δὲ ἀπόδειξις τοῦτου φανερά.

Ibid., Prop. 1, Archim. ed. Heiberg i. 260. 26-261. 22

Εἴ κα μεγέθεα ὑποσαοῦν τῷ πλήθει ἄλλοις μεγέθεσιν ἵσοι τῷ πλήθει κατὰ δύο τὸν αὐτὸν λόγον ἔχοντες τὰ ὁμολόγα τεταγμένα, λέγηται δὲ τὰ τε πρῶτα μεγέθεα ποτ' ἄλλα μεγέθεα ἡ πάντα ἡ ἑναὐτῶν ἐν λόγοις ὑποισοῦσίν, καὶ τὰ ὑστερον ποτ' ἄλλα μεγέθεα τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, πάντα τὰ πρῶτα μεγέθεα ποτὶ πάντα, ἢ λέγονται, τῶν αὐτῶν ἐξούντες λόγον, ὅπερ ἔχοντες πάντα τὰ ὑστερον μεγέθεα ποτὶ πάντα, ἢ λέγονται.

*Εστώ τινὰ μεγέθεα τὰ Α, Β, Γ, Δ, Ε, Ζ ἄλλοις μεγέθεσιν ἱσοὺς τῷ πλήθει τοὺς, Η, Θ, Ι, Κ, Λ, Μ

* If \( h \) is the common difference, the first series is \( h, 2h, 3h \ldots nh \), and the second series is \( nh, nh \ldots \) to \( n \) terms, its sum obviously being \( n^2h \). The lemma asserts that

\[
2(h + 2h + 3h + \ldots n - 1h) < n^2h < 2(h + 2h + 3h + \ldots nh).
\]

It is proved in the book On Spirals, Prop. 11. The proof is geometrical, lines being placed side by side to represent the

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equal quantity, which common difference is equal to the least term, and if there be a second series of magnitudes, equal to the first in number, in which each term is equal to the greatest term [in the first series], the sum of the magnitudes in the series in which each term is equal to the greatest term is less than double of the sum of the magnitudes differing by an equal quantity, but greater than double of the sum of those magnitudes less the greatest term. The proof of this is clear.\(^3\)


If there be a series of magnitudes, as many as you please, and it be equal in number to another series of magnitudes, and the terms have the same ratio two by two, and if any or all of the first series of magnitudes form any proportion with another series of magnitudes, and if the second series of magnitudes form the same proportion with the corresponding terms of another series of magnitudes, the sum of the first series of magnitudes bears to the sum of those with which they are in proportion the same ratio as the sum of the second series of magnitudes bears to the sums of the terms with which they are in proportion.

Let the series of magnitudes A, B, Γ, Δ, E, Z be equal in number to the series of magnitudes H, Θ, I, terms of the arithmetical progression and produced until each is equal to the greatest term. It is equivalent to this algebraical proof:

Let \( S_n = h + 2h + 3h + \ldots + nh \).

Then \( S_n = nh + (n - 1)h + (n - 2)h + \ldots + h \).

Adding, \( 2S_n = n(n + 1)h \),

and so \( 2S_{n-1} = (n - 1)nh \).

Therefore \( 2S_{n-1} < n^2h < 2Sn \).
κατὰ δύο τὸν αὐτὸν ἔχοντα λόγον, καὶ ἐξέτω τὸ μὲν Α ποτὶ τὸ Β τὸν αὐτὸν λόγον, ὡς τὸ Η ποτὶ τὸ Θ, τὸ δὲ Β ποτὶ τὸ Γ, ὡς τὸ Θ ποτὶ τὸ Ι, καὶ τὰ ἄλλα ὀμοίως τούτοις, λεγέσθω δὲ τὰ μὲν Α, Β, Γ, Δ, Ε, Ζ μεγέθεα ποτ’ ἄλλα μεγέθεα τὰ Ν, Ξ, Ο, Π, Ρ, Σ ἐν λόγοις ὀποιοσοῦν, τὰ δὲ Η, Θ, Ι, Κ, Λ, Μ ποτ’ ἄλλα τὰ Τ, Υ, Φ, Χ, Ψ, Ω, τὰ ὀμόλογα ἐν τοῖς αὐτοῖς λόγοις, καὶ ὡς μὲν ἔχει λόγον τὸ Α ποτὶ τὸ Ν, τὸ Η ἐξέτω ποτὶ τὸ Τ, ὡς δὲ λόγον ἔχει τὸ Β ποτὶ τὸ Ξ, τὸ Θ ἐξέτω ποτὶ τὸ Υ, καὶ τὰ ἄλλα ὀμοίως τούτοις δείκτεον, ὅτι πάντα τὰ Α, Β, Γ, Δ, Ε, Ζ ποτὶ πάντα τὰ Ν, Ξ, Ο, Π, Ρ, Σ τὸν αὐτὸν ἔχοντι λόγον, ὡς πάντα τὰ Η, Θ, Ι, Κ, Λ, Μ ποτὶ πάντα τὰ Τ, Υ, Φ, Χ, Ψ, Ω.


•• ex aequo

N : B = T : Θ. [Eucl. v. 22]

But B : Ξ = Θ : Υ; [ex hyp.]

•• ex aequo

N : Ξ = T : Υ. [Eucl. v. 22]

Similarly


Now since A : B = H : Θ, [ex hyp.]

•• componendo

A + B : A = H + Θ : H, [Eucl. v. 18]

i.e., permutando A + B : H + Θ = A : H. [Eucl. v. 16]

But since N : A = T : H, [ex hyp.]

••

A : H = N : T [Eucl. v. 16]

= Ξ : Υ [ibid.]

= Ο : Φ [ibid.]

= Γ : Ι. [ibid.]

••

A + B : H + Θ = Γ : Ι. [Eucl. v. 18]

••

A + B + Γ : H + Θ + I = Γ : Ι. [Eucl. v. 16]
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K, \Lambda, M, and let them have the same ratio two by two, so that

\[ A : B = H : \Theta, B : \Gamma = \Theta : I, \]

and so on, and let the series of magnitudes A, B, \Gamma, \Delta, E, Z form any proportion with another series of magnitudes N, \Xi, O, P, \Sigma, and let H, \Theta, I, K, \Lambda, M form the same proportion with the corresponding terms of another series, T, \Upsilon, \Phi, X, \Psi, \Omega so that

\[ A : N = H : T, B : \Xi = \Theta : \Upsilon, \]

and so on; it is required to prove that

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + \Theta + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega} \]

\[ = \Pi : X \quad \text{[ibid.]} \]

\[ = \Delta : K. \quad \text{[ibid.]} \]

By pursuing this method it may be proved that

\[ A + B + \Gamma + \Delta + E + Z : H + \Theta + I + K + \Lambda + M = A : H, \]

or, permutando,

\[ A + B + \Gamma + \Delta + E + Z : A = H + \Theta + I + K + \Lambda + M : H. \quad (1) \]

Now

\[ N : \Xi = T : \Upsilon; \]

\[ \therefore \text{componendo et permutando,} \]

\[ N + \Xi : T + \Upsilon = \Xi : \Upsilon \quad \text{[Eucl. v. 18, v. 16]} \]

\[ = \Theta : \Phi; \quad \text{[Eucl. v. 16]} \]

whence

\[ N + \Xi + O : T + \Upsilon + \Phi = O : \Phi, \quad \text{[Eucl. v. 18]} \]

and so on until we obtain

\[ N + \Xi + O + \Pi + \Theta + \Sigma : T + \Upsilon + \Phi + X + \Psi + \Omega = N : T. \quad \text{. (2)} \]

But

\[ A : N = H : T; \quad \text{[ex hyp.]} \]

\[ \therefore \text{by (1) and (2),} \]

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + \Theta + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}. \]

Q.E.D.
Πάν τμάμα ὀρθογωνίου κωνοειδέος ἀποτετμαμένον ἐπιπέδῳ ὀρθῷ ποτὶ τὸν ἄξονα ἡμιόλιον ἐστὶ τοῦ κώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν τῷ τμάματι καὶ ἄξονα.

"Εστω γὰρ τμάμα ὀρθογωνίου κωνοειδέος ἀποτετμαμένον ὀρθῷ ἐπιπέδῳ ποτὶ τὸν ἄξονα, καὶ τμαθέντος αὐτοῦ ἐπιπέδῳ ἄλλῳ διὰ τοῦ ἄξονος τὰς μὲν ἑπιφανείας τομὰ ἐστώ τὰ ΑΒΓ ὀρθογωνίου κώνου τομὰ, τοῦ δὲ ἐπιπέδου τοῦ ἀποτέμνοντος τὸ τμάμα τὰ ΓΑ εὐθεία, ἂξων δὲ ἐστω τοῦ τμάματος τὸ ΒΔ, ἐστώ δὲ καὶ κώνος τὰν αὐτὰν βάσιν ἔχων τῷ τμάματι καὶ ἄξονα τὸν αὐτοῦ, οὗ κορυφὰ τὸ Β. δεικτέον, ὅτι τὸ τμάμα τοῦ κωνοειδέος ἡμιόλιον ἐστὶ τοῦ κώνου τούτου.

Ἐκκείσθω γὰρ κώνος ὁ Ψ ἡμιόλιος ἔως τοῦ κώνου, οὗ βάσις ὁ περὶ διάμετρον τὰν ΑΓ, ἄξων δὲ τὸ ΒΔ, ἐστώ δὲ καὶ κύλινδρος βάσιν μὲν ἐχων
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(iii.) Volume of a Segment of a Paraboloid of Revolution

Ibid., Prop. 21, Archim. ed, Heiberg i. 344. 21–354. 20

Any segment of a right-angled conoid cut off by a plane perpendicular to the axis is one-and-a-half times the cone having the same base as the segment and the same axis.

For let there be a segment of a right-angled conoid cut off by a plane perpendicular to the axis, and let it be cut by another plane through the axis, and let the section be the section of a right-angled cone $A\Gamma \Pi$, and let $\Gamma \Lambda$ be a straight line in the plane cutting off the segment, and let $B\Lambda$ be the axis of the segment, and let there be a cone, with vertex $B$, having the same base and the same axis as the segment. It is required to prove that the segment of the conoid is one-and-a-half times this cone.

For let there be set out a cone $\Psi$ one-and-a-half times as great as the cone with base about the diameter $A\Pi$ and with axis $B\Lambda$, and let there be a

* It is proved in Prop. 11 that the section will be a parabola.
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tὸν κύκλον τὸν περὶ διάμετρον τὰν ἈΓ, ἡξονα δὲ τὰν ΒΔ· ἐσσεῖται οὖν ὁ Ἡ ἱμίσεος τοῦ κυλίνδρου [ἐπειτερ ἠμιόλιος ἐστιν ὁ Ἡ κῶνος τοῦ αὐτοῦ κῶνου].¹ λέγω, ὅτι τὸ τμῆμα τοῦ κωνοειδεὸς ἰσον ἐστὶ τῷ Ἡ κῶνῳ.

Εἰ γάρ μὴ ἐστιν ἰσον, ἦτοι μεῖζον ἐντὸ ἡ ἠλάσσον. ἐστὶ δὴ πρῶτον, εἰ δυνατόν, μεῖζον. ἐγγεγράφθω δὴ σχήμα στερεόν εἰς τὸ τμῆμα, καὶ ἄλλο περιγεγράφθω ἐκ κυλίνδρων υψὸς ἰσον ἐχόντων συγκειμενον, ὡστε τὸ περιγραφέν σχῆμα τοῦ ἐγγραφέντος υπερέχειν ἠλάσσον, ἡ ἀλίκω ὑπερέχει τὸ τοῦ κωνοειδεὸς τμῆμα τοῦ Ἡ κῶνου, καὶ ἐστὶν τῶν κυλίνδρων, ἐξ ὧν σύγκειται τὸ περιγραφέν σχῆμα, μέγιστος μὲν ὁ βάσις ἐχὼν τὸν κύκλον τὸν περὶ διάμετρον τὰν ἈΓ, ἡξονα δὲ τὰν ΕΔ, ἠλάχιστος δὲ ὁ βάσις μὲν ἐχὼν τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἡξονα δὲ τὰν ΒΙ, τῶν δὲ κυλίνδρων, ἐξ ὧν σύγκειται τὸ ἐγγραφέν σχῆμα, μέγιστος μὲν ἐστὶν ὁ βάσις ἐχὼν τὸν κύκλον τὸν περὶ διάμετρον τὰν ΚΔ, ἡξονα δὲ τὰν ΔΕ, ἠλάχιστος δὲ ὁ βάσις μὲν ἐχὼν τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἡξονα δὲ τὰν ΘΙ, ἐκβεβλήσθω δὲ τὰ ἐπὶπεδα πάντων τῶν κυλίνδρων ποτὶ τὰν

¹ ἐπειτερ ... κῶνου om. Heiberg.

¹ For the cylinder is three times, and the cone Ἡ one-and-a-

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cylinder having for its base the circle about the
diameter $\Delta\Gamma$ and for its axis $\Delta\Lambda$; then the cone $\Psi$ is
one-half of the cylinder $\alpha$; I say that the segment of
the conoid is equal to the cone $\Psi$.

If it be not equal, it is either greater or less. Let
it first be, if possible, greater. Then let there be
inscribed in the segment a solid figure and let there
be circumscribed another solid figure made up of
cylinders having an equal altitude, $b$ in such a way that
the circumscribed figure exceeds the inscribed figure
by a quantity less than that by which the segment
of the conoid exceeds the cone $\Psi$ [Prop. 19]; and let
the greatest of the cylinders composing the circum-
scribed figure be that having for its base the circle
about the diameter $\Delta\Gamma$ and for axis $\Delta\Lambda$, and let the
least be that having for its base the circle about
the diameter $\Sigma\Theta$ and for axis $\Theta\Lambda$; and let the greatest
of the cylinders composing the inscribed figure be
that having for its base the circle about the diameter
$\Lambda\Delta$ and for axis $\Delta\Theta$, and let the least be that having
for its base the circle about the diameter $\Sigma\Theta$ and for
axis $\Theta\Lambda$; and let the planes of all the cylinders be
half times, as great as the same cone; but because τοῦ $\alpha$ $\delta$ $\kappa$ $\omega$ $\nu$ is obscure and επιστευô often introduces an interpola-
tion, Heiberg rejects the explanation to this effect in the text.

$^b$ Archimedes has used those inscribed and circumscribed
figures in previous propositions. The paraboloid is gen-
erated by the revolution of the parabola $\Delta\Gamma$ about its axis $\Delta\Lambda$.
Chords $\Delta\Lambda\ldots\Sigma\Theta$ are drawn in the parabola at right angles
to the axis and at equal intervals from each other. From
the points where they meet the parabola, perpendiculars
are drawn to the next chords. In this way there are built up
inside and outside the parabola "staggered" figures con-
sisting of decreasing rectangles. When the parabola re-
volves, the rectangles become cylinders, and the segment of
the paraboloid lies between the inscribed set of cylinders and
the circumscribed set of cylinders.
ἐπιφάνειαν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τοῦ κύκλου τὸν περὶ διάμετρον τῶν ΑΓ, ἄξονα δὲ τῶν ΒΔ· ἐσεῖται δὴ ὁ ὅλος κυλίνδρος διηρήμενος εἰς κυλίνδρους τῷ μὲν πλήθει ἰσούς τοῖς κυλίνδροις τοῖς ἐν τῷ περιγεγραμμένῳ σχήματι, τῷ δὲ με-γέθει ἰσούς τῷ μεγίστῳ αὐτῶν. καὶ ἐπεὶ τὸ περιγεγραμμένον σχῆμα περὶ τὸ τμῆμα ἔλασσον ὑπερέχει τοῦ ἐγγεγραμμένου σχήματος ἢ τὸ τμῆμα τοῦ κόνου, δήλον, ὅτι καὶ τὸ ἐγγεγραμμένον σχῆμα ἐν τῷ τμῆματι μεῖζον ἐστὶ τοῦ Ψ κόνου. ὁ δὲ πρῶτος κυλίνδρος τῶν ἐν τῷ ὅλω κυλίνδρῳ ὁ ἐχων ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κυλίνδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν ἔχοντα ἄξονα τὰν ΔΕ τὸν αὐτὸν ἔχει λόγον, ὥν ἁ ΔA ποτὶ τὰν KE δυνάμει· οὕτως δὲ ἐστὶν ὁ αὐτὸς τῷ, ὅν ἔχει ἁ ΒΔ ποτὶ τὰν BE, καὶ τῷ, ὅν ἔχει ἁ ΔΑ ποτὶ τὰν ΕΣ. ὁμοίως δὲ δειχθῆσαι καὶ ὁ δεύτερος κυλίνδρος τῶν ἐν τῷ ὅλω κυλίνδρῳ ὁ ἐχων ἄξονα τὸν EZ ποτὶ τὸν δεύτερον κυλίνδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν αὐτὸν ἔχειν λόγον, ὅν ἁ ΠΕ, τούτεστιν ἁ ΔΑ, ποτὶ τὰν ΖΟ, καὶ τῶν ἀλλῶν κυλίνδρων ἐκαστός τῶν ἐν τῷ ὅλω κυλίνδρῳ ἄξονα ἔχοντων ἵσον τὰ ΔΕ ποτὶ ἐκαστὸν τῶν κυλίνδρων τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι ἄξονα ἔχοντων τοῦ αὐτοῦ ἔχει τούτον τὸν λόγον, ὅν ἁ ἡμίσεια τὰς διαμέτρου ταῖς βάσισις αὐτοῦ ποτὶ τὰν ἀπολειμμέναν ἀπ’ αὐτῶς μεταξὺ τῶν ΑΒ, ΒΔ εὐθείαν· καὶ πάντες οὖν οἱ κυλίνδροι οἱ ἐν τῷ κυλίνδρῳ, οὐ βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ διάμετρον τῶν ΑΓ, ἄξων δὲ [ἔστιν]1 ἁ ΔΙ εὐθεία, ποτὶ πάντος τοῦς κυλίνδρους τοὺς ἐν τῷ ἐγ-γεγραμμένῳ σχήματι τοῦ αὐτοῦ ἐξοῦντε λόγον, ὥν

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produced to the surface of the cylinder having for its base the circle about the diameter $\Delta \Gamma$ and for axis $B\Delta$; then the whole cylinder is divided into cylinders equal in number to the cylinders in the circumscribed figure and in magnitude equal to the greatest of them. And since the figure circumscribed about the segment exceeds the inscribed figure by a quantity less than that by which the segment exceeds the cone, it is clear that the figure inscribed in the segment is greater than the cone $\Psi$. Now the first cylinder of those in the whole cylinder, that having $\Delta E$ for its axis, bears to the first cylinder in the inscribed figure, which also has $\Delta E$ for its axis, the ratio $\Delta A^2 : KE^2$ [Eucl. xii. 11 and xii. 2]; but $\Delta A^2 : KE^2 = B\Delta : BE$. Similarly it may be proved that the second cylinder of those in the whole cylinder, that having $EZ$ for its axis, bears to the second cylinder in the inscribed figure the ratio $\Pi E : ZO$, that is, $\Delta A : ZO$, and each of the other cylinders in the whole cylinder, having its axis equal to $\Delta E$, bears to each of the cylinders in the inscribed figure, having the same axis in order, the same ratio as half the diameter of the base bears to the part cut off between the straight lines $AB$, $B\Delta$; and therefore the sum of the cylinders in the cylinder having for its base the circle about the diameter $\Delta \Gamma$ and for axis the straight line $\Delta I$ bears to the sum of the cylinders in the inscribed figure the same ratio as the sum of

* Because the circumscribed figure is greater than the segment.
* By the property of the parabola; v. Quadr. parab. 3.
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πᾶσαι αἱ εὐθείαι αἱ ἐκ τῶν κέντρων τῶν κύκλων, οἵ ἐντε βάσισε τῶν εἰρημένων κυλίνδρων, ποτε πᾶσας τὰς εὐθείας τὰς ἀπολελαμμένας ἀπ’ αὐτῶν μεταξὺ τῶν ΑΒ, ΒΔ. αἱ δὲ εἰρημέναι εὐθείαι τῶν εἰρημένων χωρίς τὰς ΑΔ μεῖζονες ἐντε ἦ διπλάσιαι: ὅστε καὶ οἱ κυλίνδροι πάντες οἱ ἐν τῷ κυλίνδρῳ, οὗ ἄξων ὁ ΔΙ, μεῖζονες ἐντε ἦ διπλασίων τοῦ ἐγγεγραμμένου σχήματος· πολλῷ ἄρα καὶ ὁ ὁλος κυλίνδρος, οὗ ἄξων ἀ ΔΒ, μεῖζον εντε ἦ διπλασίων τοῦ ἐγγεγραμμένου σχήματος. τοῦ δὲ Ψ κώνου ἦν διπλασίων ἔλασσον ἄρα τὸ ἐγγεγραμμένον σχῆμα τοῦ Ψ κώνου· ὅπερ ἀδύνατον· ἐδείχθη γὰρ μεῖζον. οὐκ ἀρα ἐστὶν μεῖζον τὸ κωνοείδες τοῦ Ψ κώνου.

"Ὅμως δέ οὐδὲ ἔλασσον· πάλιν γὰρ ἐγγεγράφθη τὸ σχῆμα καὶ περιγεγράφθη, ὡστε ὑπερέχειν [ἐκαστὸν] ἔλασσον, ἦ ἀλώ αὐτὸς ὑπερέχει ὁ Ψ κώνος τοῦ κωνοείδος, καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρότερον κατεσκευάσθω. ἐπεὶ οὖν ἔλασσον ἐστὶ τὸ ἐγγεγραμμένον σχῆμα τοῦ τμάματος, καὶ τὸ ἐγγραφέν τοῦ περιγραφέντος ἔλασσον λειτουργεῖ ἦ τὸ τμάμα τοῦ Ψ κώνου, δῆλον, ὡς ἔλασσον ἐστὶ τὸ περιγραφέν σχῆμα τοῦ Ψ κώνου. πάλιν δὲ ὁ

1. ἐκαστὸν om. Heiberg, ἐκαστὸν ἐκαστὸν Torelli (for ἐκάτερον ἐκατέρου).

<table>
<thead>
<tr>
<th>First cylinder in whole cylinder</th>
<th>ΔΑ</th>
</tr>
</thead>
<tbody>
<tr>
<td>First cylinder in inscribed figure</td>
<td>ΕΞ'</td>
</tr>
<tr>
<td>Second cylinder in whole cylinder</td>
<td>ΕΠ</td>
</tr>
<tr>
<td>Second cylinder in inscribed figure</td>
<td>ΖΟ'</td>
</tr>
</tbody>
</table>

and so on.

Whole cylinder = ΔΑ + ΕΠ + ... 
Inscribed figure = ΕΞ + ΖΟ + ...

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the radii of the circles, which are the bases of the aforesaid cylinders, bears to the sum of the straight lines cut off from them between AB, BD. But the sum of the aforesaid straight lines is greater than double of the aforesaid straight lines without AD; so that the sum of the cylinders in the cylinder whose axis is DI is greater than double of the inscribed figure; therefore the whole cylinder, whose axis is DB, is greater by far than double of the inscribed figure. But it was double of the cone Ψ; therefore the inscribed figure is less than the cone Ψ; which is impossible, for it was proved to be greater. Therefore the conoid is not greater than the cone Ψ.

Similarly [it can be shown] not to be less; for let the figure be again inscribed and another circumscribed so that the excess is less than that by which the cone Ψ exceeds the conoid, and let the rest of the construction be as before. Then because the inscribed figure is less than the segment, and the inscribed figure is less than the circumscribed by some quantity less than the difference between the segment and the cone Ψ, it is clear that the circumscribed figure is less than the cone Ψ. Again, the first

This follows from Prop. 1, for

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = \frac{\Delta A}{EΠ},
\]

and so on, and thus the other condition of the theorem is satisfied.

For ΔA, EΞ, ZO... is a series diminishing in arithmetical progression, and ΔA, EΠ... is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression. Therefore, by the Lemma to Prop. 1,

\[
ΔA + EΠ + ... > 2(ΕΞ + ZO + ...).
\]
πρώτος κύλινδρος τῶν ἐν τῷ ὀλλῳ κυλίνδρῳ ὁ ἐχὼν ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχήματι τὸν τῶν αὐτῶν ἔχοντα ἄξονα τὰν ΔΕ τὸν αὐτῶν ἔχει λόγον, ὅν τῷ ἀπὸ τὰς ΑΔ τετράγωνον ποτὶ τὸ αὐτό, ὁ δὲ δεύτερος κύλινδρος τῶν ἐν τῷ ὀλλῳ κυλίνδρῳ ὁ ἐχὼν ἄξονα τὰν ΔΕ ποτὶ τὸν δεύτερον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχήματι τὸν ἐχοντα ἄξονα τὰν ΔΕ τὸν αὐτῶν ἔχει λόγον, ὅν ἄ ΔΑ ποτὶ τὰν ΚΕ δυνάμει: οὐτος δὲ ἐστιν ὁ αὐτὸς τῷ, ὅν ἔχει ἄ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, ὅν ἔχει ἄ ΔΑ ποτὶ τὰν ΕΣ· καὶ τῶν ἀλλων κυλίνδρων ἐκαστὸς τῶν ἐν τῷ ὀλλῳ κυλίνδρῳ ἄξονα ἔχοντων ἵσον τὰ ΔΕ ποτὶ ἐκαστὸν τῶν κυλίνδρων τῶν ἐν τῷ περιγεγραμμένῳ σχήματι ἄξονα ἔχοντων τὸν αὐτῶν, ἔχει τούτον τὸν λόγον, ὅν ἄ ἡμίσεια τὰς διαμέτρου τὰς βάσιας αὐτῶν ποτὶ τὰν ἀπολελαμμέναν ἀπ’ αὐτᾶς μεταξὺ τῶν ἈΒ, ΒΔ εὐθείαν· καὶ πάντες οὖν οἱ κυλίνδροι οἱ ἐν τῷ ὀλλῳ κυλίνδρῳ, οὐ ἄξων ἐστιν ἄ ΒΔ εὐθεία, ποτὶ πάντας τοὺς κυλίνδρους τοὺς ἐν τῷ περιγεγραμμένῳ σχήματι τὸν αὐτὸν ἔχοντι λόγον, ὅν πᾶσα οἱ εὐθείαι ποτὶ πᾶσας τὰς εὐθείας· αἱ δὲ εὐθείαι πᾶσαι αἱ ἐκ τῶν κέντρων τῶν κύλων, οἱ βάσιες ἐντὶ τῶν κυλίνδρων, τῶν εὐθείαις πᾶσας τὰν ἀπολελαμμέναν ἀπ’ αὐτῶν σὺν τῷ ΔΑ ἐλάσσονες ἐντὶ

* As before,

First cylinder in whole cylinder \(=\) \(\Delta A\)
First cylinder in circumscribed figure \(=\) \(\Delta A\)
Second cylinder in whole cylinder \(=\) \(\Delta A\) \(\varepsilon\gamma\)
Second cylinder in circumscribed figure \(=\) \(\varepsilon\gamma\) \(\varepsilon\gamma\)

and so on.

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cylinder of those in the whole cylinder, having $\Delta E$ for its axis, bears to the first cylinder of those in the circumscribed figure, having the same axis $E\Delta$, the ratio $\Delta A^2 : \Delta A^2$; the second cylinder in the whole cylinder, having $EZ$ for its axis, bears to the second cylinder in the circumscribed figure, having $EZ$ also for its axis, the ratio $\Delta A^2 : KE^2$; this is the same as $BD : BE$, and this is the same as $\Delta A : E\Xi$; and each of the other cylinders in the whole cylinder, having its axis equal to $\Delta E$, will bear to the corresponding cylinder in the circumscribed figure, having the same axis, the same ratio as half the diameter of the base bears to the portion cut off from it between the straight lines $AB$, $BD$; and therefore the sum of the cylinders in the whole cylinder, whose axis is the straight line $BD$, bears to the sum of the cylinders in the circumscribed figure the same ratio as the sum of the one set of straight lines bears to the sum of the other set of straight lines.\textsuperscript{a} But the sum of the radii of the circles which are the bases of the cylinders is less than double of the sum of the straight lines cut off from them together with $\Delta A$\textsuperscript{b}; it is therefore clear

And

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = 1 = \frac{\Delta A}{E\Xi}.
\]

and so on.

Therefore the conditions of Prop. 1 are satisfied and

\[
\frac{\Delta A + E\Pi + \ldots}{\text{Whole cylinder}} = \frac{\Delta A + E\Xi + \ldots}{\text{Circumscribed figure}}.
\]

\textsuperscript{a} As before, $\Delta A$, $E\Xi$ \ldots is a series diminishing in arithmetical progression, and $\Delta A$, $E\Pi$ \ldots is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression.

Therefore, by the Lemma to Prop. 1,

\[
\Delta A + E\Pi + \ldots < 2(\Delta A + E\Xi + \ldots).
\]
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η διπλάσιαι: δήλον οὖν, οτι και οι κυλίνδροι πάντες οἱ ἐν τῷ ὅλῳ κυλίνδρῳ ἐλάσσονες ἐντὸ ἡ διπλάσιοι τῶν κυλίνδρων τῶν ἐν τῷ περιγεγραμμένῳ σχήματι· ὁ ἀρα κύλινδρος ὁ βάσις ἔχων τὸν κύκλον τὸν περὶ διάμετρον τῶν ΑΓ, ἀξόνα δὲ τὰν ΒΔ, ἐλάσσων ἐστὶν ἡ διπλάσιοι τοῦ περιγεγραμμένου σχήματος. οὐκ ἐστὶ δὲ, ἀλλὰ μείζων η διπλάσιος· τοῦ γὰρ Ψ κώνου διπλάσιον ἐστί, τὸ δὲ περιγεγ-γραμμένον σχῆμα ἐλάσσον ἐδείχθη τοῦ Ψ κώνου. οὐκ ἀρα ἐστὶν οὐδὲ ἐλάσσων τὸ τοῦ κωνοειδέος τμῆμα τοῦ Ψ κώνου. ἐδείχθη δὲ, ὅτι οὐδὲ μείζον· ἡμιόλιον ἀρα ἐστὶν τοῦ κώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν τῷ τμῆματι καὶ ἀξόνα τὸν αὐτὸν.

* Archimedes’ proof may be shown to be equivalent to an integration, as Heath has done (The Works of Archimedes, cxivii-cxlviii).

For, if n be the number of cylinders in the whole cylinder, and \( AΔ = nh \), Archimedes has shown that

\[
\text{Whole cylinder} \quad = \quad \frac{n^2h}{h + 2h + 3h + \ldots + (n-1)h} > 2, \quad \text{[Lemma to Prop. 1]}
\]

and

\[
\text{Circumscribed figure} \quad = \quad \frac{n^2h}{n + 2h + 3h + \ldots + nh} < 2. \quad \text{[ibid.}]
\]

In Props. 19 and 20 he has meanwhile shown that, by increasing a sufficiently, the inscribed and circumscribed figures can be made to differ by less than any assigned volume.

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that the sum of all the cylinders in the whole cylinder is less than double of the cylinders in the circumscribed figure; therefore the cylinder having for its base the circle about the diameter $\Delta \Gamma$ and for axis $\Delta \Delta$ is less than double of the circumscribed figure. But it is not, for it is greater than double; for it is double of the cone $\Psi$, and the circumscribed figure was proved to be less than the cone $\Psi$. Therefore the segment of the conoid is not less than the cone $\Psi$. But it was proved not to be greater; therefore it is one-and-a-half times the cone having the same base as the segment and the same axis.$^a$

When $n$ is increased, $h$ is diminished, but their product remains constant; let $nh = c$.

Then the proof is equivalent to an assertion that, when $n$ is indefinitely increased,

$$\lim_{n \to \infty} h(nh + 2h + 3h + \ldots + (n - 1)h) = \frac{1}{2} c^2,$$

which, in the notation of the integral calculus reads,

$$\int_0^c xdx = \frac{1}{2} c^2.$$

If the paraboloid is formed by the revolution of the parabola $y^2 = ax$ about its axis, we should express the volume of a segment as

$$\int_0^c \pi y^2 dx,$$

or

$$\pi a \int_0^c xdx.$$

The constant does not appear in Archimedes' proof because he merely compares the volume of the segment with the cone, and does not give its absolute value. But his method is seen to be equivalent to a genuine integration.

As in other cases, Archimedes refrains from the final step of making the divisions in his circumscribed and inscribed figures indefinitely large; he proceeds by the orthodox method of *reductio ad absurdum*.
(e) The Spiral of Archimedes

(i.) Definitions

44. 17–46. 21

α'. Εἰ κα εὐθεία ἐπὶ ἐπάνω γραμμὴ ἐν ἐπιπέδῳ καὶ μένοντος τοῦ ἐτέρου πέρατος αὐτᾶς ἱσοταχέως περιενεχθεῖσα ὀσακισοῦν ἀποκατασταθῇ πάλιν, οὐκ ἐξήλθασαν, ἀμα δὲ τὰ γραμμά περιαγομένα φέρηται τι σαμεῖον ἱσοταχέως αὐτὸ ἑαυτῷ κατὰ τὰς εὐθείας ἀρξάμενον ἀπὸ τοῦ μένοντος πέρατος, τὸ σαμεῖον ἔλικα γράφει ἐν τῷ ἐπιπέδῳ.

β'. Καλείσθω ὁν τὸ μὲν πέρας τὰς εὐθείας τὸ μένον περιαγομένας αὐτᾶς ἀρχὰ τὰς ἔλικος.

γ'. 'Α δὲ θέσις τὰς γραμμᾶς, ἂφ' ἂς ἄρξατο αἱ εὐθεία περιφέρεσθαι, ἀρχὰ τῆς περιφορᾶς.

δ'. Εὐθεία, ἅν μὲν ἐν τὰ πρῶτα περιφορᾷ διαπερευθῆ τὸ σαμεῖον τὸ κατὰ τὰς εὐθείας φερόμενον, πρῶτα καλείσθω, ἃν δὲ ἐν τὰ δευτέρα περιφορᾶ τὸ αὐτὸ σαμεῖον διανύσῃ, δευτέρα, καὶ αἱ ἀλλαι ὁμοίως ταύταις ὁμονύμως ταῖς περιφοραῖς καλείσθωσαν.

ε'. Τὸ δὲ χωρίον τὸ περιλαβθένῃ ὑπὸ τε τὰς ἔλικος τὰς ἐν τὰ πρῶτα περιφορᾶ γραφεῖσας καὶ τὰς εὐθείας, ἃ ἐστὶν πρῶτα, πρῶτον καλείσθω, τὸ δὲ περιλαβθένῃ ὑπὸ τέτας ἔλικος τὰς ἐν τὰ δευτέρα περιφορᾶ γραφεῖσας καὶ τὰς εὐθείας τὰς δευτέρας δεύτερον καλείσθω, καὶ τὰ ἀλλα ἔξη ὄντως καλείσθω.

ε'. Καὶ εἰ κα ἀπὸ τοῦ σαμεῖου, ὁ ἐστὶν ἀρχὰ τὰς ἔλικος, ἀχθῆ τις εὐθεία γραμμả, τὰς εὐθείας ταύτας 182
ARCHIMEDES

(e) The Spiral of Archimedes

(i.) Definitions

Archimedes, On Spirals, Definitions, Archim. ed. Heiberg ii. 44. 17–46. 21

1. If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a spiral in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the origin of the spiral.

3. Let the position of the line, from which the straight line began to revolve, be called the initial line of the revolution.

4. Let the distance along the straight line which the point moving along the straight line traverses in the first turn be called the first distance, let the distance which the same point traverses in the second turn be called the second distance, and in the same way let the other distances be called according to the number of turns.

5. Let the area comprised between the first turn of the spiral and the first distance be called the first area, let the area comprised between the second turn of the spiral and the second distance be called the second area, and let the remaining areas be so called in order.

6. And if any straight line be drawn from the origin, let [points] on the side of this straight line in
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tα ἐπὶ τὰ αὐτά, ἐφ' ἃ καὶ ἃ περιφορὰ γένηται, προαγούμενα καλείσθω, τὰ δὲ ἐπὶ θάτερα ἐπόμενα.

ζ'. Ὁ τε γραφεῖς κύκλος κέντρῳ μὲν τῶ σαμείῳ, ὃ ἐστὶν ἄρχα τὰς ἔλικος, διαστήματι δὲ τὰ εὐθεία, ἃ ἐστὶν πρῶτα, πρῶτος καλείσθω, ὃ δὲ γραφεῖς κέντρῳ μὲν τῷ αὐτῷ, διαστήματι δὲ τὰ διπλασία εὐθεία δεύτερος καλείσθω, καὶ οἱ ἄλλοι δὲ ἔξης τούτοις τὸν αὐτὸν τρόπον.

(ii.) Fundamental Property

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9-52. 15

Εἴ κα ποτὶ τὰν ἔλικα τὰν ἐν τὰ πρῶτα περιφορὰ γεγραμμέναν ποτιπεσῶντι δύο εὐθείαι απὸ τοῦ σαμείου, ὃ ἐστὶν ἄρχα τὰς ἔλικος, καὶ ἐκβληθέωντι ποτὶ τὰν τοῦ πρῶτου κύκλου περιφέρειαν, τὸν αὐτὸν ἐξούντι λόγον αἱ ποτὶ τὰν ἔλικα ποτιπεστοῦσαι ποτὶ ἄλλας, ὅν αἱ περιφέρειαι τοῦ κύκλου αἱ μεταξὺ τοῦ πέρατος τὰς ἔλικος καὶ τῶν περάτων τὰν ἐκβληθεισῶν εὐθείαιν τῶν ἐπὶ τὰς περιφερείας γνωμένων, ἐπὶ τὰ προαγούμενα λαμβανομενάν τὰν περιφερειάν ἀπὸ τοῦ πέρατος τὰς ἔλικος.

'Εστω ἔλιξ ἃ ΑΒΓΔΕΘ ἐν τὰ πρῶτα περιφορὰ γεγραμμένα, ἄρχα δὲ τὰς μὲν ἔλικος ἐστὶν τὸ Ἄ σαμείου, ἃ δὲ ΘΑ εὐθεία ἄρχα τὰς περιφορὰς ἐστὼ, καὶ κύκλος ὃ ΘΚΗ ἐστὼ ὁ πρῶτος, ποτιπεστοῦντων δὲ ἀπὸ τοῦ Α σαμείου ποτὶ τὰν ἔλικα αἱ ΑΕ, ΑΔ καὶ ἐκπεπτοτῶν ποτὶ τὰν τοῦ κύκλου περιφέρειαν ἐπὶ τὰ Z, H. δευκτέον, ὅτι τὸν αὐτὸν ἔγοντι λόγον ἃ ΑΕ ποτὶ τὰν ΑΔ, ὅν ἃ ΘΚΖ περιφέρεια ποτὶ τὰν ΘΚΗ περιφέρειαν.

Περιαγομενάς γὰρ τὰς ΑΘ γραμμᾶς δῆλον, ὡσ

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the direction of the revolution be called forward, and let those on the other side be called rearward.

7. Let the circle described with the origin as centre and the first distance as radius be called the first circle, let the circle described with the same centre and double of the radius of the first circle be called the second circle, and let the remaining circles in order be called after the same manner.

(ii.) Fundamental Property

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9–52. 15

If, from the origin of the spiral, two straight lines be drawn to meet the first turn of the spiral and produced to meet the circumference of the first circle, the lines drawn to the spiral will have the same ratio one to the other as the arcs of the circle between the extremity of the spiral and the extremities of the straight lines produced to meet the circumference, the arcs being measured in a forward direction from the extremity of the spiral.

Let $\Lambda \Theta \Gamma \Delta \Theta$ be the first turn of a spiral, let the point $A$ be the origin of the spiral, let $\Theta A$ be the initial line, let $\Theta K H$ be the first circle, and from the point $A$ let $\Lambda E$, $\Lambda \Delta$ be drawn to meet the spiral and be produced to meet the circumference of the circle at $Z$, $H$. It is required to prove that $\Lambda E : \Lambda \Delta = \text{arc } \Theta K Z : \text{arc } \Theta K H$.

When the line $A \Theta$ revolves it is clear that the point

* i.e., with radius equal to the sum of the radii of the first and second circles.
Α κατά τάς εὐθείας φερόμενον τάν ΑΘ γραμμάν πορεύεται, καὶ τὸ Θ σαμεῖον κατὰ τάς τοῦ κύκλου περιφερείας φερόμενον τάν ΘΚΖ περιφέρειαν, τὸ δὲ Α τῶν ΑΕ εὐθείαν, καὶ πάλιν τὸ τε Α σαμεῖον τάν ΑΔ γραμμάν καὶ τὸ Θ τῶν ΘΚΗ περιφέρειαν, ἐκάτερον ἱσοταχέως αὐτὸ ἑαυτῶ φερόμενον δῆλον οὖν, ὅτι τὸν αὐτῶν ἔχοντι λόγον ἀν ΑΕ ποτὶ τῶν ΑΔ, δὴν ἀ ΘΚΖ περιφέρεια ποτὶ τῶν ΘΚΗ περιφέρειαν [δεδεικται γὰρ τούτο ἐξω ἐν τοῖς πρώτοις].

Ομοίως δὲ δειχθῆσεται, καὶ εἰ καὶ ἀ ἔτερα τῶν ποτιπιπτοῦσαν ἐπὶ τὸ πέρας τάς ἑλικὸς ποτιπίτης, ὅτι τὸ αὐτὸ συμβαίνει.

(iii.) A Verging

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

Τῶν αὐτῶν δεδομένων καὶ τάς ἐν τῶ κύκλῳ εὐθείας ἐκβεβλημένας δυνατὸν ἐστὶν ἀπὸ τοῦ

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\( \Theta \) moves uniformly round the circumference \( \Theta \text{KH} \) of the circle while the point \( A \), which moves along the straight line, traverses the line \( \Lambda \Theta \); the point \( \Theta \) which moves round the circumference of the circle traverses the arc \( \Theta KZ \) while \( A \) traverses the straight line \( \Lambda E \); and furthermore the point \( A \) traverses the line \( A \Delta \) in the same time as \( \Theta \) traverses the arc \( \Theta \text{KH} \), each moving uniformly; it is clear, therefore, that \( AE : A \Delta = \text{arc } \Theta KZ : \text{arc } \Theta \text{KH} \) [Prop. 2].

Similarly it may be shown that if one of the straight lines be drawn to the extremity of the spiral the same conclusion follows.a

(iii.) A Verging b

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

With the same data and the chord in the circle produced,c it is possible to draw a line from the centre to meet

In Prop. 15 Archimedes shows (using different letters, however) that if \( AE, A \Delta \) are drawn to meet the second turn of the spiral, while \( AZ, AH \) are drawn, as before, to meet the circumference of the first circle, then

\[ AE : A \Delta = \text{arc } \Theta KZ + \text{circumference of first circle} : \text{arc } \Theta \text{KH} + \text{circumference of first circle} ,\]

and so on for higher turns.

In general, if \( E, \Delta \) lie on the \( n \)th turn of the spiral, and the circumference of the first circle is \( c \), then

\[ AE : A \Delta = \text{arc } \Theta KZ + n \cdot 1c : \text{arc } \Theta \text{KH} + n \cdot 1c .\]

These theorems correspond to the equation of the curve \( r = a \theta \) in polar co-ordinates.

b This theorem is essential to the one that follows.

c See n. a on this page.

1 δεικτιο... πρωτος om. Heiberg.
κέντρου ποτιβαλείν ποτι ταν ἐκβεβλημέναν, ὡστε ταν μεταξὺ τας περιφερείας και τας ἐκβεβλημένας ποτὶ τὰν ἐπιζευγθείσαν ἀπὸ τοῦ περάτου τᾶς ἐναπολαφθείσας ποτὶ τὸ πέρας τᾶς ἐκβεβλημένας τὸν ταχθέντα λόγον ἔχειν, εἰ καὶ ο ὁ δοθεὶς λόγος μεῖζων ἢ τοῦ, ὅν ἔχει ἀ ἡμίσεια τᾶς ἐν τῷ κύκλῳ δεδομένα ποτὶ τὰν ἀπὸ τοῦ κέντρου κάθετον ἐπ᾽ αὐτὰν ἀγμέναν.

Δεδόσθω τὰ αὐτὰ, καὶ ἔστω ἃ ἐν τῷ κύκλῳ γραμμά ἐκβεβλημένα, ὃ δὲ δοθεὶς λόγος ἔστω, ὅν ἔχει ἀ Ζ ποτὶ τὰν Η, μεῖζων τοῦ, ὅν ἔχει ἀ ΓΘ ποτὶ τὰν ΘΚ. μεῖζων οὖν ἐσσεῖται καὶ τοῦ, ὅν ἔχει ἀ ΚΓ ποτὶ ΓΛ. ὅν δὴ λόγον ἔχει ἀ Ζ ποτὶ Ἡ τούτων ἐξεῖ ἀ ΚΓ ποτὶ ἐλάσσονα τὰς ΓΛ. ἐξέτα ποτὶ ΙΝ νεύουσαν ἐπὶ τὸ Γ—δυνατὸν δὲ ἐστὶν οὔτως τέμνειν—καὶ πεσεῖται ἐντὸς τὰς ΓΛ, ἐπειδὴ ἐλάσσων ἐστὶ τὰς ΓΛ. ἐπεὶ οὖν τὸν αὐτὸν ἔχει λόγον ἀ ΚΓ ποτὶ ΙΝ, ὅν ἀ Ζ ποτὶ Ἡ, καὶ ἀ ΕΓ ποτὶ ΠΓ τὸν αὐτὸν ἐξει λόγον, ὅν ἀ Ζ ποτὶ τὰν Η.

* ΑΓ is a chord in a circle of centre K, and BN is the diameter drawn parallel to AG and produced. From Κ, ΚΘ is drawn perpendicular to ΑΓ, and ΓΛ is drawn perpendicular to ΚΓ so as to meet the diameter in Λ. Archimedes asserts that it is possible to draw KE to meet the circle in I and AG produced in E so that EI : Ig = Z : Η, an assigned ratio, provided that Z : H > ΓΘ : ΘΚ. The straight line ΓΙ meets ΒΛ in Ν. In Prop. 5 Archimedes has proved a similar proposition when ΑΓ is a tangent, and in Prop. 6 he has proved the proposition for the case where the positions of Ι, Γ are reversed.

* For triangle ΤΕΙ is similar to triangle ΚΙΝ, and therefore ΚΙ : IN = EI : IG [Eucl. vi. 4]; and ΚΙ = KG.

* The type of problem known as νέωσις, verging, has already been encountered (vol. i. p. 244 n. a). In this proposition, as in Props. 5 and 6, Archimedes gives no hint how
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the produced chord so that the distance between the circumference and the produced chord shall bear to the distance between the extremity of the line intercepted [by the circle] and the extremity of the produced chord an assigned ratio, provided that the given ratio is greater than that which half of the given chord in the circle bears to the perpendicular drawn to it from the centre.

Let the same things be given, and let the chord in the circle be produced, and let the given ratio be \( Z : H \), and let it be greater than \( \Gamma \Theta : OK \); therefore it will be greater than \( KG : \Gamma \Lambda \) [Eucl. vi. 4]. Then \( Z : H \) is equal to the ratio of \( KG \) to some line less than \( \Gamma \Lambda \) [Eucl. v. 10]. Let it be to \( IN \) verging upon \( \Gamma \)—for it is possible to make such an intercept—and \( IN \) will fall within \( \Gamma \Lambda \), since it is less than \( \Gamma \Lambda \). Then since

\[ KG : IN = Z : H, \]

therefore

\[ EI : II' = Z : H. \]

the construction is to be accomplished, though he was presumably familiar with a solution.

In the figure of the text, let \( T \) be the foot of the perpen-
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(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

Εἴ κα τάς ἐλικῶς τάς ἐν τὰ πρώτα περιφορὰ γεγραμμένας εὐθεία γραμμά ἐπισαύη μὴ κατά τὸ πέρας τάς ἐλικῶς, ἀπό δὲ τάς ἀφάς ἐπὶ τάν ἀρχὰν τάς ἐλικῶς εὐθεία ἐπιζευγθή, καὶ κέντρω μὲν τὰ ἀρχαὶ τάς ἐλικῶς, διαστήματι δὲ τὰ ἐπιζευγθείσα κύκλος γραφῆ, ἀπὸ δὲ τὰς ἀρχαὶς τάς ἐλικῶς ἀχθῆ τις ποτ’ ὀρθάς τὰ ἀπὸ τὰς ἀφάς ἐπὶ τὰν ἀρχαν τὰς ἐλικῶς ἐπιζευγθείσα, συμπεςεῖται αὐτὰ ποτὶ τάν ἐπισαύνουσαν, καὶ ἐσσεῖται ἀ μεταξὺ εὐθεία τάς τε συμπτώσιος καὶ τάς ἀρχαὶς τάς ἐλικῶς ὅσα τὰ περιφερεῖα τοῦ γραφέντος κύκλον τὰ μεταξὺ τὰς ἀφάς καὶ τὰς τομάς, καθ’ ἄν τέμνει ὁ γραφεῖς κύκλος τὰν ἀρχὰν τὰς περιφορᾶς, ἐπὶ τὰ προαγοῦμενα λαμβανομένας τὰς περιφερείας ἀπὸ τοῦ σαμείου τοῦ ἐν τὰ ἀρχαὶ τὰς περιφορᾶς.

"Εστω ἐλιξ, ἐφ’ ὡς ἀ ΑΒΓΔ, ἐν τὰ πρώτα περιφορὰ γεγραμμένα, καὶ ἐπισαύετω τις αὐτάς εὐθεία ἀ ΕΖ κατὰ τὸ Δ, ἀπὸ δὲ τοῦ Δ ποτὶ τὰν

dicular from Γ to ΒΛ, and let Δ be the other extremity of the diameter through B. Let the unknown length KN=x, let ΓT=a, KT=b, BD=2c, and let IN=k, a given length. Then

\[ \text{NI} \cdot \text{NG}=\text{ND} \cdot \text{NB}, \]

i.e., \[ k\sqrt{a^2+(x-b)^2}=(x-c)(x+c), \]

which, after rationalization, is an equation of the fourth degree in x.

Alternatively, if we denote NG by y, we can determine x and y by the two equations

\[ y^2=a^2+(x-b)^2, \]
\[ ky=x^2-c^2, \]
(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

If a straight line touch the first turn of the spiral other than at the extremity of the spiral, and from the point of contact a straight line be drawn to the origin, and with the origin as centre and this connecting line as radius a circle be drawn, and from the origin a straight line be drawn at right angles to the straight line joining the point of contact to the origin, it will meet the tangent, and the straight line between the point of meeting and the origin will be equal to the arc of the circle between the point of contact and the point in which the circle cuts the initial line, the arc being measured in the forward direction from the point on the initial line.

Let $AB\Gamma\Delta$ lie on the first turn of a spiral, and let

the straight line $EZ$ touch it at $\Delta$, and from $\Delta$ let $\Delta\Delta$

so that values of $x$ and $y$ satisfying the conditions of the problem are given by the points of intersection of a certain parabola and a certain hyperbola.

The whole question of *vergings*, including this problem, is admirably discussed by Heath, *The Works of Archimedes*, c-cxxii.
ἀρχάν τὰς ἐλικός ἐπεξεύχθω ἃ ΛΔ, καὶ κέντρῳ μὲν τῷ Α, διαστήματι δὲ τῷ ΛΔ κύκλος γεγράφθω ὁ ΔΜΝ, τεμνέτω δ’ οὕτος τὰν ἀρχάν τὰς περιφοράς κατὰ τὸ Κ, ἄχθω δὲ ἃ ΖΑ ποτὶ τὰν ΑΔ ὀρθά. ὅτι μὲν οὖν αὐτὰ συμπίπτει, δῆλον· ὅτι δὲ καὶ ᾧσα ἐστὶν ἃ ΖΑ εὐθεία τὰ ΚΜΝΔ περιφερεία, δεικτέον.

Εἰ γὰρ μὴ, ἦτοι μείζων ἐστὶν ἡ ἐλάσσων. ἐστώ, εἰ δυνατὸν, πρῶτερον μείζων, λελάφθω δὲ τις ἃ ΛΑ τὰς μὲν ΖΑ εὐθείας ἐλάσσων, τὰς δὲ ΚΜΝΔ περιφερειας μείζων, πάλιν δὴ κύκλος ἐστίν ὁ ΚΜΝ καὶ ἐν τῷ κύκλῳ γραμμὰ ἐλάσσων τὰς διαμέτρου ἃ ΝΔ καὶ λόγος, ὅπερ ἐκεῖ ἃ ΔΑ ποτὶ ΑΛ, μείζων τοῦ, δὲν ἐκεῖ ἀ ἡμίσεια τὰς ΝΔ ποτὶ τὰν ἀπὸ τοῦ Α κάθετον ἐν αὐτὰν ἁγμέναν· δυνατὸν οὖν ἐστιν ἀπὸ τοῦ Α ποτὶ βαλείν τὰς ΑΕ ποτὶ τὰς ΝΔ ἐκβεβλημέναν, ὡστε τὰς ΕΡ ποτὶ τὰς ΔΡ τὸν αὐτὸν ἐξείν λόγον, ὅπερ ἐκεῖ ἃ ΔΑ ποτὶ τὰς ΑΛ· δεδεικτα γὰρ τούτῳ δυνατὸν ἐν τοῖς ἐξεὶ οὖν καὶ ἃ ΕΡ ποτὶ τὰς ΔΡ τὸν αὐτὸν λόγον, ὅπερ ἐκεῖ ἃ ΔΡ ποτὶ τὰς ΑΛ. ἀ δὲ ΔΡ ποτὶ τὰς ΑΛ ἐλάσσωνα λόγον ἐχει ἃ ἃ ΔΡ περιφερεία ποτὶ τὰς ΚΜΔ περιφερείαν, ἐπεὶ ἃ μὲν ΔΡ ἐλάσσων ἐστὶ ς ΔΡ περιφερειας, ἀ δὲ ΑΛ μείζων τὰς ΚΜΔ περιφερειας· ἐλάσσων αὐτὸν λόγον ἐχει ἃ ΕΡ εὐθεία ποτὶ ΡΑ ἃ ἃ ΔΡ περιφερεία ποτὶ τὰς ΚΜΔ περιφερειαν· ὡστε καὶ ἃ ΑΕ ποτὶ ΔΡ ἐλάσσωνα λόγον ἐχει ἃ ἃ ΚΜΔ περιθ-

* For in Prop. 16 the angle ΛΔΖ was shown to be acute.
* For ΔΝ touches the spiral and so can have no part within the spiral, and therefore cannot pass through A; therefore it is a chord of the circle and less than the diameter.
* For, if a perpendicular be drawn from Α to ΔΝ, it bisects
be drawn to the origin, and with centre $A$ and radius $AA$ let the circle $AMN$ be described, and let this circle cut the initial line at $K$, and let $ZA$ be drawn at right angles to $AA$. That it will meet $[ZA]$ is clear\(^a\); it is required to prove that the straight line $ZA$ is equal to the arc $KMNA$.

If not, it is either greater or less. Let it first be, if possible, greater, and let $AA$ be taken less than the straight line $ZA$, but greater than the arc $KMNA$ [Prop. 4]. Again, $KMN$ is a circle, and in this circle $AN$ is a line less than the diameter,\(^b\) and the ratio $AA : AA$ is greater than the ratio of half $AN$ to the perpendicular drawn to it from $A$\(^c\); it is therefore possible to draw from $A$ a straight line $AE$ meeting $NA$ produced in such a way that

$$EP : AP = \Delta A : AA$$

for this has been proved possible [Prop. 7]; therefore

$$EP : AP = \Delta P : AA.$$

But

$$\Delta P : AA < \text{arc } \Delta P : \text{arc } KM\Delta,$$

since $\Delta P$ is less than the arc $\Delta P$, and $AA$ is greater than the arc $KM\Delta$;

$$\therefore \quad EP : PA < \text{arc } \Delta P : \text{arc } KM\Delta;$$

$$\therefore \quad AE : AP < \text{arc } KMP : \text{arc } KM\Delta.$$

[Eucl. v. 18]

$AN$ [Eucl. iii. 8] and divides triangle $\Delta AZ$ into two triangles of which one is similar to triangle $\Delta AZ$ [Eucl. vi. 8]; therefore

$$\Delta A : AZ = \frac{1}{2} NA : (\text{perpendicular from } A \text{ to } NA).$$

[Eucl. vi. 4]

But $AZ > AA$;

$$\therefore \quad \Delta A : AA > \frac{1}{2} NA : (\text{perpendicular from } A \text{ to } NA).$$

\(^a\) For $\Delta A = AP$, being a radius of the same circle; and the proportion follows *permutando*.
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φέρειαν. οὖν δὲ λόγον ἔχει ἀ ΚΜΡ ποτὶ τὰν ΚΜΔ περιφέρειαν, τούτον ἔχει ἀ ΧΑ ποτὶ ΛΔ· ἐλάσσουν ἀρα λόγον ἔχει ἀ ΕΑ ποτὶ ΑΡ ᾗ ἀ ΛΧ ποτὶ ΔΑ· ὑπὲρ ἐστὶν ἀδύνατον. οὐκ ἀρα μεῖζων ἀ ΖΑ τὰς ΚΜΔ περιφερείας. ὀμοίως δὲ τοῖς πρῶτοι δειχθῆσεται, ὅτι οὐδὲ ἐλάσσον ἐστίν· ἦσα ἀρα.

(f) SEMI-REGULAR SOLIDS

Papp. Coll. v. 19, ed. Hultsch i. 352. 7–354. 10

Πολλά γὰρ ἐπινοήσαι δυνατὸν στερέα σχήματα παντοῖα ἐπιφάνειας ἔχοντα, μᾶλλον δ’ ἀν τις αξιώσει λόγον τὰ τετάχθαι δοκοῦντα [καὶ τούτων πολὺ πλέον τοὺς τέ κάνους καὶ κυλίνδρους καὶ τὰ καλούμενα πολύεδρα].

ταῦτα δ’ ἐστὶν οὔ μόνον τὰ παρὰ τῷ θεωτάτῳ Πλάτωνι πέντε σχήματα, τουτέστιν τετράεδρόν τε καὶ ἕξαεδρόν, ὀκτάεδρόν τε καὶ δωδεκάεδρον, πέμπτον δ’ εἰκοσάεδρον, ἄλλα καὶ τὰ ὑπὸ Ἀρχιμήδους εὑρεθέντα τρισκαίδεκα τὸν ἄριθμὸν ὑπὸ ἰσοπλεύρων μὲν καὶ ἱσογωνίων οὐχ ὀμοίων δὲ πολυγώνων περιεχόμενα.

*καὶ...πολύεδρα om. Hultsch.*

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* This part of the proof involves a *verging* assumed in Prop. 8, just as the earlier part assumed the *verging* of Prop. 7. The verging of Prop. 8 has already been described (vol. i. p. 350 n. b) in connexion with Pappus's comments on it.

* Archimedes goes on to show that the theorem is true even if the tangent touches the spiral in its second or some higher turn, not at the extremity of the turn; and in Props. 18 and 19 he has shown that the theorem is true if the tangent should touch at an extremity of a turn.

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Now \[\text{arc KMP} : \text{arc KM} \Delta = \text{XA} : \text{A} \Delta;\] [Prop. 14

\[\therefore \quad \text{EA} : \text{AP} < \text{AX} : \text{AA};\]

which is impossible. Therefore \(ZA\) is not greater than the arc \(KM\Delta\). In the same way as above it may be shown to be not less\(^a\); therefore it is equal.\(^b\)

\[(f)\] Semi-Regular Solids

Pappus, Collection v. 19, ed. Hultsch i. 352. 7-354. 10

Although many solid figures having all kinds of surfaces can be conceived, those which appear to be regularly formed are most deserving of attention. Those include not only the five figures found in the godlike Plato, that is, the tetrahedron and the cube, the octahedron and the dodecahedron, and fifthly the icosahedron,\(^c\) but also the solids, thirteen in number, which were discovered by Archimedes\(^d\) and are contained by equilateral and equiangular, but not similar, polygons.

As Pappus (ed. Hultsch 302. 14-18) notes, the theorem can be established without recourse to propositions involving solid loci (for the meaning of which see vol. i. pp. 348-349), and proofs involving only “plane” methods have been developed by Tannery, Mémoires scientifiques, i., 1912, pp. 300-316 and Heath, H.G.M. ii. 556-561. It must remain a puzzle why Archimedes chose his particular method of proof, especially as Heath’s proof is suggested by the figures of Props. 6 and 9; Heath (loc. cit., p. 557) says “it is scarcely possible to assign any reason except his definite predilection for the form of proof by reductio ad absurdum based ultimately on his famous ‘Lemma’ or Axiom.”

\(^c\) For the five regular solids, see vol. i. pp. 216-225.

\(^d\) Heron (Definitions 104, ed. Heiberg 66. 1-9) asserts that two were known to Plato. One is that described as \(P_2\) below, but the other, said to be bounded by eight squares and six triangles, is wrongly given.
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Τὸ μὲν γὰρ πρῶτον ὀκτάεδρον ἐστὶν περιεχόμενον ὑπὸ τριγώνων δ καὶ ἕξαγωνων δ.

Τρία δὲ μετὰ τούτο τεσσαρεσκαίδεκαέδρα, ὡν τὸ μὲν πρῶτον περιέχεται τριγώνοις ἦ καὶ τετραγώνοις ἕ, τὸ δὲ δεύτερον τετραγώνοις ἕ καὶ ἕξαγωνοις ἦ, τὸ δὲ τρίτον τριγώνοις ἦ καὶ ὀκταγώνοις ἐ.

Μετὰ δὲ ταῦτα ἐκκατεικοσάεδρα ἐστὶν δύο, ὡν τὸ μὲν πρῶτον περιέχεται τριγώνοις ἦ καὶ τετραγώνοις ἦ, τὸ δὲ δεύτερον τετραγώνοις ἐβ, ἕξαγωνοις ἦ καὶ ὀκταγώνοις ἐ.

Μετὰ δὲ ταῦτα δυοκαταρικοτάεδρα ἐστὶν τρία, ὡν τὸ μὲν πρῶτον περιέχεται τριγώνοις κ καὶ πενταγώνοις β, τὸ δὲ δεύτερον πενταγώνοις β καὶ ἕξαγωνοις κ, τὸ δὲ τρίτον τριγώνοις κ καὶ δεκαγώνοις β.

Μετὰ δὲ ταῦτα ἐν ἐστὶν οκτωκαταρικοτάεδρον περιεχόμενον ὑπὸ τριγώνων β καὶ τετραγώνων ε.

Μετὰ δὲ τούτῳ δυοκαταεξηκοτάεδρα ἐστὶ δύο, ὡν τὸ μὲν πρῶτον περιέχεται τριγώνοις κ καὶ τετραγώνοις λ καὶ πενταγώνοις β, τὸ δὲ δεύτερον τετραγώνοις λ καὶ ἕξαγωνοις κ καὶ δεκαγώνοις β.

Μετὰ δὲ ταῦτα τελευταῖον ἐστὶν δυοκαταιεκατονταέδρον, δ περιέχεται τριγώνοις π καὶ πενταγώνοις β.

a For the purposes of n, b, the thirteen polyhedra will be designated as $P_1, P_2, \ldots, P_{13}$.

b Kepler, in his Harmonice mundi (Opera, 1864, v. 123-126), appears to have been the first to examine these figures systematically, though a method of obtaining some is given in a scholium to the Vatican ms. of Pappus. If a solid angle of a regular solid be cut by a plane so that the same length is cut off from each of the edges meeting at the solid angle,
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The first is a figure of eight bases, being contained by four triangles and four hexagons \([P_1]\).\(^6\)

After this come three figures of fourteen bases, the first contained by eight triangles and six squares \([P_2]\), the second by six squares and eight hexagons \([P_3]\), and the third by eight triangles and six octagons \([P_4]\).

After these come two figures of twenty-six bases, the first contained by eight triangles and eighteen squares \([P_5]\), the second by twelve squares, eight hexagons and six octagons \([P_6]\).

After these come three figures of thirty-two bases, the first contained by twenty triangles and twelve pentagons \([P_7]\), the second by twelve pentagons and twenty hexagons \([P_8]\), and the third by twenty triangles and twelve decagons \([P_9]\).

After these comes one figure of thirty-eight bases, being contained by thirty-two triangles and six squares \([P_{10}]\).

After this come two figures of sixty-two bases, the first contained by twenty triangles, thirty squares and twelve pentagons \([P_{11}]\), the second by thirty squares, twenty hexagons and twelve decagons \([P_{12}]\).

After these there comes lastly a figure of ninety-two bases, which is contained by eighty triangles and twelve pentagons \([P_{13}]\).\(^6\)

the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is composed of three, four or five plane angles. If certain equal lengths be cut off in this way from all the solid angles, regular polygons will also be left in the faces of the solid. This happens (i) obviously when the cutting planes bisect the edges of the solid, and (ii) when the cutting planes cut off a smaller length from each edge in such a way that a regular polygon is left in each face with double the number of sides. This method gives (1) from the tetrahedron, \(P_1\); (2) from the

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(g) System of expressing Large Numbers

Archim. Arith. 3, Archim. ed. Heiberg ii. 236. 17–240. 1

"Α μὲν οὖν ύποτίθεμαι, τάτα: χρήσιμον δὲ εἶμεν ύπολαμβάνον τὰν κατονόμασιν τῶν ἀριθμῶν ἑκάτερον, ὅπως καί τῶν ἄλλων οἱ τῷ βιβλίῳ μὴ περιτετευχότες τῷ ποτὶ Ζευξίππου γεγραμμένω μὴ πλανώνται διὰ τὸ μηδὲν εἴμεν ύπέρ αὐτᾶς ἐν τῷ τῷ βιβλίῳ προειρημένον. συμβαίνει δὴ τὰ ὀνόματα τῶν ἀριθμῶν ἐστὶ τὸ μὲν τῶν μυρίων ύπάρχειν ἀμῖν παραδεδομένα, καὶ ύπὲρ τὸ τῶν μυρίων [μὲν] ἀποχρεόντως γιγνώσκομες μυριάδων ἀριθμῶν λέγοντες ἐστε ποτὶ τὰς μυρίας μυριάδας. ἐστών οὖν ἀμῖν οἱ μὲν τῶν εἰρήμενοι ἀριθμοὶ ἐστὶ τὰς μυρίας μυριάδας πρῶτοι καλομένοι, τῶν δὲ πρῶτων ἀριθμῶν αἱ μύριαι μυριάδες μονᾶς καλεῖσθω δευτέρων ἀριθμῶν, καὶ ἀριθμεύον τῶν δευτέρων μονάδων καὶ ἕκ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐστὶ τὰς μυρίας μυριάδας. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τῶν δευτέρων ἀριθμῶν μονᾶς καλεῖσθω τριῶν ἀριθμῶν, καὶ ἀριθμεύον τῶν τριῶν ἀριθμῶν μονάδων καὶ ἀπὸ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐστὶ τὰς μυρίας μυριάδας. τοὺς δὲ τῶν τριῶν ἀριθμῶν μύριαι μυριάδες μονᾶς καλεῖσθω τετράτων ἀριθμῶν, 1 μὲν om. Heiberg.

cube, \( P_3 \) and \( P_4 \); (3) from the octahedron, \( P_3 \) and \( P_5 \); (4) from the icosahedron, \( P_3 \) and \( P_5 \); (5) from the dodecahedron, \( P_7 \) and \( P_5 \). It was probably the method used by Plato.

Four more of the semi-regular solids are obtained by first cutting all the edges symmetrically and equally by planes parallel to the edges, and then cutting off angles. This

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Such are then the assumptions I make; but I think it would be useful to explain the naming of the numbers, in order that, as in other matters, those who have not come across the book sent to Zeuxippus may not find themselves in difficulty through the fact that there had been no preliminary discussion of it in this book. Now we already have names for the numbers up to a myriad \(10^4\), and beyond a myriad we can count in myriads up to a myriad myriads \(10^8\). Therefore, let the aforesaid numbers up to a myriad myriads be called numbers of the first order [numbers from 1 to \(10^8\)], and let a myriad myriads of numbers of the first order be called a unit of numbers of the second order [numbers from \(10^8\) to \(10^{16}\)], and let units of the numbers of the second order be enumerable, and out of the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. Again, let a myriad myriads of numbers of the second order be called a unit of numbers of the third order [numbers from \(10^{16}\) to \(10^{24}\)], and let units of numbers of the third order be enumerable, and from the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. In the same manner, let a myriad myriads of numbers of the third order be given (1) from the cube, \(P_5\) and \(P_6\); (2) from the icosahedron, \(P_{11}\); (3) from the dodecahedron, \(P_{12}\).

The two remaining solids are more difficult to obtain; \(P_{19}\) is the snub cube in which each solid angle is formed by the angles of four equilateral triangles and one square; \(P_{19}\) is the snub dodecahedron in which each solid angle is formed by the angles of four equilateral triangles and one regular pentagon.
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καὶ αἱ τῶν τετάρτων ἀριθμῶν μύριαι μυριάδες μονάς καλείσθω πέμπτων ἀριθμῶν, καὶ ἀεί οὕτως προάγοντες οἱ ἀριθμοὶ τὰ ὀνόματα ἔχοντων ἐς τὰς μυριακισμυριστῶν ἀριθμῶν μυρίας μυριάδας.

Ἀποχρέοντι μὲν οὖν καὶ ἐπὶ τοσοῦτον οἱ ἀριθμοὶ γυγνωσκομένοι, ἔξεστι δὲ καὶ ἐπὶ πλέον προάγειν. ἐστων γὰρ οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ πρώτας περιόδου καλομένοι, ὁ δὲ ἐσχάτος ἀριθμὸς τὰς πρώτας περιόδου μονάς καλείσθω δευτέρας περιόδου πρώτων ἀριθμῶν. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τὰς δευτέρας περιόδου πρώτων ἀριθμῶν μονάς καλείσθω τὰς δευτέρας περιόδου δευτέρων ἀριθμῶν. ὁμοίως δὲ καὶ τούτων οἱ ἐσχάτοις μονάς καλείσθω δευτέρας περιόδου τρίτων ἀριθμῶν, καὶ ἀεὶ οὕτως οἱ ἀριθμοὶ προάγοντες τὰ ὀνόματα ἔχοντων τὰς δευτέρας περιόδου ἐς τὰς μυριακισμυριστῶν ἀριθμῶν μυρίας μυριάδας.

Πάλιν δὲ καὶ ὁ ἐσχάτος ἀριθμὸς τὰς δευτέρας περιόδου μονάς καλείσθω τρίτας περιόδου πρώτων ἀριθμῶν, καὶ ἀεὶ οὕτως προαγόντων ἐς τὰς μυριακισμυριστῶς περιόδου μυριακισμυριστῶν ἀριθμῶν μυρίας μυριάδας.

* Expressed in full, the last number would be 1 followed by 80,000 million millions of ciphers. Archimedes uses this system to show that it is more than sufficient to express the number of grains of sand which it would take to fill the universe, basing his argument on estimates by astronomers of the sizes and distances of the sun and moon and their relation to the size of the universe and allowing a wide margin for safety. Assuming that a poppy-head (for so μῦχον is here to be understood, not "poppy-seed," r. D'Arcy W. Thompson, The Classical Review, lvi. (1942), p. 75) would contain not more than 10,000 grains of sand, and that its diameter is not less than a finger's breadth, and having proved that the
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called a unit of *numbers of the fourth order* [numbers from \(10^{24}\) to \(10^{32}\)], and let a myriad myriads of numbers of the fourth order be called a unit of *numbers of the fifth order* [numbers from \(10^{32}\) to \(10^{40}\)], and let the process continue in this way until the designations reach a myriad myriads taken a myriad myriad times \([10^{8} \cdot 10^{5}]\).

It is sufficient to know the numbers up to this point, but we may go beyond it. For let the numbers now mentioned be called *numbers of the first period* \([1\) to \(10^{8} \cdot 10^{5}]\), and let the last number of the first period be called a unit of *numbers of the first order of the second period* \([10^{8} \cdot 10^{5}\) to \(10^{8} \cdot 10^{5} \cdot 10^{3}]\). And again, let a myriad myriads of numbers of the first order of the second period be called a unit of *numbers of the second order of the second period* \([10^{9} \cdot 10^{5} \cdot 10^{8}\) to \(10^{9} \cdot 10^{5} \cdot 10^{8} \cdot 10^{16}]\). Similarly let the last of these numbers be called a unit of *numbers of the third order of the second period* \([10^{8} \cdot 10^{5} \cdot 10^{16}\) to \(10^{8} \cdot 10^{5} \cdot 10^{24}]\), and let the process continue in this way until the designations of numbers in the second period reach a myriad myriads taken a myriad myriad times \([10^{8} \cdot 10^{5} \cdot 10^{8} \cdot 10^{5}\), or \((10^{8} \cdot 10^{5})^{3}]\).

Again, let the last number of the second period be called a unit of *numbers of the first order of the third period* \([10^{8} \cdot 10^{5})^{2}\) to \((10^{8} \cdot 10^{5})^{3} \cdot 10^{8}]\), and let the process continue in this way up to a myriad myriad units of *numbers of the myriad myriadth order of the myriad myriadth period* \([10^{8} \cdot 10^{5})^{10^{8}}\) or \(10^{8} \cdot 10^{16}]\).

sphere of the fixed stars is less than \(10^{7}\) times the sphere in which the sun's orbit is a great circle. Archimedes shows that the number of grains of sand which would fill the universe is less than "10,000,000 units of the eighth order of numbers," or \(10^{48}\). The work contains several references important for the history of astronomy.
Πρόβλημα

διπερ Ἀρχιμήδης ἐν ἐπιγράμμασιν εὐρών τοῖς ἐν Ἀλεξανδρείᾳ περὶ ταῦτα πραγματευομένοις ζητεῖν ἀπέστειλεν ἐν τῇ πρὸς Ἑρατοσθένην τῶν Κυρηναίων ἐπιστολῇ.

Πλήθου 'Ἡλίου βοῶν, ὁ ξείνε, μέτρησον φροντίδ' ἐπιστήσας, εἰ μετέχεις σοφίς, πόση ἃρ' ἐν πεδίοις Σικελίης ποτ' ἐβόακετο νήσου Θρυμακίης τετραχτῇ στίφεα δασσαμένη χρυίν ἀλλάσσοντα: τὸ μὲν λευκὸ γάλακτος, κυνάετο δ' ἐτερον χρώματι λαμπόμενον, ἄλλο γε μὲν ξανθόν, τὸ δὲ ποικίλω. ἐν δὲ ἐκάστῳ στίφει ἐσαν ταύροι πλήθειε βριθόμενοι συμμετρίης τοιῆδε τετευχότες: ἀργότριχας μὲν κυνάεν ταύρων ἡμίσει ἢδὲ τρίτω καὶ ξανθοῖς συμπασιν ἵσον, ὁ ξείνε, νόησον, αὐτὰρ κυνάεσσι τῷ τετράτῳ τε μέρει μικτοχρόνων καὶ πέμπτῳ, ἐτί ξανθοῦσι τε πᾶσιν.

τῶς δ' ὑπολειπομένους ποικιλόχρωσις ἀθρεί ἄργεννων ταύρων ἐκτῷ μέρει ἐβδομάτῳ τε καὶ ξανθοῖς αὐτοὺς πᾶσιν ἵσαζομένους. θηλείαιν δὲ βουσὶ τἀδ' ἐπλετο: λευκότριχας μὲν ἡσαν συμπάσις κυνάες ἀγέλης τῷ τριτῶ τε μέρει καὶ τετράτῳ ἀτρεκὲς ἱσαί. αὐτὰρ κυνάεσσι τῷ τετράτῳ τε πάλιν μικτοχρόνων καὶ πέμπτῳ ὅμοι χρυσόντο σὺν ταύροις πάσαις εἰς νομὸν ἐρχομέναις.
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(h) **Indeterminate Analysis: The Cattle Problem**

Archimedes (?), *Cattle Problem,* Archim. ed. Heiberg
ii. 528. 1–532. 9

**A Problem**

which Archimedes solved in epigrams, and which he communicated to students of such matters at Alexandria in a letter to Eratosthenes of Cyrene.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now

*It is unlikely that the epigram itself, first edited by G. E. Lessing in 1773, is the work of Archimedes, but there is ample evidence from antiquity that he studied the actual problem. The most important papers bearing on the subject have already been mentioned (vol. i. p. 16 n. c), and further references to the literature are given by Heiberg ad loc.*
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εανθετρίχων δ’ ἀγέλης πέμπτω μέρει ἢδε καὶ ἐκτω
ποικίλαι ἰσάρθμων πλῆθως ἔχουν τετραχί.
εανθαί δ’ ἡρυμεντὸ μέρος τρίτου ἡμίσει ἵσαι
ἀργεννῆς ἀγέλης ἐβδομάτῳ τε μέρει.
ζείνε, σὺ δ’, Ἡλίουο βόες πόσαι, ἀτρεκές εἰσών,
χωρὶς μὲν ταῦρων ζατρεφέων ἀριθμῶν,
χωρὶς δ’ αὐ, θήλεαι ὅσαι κατὰ χρούν ἐκασταὶ,
οὐκ ἀνδρίς κε λέγοι οὐδ’ ἀριθμῶν ἀδας,
οὐ μὴν πώ γε σοφοῖς ἑναριθμός. ἀλλ’ ἰθι φράζευ
καὶ τάδε πάντα βοῶν Ἡλίουο πάθη.
ἀργότρυχες ταῦροι μὲν ἐπεὶ μεξαίατο πληθὼν
κυανεώς, ἵσταντ’ ἐμπεδον ἴσομετροι
eἰς βάθος εἰς εὐρὸς τε, τὰ δ’ αὖ περιμήκεα πάντῃ
πύρπλαντο πλῆθους. Ὀρυκαίης πεδία.
ζεαθοὶ δ’ αὐτ’ εἰς ἐν καὶ ποικίλαι ἀθροισθέντες
ἱσταντ’ ἄμβολάδην ἐξ ἐνὸς ἀρχόμενου
σχήμα τελειοῦντες τὸ τρικράσπεδον οὔτε προσόντων
ἀλλοχρῶν ταῦρων οὔτ’ ἐπιειπομένων.
ταῦτα συνεξευρόν καὶ ἐν πραπίδεσσι πάθοσα
καὶ πληθέων ἀποδοὺς, ζείνε, τὰ πάντα μέτρα
ἐρχεο κυδιόνυ μυκηφόρος ἵσθι τε πάντως
κεκριμένος ταύτῃ γ’ ὀμπνίοι ἐν σοφίᾳ.

1 πλῆθως Krumbiegel, πλῆθων cod.

* a fifth and a sixth both of the males and of the females.

b At a first glance this would appear to mean that the sum of the number of white and black bulls is a square, but this makes the solution of the problem intolerably difficult. There is, however, an easier interpretation. If the bulls are packed together so as to form a square figure, their number need not be a square, since each bull is longer than it is broad. The simplified condition is that the sum of the number of white and black bulls shall be a rectangle.

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the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

If

\[ X, x \] are the numbers of white bulls and cows respectively,
\[ Y, y \] black
\[ Z, z \] yellow
\[ W, w \] dappled

the first part of the epigram states that

\[ X = \left(\frac{1}{4} + \frac{1}{4}\right) Y + Z \] \hspace{1cm} (1)
\[ Y = \left(\frac{1}{4} + \frac{1}{4}\right) W + Z \] \hspace{1cm} (2)
\[ W = \left(\frac{1}{4} + \frac{1}{4}\right) X + Z \] \hspace{1cm} (3)

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(i) Mechanics: Centres of Gravity

(i.) Postulates

Heiberg ii. 124. 3–126. 3

α'. Αὐτοῦμεθα τὰ ἵσα βάρεα ἀπὸ ἴσων μακέων ἰσορροπεῖν, τὰ δὲ ἴσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μεῖζονος μάκεος.

(b)

\[
x = \left(\frac{1}{2} + \frac{1}{3}\right)(Y + y) \quad \cdots \cdots \quad (4)
\]
\[
y = \left(\frac{1}{4} + \frac{1}{5}\right)(W + w) \quad \cdots \cdots \quad (3)
\]
\[
w = \left(\frac{1}{6} + \frac{1}{7}\right)(Z + z) \quad \cdots \cdots \quad (6)
\]
\[
z = \left(\frac{1}{8} + \frac{1}{9}\right)(X + x) \quad \cdots \cdots \quad (7)
\]

The second part of the epigram states that

\[
X + Y = \text{a rectangular number} \quad \cdots \cdots \quad (8)
\]
\[
Z + W = \text{a triangular number} \quad \cdots \cdots \quad (9)
\]

This was solved by J. F. Wurm, and the solution is given by A. Amthor, Zeitschrift für Math. u. Physik. (Hist.-litt. Abtheilung), xxv. (1880), pp. 153-171, and by Heath, The Works of Archimedes, pp. 319-326. For reasons of space, only the results can be noted here.

Equations (1) to (7) give the following as the values of the unknowns in terms of an unknown integer \( n \):

\[
X = 10366482n \quad x = 7206360n
\]
\[
Y = 7460514n \quad y = 4893246n
\]
\[
Z = 4149387n \quad z = 5439313n
\]
\[
W = 7358060n \quad w = 3515820n.
\]

We have now to find a value of \( n \) such that equation (9) is also satisfied—equation (8) will then be simultaneously satisfied. Equation (9) means that

\[
Z + W = \frac{p(p + 1)}{2},
\]

where \( p \) is some positive integer, or

\[
(4149387 + 7358060)n = \frac{p(p + 1)}{2},
\]

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(i) Mechanics: Centres of Gravity

(i.) Postulates

Archimedes, On Plane Equilibrium, Definitions, Archim. ed. Heiberg ii. 124. 3-126. 3

1. I postulate that equal weights at equal distances balance, and equal weights at unequal distances do not balance, but incline towards the weight which is at the greater distance.

\[ 2471 \cdot 4657n = \frac{p(p + 1)}{2} \]

This is found to be satisfied by \( n = 3^3 \cdot 4349 \), and the final solution is

\[
\begin{align*}
X &= 1217263415886 \\
Y &= 876035935422 \\
Z &= 487233469701 \\
W &= 864005479380
\end{align*}
\]

and the total is 5916837175686.

If equation (8) is taken to be that \( X + Y = a \) square number, the solution is much more arduous; Amthor found that in this case,

\[ W = 1598 \langle 206541 \rangle, \]

where \( \langle 206541 \rangle \) means that there are 206541 more digits to follow, and the whole number of cattle = 7766 \( \langle 206541 \rangle \). Merely to write out the eight numbers, Amthor calculates, would require a volume of 660 pages, so we may reasonably doubt whether the problem was really framed in this more difficult form, or, if it were, whether Archimedes solved it.

* This is the earliest surviving treatise on mechanics; it presumably had predecessors, but we may doubt whether mechanics had previously been developed by rigorous geometrical principles from a small number of assumptions. References to the principle of the lever and the parallelogram of velocities in the Aristotelian Mechanics have already been given (vol. i. pp. 430-433).
β'. εἰ καὶ βαρέων ἱσορροπεόντων ἀπό τινων μακέων ποτὶ τὸ ἑτέρον τῶν βαρέων ποτιτεθη, μὴ ἱσορροπεῖν, ἀλλὰ ρέπειν ἐπὶ τὸ βάρος ἕκεῖνο, ὃ ποτετέθη.

γ'. ὶμοιοὶς δὲ καὶ, εἰ καὶ ἀπὸ τοῦ ἑτέρου τῶν βαρέων ἀφαιρεθῇ τι, μὴ ἱσορροπεῖν, ἀλλὰ ρέπειν ἐπὶ τὸ βάρος, ἀφ' οὐ οὐκ ἀφηρέθη.

δ'. Τῶν ἰσων καὶ ὅμοιων σχήματων ἐπιπέδων ἐφαρμοζομένων ἐπὶ ἀλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπὶ ἀλλαλα.

ε'. Τῶν δὲ ἀνίσων, ὅμοιων δὲ, τὰ κέντρα τῶν βαρέων ὅμοιως ἔσσείται κείμενα. ὅμοιως δὲ λέγομες σαμεία κέεσθαι ποτὶ τὰ ὅμοια σχήματα, ἀφ' ὧν ἐπὶ τὰς ἱσας γινιας ἀγόμεναι εὐθείαι ποιεόντι γινιας ἱσας ποτὶ τὰς ὁμολογους πλευράς.

ζ'. Εἰ καὶ μεγέθεα ἀπὸ τινων μακέων ἱσορροπεόντων, καὶ τὰ ἱσα αὐτοῖς ἀπὸ τῶν αὐτῶν μακέων ἱσορροπήσει.

ξ'. Πάντως σχήματος, οὐ καὶ ἀ περίμετρος ἐπὶ τὰ αὐτὰ κοῖλα ἦ, τὸ κέντρον τοῦ βάρεος ἐντός ἐλμέν δεῖ τοῦ σχήματος.

(ii.) Principle of the Lever

Ibid., Props. 6 et 7, Archim. ed. Heiberg ii. 132. 13–138. 8

σ'

Τὰ σύμμετρα μεγέθεα ἱσορροπεόντι ἀπὸ μακέων ἀντιπεπουθότως τὸν αὐτῶν λόγον ἐχόντων τοῖς βάρεσιν.

Ἔστω σύμμετρα μεγέθεα τὰ Α, Β, ὅν κέντρα τὰ Α, Β, καὶ μάκος ἐστώ τι τὸ ΕΔ, καὶ ἕστω, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ μάκος ποτὶ τὸ ΓΕ 208
2. If weights at certain distances balance, and something is added to one of the weights, they will not remain in equilibrium, but will incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they will not remain in equilibrium, but will incline towards the weight from which nothing was subtracted.

4. When equal and similar plane figures are applied one to the other, their centres of gravity also coincide.

5. In unequal but similar figures, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances balance, magnitudes equal to them will also balance at the same distances.

7. In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

(ii.) Principle of the Lever

*Ibid.*, Props. 6 and 7, Archim, ed. Heiberg
ii. 132. 13-138. 8

Prop. 6

Commensurable magnitudes balance at distances reciprocally proportional to their weights.

Let A, B be commensurable magnitudes with centres [of gravity] A, B, and let $EA$ be any distance, and let $A : B = \Delta \Gamma : \Gamma E$;
μάκος· δεικτέον, ὅτι τοῦ εὖ ἀμφοτέρων τῶν Α, Β συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεως τὸ Γ.

Επεὶ γὰρ ἐστίν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ ποτὶ τὸ ΓΕ, τὸ δὲ Α τῷ Β σύμμετρον, καὶ τὸ ΓΔ ἀρα τῷ ΓΕ σύμμετρον, τούτεστι εὐθεία τῇ εὐθείᾳ· ὡστε τῶν ΕΓ, ΓΔ ἐστὶ κοινὸν μέτρον. ἐστὶ δὴ τὸ Ν, καὶ κείσθω τὰ μὲν ΕΓ ἵσα ἐκατέρα τῶν ΔΗ, ΔΚ, τὰ δὲ ΔΓ ἵσα αἱ ΕΛ. καὶ ἐπεὶ ἵσα

\[
\begin{array}{c}
\text{A} \\
\text{B} \\
\text{Z}
\end{array}
\]

\[
\text{Α Ε Γ Η Δ Κ}
\]

ἀ ΔΗ τὰ ΓΕ, ἵσα καὶ ἀ ΔΓ τὰ ΕΗ· ὡστε καὶ ἀ ΛΕ ἵσα τὰ ΕΗ. διπλασία ἀρα ἀ μὲν ΛΗ τὰς ΔΓ, ἀ δὲ ΗΚ τὰς ΓΕ. ὡστε τὸ Ν καὶ ἐκατέρα τῶν ΛΗ, ΗΚ μετρεῖ, ἐπειδὴ περὶ καὶ τὰ ἡμίσει αὐτῶν. καὶ ἐπεὶ ἐστίν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως ἀ ΔΓ ποτὶ ΓΕ, ὡς δὲ ἀ ΔΓ ποτὶ ΓΕ, οὕτως ἀ ΛΗ ποτὶ ΗΚ—διπλασία γὰρ ἐκατέρα ἐκατέρας—καὶ ὡς ἀρα τὸ Α ποτὶ τὸ Β, οὕτως ἀ ΛΗ ποτὶ ΗΚ. ὀσαπλασίων δὲ ἐστὶν ἀ ΛΗ τὰς Ν, τοσαν—210
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it is required to prove that the centre of gravity of
the magnitude composed of both A, B is \( \Gamma \).

Since \( A : B = \Delta \Gamma : \Gamma E \),
and A is commensurate with B, therefore \( \Gamma \Delta \) is com-
mensurate with \( \Gamma E \), that is, a straight line with a
straight line [Eucl. x. 11]; so that \( \Gamma E, \Gamma \Delta \) have a
common measure. Let it be \( N \), and let \( \Delta H, \Delta K \) be
each equal to \( \Gamma E \), and let \( \Gamma A \) be equal to \( \Delta \Gamma \). Then
since \( \Delta H = \Gamma E \), it follows that \( \Delta \Gamma = \Gamma H \); so that
\( \Delta E E = H \). Therefore \( \Delta H = 2 \Delta \Gamma \) and \( \Delta K = 2 \Gamma E \); so
that \( N \) measures both \( \Delta H \) and \( \Delta K \), since it measures
their halves [Eucl. x. 12]. And since

\[ A : B = \Delta \Gamma : \Gamma E, \]

while \( \Delta \Gamma : \Gamma E = \Delta H : \Delta K \)

for each is double of the other—

therefore \( A : B = \Delta H : \Delta K \).

Now let \( Z \) be the same part of \( A \) as \( N \) is of \( \Delta H \);
ταπλασίων ἦστω καὶ τὸ Α τοῦ Ζ. ἦστω ἄρα, ὡς ἀ πολ. Ν, οὕτως τὸ Α πολ. Ζ. ἦστι δὲ καὶ, ὡς ἀ ΚΗ πολ. ΛΗ, οὕτως τὸ Β πολ. Α· δὲ ἰσον ἁρα ἦστων, ὡς ἀ ΚΗ πολ. Ν, οὕτως τὸ Β πολ. Ζ· ἵσακισ ἁρα πολλαπλασίων ἦστιν ἀ ΚΗ τᾶ · Ν καὶ τὸ Β τοῦ Ζ. ἐ ἔδειξθη δὲ τοῦ Ζ καὶ τὸ Α πολλαπλασίων ἐ ου· ὡστε τὸ Ζ τῶν Α, Β κοινών ἑστι μέτρον. διαιρεθείσας οὖν τὰς μὲν ΛΗ εἰς τὰς τὰ · Ν ἵσας, τοῦ δὲ Α εἰς τὰ τῶ Ζ ἵσα, τὰ ἐν τὰ · ΛΗ τμάματα ἰσομεγέθεα τὰ · Ν ἵσα ἐσσεῖται τῷ πλήθει τοῖς ἐν τῶ Α τμαμάτσιον ἰσου εὐθον τῶ Ζ. ὡστε, ἂν ἐφ' ἐκαστον τῶν τμαμάτων τῶν ἐν τὰ · ΛΗ ἐπιτεθῆ μέγεθος ἱσον τῶ Ζ τὸ κέντρον τοῦ βάρεος ἐχον ἐπὶ μέσου τοῦ τμάματος, τὰ τε πάντα μεγέθεα ἱσα ἐντὶ τῶ Α, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρου ἐσσεῖται τοῦ βάρεος τὸ Ε· ἀρτιά τε γὰρ ἑστὶ τὰ πάντα τῶ πλήθει, καὶ τὰ ἐφ' ἐκάτερα τοῦ Ε ἱσα τῶ πλήθει διὰ τὸ ἱσαν ἐμεν τῶν ΛΕ τὰ · ΗΕ.

'Ομοίως δὲ δειχθῆσαι, ὅτι κἀν, εἰ κα ἐφ' ἐκαστον τῶν ἐν τὰ · ΚΗ τμαμάτων ἐπιτεθῆ μέγεθος ἱσον τῶ Ζ κέντρον τοῦ βάρεος ἐχον ἐπὶ τοῦ μέσου τοῦ τμάματος, τὰ τε πάντα μεγέθεα ἱσα ἐσσεῖται τῶ Β, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρου τοῦ βάρεος ἐσσεῖται τὸ Δ· ἐσσεῖται οὖν τὸ μὲν Α ἐπικείμενον κατὰ τὸ Ε, τὸ δὲ Β κατὰ τὸ Δ. ἐσσεῖται δὴ μεγέθεα ἱσα ἀλλάζουσ ἐπ' εὐθείας κείμενα, ὅν τὰ κέντρα τοῦ βάρεος ἱσα ἀπ' ἀλλάζων διέστακεν, [συγκείμενα]: ἀρτια τῶ πλήθει· δήλον οὖν, ὅτι τοῦ ἐκ πάντων συγκειμένου μεγέθεως κέντρον ἔστι τοῦ βάρεος ἀ διχοτομία τᾶς εὐθείας τᾶς ἐχούσας τὰ κέντρα τῶν μέσων μεγεθέων. ἐπεὶ δ' ἱσα ἐντὶ
then \( \Delta H : N = A : Z \). [Eucl. v., Def. 5]

And \( KH : \Delta H = B : A \); [Eucl. v. 7, coroll.]

therefore, \textit{ex aequo},

\( KH : N = B : Z \); [Eucl. v. 22]

therefore \( Z \) is the same part of \( B \) as \( N \) is of \( KH \). Now \( A \) was proved to be a multiple of \( Z \); therefore \( Z \) is a common measure of \( A, B \). Therefore, if \( \Delta H \) is divided into segments equal to \( N \) and \( A \) into segments equal to \( Z \), the segments in \( \Delta H \) equal in magnitude to \( N \) will be equal in number to the segments of \( A \) equal to \( Z \). It follows that, if there be placed on each of the segments in \( \Delta H \) a magnitude equal to \( Z \), having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to \( A \), and the centre of gravity of the figure compounded of them all will be \( E \); for they are even in number, and the numbers on either side of \( E \) will be equal because \( \Delta E = HE \). [Prop. 5, coroll. 2.]

Similarly it may be proved that, if a magnitude equal to \( Z \) be placed on each of the segments [equal to \( N \)] in \( KH \), having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to \( B \), and the centre of gravity of the figure compounded of them all will be \( \Delta \) [Prop. 5, coroll. 2]. Therefore \( A \) may be regarded as placed at \( E \), and \( B \) at \( \Delta \). But they will be a set of magnitudes lying on a straight line, equal one to another, with their centres of gravity at equal intervals, and even in number; it is therefore clear that the centre of gravity of the magnitude compounded of them all is the point of bisection of the line containing the centres [of gravity] of the middle magnitudes [from Prop. 5, coroll. 2].

\[1\] \textit{συγκείμενα} om. Heiberg.
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ά μὲν ΔΕ τὰ ΓΔ, ἀ δὲ ΕΓ τὰ ΔΚ, καὶ ὅλα ἄρα ἀ ΛΓ ἵσα τὰ ΓΚ. ὡστε τοῦ ἐκ πάντων μεγέθεως κέντρων τοῦ βάρεος τοῦ Γ σαμεῖον, τοῦ μὲν ἄρα Α χειμένου κατὰ τὸ Ε, τοῦ δὲ Β κατὰ τὸ Δ, ἰσορροπησοῦντι κατὰ τὸ Γ.

ζ’

Καὶ τοίνυν, εἰ κα ἀσύμμετρα ἐσοντι τὰ μεγέθεα, ὀμοίως ἰσορροπησοῦντι ἀπὸ μακέων ἀντιπεπουθό- τως τῶν αὐτῶν λόγων ἐχόντων τοῖς μεγέθεσιν.

"Εστώ ἀσύμμετρα μεγέθεα τὰ AB, Γ, μάκεα δὲ τὰ ΔΕ, EZ, ἐκέτω δὲ τὸ AB ποτὲ τὸ Γ τῶν αὐτῶν λόγων, ὅν καὶ τὸ ED ποτὲ τὸ EZ μάκος· λέγω, ὅτι τοῦ εὖ ἀμφοτέρων τῶν AB, Γ κέντρων τοῦ βάρεος ἢστι τὸ E.

Εἰ γὰρ μὴ ἰσορροπήσει τὸ AB τεθεῖν ἐπὶ τῷ Ζ τῶ Γ τεθέντι ἐπὶ τῷ Δ, ἦτοι μειζόν ἢστι τὸ AB

![Diagram](image1)

τοῦ Γ ἦ ὡστε ἰσορροπεῖν [τῷ Γ] ἦ οὐ. ἔστω μειζόν, καὶ ἀφημήσω οὖν τοῦ AB ἔλασσον τὰς υπεροχάς, ἣ μειζόν ἢστι τὸ AB τοῦ Γ ἦ ὡστε ἰσορροπεῖν, ὡστε [τῷ] ὑπὸ τὸ Δ σύμμετρον

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And since $\Delta E = \Gamma \Delta$ and $E \Gamma = \Delta K$, therefore $\Delta \Gamma = \Gamma K$; so that the centre of gravity of the magnitude compounded of them all is the point $\Gamma$. Therefore if $\Lambda$ is placed at $E$ and $B$ at $\Delta$, they will balance about $\Gamma$.

Prop. 7

And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

Let $(A + B)$, $\Gamma$ be incommensurable magnitudes, and let $\Delta E$, $EZ$ be distances, and let

$$(A + B) : \Gamma = E\Delta : EZ;$$

I say that the centre of gravity of the magnitude composed of both $(A + B)$, $\Gamma$ is E.

For if $(A + B)$ placed at $Z$ do not balance $\Gamma$ placed at $\Delta$, either $(A + B)$ is too much greater than $\Gamma$ to balance or less. Let it [first] be too much greater, and let there be subtracted from $(A + B)$ a magnitude less than the excess by which $(A + B)$ is too much greater than $\Gamma$ to balance, so that the remainder $A$ is

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* As becomes clear later in the proof, the first magnitude is regarded as made up of two parts—$A$, which is commensurate with $\Gamma$ and $B$, which is not commensurate; if $(A + B)$ is too big for equilibrium with $\Gamma$, then $B$ is so chosen that, when it is taken away, the remainder $A$ is still too big for equilibrium with $\Gamma$. Similarly if $(A + B)$ is too small for equilibrium.

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1 $\tau \varphi$ $\Gamma$ om. Eutocius.
2 $\tau \delta$ om. Eutocius.
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...tô Π. ἐπεὶ οὖν σύμμετρά ἦστι τὰ A, Γ μεγέθεα, καὶ ἐλάσσονα λόγον ἔχει τὸ A ποτὲ τὸ Γ ἢ ᾧ ΔΕ ποτὶ EZ, οὐκ ἰσορροπησοῦντι τὰ A, Γ ἀπὸ τῶν ΔΕ, EZ μακέων, τεθέντος τοῦ μὲν A ἐπὶ τῷ Z, τοῦ δὲ Γ ἐπὶ τῷ Δ. διὰ ταύτα δ', οὐδὲν εἰ τὸ Γ μεῖζὸν ἦστιν ἢ ὅστε ἰσορροπεῖν τῷ AB.

(iii.) Centre of Gravity of a Parallelogram

Ibid., Props. 9 et 10, Archim. ed. Heiberg ii. 140. 16–144. 4

θ'

Παντὸς παράλληλογράμμου τὸ κέντρον τοῦ βάρεός ἦστιν ἐπὶ τᾶς εὐθείας τὰς ἐπὶ ἐξενυγνούσας τὰς διχοτομίας τῶν κατ' ἐναντίον τοῦ παράλληλογράμμου πλευράν.

'Εστω παράλληλόγραμμον τὸ ABΓΔ, ἐπὶ δὲ τῶν διχοτομιῶν τῶν AB, ΓΔ ἢ EZ· φαμί δὴ, ὅτι τοῦ ABΓΔ παράλληλογράμμου τὸ κέντρον τοῦ βάρεος ἔσσεται ἐπὶ τᾶς EZ.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἐστω τὸ Θ, καὶ ἄχρω παρὰ τῶν AB ἢ ΘΙ. τὰς [δὲ]¹ δὴ EB διχοτομομένας αἰεὶ ἔσσεταί ποικά ἡ καταλειπομένα ἐλάσσων

¹ δὲ om. Heiberg.

* The proof is incomplete and obscure; it may be thus completed.

Since A : Γ<ΔE : EZ,

Δ will be depressed, which is impossible, since there has been taken away from (A+B) a magnitude less than the deduc-
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commensurate with \( \Gamma \). Then, since \( A, \Gamma \) are commensurable magnitudes, and

\[
A : \Gamma < \Delta E : EZ,
\]

\( A, \Gamma \) will not balance at the distances \( \Delta E, EZ \), \( A \) being placed at \( Z \) and \( \Gamma \) at \( \Delta \). By the same reasoning, they will not do so if \( \Gamma \) is greater than the magnitude necessary to balance \( (A + B) \).^{a}

(iii. ) Centre of Gravity of a Parallelogram \(^{b}\)


ii. 140. 16–144. 4

Prop. 9

*The centre of gravity of any parallelogram is on the straight line joining the points of bisection of opposite sides of the parallelogram.*

Let \( AB \Gamma \Delta \) be a parallelogram, and let \( EZ \) be the straight line joining the mid-points of \( AB, \Gamma \Delta \); then I say that the centre of gravity of the parallelogram \( AB \Gamma \Delta \) will be on \( EZ \).

For if it be not, let it, if possible, be \( \Theta \), and let \( \Theta I \) be drawn parallel to \( AB \). Now if \( EB \) be bisected, and the half be bisected, and so on continually, there will be left some line less than \( IO \); [let \( EK \) be less than

*The centres of gravity of a triangle and a trapezium are also found by Archimedes in the first book; the second book is wholly devoted to finding the centres of gravity of a parabolic segment and of a portion of it cut off by a parallel to the base.
σαμείων ἄχθωσαν παρὰ τὰς ΕΖ· διαμεθήσεται δὴ τὸ ὀλὸν παράλληλόγραμμον εἰς παράλληλόγραμμα τὰ ἵσα καὶ ὁμοία τῷ ΚΖ. τῶν οὖν παράλληλο-
γράμμων τῶν ἵσων καὶ ὁμοίων τῷ ΚΖ ἐφαρμοζο-
μένων ἐπ' ἄλλα καὶ τὰ κέντρα τοῦ βάρεως αὐτῶν ἐπ' ἄλλα καὶ παραλληλόγραμμα ἵσα τῷ ΚΖ, ἀρτία τῷ πλήθεις, καὶ τὰ κέντρα τῶν βάρεως αὐτῶν ἐπ' εὐθείᾳς κείμενα, καὶ τὰ μέσα ἵσα, καὶ πάντα τὰ ἐφ' ἐκάτερα τῶν μέσων αὐτά τε ἵσα ἐντι καὶ αἱ μεταξὺ τῶν κέντρων ἐθείαι ἵσαι τοῦ ἐκ πάντων αὐτῶν ἀρα συγκειμένου μεγέθεος τὸ κέντρον ἐσοεῖται τοῦ βάρεως ἐπὶ τᾶς εὐθείας τᾶς ἐπιζευγνωμοῦσας τὰ κέντρα τοῦ βάρεως τῶν μέσων χωρίων. οὐκ ἔστι δὲ τὸ γὰρ Θ ἐκτὸς ἐστὶ τῶν μέσων παράλληλο-
γράμμων. φανερὸν οὖν, ὅτι ἐπὶ τᾶς ΕΖ εὐθείας τὸ κέντρον ἐστὶ τοῦ βάρεως τοῦ ΑΒΓΔ παράλληλο-
γράμμου.

Παντὸς παράλληλόγραμμον τὸ κέντρον τοῦ βάρεως ἐστὶ τὸ σαμεῖον, καθ' ὁ αἱ διαμέτροι συμ-
πιπτοῦντι.
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10] and let each of AE, EB be divided into parts equal to EK, and from the points of division let straight lines be drawn parallel to EZ; then the whole parallelogram will be divided into parallelograms equal and similar to KZ. Therefore, if these parallelograms equal and similar to KZ be applied to each other, their centres of gravity will also coincide [Post. 4]. Thus there will be a set of magnitudes, being parallelograms equal to KZ, which are even in number and whose centres of gravity lie on a straight line, and the middle magnitudes will be equal, and the magnitudes on either side of the middle magnitudes will also be equal, and the straight lines between their centres [of gravity] will be equal; therefore the centre of gravity of the magnitude compounded of them all will be on the straight line joining the centres of gravity of the middle areas [Prop. 5, coroll. 2]. But it is not; for Θ lies without the middle parallelograms. It is therefore manifest that the centre of gravity of the parallelogram ABΓΔ will be on the straight line EZ.

Prop. 10

The centre of gravity of any parallelogram is the point in which the diagonals meet.
μαθήμασιν κατά τό ὑποπίπτον θεωρίαν τετμηκότα ἐδοκίμασα γράψαι σοι καὶ εἰς τὸ αὐτὸ βιβλίον ἔξορισαι τρόπον τινὸς ἱδιότητα, καθ’ ὅν σοι παρεχόμενον ἔσται λαμβάνειν ἀφορμᾶς εἰς τὸ δύνασθαι τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν. τούτῳ δὲ πεπεσμαῖ χρήσιμον εἶναι ουδὲν ἠσον καὶ εἰς τὴν ἀπόδειξιν αὐτῶν τῶν θεωρημάτων. καὶ γάρ τινα τῶν πρῶτων μοι φανέντων μηχανικῶς υστερον γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρίς ἀποδείξεως εἶναι τὴν διὰ τοῦτον τοῦ τρόπου θεωρίαν ἐτοιμότερον γάρ ἐστὶ προλαβόντα διὰ τοῦ τρόπου γνώσιν τινα τῶν ζητημάτων πορίσασθαι τὴν ἀπόδειξιν μᾶλλον ἡ μηδενὸς ἐγνωσμένον ξητεῖν... γράφομεν οὖν πρῶτον τὸ καὶ πρῶτον φανέν διὰ τῶν μηχανικῶν, ὅτι πᾶν τμῆμα ὀρθογώνιον κώνου τομῆς ἐπίτροτον ἐστὶ τριγόνων τοῦ βασιν ἔχοντος τὴν αὐτὴν καὶ ύψος ἴσον.

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

"Εστῶ τμήμα τὸ ΑΒΓ περιεχόμενον ὑπὸ εὐθείας τῆς ΑΓ καὶ ὀρθογώνιου κώνου τομῆς τῆς ΑΒΓ, καὶ τετμήσθω διὰ ἡ ἈΓ τῷ Δ, καὶ παρὰ τὴν διάμετρον ἡχθων ἡ ΔΒΕ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ.

Λέγω, ὅτι ἐπίτροτον ἐστὶ τὸ ΑΒΓ τμήμα τοῦ ΑΒΓ τριγώνου.

"Ηχθωσαν ἀπὸ τῶν Α, Γ σημείων ἡ μὲν ΑΖ παρὰ τὴν ΔΒΕ, ἡ δὲ ΓΖ ἐπιευμοῦσα τῆς τομῆς, καὶ ἐκβεβλήσθω ἡ ΓΒ ἐπὶ τῷ Κ, καὶ κεῖσθω τῇ ΓΚ ἴση ἡ ΚΘ. νοεῖσθω ζυγὸς ὁ ΓΘ καὶ μέσου
who gives due honour to mathematical inquiries when they arise, I have thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, with which furnished you will be able to make a beginning in the investigation by mechanics of some of the problems in mathematics. I am persuaded that this method is no less useful even for the proof of the theorems themselves. For some things first became clear to me by mechanics, though they had later to be proved geometrically owing to the fact that investigation by this method does not amount to actual proof; but it is, of course, easier to provide the proof when some knowledge of the things sought has been acquired by this method rather than to seek it with no prior knowledge. . . . At the outset therefore I will write out the very first theorem that became clear to me through mechanics, that any segment of a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

Let \( AB \Gamma \) be a segment bounded by the straight line \( A \Gamma \) and the section \( AB \Gamma \) of a right-angled cone, and let \( A \Gamma \) be bisected at \( \Delta \), and let \( ABE \) be drawn parallel to the axis, and let \( AB \), \( B \Gamma \) be joined.

I say that the segment \( AB \Gamma \) is four-thirds of the triangle \( AB \Gamma \).

From the points \( A \), \( \Gamma \) let \( AZ \) be drawn parallel to \( ABE \), and let \( \Gamma Z \) be drawn to touch the section, and let \( \Gamma B \) be produced to \( K \), and let \( K \Theta \) be placed equal to \( \Gamma K \). Let \( \Gamma \Theta \) be imagined to be a balance
αὐτοῦ τὸ Κ καὶ τῇ ΕΔ παράλληλος τυχοῦσα ἡ ΜΕ.

'Επεὶ οὖν παραβολή ἐστιν ἡ ΓΒΑ, καὶ ἐφάπτεται η ΓΖ, καὶ τεταγμένως ἡ ΓΔ, ἵση ἔστιν ἡ EB τῇ BD ὑστο γὰρ ἐν τοῖς στοιχείοις δείκνυται διὰ δὴ τούτῳ, καὶ διότι παράλληλοι εἰσίν αἱ ΖΑ, ΜΕ τῇ ΕΔ, ἦση ἔστιν καὶ ἡ μὲν MN τῇ ΝΕ, ἡ δὲ ΖΚ τῇ KA. καὶ ἐπεὶ ἔστιν, ὡς ἡ ΓΑ πρὸς ΑΞ, οὕτως ἡ ΜΕ πρὸς ΞΩ [τούτῳ γὰρ ἐν λήμματι δείκνυται], ὡς δὲ ἡ ΓΑ πρὸς ΑΞ, οὕτως ἡ ΓΚ πρὸς ΚΝ, καὶ ἦση ἔστιν ἡ ΓΚ τῇ ΚΘ, ὡς ἄρα ἡ ΘΚ πρὸς ΚΝ, οὕτως ἡ ΜΕ πρὸς ΞΩ. καὶ ἐπεὶ τὸ Ν σημεῖον κέντρον τοῦ βάρους τῆς ΜΕ εὐθείας ἐστίν, ἐπείπερ ἦση ἔστιν ἡ MN τῇ ΝΕ, ἐὰν ἄρα τῇ ΞΩ ἤση βάμεν τῆς TH καὶ κέντρον τοῦ βάρους αὐτῆς τὸ Θ, ὡς ἦση ἡ ΘΩ τῇ ΘΗ, ἵσορροπησε ἡ θOH τῇ ΜΕ αὐτοῦ μενοῦσα διὰ τὸ ἀντιπεπονθότως τετμῆσθαι.
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with mid-point $K$, and let $M\Xi$ be drawn parallel to $E\Delta$.

Then since $\Gamma'BA$ is a parabola, and $\Gamma'Z$ touches it, and $\Gamma\Delta$ is a semi-ordinate, $EB = B\Delta$—for this is proved in the elements; for this reason, and because $ZA, M\Xi$ are parallel to $E\Delta$, $MN = N\Xi$ and $ZK = KA$ [Eucl. vi. 4, v. 9]. And since

$$\Gamma A : A\Xi = M\Xi : \Xi O,$$  
[Quad. parab. 5, Eucl. v. 18]

and

$$\Gamma A : A\Xi = \Gamma K : KN,$$  
[Eucl. vi. 2, v. 18]

while

$$\Gamma K = K\Theta,$$

therefore

$$\Theta K : KN = M\Xi : \Xi O.$$

And since the point $N$ is the centre of gravity of the straight line $M\Xi$, inasmuch as $MN = N\Xi$ [Lemma 4], if we place $TH = \Xi O$, with $\Theta$ for its centre of gravity, so that $T\Theta = \Theta H$ [Lemma 4], then $T\Theta H$ will balance $M\Xi$ in its present position, because $\Theta N$ is cut

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$^a$ Archimedes would have said “section of a right-angled cone”—δρογωνίου κόνου τομή.

$^b$ The reference will be to the *Elements of Conics* by Euclid and Aristaeus for which v. vol. i. pp. 486-491 and infra, p. 280 n. a; cf. similar expressions in *On Conoids and Spheroids*, Prop. 3 and *Quadrature of a Parabola*, Prop. 3; the theorem is *Quadrature of a Parabola*, Prop. 2.

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$^1$ τούτο ... δείκνυται om. Heiberg. It is probably an interpolator’s reference to a marginal lemma.

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tην ΘΝ τοις ΤΗ, ΜΕ βάρεσιν, καὶ ὡς την ΘΚ πρὸς ΚΝ, οὖτως την ΜΕ πρὸς την ΗΤ. ὡστε τοῦ ἐκ ἀμφοτέρων βάρους κέντρον ἐστὶν τοῦ βάρους τὸ Κ. ὁμοίως δὲ καὶ ὡστε ἄν ἀκριβῶς ἐν τῷ ΖΑΓ τρίγωνῳ παράλληλοι τῇ ΕΔ, ἰσορροπήσουσιν αὐτοῦ μένουσι ταῖς ἀπολαμβανομέναις ἀπ’ αὐτῶν ὑπὸ τῆς τομῆς μετενεχθεῖσαι ἐπὶ τὸ Θ, ὡστε εἶναι τοῦ ἐκ ἀμφοτέρων κέντρον τοῦ βάρους τὸ Κ. καὶ ἐπεὶ ἐκ μὲν τῶν ἐν τῷ ΓΖΑ τριγώνῳ τὸ ΓΖΑ τρίγωνον συνέστηκεν, ἐκ δὲ τῶν ἐν τῇ τομῇ ὁμοίως τῇ ΞΟ λαμβανομένων συνέστηκε τὸ ΑΒΓ τμῆμα, ἰσορροπήσει ἂρα τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ τμῆματι τῆς τομῆς τεθέντι περὶ κέντρον τοῦ βάρους τὸ Θ κατὰ τὸ Κ σημείον, ὡστε τοῦ ἐκ ἀμφοτέρων κέντρον εἶναι τοῦ βάρους τὸ Κ. τετμῆσθαι δὴ ἡ ΓΚ τῷ Χ, ὡστε τριπλασίαιν εἶναι την ΓΚ τῆς ΚΧ. ἐσται ἂρα τὸ Χ σημείον κέντρον βάρους τοῦ ΑΖΓ τριγώνου· δεδεικται γὰρ ἐν τοῖς ἰσορροπικοῖς. ἐπεὶ οὖν ἰσορροποῦ τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ ΒΑΓ τμῆματι κατὰ τὸ Κ τεθέντι περὶ τὸ Θ κέντρον τοῦ βάρους, καὶ ἐστὶν τοῦ ΖΑΓ τρίγωνον κέντρον βάρους τὸ Χ, ἐστὶν ἂρα, ὡς τὸ ΑΖΓ τρίγωνον πρὸς τὸ ΑΒΓ τμῆμα κείμενον περὶ τὸ Θ κέντρον, οὖτως ἡ ΘΚ πρὸς ΧΚ. τριπλασία δὲ ἐστὶν ἡ ΘΚ τῆς ΚΧ. τριπλάσιον ἂρα καὶ τὸ ΑΖΓ τρίγωνον τοῦ ΑΒΓ τμήματος. ἐστὶ δὲ καὶ τὸ ΖΑΓ τρίγωνον τετραπλάσιον τοῦ ΑΒΓ τριγώνου διὰ τὸ ἵσην εἶναι την μὲν ΖΚ τῆς ΚΑ, τῆν δὲ ΑΔ τῆς ΔΓ· ἐπίτριτον ἂρα ἐστὶν τὸ ΑΒΓ τμῆμα τοῦ ΑΒΓ τριγώνου. [τούτῳ οὖν φανερὸν ἐστιν].

1 τούτο . . . ἐστιν om. Heiberg.
in the inverse proportion of the weights $TH$, $ME$, and $OK : KN = ME : HT$;

therefore the centre of gravity of both $[TH, ME]$ taken together is $K$. In the same way, as often as parallels to $E\Delta$ are drawn in the triangle $Z\Delta\Gamma$, these parallels, remaining in the same position, will balance the parts cut off from them by the section and transferred to $\Theta$, so that the centre of gravity of both together is $K$. And since the triangle $\Gamma Z\Delta$ is composed of the [straight lines drawn] in $\Gamma Z\Delta$, and the segment $AB\Gamma$ is composed of the lines in the section formed in the same way as $\Xi \Theta$, therefore the triangle $Z\Delta\Gamma$ in its present position will be balanced about $K$ by the segment of the section placed with $\Theta$ for its centre of gravity, so that the centre of gravity of both combined is $K$. Now let $\Gamma K$ be cut at $X$ so that $\Gamma K = 3KX$; then the point $X$ will be the centre of gravity of the triangle $AZ\Gamma$; for this has been proved in the books $On\ Equilibriums$. Then since the triangle $Z\Delta\Gamma$ in its present position is balanced about $K$ by the segment $BA\Gamma$ placed so as to have $\Theta$ for its centre of gravity, and since the centre of gravity of the triangle $Z\Delta\Gamma$ is $X$, therefore the ratio of the triangle $AZ\Gamma$ to the segment $AB\Gamma$ placed about $\Theta$ as its centre [of gravity] is equal to $OK : XK$. But $OK = 3KX$; therefore

triangle $AZ\Gamma' = 3 \cdot$ segment $AB\Gamma'$.

And triangle $Z\Delta\Gamma = 4 \cdot$ triangle $AB\Gamma$,

because $ZK = KA$ and $\Lambda \Delta = \Delta \Gamma$;

therefore segment $AB\Gamma' = \frac{4}{3}$ triangle $AB\Gamma$.

* Cf. De Plan. Equil. 1. 15.
Τούτο δὴ διὰ μὲν τῶν νῦν εἰρημένων οὐκ ἀποδε- 
δεικται, ἐμφασιν δὲ τινα πεποίηκε τὸ συμπέρασμα 
ἀληθὲς εἶναι: διόπερ ἤμεισ ὀρῶντες μὲν οὐκ ἀποδε-
δειγμένον, ὑπονοοῦντες δὲ τὸ συμπέρασμα ἀληθὲς 
eἶναι, τάξομεν τὴν γεωμετρουμένην ἀπόδειξιν ἔξευ-
ρόντες αὐτὸ τὴν ἐκδοθείσαιν πρότερον. ¹

262. 2–266. 4

'Αρχιμήδης Δοσιθέω εἰς πράττειν. 
'Ακούσας Κόνωνα μὲν τετελευτηκέναι, ὃς ἦν 
oυδὲν ἐπιλείπον ἀμῖν ἐν φιλίᾳ, τὸν δὲ Κόνωνος 
γνώριμον γεγενήθαι καὶ γεωμετρίας οἰκεῖον εἰμὲν 
tοῦ μὲν τετελευτηκότος εἶνεκεν ἑλπίζῃμεν ὡς 
καὶ φιλοῦ τοῦ ἀνδρὸς γεναμένου καὶ ἐν τοῖς μαθη-
mάτεσιν θαυμαστοῦ τυποῖς, ἐπροχειριζόμεθα δὲ 
ἀποστείλαι τοις γράφαντες, ὡς Κόνωνι γράφει 
ἐγνωκότες ἤμες, γεωμετρικῶν θεωρημάτων, ὥ 
πρότερον μὲν οὐκ ἦν τεθεωρημένον, νῦν δὲ ύφ 
ἀμῖν τεθεώρηταί, πρότερον μὲν διὰ μηχανικῶν 
eὑρεθέν, ἔπειτα δὲ καὶ διὰ τῶν γεωμετρικῶν ἐπι-
δειχθέν. τῶν μὲν οὖν πρότερον περὶ γεωμετρίαν 
πραγματευθέντων ἐπεχείρησάν τινες γράφειν ὡς 
dυνατὸν ἐνοῦ κύκλω τῷ δοθέντι καὶ κύκλου τμάματι 
tῷ δοθέντι χωρίον εὑρέθην εὐσάρκες ὑπό τε τὰς

¹ τούτο ... πρότερον. In the ms. the whole paragraph 
from τούτο to πρότερον comes at the beginning of Prop. 2; 
it is more appropriate at the end of Prop. 1.

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This, indeed, has not been actually demonstrated by the arguments now used, but they have given some indication that the conclusion is true; seeing, therefore, that the theorem is not demonstrated, but suspecting that the conclusion is true, we shall have recourse to the geometrical proof which I myself discovered and have already published.

Archimedes, *Quadrature of a Parabola*, Preface, Archim. ed. Heiberg ii. 262. 2-266. 4

Archimedes to Dositheus greeting.

On hearing that Conon, who fulfilled in the highest degree the obligations of friendship, was dead, but that you were an acquaintance of Conon and also versed in geometry, while I grieved for the death of a friend and an excellent mathematician, I set myself the task of communicating to you, as I had determined to communicate to Conon, a certain geometrical theorem, which had not been investigated before, but has now been investigated by me, and which I first discovered by means of mechanics and later proved by means of geometry. Now some of those who in former times engaged in mathematics tried to find a rectilinear area equal to a given circle and to a given segment of a circle, and afterwards they tried to square the area bounded by the section

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*a* I have followed Heath's rendering of τῦκομεν, which seems more probable than Heiberg's "suo loco proponemus," though it is a difficult meaning to extract from τῦκομεν.

*b* Presumably *Quadr. Parab.* 24, the second of the proofs now to be given. The theorem has not been demonstrated, of course, because the triangle and the segment may not be supposed to be composed of straight lines.

*c* This seems to indicate that Archimedes had not at this time written his own book *On the Measurement of a Circle*. For attempts to square the circle, v. vol. i. pp. 308-347.
όλου τοῦ κόνου τομᾶς καὶ εὐθείας τετραγωνίζειν ἑπειρῶντο λαμβάνοντες οὐκ εὐπαραχώρητα λήμματα, διόπερ αὐτοῖς ὑπὸ τῶν πλείστων οὐκ εὑρισκόμενα ταῦτα κατεγνώσθεν. τὸ δὲ ὑπ’ εὐθείας τε καὶ ὀρθογωνίου κόνου τομᾶς τμᾶμα περιεχόμενον οὐδένα τῶν προτέρων ἐγχειρήσαντα τετραγωνίζειν ἐπιστάμεθα, ὅ ὅτι οὐν ψε ἀμών εὐρηται· δεικνύει γάρ, ὅτι πᾶν τμῆμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κόνου τομᾶς ἐπίτριτον ἐστὶ τοῦ τριγώνου τοῦ βάσιν ἔχοντος τῶν αὐτῶν καὶ ψε οἷον τῷ τμῆματι λαμβανομένου τούτῳ τοῦ λήμματος ἐστὶν ἀποδείξεως αὐτοῦ τῶν ἁπάντων χωρίων τῶν ὑπεροχῶν, ὃ ὑπερέχει τὸ μείζον τοῦ ἐλάσσονος, δυνατὸν εἰμεν αὐτῶν ἑαυτά συντιθεμέναν παντὸς ὑπερέχειν τοῦ προτεθέντος πεπερασμένου χωρίου. κεχρήσται δὲ καὶ οἱ πρότερον γεωμέτραι τῶν τῷ λήμματι· τοὺς τε γὰρ κύκλους διπλασίονα λόγον ἔχειν ποτ’ ἀλλάλους τὰν διαμέτρων ἀποδεδείχσαιν αὐτῷ τούτῳ τῷ λήμματι χρωμένου, καὶ τὰς σφαίρας ὅτι τριπλασίονα λόγον ἔχοντι ποτ’ ἀλλάλας τὰν διαμέτρου, ἔτη δὲ καὶ ὅτι πάσα πυραμίς τρίτων μέρος ἐστὶ τοῦ πρίσματος τοῦ τῶν αὐτῶν βάσιν ἔχοντος τὰ πυραμίδι καὶ ἦλιον οἷον· καὶ διότι πᾶσα σῶν τρίτων μέρος ἐστὶ τοῦ κυλίνδρου τοῦ τῶν αὐτῶν βάσιν ἔχοντος τῷ κόνω καὶ ἦλιον οἷον, ὁμοίων τῶν προειρημένων λήμματος λαμβάνοντες ἡγαφον. συμβαίνει δὲ τῶν προειρημένων ὑποθημάτων ἐκαστοῦ μηδενὸς ἄρσου τῶν ἀνευ τούτου τοῦ λήμματος ἀποδεειγμένων πεπεστευκέναι· ἀρκεῖ δὲ ἐς τὰν ὁμοίων πῦταν τούτοις ἀναγμένων τῶν ὑπ’ ἀμών ἔκδιδομένων. ἀναγράφαντες οὖν αὐτοῦ τὰς ἀποδείξεις ἀποστέλλομεν 230
of the whole cone and a straight line,\textsuperscript{a} assuming lemmas far from obvious, so that it was recognized by most people that the problem had not been solved. But I do not know that any of my predecessors has attempted to square the area bounded by a straight line and a section of a right-angled cone, the solution of which problem I have now discovered; for it is shown that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle which has the same base and height equal to the segment, and for the proof this lemma is assumed: given [two] unequal areas, the excess by which the greater exceeds the less can, by being added to itself, be made to exceed any given finite area. Earlier geometers have also used this lemma: for, by using this same lemma, they proved that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and also that any pyramid is a third part of the prism having the same base as the pyramid and equal height; and, further, by assuming a lemma similar to that aforesaid, they proved that any cone is a third part of the cylinder having the same base as the cone and equal height.\textsuperscript{b} In the event, each of the aforesaid theorems has been accepted, no less than those proved without this lemma; and it will satisfy me if the theorems now published by me obtain the same degree of acceptance. I have therefore written out the proofs, and now send them, first

\textsuperscript{a} A "section of the whole cone" is probably a section cutting right through it, i.e., an ellipse, but the expression is odd.

\textsuperscript{b} For this lemma, \textit{v. supra}, p. 46 n. a.
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πρῶτον μὲν, ὡς διὰ τῶν μηχανικῶν ἑδεωρῆθη, μετὰ ταῦτα δὲ καὶ, ὡς διὰ τῶν γεωμετρουμένων ἀποδείκνυται, προγράφεται δὲ καὶ στοιχεία κωνικά χρείαν ἔχοντα ἐς τὰς ἀπόδειξιν. ἐρρωσο.


"Εστω τμῆμα τὸ ΒΘΓ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς. ἔστω δὴ πρῶτον

α ἘΒ Γ ποτ’ ὀρθὰς τὰ διαμέτρω, καὶ ἄχθω ἀπὸ μὲν τοῦ Β σαμείου ᾧ ΒΔ παρὰ τὰν διάμετρον, ἀπὸ δὲ τοῦ Γ ᾧ ΓΔ ἐπισφάλωσα τὰς τοῦ κώνου τομᾶς κατὰ τὸ Γ· ἐσσείται δὴ τὸ ΒΓΔ τρίγωνον ὀρθογώνιον. διηρήσω δὴ ἀ ΒΓ ἐς ἴσα τμάματα ὀποῖαν τὰ BE, EZ, ZH, HI, IG, καὶ ἀπὸ τὰν τομᾶν ἄχθωσαν παρὰ τὰν διάμετρον αἱ ΕΣ, ΖΤ, ΗΥ, ΙΕ, ἀπὸ δὲ τῶν σαμείων, καθ’ ἀ τέμνοντι 232
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as they were investigated by means of mechanics, and also as they may be proved by means of geometry. By way of preface are included the elements of conics which are needed in the demonstration. Farewell.

Ibid., Prop. 14, Archim. ed. Heiberg ii. 284. 24-290. 17

Let \( \Theta \Gamma \) be a segment bounded by a straight line and a section of a right-angled cone. First let \( \Gamma \) be at right angles to the axis, and from \( B \) let \( B \Delta \) be drawn parallel to the axis, and from \( \Gamma \) let \( \Gamma \Delta \) be drawn touching the section of the cone at \( \Gamma \); then the triangle \( B \Gamma \Delta \) will be right-angled [Eucl. i. 29]. Let \( B \Gamma \) be divided into any number of equal segments \( BE, EZ, ZH, HI, I \Gamma \), and from the points of section let \( E \Xi, ZT, HY, I \Xi \) be drawn parallel to the axis, and from the points in which these cut the
αὐταὶ τὰν τοῦ κώνου τομάν, ἐπεξεύχθωσαν ἐπὶ τὸ Γ καὶ ἐκβεβλήσθωσαν. φαμὶ δὴ τὸ τρίγωνον τὸ ΒΔΓ τῶν μὲν τραπεζίων τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ καὶ τοῦ ΞΙΓ τριγώνου ἐλασσὸν εἶμεν ἡ τριπλάσιον, τῶν δὲ τραπεζίων τῶν ΖΦ, ΗΘ, ΙΠ καὶ τοῦ ΙΟΓ τριγώνου μεῖζόν [ἔστιν] ἡ τριπλάσιον.

Διάχῳ γὰρ εὐθεία ἀ ΑΒΓ, καὶ ἀπολελάφθω καὶ ΑΒ ᾧν τὰ ΒΓ, καὶ νοεῖσθω ζύγιον τὸ ΑΓ· μέσον δὲ αὐτοῦ ἔσειται τὸ B· καὶ κρεμάσθω ἐκ τοῦ Β, κρεμάσθω δὲ καὶ τὸ ΒΔΓ ἐκ τοῦ ζυγοῦ κατὰ τὰ Β, Γ, ἐκ δὲ τοῦ βατέρου μέρος τοῦ ζυγοῦ κρεμάσθω τὰ Ρ, Χ, Ψ, Ω, Δ χωρία κατὰ τὸ Α, καὶ ἱσορροπεῖτω τὸ μὲν Ρ χωρίον τὸ ΔΕ τραπεζίῳ ὀυτως ἐχοντι, τὸ δὲ Χ τῶν ΖΣ τραπεζίων, τὸ δὲ Ψ τῶν ΘΗ, τὸ δὲ Ω τῶν ΥΙ, τὸ δὲ Δ τῶν ΞΙΓ τριγώνων· ἱσορροπήσει δὴ καὶ τὸ ὁλον τῷ ὅλῳ· ὡστε τριπλάσιον ἀν εἰη τὸ ΒΔΓ τριγώνου τοῦ ΡΧΨΩΔ χωρίου. καὶ ἑπεὶ ἔστιν τμῆμα τὸ ΒΓΘ, δ' περιέχεται ὑπὸ τε εὐθείας καὶ ὀρθογώνιου κώνου τομᾶς, καὶ ἀπὸ μὲν τοῦ Β παρὰ τὰν διάμετρον ἀκταὶ ἂ ΒΔ, ἀπὸ δὲ τοῦ Γ ἂ ΓΔ ἐπιμακουσά τάς τοῦ κώνου τομᾶς κατὰ τὸ Γ, ἀκταὶ δὲ τις καὶ ἄλλα παρὰ τὰν διάμετρον ἂ ΣΕ, τῶν αὐτῶν ἔχει λόγον ἂ ΒΓ ποτὶ τὰν ΒΕ, ὅν ἂ ΣΕ ποτὶ τὰν ΕΦ· ὡστε καὶ ἂ ΒΑ ποτὶ τὰν ΒΕ τῶν αὐτῶν ἔχει λόγον, ὅν τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ. ὀμοίως δὲ δευτερέσεται ἂ ΑΒ ποτὶ τὰν ΒΖ τῶν αὐτῶν ἔχουσα λόγον, ὅν τὸ ΣΖ τραπεζίον ποτὶ τὸ ΛΖ, ποτὶ δὲ τὰν ΒΗ, ὅν τὸ ΘΗ ποτὶ τὸ ΜΗ, ποτὶ δὲ τὰν ΒΙ, ὅν τὸ ΥΙ ποτὶ τὸ ΝΙ. ἑπεὶ οὖν ἔστι τραπεζίου τὸ ΔΕ τὰς

1 ἔστιν om. Heiberg.
section of the cone let straight lines be drawn to Γ and produced. Then I say that the triangle ΒΔΓ is less than three times the trapezia ΚΕ, ΔΖ, ΜΗ, ΝΙ and the triangle ΞΙΓ, but greater than three times the trapezia ΖΦ, ΗΘ, ΙΠ and the triangle ΙΟΓ.

For let the straight line ΑΒΓ be drawn, and let ΑΒ be cut off equal to ΒΓ, and let ΑΓ be imagined to be a balance; its middle point will be Β; let it be suspended from Β, and let the triangle ΒΔΓ be suspended from the balance at Β, Γ, and from the other part of the balance let the areas Ρ, Χ, Ψ, Ω, Δ be suspended at Α, and let the area Ρ balance the trapezium ΔΕ in this position, let Χ balance the trapezium ΖΣ, let Ψ balance ΤΗ, let Ω balance ΥΙ, and let Δ balance the triangle ΞΙΓ; then the whole will balance the whole; so that the triangle ΒΔΓ will be three times the area Ρ + Χ + Ψ + Ω + Δ [Prop. 6]. And since ΒΓΘ is a segment bounded by a straight line and a section of a right-angled cone, and ΒΔ has been drawn from Β parallel to the axis, and ΓΔ has been drawn from Γ touching the section of a cone at Γ, and another straight line ΖΕ has been drawn parallel to the axis,

\[ \text{BG : BE} = \Sigma E : E \Phi; \quad \text{[Prop. 5] } \]

therefore \( \text{BA : BE} = \text{trapezium } ΔE : \text{trapezium } KE. \)

Similarly it may be proved that

\[ \text{AB : BZ} = \Sigma Z : ΔZ, \]
\[ \text{AB : BH} = \text{TH : MH}, \]
\[ \text{AB : BI} = \text{YI : NI}. \]

Therefore, since ΔΕ is a trapezium with right angles

\[ * \text{For } BA = BG \text{ and } ΔE : KE = \Sigma E : E \Phi. \]

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μὲν ποτὶ τοῖς Β, Ε σαμείους γνωνίας ὀρθὰς ἔχουν, τὰς δὲ πλευρὰς ἐπὶ τὸ Γ νεούσας, ἱσορροπεῖ δὲ τὶ χωρίον αὐτῷ τὸ Ρ κρεμάμενον ἐκ τοῦ ξυγοῦ κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπέζιου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς ἂ ΒΑ ποτὶ τὰν ΒΕ, οὕτως τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ, μεῖζον ἀρα ἐστὶν τὸ ΚΕ χωρίον τοῦ Ρ χωρίου. δέδεικται γὰρ τοῦτο, πάλιν δὲ καὶ τὸ ΖΣ τραπέζιον τὰς μὲν ποτὶ τοῖς Ζ, Ε γνωνίας ὀρθὰς ἔχουν, τὰν δὲ ΣΤ νεούσας ἐπὶ τὸ Γ, ἱσορροπεῖ δὲ αὐτῷ χωρίον τὸ Χ ἐκ τοῦ ξυγοῦ κρεμάμενον κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπέζιου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς μὲν ἂ ΑΒ ποτὶ τὰν ΒΕ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΖΦ, ὡς δὲ ἂ ΑΒ ποτὶ τὰν ΒΖ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΑΖ· εἰη οὖν καὶ τὸ Χ χωρίον τοῦ μὲν ΛΖ τραπέζιον ἐλασσόν, τοῦ δὲ ΖΦ μείζον· δέδεικται γὰρ καὶ τοῦτο. διὰ τὰ αὐτὰ δὴ καὶ τὸ Ψ χωρίον τοῦ μὲν ΜΗ τραπέζιον ἐλασσόν, τοῦ δὲ ΘΗ μείζον, καὶ τὸ Ω χωρίον τοῦ μὲν ΝΟΙΗ τραπέζιον ἐλασσόν, τοῦ δὲ ΠΙ μείζον, ὡμοίως δὲ καὶ τὸ Δ χωρίον τοῦ μὲν ΕΙΓ τριγώνου ἐλασσόν, τοῦ δὲ ΠΙΟ μείζον. ἐπεὶ οὖν τὸ μὲν ΚΕ τραπέζιον μείζον ἐστὶ τοῦ Ρ χωρίου, τὸ δὲ ΛΖ τοῦ Χ, τὸ δὲ ΜΗ τοῦ Ψ, τὸ δὲ ΝΙ τοῦ Ω, τὸ δὲ ΕΙΓ τριγώνου τοῦ Δ, φανερὸν, ὅτι καὶ πάντα τὰ εἰρημένα χωρία μείζονα ἐστὶ τοῖς ΡΧΨΩΔ χωρίοις. ἐστιν δὲ τὸ ΡΧΨΩΔ τρίτον μέρος τοῦ ΒΓΔ τριγώνου· δήλου ἀρα, ὅτι τὸ ΒΓΔ τριγώνου ἐλασσόν ἐστιν η τριπλάσιον τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ τραπέζιων καὶ τοῦ ΕΙΓ τριγώνου. πάλιν, ἐπεὶ τὸ μὲν ΖΦ τραπέζιον ἐλασσόν ἐστὶ τοῦ Χ χωρίου, τὸ δὲ ΘΗ τοῦ Ψ, τὸ δὲ ΠΙ τοῦ Ω, τὸ δὲ ΙΟΓ τριγώνου τοῦ Δ, φανερὸν, ὅτι καὶ πάντα 236
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at the points B, E and with sides converging on Γ, and it balances the area P suspended from the balance at A, if the trapezium be in its present position, while

\[ \frac{BA}{BE} = \frac{\Delta E}{KE}, \]

therefore

\[ KE > P; \]

for this has been proved [Prop. 10]. Again, since \( \Sigma \) is a trapezium with right angles at the points Z, E and with ΣT converging on Γ, and it balances the area X suspended from the balance at A, if the trapezium be in its present position, while

\[ \frac{AB}{BE} = \frac{\Sigma}{Z\Phi}, \]

\[ \frac{AB}{BZ} = \frac{\Sigma}{\Lambda Z}, \]

therefore

\[ \Lambda Z > X > Z\Phi; \]

for this also has been proved [Prop. 12]. By the same reasoning

\[ MH > \Psi > \Theta H, \]

and

\[ NOI H > \Omega > \Pi I, \]

and similarly

\[ \Xi I \Gamma > \Delta > \Gamma O. \]

Then, since \( KE > P, \Delta Z > X, MH > \Psi, NI > \Omega, \Xi I \Gamma > \Delta, \) it is clear that the sum of the aforesaid areas is greater than the area \( P + X + \Psi + \Omega + \Delta. \) But

\[ P + X + \Psi + \Omega + \Delta = \frac{1}{2} \, B\Gamma \Delta; \quad \text{[Prop. 6]} \]

it is therefore plain that

\[ B\Gamma \Delta < S(KE + \Delta Z + MH + NI + \Xi I \Gamma). \]

Again, since \( Z\Phi < X, \Theta H < \Psi, \Pi I < \Omega, \Theta O \Gamma < \Delta, \) it is

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τὰ εἰρημένα ἐλάσσονά ἐστὶ τοῦ ΔΩΨΧ χωρίου φανερῶν οὖν, ὅτι καὶ τὸ ΒΔΓ τριγώνου μεῖζὸν ἐστὶν ἡ τριπλάσιον τῶν ΦΖ, ΘΗ, ΠΙ τραπεζίων καὶ τοῦ ΙΓΘ τριγώνου, ἐλάσσον δὲ ἡ τριπλάσιον τῶν προγεγραμμένων.

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Πᾶν τμῆμα τὸ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κόνου τομᾶς ἐπίτριτον ἐστὶ τριγώνου τοῦ τῶν αὐτῶν βάσιν ἔχοντος αὐτῷ καὶ ὑψὸς ἴσον.

"Εστω γὰρ τὸ ΑΔΒΕΓ τμῆμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κόνου τομᾶς, τὸ δὲ ΑΒΓ τριγώνον ἐστὶ τῶν αὐτῶν βάσιν ἔχον τὰ τμάματι καὶ ὑψὸς ἴσον, τοῦ δὲ ΑΒΓ τριγώνου ἐστὶν ἐπίτριτον τὸ Κ χωρίον. δεικτέον, ὅτι ἴσον ἐστὶ τῷ ΑΔΒΕΓ τμάματι.

Εἰ γὰρ μὴ ἐστὶν ἴσον, ἦτοι μεῖζὸν ἐστὶν ἡ ἐλασσον. ἐστὶ πρῶτον, εἴ δυνατόν, μεῖζον τὸ ΑΔΒΕΓ τμῆμα τοῦ Κ χωρίου. ἐνέγραψα δὴ τὰ ΑΔΒ, ΒΕΓ τρίγωνα, ὡς εἰρηται, ἐνέγραψα δὲ καὶ εἰς τὰ περιείπομενα τμάματα ἄλλα τρίγωνα τῶν αὐτῶν.
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clear that the sum of the aforesaid areas is greater than the area $\Delta + \Omega + \Psi + X$;
it is therefore manifest that

$$B\Delta \Gamma > 3(\Phi Z + \Theta H + \Pi + \Gamma H),$$

but is less than thrice the aforementioned areas.\footnote{In Prop. 15 Archimedes shows that the same theorem holds good even if $\Gamma T$ is not at right angles to the axis. It is then proved in Prop. 16, by the method of exhaustion, that the segment is equal to one-third of the triangle $BT\Gamma$. This is done by showing, on the basis of the "Axiom of Archimedes," that by taking enough parts the difference between the circumscribed and the inscribed figures can be made as small as we please. It is equivalent to integration. From this it is easily proved that the segment is equal to four-thirds of a triangle with the same base and equal height (Prop. 17).}


*Any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.*

For let $\Delta\Delta\Gamma\Gamma$ be a segment bounded by a straight line and a section of a right-angled cone, and let $\Delta\Gamma\Gamma$ be a triangle having the same base as the segment and equal height, and let the area $K$ be four-thirds of the triangle $\Delta\Gamma\Gamma$. It is required to prove that it is equal to the segment $\Delta\Delta\Gamma\Gamma$.

For if it is not equal, it is either greater or less. Let the segment $\Delta\Delta\Gamma\Gamma$ first be, if possible, greater than the area $K$. Now I have inscribed the triangles $\Delta\Delta\Gamma$, $\Gamma\Gamma\Gamma$, as aforesaid,\footnote{In earlier propositions Archimedes has used the same procedure as he now describes. $\Delta$, $\Gamma$ are the points in which the diameter through the mid-points of $AB$, $BT$ meet the curve.} and I have inscribed in the remaining segments other triangles having the same

\footnote{For $B\Delta \Gamma = 3(P + X + \Psi + \Omega + \Delta) > 3(\Delta + \Omega + \Psi + X)$.}
βάσιν ἔχοντα τοῖς τμαμάτεσσιν καὶ ύφος τὸ αὐτὸ, καὶ ἀεὶ εἰς τὰ ὑστερον γνώμενα τμάματα ἐγγράφω δύο τρίγωνα τὰν αὐτάν βάσιν ἔχοντα τοῖς τμαμά
tεσσιν καὶ ύφος τὸ αὐτὸ· ἐσσοῦνται δὴ τὰ κατα-
λειπόμενα τμάματα ἐλάσσονα τὰς ὑπεροχὰς, ἕ
ὑπερέχει τὸ ΑΔΒΕΓ τμῆμα τοῦ Κ χωρίου. ᾧστε
τὸ ἐγγραφόμενον πολύγωνον μεῖζον ἐσσείται τοῦ
Κ. ὅπερ ἀδύνατον. ἐπεὶ γάρ ἐστιν εξῆς κείμενα
χωρία ἐν τῷ τετραπλάσιον λόγῳ, πρῶτον μὲν τὸ
ΑΒΓ τρίγωνον τετραπλάσιον τῶν ΑΔΒ, ΒΕΓ
tριγώνων, ἔπειτα δὲ αὐτά ταῦτα τετραπλάσια τῶν
eἰς τὰ ἐπόμενα τμάματα ἐγγραφέντων καὶ ἀεὶ
οὗτο, δὴ λοι, ὡς σύμπαντα τὰ χωρία ἐλάσσονα
ἐστιν ἢ ἐπίτριτα τοῦ μεγίστου, τὸ δὲ Κ ἐπίτριτον
ἐστὶ τοῦ μεγίστου χωρίου. οὐκ ἄρα ἐστίν μεῖζον
τὸ ΑΔΒΕΓ τμῆμα τοῦ Κ χωρίου.

"Εστὶ δὲ, εἰ δυνατὸν, ἔλασσον. κείσθω δὴ τὸ
μὲν ΑΒΓ τρίγωνον ἵσον τῷ Ζ, τοῦ δὲ Ζ τέταρτον
τὸ Η, καὶ ὁμοίως τοῦ Η τὸ Θ, καὶ ἀεὶ εξῆς
tιθέοις, ἔως καὶ γένηται τὸ ἐσχατὸν ἔλασσον τᾶς

* This was proved geometrically in Prop. 23, and is proved
 generally in Eucl. ix. 35. It is equivalent to the summation

\[ 1 + (\frac{1}{4}) + (\frac{1}{4})^2 + \cdots = \frac{4}{3} \]

\[ = \frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}}. \]
base as the segments and equal height, and so on continually I inscribe in the resulting segments two

triangles having the same base as the segments and equal height; then there will be left [at some time] segments less than the excess by which the segment \( A\Delta B\Gamma \) exceeds the area \( K \) [Prop. 20, coroll.]. Therefore the inscribed polygon will be greater than \( K \); which is impossible. For since the areas successively formed are each four times as great as the next, the triangle \( A\Gamma I \) being four times the triangles \( A\Delta B, B\Gamma \) [Prop. 21], then these last triangles four times the triangles inscribed in the succeeding segments, and so on continually, it is clear that the sum of all the areas is less than four-thirds of the greatest [Prop. 23], and \( K \) is equal to four-thirds of the greatest area. Therefore the segment \( A\Delta B\Gamma \) is not greater than the area \( K \).

Now let it be, if possible, less. Then let

\[ Z = A\Gamma I, \quad H = \frac{1}{4}Z, \quad \Theta = \frac{1}{4}H, \]

and so on continually, until the last [area] is less than

(k) Hydrostatics

(i.) Postulates

Archim. De Corpore. Fluit. i., Archim. ed. Heiberg ii. 318. 2-8

Ὑποκείσθω τὸ υγρὸν φύσιν ἔχον τοιαύταν, ὡστε τῶν μερέων αὐτοῦ τῶν ἐξ ἵσον κείμενων καὶ συν-

*The Greek text of the book On Floating Bodies, the earliest extant treatise on hydrostatics, first became available in 1906 when Heiberg discovered at Constantinople the ms. which he terms C. Unfortunately many of the readings are doubtful, and those who are interested in the text should consult the Teubner edition. Still more unfortunately, it is incomplete; but, as the whole treatise was translated into Latin in 1269 by William of Moerbeke from a Greek ms.
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the excess by which the area $K$ exceeds the segment [Eucl. x. 1], and let $I$ be [the area] less [than this excess]. Now

$$Z + H + \Theta + I + \frac{1}{3}I = \frac{4}{3}Z.$$  [Prop. 23]

But

$$K = \frac{4}{3}Z;$$

therefore

$$K = Z + H + \Theta + I + \frac{1}{3}I.$$

Therefore since the area $K$ exceeds the areas $Z$, $H$, $\Theta$, $I$ by an excess less than $I$, and exceeds the segment by an excess greater than $I$, it is clear that the areas $Z$, $H$, $\Theta$, $I$ are greater than the segment; which is impossible; for it was proved that, if there be any number of areas in succession such that each is four times the next, and the greatest be equal to the triangle inscribed in the segment, then the sum of the areas will be less than the segment [Prop. 22]. Therefore the segment $\Delta \triangle E \Gamma$ is not less than the area $K$. And it was proved not to be greater; therefore it is equal to $K$. But the area $K$ is four-thirds of the triangle $\triangle B \Gamma$; and therefore the segment $\Delta \triangle E \Gamma$ is four-thirds of the triangle $\triangle B \Gamma$.

(k) HYDROSTATICS

(i.) Postulates

Heiberg ii. 318. 2-8

Let the nature of a fluid be assumed to be such that, of its parts which lie evenly and are continuous, since lost, it is possible to supply the missing parts in Latin, as is done for part of Prop. 2. From a comparison with the Greek, where it survives, William's translation is seen to be so literal as to be virtually equivalent to the original. In each case Heiberg's figures are taken from William's translation, as they are almost unrecognizable in C; for convenience in reading the Greek, the figures are given the appropriate Greek letters in this edition.
GREEK MATHEMATICS

εχέων ἑντὼν ἐξωθείσθαι τὸ ἰσον θλιβώμενον ὑπὸ τοῦ μᾶλλον θλιβομένου, καὶ ἐκαστὸν ἐκ τῶν μερέων αὐτοῦ θλίβεσθαι τῷ ὑπέρανῳ αὐτοῦ ὕγρῳ κατὰ κάθετον ἑντὶ, εἰ καὶ μη τὸ ὕγρον ἢ καθεργεμένον ἐν τινί καὶ ὑπὸ ἄλλου τινὸς θλιβομένου.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

Ὑποκείσθω, τῶν ἐν τῷ ὕγρῳ ἀνω φερομένων ἐκαστὸν ἀναφέρεσθαι κατὰ τὰν κάθετον τὰν διὰ τοῦ κέντρου τοῦ βάρεος αὐτοῦ ἀγμέναν.

(ii.) Surface of Fluid at Rest

Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7-320. 30

Omnis humidum consistens ita, ut maneat in motum, superficies habebit figuram sperae habentis centrum idem cum terra.

Intelligatur enim humidum consistens ita, ut maneat non motum, et secetur ipsius superficies plano per centrum terrae, sit autem terrae centrum K, superficieln autem sectio linea ABGD. Dico itaque,

lineam ABGD circuli esse periferiam, centrum autem ipsius K.

Si enim non est, rectae a K ad lineam ABGD 244
that which is under the lesser pressure is driven along by that under the greater pressure, and each of its parts is under pressure from the fluid which is perpendicularly above it, except when the fluid is enclosed in something and is under pressure from something else.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

Let it be assumed that, of bodies which are borne upwards in a fluid, each is borne upwards along the perpendicular drawn through its centre of gravity.\(^a\)

(ii.) Surface of Fluid at Rest

Ibid. i., Prop. 2, \(^b\) Archim. ed. Heiberg ii. 319. 7-320. 30

The surface of any fluid at rest is the surface of a sphere having the same centre as the earth.

For let there be conceived a fluid at rest, and let its surface be cut by a plane through the centre of the earth, and let the centre of the earth be K, and let the section of the surface be the curve ABΓΔ. Then I say that the curve ABΓΔ is an arc of a circle whose centre is K.

For if it is not, straight lines drawn from K to the

\(^a\) These are the only assumptions, other than the assumptions of Euclidean geometry, made in this book by Archimedes; if the object of mathematics be to base the conclusions on the fewest and most "self-evident" axioms, Archimedes' treatise On Floating Bodies must indeed be ranked highly.

\(^b\) The earlier part of this proposition has to be given from William of Moerbeke's translation. The diagram is here given with the appropriate Greek letters.
occurrentes non erunt aequales. Sumatur itaque aliqua recta, quae est quarundam quidem a K occurrentium ad lineam ABGD maior, quarundam autem minor, et centro quidem K, distantia autem sumptae lineae circulus describatur; cadet igitur periferia circuli habens hoc quidem extra lineam ABGD, hoc autem intra, quoniam quae ex centro quarundam quidem a K occurrentium ad lineam ABGD est maior, quarundam autem minor. Sit igitur descripti circuli periferia quae ZBH, et a B ad K recta ducatur, et copulentur quae ZK, KEL aequales facientes angulos, describatur autem et centro K periferia quaedam quae XOP in plano et in humid ; partes itaque humili quae secundum XOP periferiam ex aequo sunt positae et continuae inuicem. Et pre- muntur quae quidem secundum XO periferiam humidio quod secundum ZB locum, quae autem secundum periferiam OP humidio quod secundum BE locum; inaequaliter igitur premuntur partes humili quae secundum periferiam XO ei quae [?]" kata tawn OP. woste ezwethsontai ta hisoun thleboimen aypo twn mallo thleboemwn ou menei ara to ygron. upkeito de kathetakos elmei woste menein akiniton anagkaion ara tawn ABGD grama- maon kyklou periferiean elmei kai kentroon autaas to K. omois eis deichytheTai kai, opous ka alles e euphaneia tou ygron epipedw thath dia tou kentron tas gws, oti a toma esseita kyklou periferieia, kai kentroon autaas esseita, o kai tas gws esti kentroon. dhlon ouv, oti a euphaneia tou ygron kathetakotos akinitou sfairas exei to schema to auto kentroon ehoussas ta gws, epeidh

1 η om. Heiberg.
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curve $AB\Gamma\Delta$ will not be equal. Let there be taken, therefore, any straight line which is greater than some of the straight lines drawn from $K$ to the curve $AB\Gamma\Delta$, but less than others, and with centre $K$ and radius equal to the straight line so taken let a circle be described; the circumference of the circle will fall partly outside the curve $AB\Gamma\Delta$, partly inside, inasmuch as its radii are greater than some of the straight lines drawn from $K$ to the curve $AB\Gamma\Delta$, but less than others. Let the arc of the circle so described be $ZBH$, and from $B$ let a straight line be drawn to $K$, and let $ZK$, $KE\Lambda$ be drawn making equal angles [with $KB$], and with centre $K$ let there be described, in the plane and in the fluid, an arc $ZO\Pi$; then the parts of the fluid along $ZO\Pi$ lie evenly and are continuous [v. supra, p. 243]. And the parts along the arc $ZO$ are under pressure from the portion of the fluid between it and $ZB$, while the parts along the arc $O\Pi$ are under pressure from the portion of the fluid between it and $BE$; therefore the parts of the fluid along $ZO$ and the parts of the fluid along $O\Pi$ are under unequal pressures; so that the parts under the lesser pressure are thrust along by the parts under the greater pressure [v. supra, p. 245]; therefore the fluid will not remain at rest. But it was postulated that the fluid would remain unmoved; therefore the curve $AB\Gamma\Delta$ must be an arc of a circle with centre $K$. Similarly it may be shown that, in whatever other manner the surface be cut by a plane through the centre of the earth, the section is an arc of a circle and its centre will also be the centre of the earth. It is therefore clear that the surface of the fluid remaining at rest has the form of a sphere with the same centre as the earth, since it is such
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toiacata estin,oste diata tou autou semeion trimbhe-
san tav toman poiein periferian kuklon kentron
exontos to semeion, di' ou teimetai to epipedo.

(iii.) Solid immersed in a Fluid

Ibid. 1., Prop. 7, Archim. ed. Heiberg ii. 332. 21-336. 13

Ta barutera tou ugrou afe神秘ta eis to ugron
oioeitai katw, est' an katabaanti, kai eposounai
koufoteran eis to ugro tosootou, oson echei to
bapos tou ugro tou talikoouton ognon exontos,
allinos estin o to ou stereon megebeos ognos.

"Oste men ou oioeitai est' to katw, est' an kata-
baanti, dhlou tis gar upokatai autou meres tou
ugrou thleitsoynai malloin touen ex isou autouis
keimenois merewon, epidei' baruteron upokeita to
stereon megebos to ugroi' oste de koufoteran
eopouynai, ois eipetai, deichtheitai.

"Estw ti megebos to A, o esti baruteren tou
ugrou, bapos de estw tou men ev o A megebeos
to BG, tou de ugroi tou isou ognon exontos tou
A to B. deikten, oste to A megebos ev to ugroi
evn bapos echei ison tou BG.

Lelepho gar ti megebos to ev o to Delta koufo-
teron tou ugroi tou isou ognon exontos autou,
estw de tou men ev o to Delta megebeos bapos isou
tou B bareai, tou de ugroi tou isou ognon exontos
tou Delta megebhe to bapos estw ison tou BG bareai.

* Or, as we should say, "lighter by the weight of fluid
  displaced."

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that, when it is cut [by a plane] always passing through the same point, the section is an arc of a circle having for centre the point through which it is cut by the plane [Prop. 1].

(iii.) Solid immersed in a Fluid


Solids heavier than a fluid will, if placed in the fluid, sink to the bottom, and they will be lighter [if weighed] in the fluid by the weight of a volume of the fluid equal to the volume of the solid.¹

That they will sink to the bottom is manifest; for the parts of the fluid under them are under greater pressure than the parts lying evenly with them, since it is postulated that the solid is heavier than water; that they will be lighter, as aforesaid will be [thus] proved.

Let A be any magnitude heavier than the fluid, let the weight of the magnitude A be B + Γ, and let the weight of fluid having the same volume as A be B. It is required to prove that in the fluid the magnitude A will have a weight equal to Γ.

For let there be taken any magnitude Δ lighter than the same volume of the fluid such that the weight of the magnitude Δ is equal to the weight B, while the weight of the fluid having the same volume as the magnitude Δ is equal to the weight B + Γ.
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συντεθέντων δὴ εσ το αυτο των μεγεθεων, εν οἷς
τα A, Δ, το των συναμμοντων μέγεθων ἴσοβαρές
ἐσσειται τῷ υγρῷ. ἔστι γὰρ τῶν μεγεθῶν συν-
αμμοντων τὸ βάρος ἴσου συναμμοντων τοῖς
βάρεσι τῷ τε ΒΓ καὶ τῷ B, τοῦ δὲ υγροῦ τοῦ
ἰσον δικον ἐχοντος ἴσομον τοῖς μεγεθει τὸ
βάρος ἴσου ἔστι τοῖς αὐτοῖς βάρεσιν. ἄφεεθεν
tῶν μεγεθῶν ἐσ τῷ υγρῷ ἴσορροπησοῦνται
tῷ υγρῷ καὶ οὕτε εἰς τὸ ἀνω οἰσιώτατοι οὕτε εἰς
tὸ κάτω διὸ τὸ μὲν ἐν ω A μέγεθος οἰσιώτατος
ἐσ τὸ κάτω καὶ τοσαῦτα βια ὑπὸ τοῦ ἐν ω Δ
μέγεθος ἀνελκεται ἐς τὸ ἀνω, τὸ δὲ ἐν ω Δ
μέγεθος, ἕπει κουφότερον ἐστὶ τοῦ υγροῦ,
ἀναισθηται εἰς τὸ ἀνω τοσαῦτα βια, ὅσον ἔστι
tὸ Γ βάρος· διδείκται γὰρ,
ὅτι τὰ κουφότερα τοῦ υγροῦ μεγεθα στερεὰ βια-
σθέντα ἐς τὸ υγρὸν ἀναφέρουται τοσαῦτα βια ἐς τὸ
ἀνω, ὅσον ἔστι τὸ βάρος, ὁ βαρύτερον ἐστὶ τοῦ
μεγεθεος τὸ υγρὸν τὸ ἴσογκον τῷ μεγεθε. ἔστι
δε τῷ Γ βάρει βαρύτερον τοῦ Δ μεγεθεος τὸ υγρὸν
tὸ ἴσον ὅγκον ἐχον τῷ Δ· δῆλον οὖν, ὅτι καὶ τὸ ἐν
ω A μέγεθος ἐσ τὸ κάτω οἰσιώτατοι τοσοῦτοι βάρει,
ὅσον ἔστι τὸ Γ.

* This proposition suggests a method, alternative to that
given by Vitruvius (e. supra, pp. 36-39, especially p. 38 n. a),
whereby Archimedes may have discovered the proportions of
gold and silver in King Hiero's crown.

Let w be the weight of the crown, and let w₁ and w₂ be the
weights of gold and silver in it respectively, so that w =
w₁ + w₂.

Take a weight w of gold and weigh it in a fluid, and let
the loss of weight be P₁. Then the loss of weight when a
weight w₁ of gold is weighed in the fluid, and consequently
the weight of fluid displaced, will be w₁ . P₁.

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Then if we combine the magnitudes $A$, $\Delta$, the combined magnitude will be equal to the weight of the same volume of the fluid; for the weight of the combined magnitudes is equal to the weight $(B + \Gamma) + B$, while the weight of the fluid having the same volume as both the magnitudes is equal to the same weight. Therefore, if the [combined] magnitudes are placed in the fluid, they will balance the fluid, and will move neither upwards nor downwards [Prop. 8]; for this reason the magnitude $A$ will move downwards, and will be subject to the same force as that by which the magnitude $\Delta$ is thrust upwards, and since $\Delta$ is lighter than the fluid it will be thrust upwards by a force equal to the weight $\Gamma$; for it has been proved that when solid magnitudes lighter than the fluid are forcibly immersed in the fluid, they will be thrust upwards by a force equal to the difference in weight between the magnitude and an equal volume of the fluid [Prop. 6]. But the fluid having the same volume as $\Delta$ is heavier than the magnitude $\Delta$ by the weight $\Gamma$; it is therefore plain that the magnitude $A$ will be borne upwards by a force equal to $\Gamma$.

Now take a weight $w$ of silver and weigh it in the fluid, and let the loss of weight be $P_2$. Then the loss of weight when a weight $w_2$ of silver is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{w_2}{w} \cdot P_2$.

Finally, weigh the crown itself in the fluid, and let the loss of weight, and consequently the weight of fluid displaced, be $P$.

It follows that $\frac{w_1}{w} \cdot P_1 + \frac{w_2}{w} \cdot P_2 = P$,

whence $\frac{w_1}{w_2} = \frac{P_2 - P}{P - P_1}$.
(iv.) Stability of a Paraboloid of Revolution


Τὸ ὀρθὸν τμῆμα τοῦ ὀρθογωνίου κωνοειδέos, ὅταν τὸν ἀξονα ἔχῃ μή μείζονα ἢ ἡμιόλιον τὰς μέχρι τοῦ ἄξονα, πάντα λόγου ἔχον ποτὶ τὸ ύγρὸν τῶν βάρει, ἀφεθὲν εἰς τὸ ύγρὸν οὕτως, ὡστε τὰν βάσιν αὐτοῦ μὴ ἀπετεθαι τοῦ ύγροῦ, τεθὲν κεκλιμένον οὐ μενεί κεκλιμένον, ἀλλὰ ἀποκαταστασεῖται ὀρθῶν. ὀρθῶν δὲ λέγω καθεστακέναι τὸ τοιοῦτο τμῆμα, ὅποταν τὸ ἀποτετμακὸς αὐτὸ ἐπίπεδον παρὰ τὰν ἐπιφάνειαν ἢ τὸν ύγροῦ.

"Εστὶν τμῆμα ὀρθογωνίου κωνοειδέος, οἷον εἰρήνηται, καὶ κεῖσθαι κεκλιμένον. δεικτέον, ὅτι οὐ μενεῖ, ἀλλ' ἀποκαταστασεῖται ὀρθῶν.

Τμαθέντος δὴ αὐτοῦ ἐπίπεδω διὰ τοῦ ἄξονος ὀρθῶ ποτὶ τὸ ἐπίπεδον τὸ ἐπὶ τὰς ἐπιφανείας τοῦ ύγροῦ τμάματος ἐστὶν τομὰ ἀ ΑΠΟΛ ὀρθογωνίου κώνου τομά, ἄξον δὲ τοῦ τμάματος καὶ διάμετρος τὰς τομὰς ἀ ΝΟ, τὰς δὲ τοῦ ύγροῦ ἐπιφανείας τομὰ ἀ ΙΣ. ἐπεὶ οὖν τὸ τμῆμα οὐκ ἔστιν ὀρθῶν, οὐκ ἂν εἴη παράλληλος ἀ ΑΛ τὸν ἐστὶν ὀρθῶν, οὐκ ἂν εἴη παράλληλος τὰ ἀ ΙΣ. ὥστε οὐ ποιήσει ὀρθῶν γωνίαν ἀ ΝΟ ποτὶ τὰν ἐστὶν ἀχθω

* Writing of the treatise On Floating Bodies, Heath (H.G.M. ii. 94–95) justly says: "Book ii., which investigates fully the conditions of stability of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and the principal parameter of the generating parabola, is a veritable tour de force which must be read in full to be appreciated."

* In this technical term the "axis" is the axis of the
(iv.) Stability of a Paraboloid of Revolution


If there be a right segment of a right-angled conoid, whose axis is not greater than one-and-a-half times the line drawn as far as the axis, and whose weight relative to the fluid may have any ratio, and if it be placed in the fluid in an inclined position in such a manner that its base do not touch the fluid, it will not remain inclined but will return to the upright position. I mean by returning to the upright position the figure formed when the plane cutting off the segment is parallel to the surface of the fluid.

Let there be a segment of a right-angled conoid, such as has been stated, and let it be placed in an inclined position. It is required to prove that it will not remain there but will return to the upright position.

Let the segment be cut by a plane through the axis perpendicular to the plane which forms the surface of the fluid, and let ΑΠΟΛ be the section of the segment, being a section of a right-angled cone [De Con. et Sphaer. 11], and let ΝΟ be the axis of the segment and the axis of the section, and let ΙΣ be the section of the surface of the liquid. Then since the segment is not upright, ΛΛ will not be parallel to ΙΣ; and therefore ΝΟ will not make a right angle right-angled cone from which the generating parabola is derived. The _latus rectum_ is "the line which is double of the line drawn as far as the axis" (ά διπλασία τᾶς μέχρι τοῦ ἀξόνος); and so the condition laid down by Archimedes is that the axis of the segment of the paraboloid of revolution shall not be greater than three-quarters of the _latus rectum_ or _principal parameter_ of the generating parabola.
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οὗν παράλληλος ἀ ἐφαπτομένα ἀ ΚΩ τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Π, καὶ ἀπὸ τοῦ Π παρὰ τὰν

ΝΩ ἅχθῷ ἀ ΠΦ· τέμνει δὴ ἀ ΠΦ δίχα τὰν ΙΣ· δεδεικται γὰρ ἐν τοῖς κωνικοῖς. τετμάσθω ἀ ΠΦ, ὡστε εἰμὲν διπλασίαν τὰν ΠΒ τᾶς ΒΦ, καὶ ἀ ΝΩ κατὰ τὸ Ρ τετμάσθω, ὡστε καὶ τὰν ὈΡ τᾶς ΡΝ διπλασίαν εἰμὲν· ἐσσεῖται δὴ τοῦ μείζονος ἀποτμάματος τοῦ στερεοῦ κέντρον τοῦ βάρεος τὸ Ρ, τοῦ δὲ κατὰ τὰν ΠΙΟΣ τὸ Β· δεδεικται γὰρ ἐν ταῖς Ἰσορροπίαις, ὅτι παντὸς ὀρθογωνίου κωνοειδέος τμάματος τὸ κέντρον τοῦ βάρεος ἐστὶν ἐπὶ τοῦ ἄξονος διηρημένου οὕτως, ὡστε τὸ ποτὶ τὰ κορυφὰ τοῦ ἄξονος τμῆς διπλάσιον εἰμὲν τοῦ λοιποῦ. ἀφαιρεθέντος δὴ τοῦ κατὰ τὰν ΠΙΟΣ τμάματος στερεοῦ ἀπὸ τοῦ ὅλου τοῦ λοιποῦ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς ΒΓ εὐθείας· δεδεικται γὰρ τούτῳ ἐν τοῖς Στοιχείοις τῶν μηχανικῶν, ὅτι, εἰ καὶ μέγεθος ἀφαιρεθῇ μὴ τὸ αὐτὸ κέντρον ἔχων τοῦ βάρεος τῷ ὅλῳ μεγέθει, τοῦ λοιποῦ τὸ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς εὐθείας τὰς ἐπὶ ὑπεργινούσας τὰ κέντρα τοῦ τε ὅλου μεγέθεος καὶ τοῦ

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with $I\Sigma$. Therefore let $K\Omega$ be drawn parallel [to $I\Sigma$] and touching the section of the cone at $\Pi$, and from $\Pi$ let $\Pi\Phi$ be drawn parallel to $NO$; then $\Pi\Phi$ bisects $I\Sigma$—for this is proved in the [*Elements of*] Conics.* Let $\Pi\Phi$ be cut so that $\Pi\beta = 2\beta\Phi$, and let $NO$ be cut at $P$ so that $OP = 2PN$; then $P$ will be the centre of gravity of the greater segment of the solid, and $B$ that of $\Pi\Omega\Sigma$; for it is proved in the books *On Equilibriums* that the centre of gravity of any segment of a right-angled conoid is at the point dividing the axis in such a manner that the segment towards the vertex of the axis is double of the remainder.\(^b\)

Now if the solid segment $\Pi\Omega\Sigma$ be taken away from the whole, the centre of gravity of the remainder will lie upon the straight line $BG$; for it has been proved in the *Elements of Mechanics* that if any magnitude be taken away not having the same centre of gravity as the whole magnitude, the centre of gravity of the remainder will be on the straight line joining the centres [of gravity] of the whole magnitude and of the part

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* Presumably in the works of Aristaeus or Euclid, but it is also *Quad. Parab.* 1.

* The proof is not in any extant work by Archimedes.
άφηρημένου ἐκβεβλημένας ἐπὶ τὰ αὐτὰ, ἐφ' ἂ τὸ κέντρον τοῦ ὄλου μεγέθεος ἐστὶν. ἐκβεβληθὼς δὴ ἡ ὙΡ ἐπὶ τὸ Γ, καὶ ἐστὶ τὸ Γ τὸ κέντρον τοῦ βάρεος τοῦ λοιποῦ μεγέθεος. ἐπεὶ οὖν ἃ ΝΟ τὰς μὲν ὙΡ ἡμιολία, τὰς δὲ μέχρι τοῦ ἄξονος οὐ μείζων ἡ ἡμιολία, δὴ λοιπὸν, ὅτι ἡ ὙΡ τὰς μέχρι τοῦ ἄξονος οὐκ ἐστὶ μείζων· ἡ ὙΡ ἀρα ποτὶ τὰν ΚΩ γωνίας ἀνίσους ποιεῖ, καὶ ἀ ὑπὸ τῶν ῬΠΩ γίνεται ὀξεία· ἀ ἀπὸ τοῦ Ὑ ἀρα κάθετος ἐπὶ τὰν ΠΩ ἀγομένα μεταξὺ πεσεῖται τῶν Π, Ω. πιστεύω ὡς ἃ ὙΡ· ἀ ὙΡ ἀρα ὑπὸ ἐστὶν ποτὶ τὸ (ἀποτετμακός) ἐπίπεδον, ἐν ὡ ἐστὶν ἀ ΣΙ, ὁ ἐστὶν ἐπὶ τὰς ἐπι- 

faneias τοῦ ύγροῦ. ἀκομίζων δὴ τινες ἀπὸ τῶν Β, Γ παρὰ τὰν ὙΡ· ἐνεχθήσεται δὴ τὸ μὲν ἐκτὸς τοῦ ύγροῦ τοῦ μεγέθεος εἰς τὸ κάτω κατὰ τὸν διὰ τοῦ Γ ἀγομέναν κάθετον ὑπόκειται γὰρ ἐκαστὸν τῶν βαρέων εἰς τὸ κάτω φέρεσθαι κατὰ τὸν κάθετον τὸν διὰ τοῦ κέντρου ἀγομέναν· τὸ δὲ ἐν τῷ ύγρῷ μέγεθος, ἐπεὶ κοινῷ τοῦ ἐπὶ τοῦ ύγροῦ, ἐνεχθήσεται εἰς τὸ ἀνω κατὰ τὸν κάθετον τὸν διὰ τοῦ Β ἀγομέναν. ἐπεὶ δὲ οὐ κατὰ τὸν αὐτὸν κάθετον ἀλλὰς ἀντιπλῆσθαι, οὐ μενεῖ τὸ σχῆμα, ἀλλὰ τὸ μὲν κατὰ τὸ Λ ἐς τὸ ἀνω ἐνεχθή

σεται, τἀ δὲ κατὰ τὸ Λ ἐς τὸ κάτω, καὶ τούτο ἀεὶ ἐσσεῖται, ἐως ἃν ὀρθὸν ἀποκατασταθῇ.

1 ἀποτετμακός, cf. supra, p. 252 line 8; Heiberg prints

..... κ. 05.

* If the normal at Η meets the axis in Μ, then ΟΜ is greater than “the line drawn as far as the axis” except in the case where Η coincides with the vertex, which case is excluded by the conditions of this proposition. Hence OM is always greater than OP; and because the angle ΟΠΠΜ is right, the angle ΟΠΠ must be acute.

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taken away, produced from the extremity which is the centre of gravity of the whole magnitude [De Plan. Aequil. i. 8]. Let BP then be produced to Γ, and let Γ be the centre of gravity of the remaining magnitude. Then, since \( NO = \frac{a}{2} \cdot OP \), and \( NO \cdot \frac{a}{2} \cdot ( \text{the line drawn as far as the axis} ) \), it is clear that \( PO \cdot ( \text{the line drawn as far as the axis} ) \); therefore \( \Pi P \) makes unequal angles with \( K \Omega \), and the angle \( \Pi P \Omega \) is acute; therefore the perpendicular drawn from \( P \) to \( \Pi \Omega \) will fall between \( \Pi, \Omega \). Let it fall as \( P \Theta \); then \( P \Theta \) is perpendicular to the cutting plane containing \( \Sigma I \), which is on the surface of the fluid. Now let lines be drawn from \( B, \Gamma \) parallel to \( P \Theta \); then the portion of the magnitude outside the fluid will be subject to a downward force along the line drawn through \( \Gamma \)—for it is postulated that each weight is subject to a downward force along the perpendicular drawn through its centre of gravity; and since the magnitude in the fluid is lighter than the fluid, it will be subject to an upward force along the perpendicular drawn through \( B \). But, since they are not subject to contrary forces along the same perpendicular, the figure will not remain at rest but the portion on the side of \( A \) will move upwards and the portion on the side of \( \Lambda \) will move downwards, and this will go on continually until it is restored to the upright position.

\(^a\) Cf. supra, p. 245: possibly a similar assumption to this effect has fallen out of the text.

\(^b\) A tacit assumption, which limits the generality of the opening statement of the proposition that the segment may have any weight relative to the fluid.

\(^d\) v. supra, p. 251.
XVIII. ERATOSTHENES
XVIII. ERATOSTHENES

(a) General

Suidas, s.v. 'Ερατοσθένης

'Ερατοσθένης, 'Αγλαοῦ, οἱ δὲ 'Αμβροσίου: Κυρηναῖος, μαθητὴς φιλοσόφου 'Αρίστωνος Χίου, γραμματικὸς δὲ Λυσανίου τοῦ Κυρηναίου καὶ Καλλιμάχου τοῦ ποιητοῦ. μετεπέμβαθε δὲ ἐξ Ἀθηνῶν ὑπὸ τοῦ τρίτου Πτολεμαίου καὶ διέτρυψε μέχρι τοῦ πέμπτου. διὰ δὲ τὸ δευτέρευν ἐν παντὶ εἶδε παιδείας τοῖς ἁκροῖς ἐγγίσαντα Ἡτα α ἐπεκλήθη. οἱ δὲ καὶ δεύτερον ἢ νέον Πλάτωνα, ἀλλοὶ Πένταβλον ἐκάλεσαν. ἐτέχθη δὲ ρκς ὁ Ολυμ-

1 ἐγγίσαντα Meursius, ἐγγίσαντι Adler.
2 Ἡτα Gloss. in Psalms, Hesych. Mil., τὰ βήματα codd.

* Several of Eratosthenes' achievements have already been described—his solution of the Delian problem (vol. i. pp. 290-297), and his sieve for finding successive odd numbers (vol. i. pp. 100-103). Archimedes, as we have seen, dedicated the Method to him, and the Cattle Problem, as we have also seen, is said to have been sent through him to the Alexandrian mathematicians. It is generally supposed that Ptolemy credits him with having calculated the distance between the tropics (or twice the obliquity of the ecliptic) at 11/83 rds. of a complete circle or 47° 29' 39", but Ptolemy’s meaning is not clear. Eratosthenes also calculated the distances of the sun and moon from the earth and the size of the sun. Fragments of an astronomical poem which he wrote under the title
XVIII. ERATOSTHENES

(a) General

Suidas, s.v. Eratosthenes

Eratosthenes, son of Aglaus, others say of Ambrosius; a Cyrenian, a pupil of the philosopher Ariston of Chios, of the grammarian Lysanias of Cyrene and of the poet Callimachus; he was sent for from Athens by the third Ptolemy and stayed till the fifth. Owing to taking second place in all branches of learning, though approaching the highest excellence, he was called Beta. Others called him a Second or New Plato, and yet others Pentathlon. He was born in the 126th Olympiad and died at the age

Hermes have survived. He was the first person to attempt a scientific chronology from the siege of Troy in two separate works, and he wrote a geographical work in three books. His writings are critically discussed in Bernhardy’s Eratosthenica (Berlin, 1829).

Callimachus, the famous poet and grammarian, was also a Cyrenian. He opened a school in the suburbs of Alexandria and was appointed by Ptolemy Philadelphus chief librarian of the Alexandrian library, a post which he held till his death c. 240 B.C. Eratosthenes later held the same post.

Euergetes I (reigned 246–221 B.C.), who sent for him to be tutor to his son and successor Philopator (v. vol. i. pp. 256, 296).

Epiphanes (reigned 204–181 B.C.). 276-273 B.C.
πιάδι καὶ ἐτελεύτησεν ἦ ἐτῶν γεγονός, ἀποσχόμενος τροφῆς διὰ τὸ ἀμβλυώτευν, μαθητὴν ἐπίσημον καταλειπὼν Ἀριστοφάνην τὸν Βυζάντεον οὖ πάλιν Ἀρισταρχος μαθητής. μαθηταὶ δὲ αὐτῶν Μνασέας καὶ Μένανδρος καὶ Ἀριστις. ἔγραψε δὲ φιλόσοφα καὶ ποιήματα καὶ ἱστορίας, Ἀστρονομίαν ἡ Καταστερισμοῦς, ἗ Περὶ τῶν κατὰ φιλοσοφίαν αἱρέσεων, Περὶ ἀλυσίας, διαλόγους πολλοὺς καὶ γραμματικὰ συχνά.

(b) On Means

Papp. Coll. vii. 3, ed. Hultsch 636. 18-25

Τῶν δὲ προειρημένων τοῦ Ἀναλυμένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη... Ἐρατοσθένους περὶ μεσοτήτων δύο.

Papp. Coll. vii. 21, ed. Hultsch 660. 18-662. 18

Τῶν τόπων καθόλου οἱ μὲν εἰσὶν ἐφεκτικοὶ, οὐ καὶ Ἀπολλώνιος πρὸ τῶν ἰδίων στοιχείων λέγει σημείον μὲν τόπον σημείον, γραμμὴς δὲ τόπον γραμμὴν, ἐπιφανείας δὲ ἐπιφάνειαν, στερεοῦ δὲ στερεοῦ, οἱ δὲ διεξοδικοὶ, ὡς σημείον μὲν γραμμὴν, γραμμῆς δὲ ἐπιφάνειαν, ἐπιφανείας δὲ στερεοῦ,

¹ Καταστερισμοῦς cons. Portus, Καταστηρίγμους codd.

¹ Not, of course, Aristarchus of Samos, the mathematician, but the celebrated Samothracian grammarian.
² Mnaseas was the author of a work entitled Περίπλους, whose three sections dealt with Europe, Asia and Africa, and a collection of oracles given at Delphi.
³ This work is extant, but is not thought to be genuine in
of eighty of voluntary starvation, having lost his sight; he left a distinguished pupil, Aristophanes of Byzantium; of whom in turn Aristarchus was a pupil. Among his pupils were Mnaseas, Menander and Aristis. He wrote philosophical works, poems and histories, Astronomy or Placings Among the Stars, On Philosophical Divisions, On Freedom from Pain, many dialogues and numerous grammatical works.

(b) On Means

Pappus, Collection vii. 3, ed. Hultsch 636. 18-95

The order of the aforesaid books in the Treasury of Analysis is as follows . . . the two books of Eratosthenes On Means.

Pappus, Collection vii. 21, ed. Hultsch 660. 18-662. 18

Loci in general are termed fixed, as when Apollonius at the beginning of his own Elements says the locus of a point is a point, the locus of a line is a line, the locus of a surface is a surface and the locus of a solid is a solid; or progressive, as when it is said that the locus of a point is a line, the locus of a line is a surface and the locus of a surface is a solid; or circumambient as its extant form; it contains a mythology and description of the constellations under forty-four heads. The general title Ἀστρονομία may be a mistake for Ἀστροθεία; elsewhere it is alluded to under the title Κατάλογος.

The inclusion of this work in the Treasury of Analysis, along with such works as those of Euclid, Aristaeus and Apollonius, shows that it was a standard treatise. It is not otherwise mentioned, but the loci with reference to means referred to in the passage from Pappus next cited were presumably discussed in it.

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οἱ δὲ ἀναστροφικοὶ, ὡς σημείου μὲν ἐπιφάνειαν, γραμμῆς δὲ στερεόν. [... οἱ δὲ ὑπὸ Ἐρατοσθένους ἐπιγραφέντες τόποι πρὸς μεσότητας ἐκ τῶν προειρημένων εἰσὶν τῷ γένει, ἀπὸ δὲ τῆς ἰδιότητος τῶν ὑποθέσεων ... ἐκεῖνοι.]

(c) THE "PLATONICUS"

Theon Smyr., ed. Hiller 81. 17–82. 5

Ἐρατοσθένης δὲ ἐν τῷ Πλατωνικῷ φησι, μὴ ταύτων εἶναι διάστημα καὶ λόγον. ἔπειθή λόγος μὲν ἐστὶ δύο μεγεθῶν ἡ πρὸς ἀλλήλα ποιὰ σχέσις, γίνεται δὲ αὐτὴ καὶ ἐν διαφόροις (καὶ ἐν ἄδιαφόροις). οἶον ὃς λόγῳ ἐστὶ τὸ αἰσθητὸν πρὸς τὸ νοητὸν, ἐν τούτῳ δόξα πρὸς ἐπιστήμην, καὶ διαφέρει καὶ τὸ νοητὸν τοῦ ἐπιστήμου ὃ καὶ ἡ δόξα τοῦ αἰσθητοῦ. διάστημα δὲ ἐν διαφέρουσι μόνον, ἡ κατὰ τὸ μέγεθος ἡ κατὰ ποιότητα ἡ κατὰ θέσιν ἡ ἄλλως ὁπωσοῦ. δῆλον δὲ καὶ ἐνεπείθεν,

1 The passage of which this forms the concluding sentence is attributed by Hultsch to an interpolator. To fill the lacuna before ἐκεῖνοι he suggests ἀνώμοιοι ἐκεῖνοι, following Halley’s rendering, "diversa sunt ab illis."

2 καὶ ἐν ἄδιαφόροις add. Hiller.

* Tannery conjectured that these were the loci of points such that their distances from three fixed lines provided a "médiété," i.e., loci (straight lines and conics) which can be represented in trilinear co-ordinates by such equations as

\[2y = x + z, \quad y^2 = xz, \quad y(x + z) = 2xz, \quad x(x - y) = z(y - z),\]

\[x(x - y) = y(y - z);\]

these represent respectively the arithmetic, geometric and harmonic means, and the means subcontrary to the harmonic and geometric means (v. vol. i. pp. 122-135). Zeuthen has

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when it is said that the locus of a point is a surface and the locus of a line is a solid. [. . .] the loci described by Eratosthenes as having reference to means belong to one of the aforesaid classes, but from a peculiarity in the assumptions are unlike them.]

(c) The "Platonicus"

Theon of Smyrna, ed. Hiller 81. 17–82. 5

Eratosthenes in the Platonicus says that interval and ratio are not the same. Inasmuch as a ratio is a sort of relationship of two magnitudes one towards the other, there exists a ratio both between terms that are different and also between terms that are not different. For example, the ratio of the perceptible to the intelligible is the same as the ratio of opinion to knowledge, and the difference between the intelligible and the known is the same as the difference of opinion from the perceptible. But there can be an interval only between terms that are different, according to magnitude or quality or position or in some other way. It is thence clear that ratio is an alternative conjecture on similar lines (Die Lehre von den Kegelschnitten im Altertum, pp. 320–321).

Theon cites this work in one other passage (ed. Hiller 2. 3-12) telling how Plato was consulted about the doubling of the cube; it has already been cited (vol. i. p. 256). Eratosthenes' own solution of the problem has already been given in vol. i. pp. 290-297, and a letter purporting to be from Eratosthenes to Ptolemy Euergetes is given in vol. i. pp. 256-261. Whether the Platonicus was a commentary on Plato or a dialogue in which Plato was an interlocutor cannot be decided.


διότι λόγος διαστήματος ἔτερον· τὸ γὰρ ἦμισυν πρὸς τὸ διπλάσιον (καὶ τὸ διπλάσιον πρὸς τὸ ἦμισυν) λόγον μὲν οὐ τὸν αὐτὸν ἔχει, διάστημα δὲ τὸ αὐτὸ.

(d) Measurement of the Earth

Cleom. De motu cire. i. 10. 52, ed. Ziegler 94. 23–100. 23

Καὶ ἡ μὲν τοῦ Ποσειδωνίου ἐφόδος περὶ τοῦ κατὰ τὴν γῆν μεγέθους τοιαύτη, ἡ δὲ τοῦ Ἐρατοσθένους γεωμετρικῆς ἐφόδου ἔχομεν, καὶ δοκούμα τι ἀσάφεστον ἔχειν. ποιήσει δὲ σαφῆ τὰ λεγόμενα ὑπ’ αὐτοῦ τάδε προϋποτιθεμένων ἦμων. ὑποκείσθω ἦμιν πρῶτον μὲν κἂνταῦθα, ὑπὸ τῷ αὐτῷ μεσημβρίνῳ κεῖσθαι Συήνην καὶ Ἀλεξάνδρειαν, καὶ δεύτερον, τὸ διάστημα τὸ μεταξὺ τῶν πόλεων πεντακισχιλίων σταδίων εἶναι, καὶ τρίτον, τὰς καταπεμπόμενα ἀκτίνας ἀπὸ διαφόρων μερῶν τοῦ ἱλίου ἐπὶ διάφορα τῆς γῆς μέρη παραλλήλους εἶναι· οὕτως γὰρ ἔχειν αὐτὰς οἱ γεωμέτραι ὑποτίθενται. τέταρτον ἐκεῖνο ὑποκείσθω, δεικνύμενον παρὰ τοῖς γεωμέτραις, τὰς εἰς παραλλήλους ἐμπυτυχοῦσας εὐθείας τὰς ἐν ἐναλλάξ γωνίας ἵσας ποιεῖν, πέμπτον, τὰς ἐπὶ ἑσών γωνιῶν βεβηκόντων περιφερείας ὁμοίας εἶναι, τούτεστι τὴν αὐτὴν ἀναλογίαν καὶ τὸν αὐτὸν λόγον ἐχεῖν πρὸς τοὺς οἰκείους κύκλους, δεικνύμενον καὶ τοῦτο παρὰ τοῖς γεωμέτραις. ὁπόταν γὰρ περιφερείας ἐπὶ ἑσών γωνιῶν ὡς βεβηκόντων, ἀν μία ἡτισοῦν

1 καὶ ... ἦμισυν add. Hiller.

* The difference between ratio and interval is explained a little more neatly by Theon himself (ed. Hiller 81. 6-9): 266
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different from interval; for the relationship of the half to the double and of the double to the half does not furnish the same ratio, but it does furnish the same interval.

(d) Measurement of the Earth

Cleomedes, On the Circular Motion of the Heavenly Bodies i. 10. 52, ed. Ziegler 94. 23–100. 23

Such then is Posidonius's method of investigating the size of the earth, but Eratosthenes' method depends on a geometrical argument, and gives the impression of being more obscure. What he says will, however, become clear if the following assumptions are made. Let us suppose, in this case also, first that Syene and Alexandria lie under the same meridian circle; secondly, that the distance between the two cities is 5000 stades; and thirdly, that the rays sent down from different parts of the sun upon different parts of the earth are parallel; for the geometers proceed on this assumption. Fourthly, let us assume that, as is proved by the geometers, straight lines falling on parallel straight lines make the alternate angles equal, and fifthly, that the arcs subtended by equal angles are similar, that is, have the same proportion and the same ratio to their proper circles—this also being proved by the geometers. For whenever arcs of circles are subtended by equal angles, if any one of these is (say) one-tenth

diapheret δὲ διάστημα καὶ λόγος, ἐπειδὴ διάστημα μὲν ἐστι τὸ μεταξὺ τῶν ὁμογενῶν τε καὶ ἄνευ ὀρῶν, λόγος δὲ ἀπλῶς ἢ τῶν ὁμογενῶν ὀρῶν πρὸς ἄλληλον σχέσις.

Cleomedes probably wrote about the middle of the first century B.C. His handbook De motu circulating corporum caelestium is largely based on Posidonius.
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αὐτῶν δέκατον ἢ μέρος τοῦ οἰκείου κύκλου, καὶ αἰ λοιπαὶ πάσαι δέκατα μέρη γενήσονται τῶν οἰκείων κύκλων.

Τούτων ὁ κατακρατήσας οὐκ ἂν χαλεπῶς τὴν ἑφῳδὸν τοῦ Ἐρατοσθένους καταμάθω ἔχουσαι οὕτως. ὑπὸ τῶν αὐτῶ κείσθαι μεσημβρινῷ φησι Συήνην καὶ Ἀλεξάνδρειαν. ἔπει οὖν μέγιστοι τῶν ἐν τῷ κόσμῳ οἱ μεσημβρινοί, δεὶ καὶ τοὺς ὑποκειμένους τούτους τῆς γῆς κύκλους μεγίστους εἶναι ἀναγκαῖοι. ὡστε ἡλικον ἃν τὸν διὰ Συήνης καὶ Ἀλεξάνδρειας ἠκούσα κύκλον τῆς γῆς ἢ ἑφῳδὸς ἀποδείξει αὐτῇ, τηλικοῦτος καὶ ὁ μέγιστος ἐσται τῆς γῆς κύκλος. φησι τοῖνυν, καὶ ἔχει οὕτως, τὴν Συήνην ὑπὸ τῷ θερινῷ τροπικῷ κείσθαι κύκλω. ὃποταν οὖν ἐν καρκίνῳ γενόμενος ὁ ἡλιος καὶ θερινῶν τοιῶν τροπῶν ἀκριβῶς μεσοπαραγή, ἀσκιομεν γίνονται οἱ τῶν ὑρολογίων γνώμονες ἀναγκαῖοι, κατὰ κάθετον ἀκριβὴ τοῦ ἡλίου ὑπερκειμένου καὶ τοῦτο γίνεσθαι λόγος ἐπὶ σταδίους τριακοσίως τῆς διάμετρος. ἐν Ἀλεξάνδρειᾳ δὲ τῇ αὐτῇ ὥρᾳ ἀποβάλλουσιν οἱ τῶν ὑρολογίων γνώμονες σκιάς, ἀπὸ πρὸς ἄρκτω μᾶλλον τῆς Συήνης ταύτης τῆς πόλεως κειμένης. ὑπὸ τῶν αὐτῶ μεσημβρινῶ τοῖνυν καὶ μεγίστῳ κύκλῳ τῶν πόλεως κειμένων, ἀν περιγάγωμεν περιφέρειαν ἀπὸ τοῦ ἀκρου τῆς τοῦ γνώμονος σκιᾶς ἐπὶ τὴν βάσιν αὐτὴν τοῦ γνώμονος τοῦ ἐν Ἀλεξάνδρειᾳ ὑρολογίου, αὐτῇ ἡ περιφέρεια τημήμα γενήσεται τοῦ μεγίστος τῶν ἐν τῇ σκάφη κύκλων, ἔπει μεγίστοι κύκλῳ ὑπο- κειται ἡ τοῦ ὑρολογίου σκάφη. εἰ οὖν ἐξῆς νοῆσαμεν εὐθεῖας διὰ τῆς γῆς ἐκβαλλομένας ἀφ’ ἐκατέρου τῶν γνωμόνων, πρὸς τῷ κέντρῳ τῆς γῆς

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of its proper circle, all the remaining ares will be tenth parts of their proper circles.

Anyone who has mastered these facts will have no difficulty in understanding the method of Eratosthenes, which is as follows. Syene and Alexandria, he asserts, are under the same meridian. Since meridian circles are great circles in the universe, the circles on the earth which lie under them are necessarily great circles also. Therefore, of whatever size this method shows the circle on the earth through Syene and Alexandria to be, this will be the size of the great circle on the earth. He then asserts, as is indeed the case, that Syene lies under the summer tropic. Therefore, whenever the sun, being in the Crab at the summer solstice, is exactly in the middle of the heavens, the pointers of the sundials necessarily throw no shadows, the sun being in the exact vertical line above them; and this is said to be true over a space 300 stades in diameter. But in Alexandria at the same hour the pointers of the sundials throw shadows, because this city lies farther to the north than Syene. As the two cities lie under the same meridian great circle, if we draw an arc from the extremity of the shadow of the pointer to the base of the pointer of the sundial in Alexandria, the arc will be a segment of a great circle in the bowl of the sundial, since the bowl lies under the great circle. If then we conceive straight lines produced in order from each of the pointers through the earth, they
συμπεσοῦνται. ἔπει οὖν τὸ ἐν Συνήθη ωρολόγιον κατὰ κάθετον ὑπόκειται τῷ ἕλιῳ, ἃν ἐπινοήσωμεν έθείαν ἀπὸ τοῦ ἕλιου ἡκουσαν ἐπὶ ἄκρου τοῦ ωρολογίου γνώμονα, μία γενήσεται έθεία ἡ ἀπὸ τοῦ ἕλιου μέχρι τοῦ κέντρου τῆς γῆς ἡκουσα. ἦν οὖν ἐτέραν έθείαν νοήσωμεν ἀπὸ τοῦ ἄκρου τῆς σκιᾶς τοῦ γνώμονος δι' ἄκρου τοῦ γνώμονος ἐπὶ τοῦ ἕλιου ἀναγομένην ἀπὸ τῆς ἐν Ἀλέξανδρεία σκάφης, αὐτῇ καὶ ἡ προειρημένη έθεία παράλληλοι γενήσονται ἀπὸ διαφόρων γε τοῦ ἕλιου μερῶν ἐπὶ διάφορα μέρη τῆς γῆς διήκουσαν. εἰς ταύτας τοῖς παραλλήλοις οὕσας ἔμπιπτει έθεία ἡ ἀπὸ τοῦ κέντρου τῆς γῆς ἐπὶ τοῦ ἐν Ἀλέξανδρεία γνώμονα ἡκουσα, ὡστε τὰς ἐναλλάξ γνώμιας ἑκατέρας ἑποίειν· ἦν η μὲν ἐστὶ πρὸς τῷ κέντρῳ τῆς γῆς κατὰ σύμπτωσιν τῶν έθείων, αἱ ἀπὸ τῶν ωρολογίων ἡχθησαν ἐπὶ τὸ κέντρον τῆς γῆς, γενόμενη, ἡ δὲ κατὰ σύμπτωσιν ἁκροῦ τοῦ ἐν Ἀλέξανδρεία γνώμονος καὶ τῆς ἀπὸ ἁκροῦ τῆς σκιᾶς αὐτοῦ ἐπὶ τοῦ ἕλιου διὰ τῆς πρὸς αὐτὸν φαύσεως ἀναψηθάς γεγενημένην· καὶ ἐπὶ μὲν ταύτης βεβηκε περιφέρεια ἡ ἀπὸ ἁκροῦ τῆς σκιᾶς τοῦ γνώμονος ἐπὶ τὴν βάσιν αὐτοῦ περιαγθεῖσα, ἐπὶ δὲ τῆς πρὸς τῷ κέντρῳ τῆς γῆς ἡ ἀπὸ Συνήθης διήκουσα εἰς Ἀλέξανδρείαν. ὅμοια τοῖς παραφέρειαι εἰς ἀλλήλαις ἐπὶ ίσων γε γνώμων βεβηκών. ὃν ἄρα λόγον ἔχει ἡ ἐν τῇ σκάφῃ πρὸς τὸν οἰκεῖον κύκλον, τούτον ἔχει τὸν λόγον καὶ ἡ ἀπὸ Συνήθης εἰς Ἀλέξανδρειαν ἡκουσα. ἡ δὲ γε ἐν τῇ σκάφῃ πεντηκοστὸν μέρος εὑρίσκεται τοῦ οἰκείου κύκλου. δεῖ οὖν ἀναγκαῖον καὶ τὸ ἀπὸ Συνήθης εἰς Ἀλέξανδρειαν διάστημα πεντηκοστὸν εἶναι μέρος τοῦ
will meet at the centre of the earth. Now since the sundial at Syene is vertically under the sun, if we conceive a straight line drawn from the sun to the top of the pointer of the sundial, the line stretching from the sun to the centre of the earth will be one straight line. If now we conceive another straight line drawn upwards from the extremity of the shadow of the pointer of the sundial in Alexandria, through the top of the pointer to the sun, this straight line and the aforesaid straight line will be parallel, being straight lines drawn through from different parts of the sun to different parts of the earth. Now on these parallel straight lines there falls the straight line drawn from the centre of the earth to the pointer at Alexandria, so that it makes the alternate angles equal; one of these is formed at the centre of the earth by the intersection of the straight lines drawn from the sundials to the centre of the earth; the other is at the intersection of the top of the pointer in Alexandria and the straight line drawn from the extremity of its shadow to the sun through the point where it meets the pointer. Now this latter angle subtends the arc carried round from the extremity of the shadow of the pointer to its base, while the angle at the centre of the earth subtends the arc stretching from Syene to Alexandria. But the arcs are similar since they are subtended by equal angles. Whatever ratio, therefore, the arc in the bowl of the sundial has to its proper circle, the arc reaching from Syene to Alexandria has the same ratio. But the arc in the bowl is found to be the fiftieth part of its proper circle. Therefore the distance from Syene to Alexandria must necessarily be a fiftieth part of the great
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μεγίστου τῆς γῆς κύκλου· καὶ ἐστὶ τούτῳ σταδίων πεντακισχιλίων. ὁ ἀρα σύμπας κύκλος γίνεται μυριάδων εἴκοσι πέντε. καὶ ἥ μὲν 'Ερατοσθένους ἐφοδιος τοιαύτη.

Heron, Dioptra 36, ed. H. Schöne 302. 10-17

Δέον δὲ ἔστω, εἰ τύχωι, τὴν μεταξὺ Ἀλεξάνδρειας καὶ Ὁρῶν ὀδὸν ἐκμετρῆσαι τὴν ἐπ' εὐθείας, τὴν γε ἐπὶ κύκλου περιφέρειας μεγίστου τοῦ ἐν τῇ γῆ, προσομολογομένου τοῦ ὅτι περίμετρος τῆς γῆς σταδίων ἐστὶ μ' καὶ ἐπὶ β', ὡς ὁ μάλιστα τῶν ἄλλων ἀκριβέστερον πεπραγματευμένος Ἐρατο-

*The attached figure will help to elucidate Cleomedes. S is Syene and A Alexandria; the centre of the earth is O. The sun's rays at the two places are represented by the broken straight lines. If α be the angle made by the sun's rays with the pointer of the sundial at Alexandria (OA produced), the angle SOA is also equal to α, or one-fiftieth of four right angles. The arc SA is known to be 5000 stades and it follows that the whole circumference of the earth must be 250000 stades.
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circle of the earth. And this distance is 5000 stades. Therefore the whole great circle is 250000 stades. Such is the method of Eratosthenes.∗

Heron, Dioptra 36, ed. H. Schöne 302. 10-17

Let it be required, perchance, to measure the distance between Alexandria and Rome along the arc of a great circle, on the assumption that the perimeter of the earth is 252000 stades, as Eratosthenes, who investigated this question more accurately than others, shows in the book which he wrote On the Measurement of the Earth.∗

∗ Lit. "along the circumference of the greatest circle on the earth."

∗ Strabo (ii. 5. 7) and Theon of Smyrna (ed. Hiller 124. 10-12) also give Eratosthenes’ measurement as 252000 stades against the 250000 of Cleomedes. "The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250000 to 252000 for some reason, perhaps in order to get a figure divisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny (N.H. xii. 13. 53) is right in saying that Eratosthenes made 40 stades equal to the Egyptian σχοή, then, taking the σχοή at 12000 Royal cubits of 0.525 metres, we get 300 such cubits, or 157.5 metres, i.e., 516.73 feet, as the length of the stade. On this basis 252000 stades works out to 24662 miles, and the diameter of the earth to about 7850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation" (Heath, H.G.M. ii. 107).
XIX. APOLLONIUS OF PERGA
XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) Relation to Previous Works


'Απολλώνιος ὁ γεωμέτρης, ὁ φίλε ἑταῖρε 'Ανθέμε, γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίᾳ ἐν χρόνοις τοῦ Εὐεργέτου Πτολεμαίου, ὥς ἱστορεῖ Ἡράκλειος ὁ τὸν βἶον 'Αρχιμήδους γράφων, ὃς καὶ φησὶ τὰ κωνικὰ κειμήλια ἐπινοοῦσαμεν μὲν πρώτον τὸν 'Αρχιμήδη, τὸν δὲ 'Απολλώνιον αὐτὰ εὑρόντα ὑπὸ 'Αρχιμήδους μὴ ἐκδοθέντα ἰδιοποιήσασθαι, οὐκ ἀληθεύον κατὰ γε τὴν ἐμὴν. ὁ τε γὰρ 'Αρχιμήδης ἐν πολλοῖς φαίνεται ὡς παλαιοτέρας τῆς στοιχεῖωσεως τῶν κωνικῶν μεμημένος, καὶ ὁ 'Απολλώνιος οὐχ ὡς ἱδίας ἐπινοίας γράφει οὐ γὰρ ἄν ἔφη ἐπὶ πλέον καὶ καθόλου μᾶλλον

* Scarcely anything more is known of the life of one of the greatest geometers of all time than is stated in this brief reference. From Pappus, Coll. vii., ed. Hultsch 67 (quoted in vol. i. p. 488), it is known that he spent much time at Alexandria with Euclid’s successors. Ptolemy Euergetes reigned 246–221 B.C., and as Ptolemaeus Chennus (apud Photii Bibli., cod. exc., ed. Bekker 151 b 18) mentions an astro-
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(a) The Conic Sections

(i.) Relation to Previous Works

Eutocius, Commentary on Apollonius’s Conics,
Apoll. Perg. ed. Heiberg ii. 168. 5–170. 26

Apollonius the geometer, my dear Anthemius, flourished at Perga in Pamphylia during the time of Ptolemy Euergetes, as is related in the life of Archimedes written by Heraclius, who also says that Archimedes first conceived the theorems in conics and that Apollonius, finding they had been discovered by Archimedes but not published, appropriated them for himself, but in my opinion he errs. For in many places Archimedes appears to refer to the elements of conics as an older work, and moreover Apollonius does not claim to be giving his own discoveries; otherwise he would not have described his purpose as “to investigate these properties more fully and more

nomer named Apollonius who flourished in the time of Ptolemy Philopator (221–204 B.C.), the great geometer is probably meant. This fits in with Apollonius’s dedication of Books iv.-viii. of his Conics to King Attalus I (247–197 B.C.). From the preface to Book i., quoted infra (p. 281), we gather that Apollonius visited Eudemus at Pergamum, and to Eudemus he dedicated the first two books of the second edition of his work.

a More probably Heraclides, v. supra, p. 18 n. a.
εξειργάσθαι ταύτα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα." ἀλλ’ ὅπερ φησίν ὁ Γέμινος ἀληθές ἐστιν, ὅτι οἱ παλαιοὶ κόων ὀριζόμενοι τὴν τοῦ ὀρθογωνίου τριγώνου περιφορὰν μενούσης μᾶς τῶν περὶ τὴν ὀρθὴν εἰκότως καὶ τοὺς κόωνοι πάντας ὀρθοῦς ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἐκάστῳ, ἐν μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλουμένην παραβολήν, ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολήν, ἐν δὲ τῷ ὀξυγωνίῳ τὴν ἐλλειψιν· καὶ ἐστὶ παρ’ αὐτοῖς εὑρεῖν οὕτως ὀνομαζομένας τὰς τομὰς. ὡσπερ οὖν τῶν ἀρχαίων ἐπὶ ἐνὸς ἐκάστου εἴδους τριγώνου θεωρησάντων τὰς δύο ὀρθὰς πρότερον ἐν τῷ ἴσοπλεύρῳ καὶ πάλιν ἐν τῷ ἴσο-σκελεί καὶ ὡσπερ ἐν τῷ σκαληνῷ οἱ μεταγενε-στεροι καθολικοῦ θεωρήματι ἀπέδειξαν τοιούτω· παντὸς τριγώνου αἱ τρεῖς γωνίαι δυσὶ ὀρθαῖς ἵσαι εἰσὶν οὕτως καὶ ἐπὶ τῶν τοῦ κόων τομῶν· τὴν μὲν γὰρ λεγομένην ὀρθογωνίου κόων τομὴν ἐν ὀρθογωνίῳ μόνον κόων ἑθεώρουν τεμνομένω ἐπιπέδῳ ὀρθῷ πρὸς μίαν πλευρὰν τοῦ κόων, τὴν δὲ τοῦ ἀμβλυγωνίου κόων τομὴν ἐν ἀμβλυγωνίῳ γινομένῃ κόων ἀπεδείκνυσαν, τὴν δὲ τοῦ ὀξυ- γωνίου ἐν ὀξυγωνίῳ, ὡμοίως ἐπὶ πάντων τῶν κόων ἀγοντες τὰ ἐπιπέδα ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κόων· δηλοὶ δὲ καὶ αὐτὰ τὰ ἀρχαία ὀνόματα τῶν γραμμῶν. ὡσπερ δὲ Ἀπολλώνιος ὁ Περ-γαῖος καθόλου τι ἐθεωρήσεν, ὅτι ἐν παντὶ κόων καὶ ὀρθῷ καὶ σκαληνῷ πᾶσαι αἱ τομαὶ εἰσὶ κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κόων προσβολῆν· ὁν καὶ θαυμάσαντες οἱ κατ’ αὐτὸν γενόμενοι διὰ τὸ θαυμάσιον τῶν ὑπ’ αὐτοῦ δεδειγμένων κοινικῶν θεωρημάτων μέγαν γεωμέτρητην ἐκάλουν. ταύτα

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generally than is done in the works of others." But what Geminus says is correct: defining a cone as the figure formed by the revolution of a right-angled triangle about one of the sides containing the right angle, the ancients naturally took all cones to be right with one section in each—in the right-angled cone the section now called the parabola, in the obtuse-angled the hyperbola, and in the acute-angled the ellipse; and in this may be found the reason for the names they gave to the sections. Just as the ancients, investigating each species of triangle separately, proved that there were two right angles first in the equilateral triangle, then in the isosceles, and finally in the scalene, whereas the more recent geometers have proved the general theorem, that in any triangle the three internal angles are equal to two right angles, so it has been with the sections of the cone; for the ancients investigated the so-called section of a right-angled cone in a right-angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the section of an obtuse-angled cone in an obtuse-angled cone and the section of an acute-angled cone in the acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone. In admiration for this, and on account of the remarkable nature of the theorems in conics proved by him, his contemporaries called him the "Great

* This comes from the preface to Book i., v. infra, p. 283.
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μὲν οὖν ὁ Γέμινος ἐν τῷ ἔκτῳ φησὶ τῆς Τῶν μαθημάτων θεωρίας.

(ii.) Scope of the Work

i. 2. 2-4. 28

'Απολλώνιος Εὐδιήμων χαίρειν.

Εἰ τῷ τε σώματι εὖ ἐπαινάγεις καὶ τὰ ἄλλα κατὰ γνώμην ἔστι σοι, καλῶς ἂν ἔχοι, μετρῶς δὲ ἔχομεν καὶ αὐτοί. καθ' ὦν δὲ καίρον ἡμῖν μετὰ σοι ἐν Περγάμῳ, ἐθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κοινών· πέριμαξάνοι σοι τὸ πρῶτον βιβλίον διορθωσάμενο, τὰ δὲ λοιπά, ἂν εὐαρεστήσουμεν, ἐξαποστελεῦμεν· οὐκ ἀμηνοεῖν γὰρ οἴμαι σε παρ' ἐμοῦ ἄκηκοτα, διότι τὴν περὶ ταῦτα ἐφοδον ἐποιησάμην ἀξιωθεὶς ὑπὸ Ναυκράτους τοῦ γεωμέτρου, καθ' ὦν καίρον ἐσχόλαζε

* Menaechmus, as shown in vol. i. pp. 278-283, and more particularly p. 283 n. a, solved the problem of the doubling of the cube by means of the intersection of a parabola with a hyperbola, and also by means of the intersection of two parabolas. This is the earliest mention of the conic sections in Greek literature, and therefore Menaechmus (β. 360-330 B.C.) is generally credited with their discovery; and as Eratosthenes' epigram (vol. i. p. 296) speaks of "cutting the cone in the triads of Menaechmus," he is given credit for discovering the ellipse as well. He may have obtained them all by the method suggested by Geminus, but Heath (H.G.M. ii. 111-116) gives cogent reasons for thinking that he may have obtained his rectangular hyperbola by a section of a right-angled cone parallel to the axis.

A passage already quoted (vol. i. pp. 486-489) from Pappus (ed. Hultsch 672. 18-678. 24) informs us that treatises on the conic sections were written by Aristaeus and Euclid. Aristaeus' work, in five books, was entitled Solid Loci; Euclid's
APOLLONIUS OF PERGA

Geometer.” Geminus relates these details in the sixth book of his Theory of Mathematics.⁶

(ii.) Scope of the Work

Apollonius, Conics i., Preface, Apoll. Perg. ed. Heiberg i. 2. 2–4. 28

Apollonius to Eudemus ᵇ greetings.
If you are in good health and matters are in other respects as you wish, it is well; I am pretty well too. During the time I spent with you at Pergamum, I noticed how eager you were to make acquaintance with my work in conics; I have therefore sent to you the first book, which I have revised, and I will send the remaining books when I am satisfied with them. I suppose you have not forgotten hearing me say that I took up this study at the request of Naucrates the geometer, at the time when he came

Conics was in four books. The work of Aristaeus was obviously more original and more specialized; that of Euclid was admittedly a compilation largely based on Aristaeus. Euclid flourished about 300 B.C. As noted in vol. i. p. 495 n. a, the focus-directrix property must have been known to Euclid, and probably to Aristaeus; curiously, it does not appear in Apollonius’s treatise.

Many properties of conics are assumed in the works of Archimedes without proof and several have been encountered in this work; they were no doubt taken from the works of Aristaeus or Euclid. As the reader will notice, Archimedes’ terminology differs in several respects from that of Apollonius, apart from the fundamental difference on which Geminus laid stress.

The history of the conic sections in antiquity is admirably treated by Zeuthen, Die Lehre von den Kegelschnitten im Altertum (1886) and Heath, Apollonius of Perga, xvii-clvi.

⁷ Not, of course, the pupil of Aristotle who wrote the famous History of Geometry, unhappily lost.
παρ' ἡμῖν παραγενθεὶς εἰς Ἑλεξάνδρειαν, καὶ διότι πραγματεύσαντες αὐτὰ ἐν ὁκτὼ βιβλίοις ἡς αὐτῆς μεταδιδόκαμεν αὐτὰ ἐις τὸ σπουδαιότερον διὰ τὸ πρὸς ἕκπλω αὐτὸν εἶναι οἷς διακαθάραντες, ἀλλὰ πάντα τὰ ὑποπίπτοντα ἡμῖν θέντες ὡς ἔσχατον ἐπελευσόμενοι. ὅθεν καίρον νῦν λαβόντες ἀεὶ τὸ τυχχάνον διορθώσεως ἐκδίδομεν. καὶ ἕπει συμβέβηκε καὶ ἄλλους τινὰς τῶν συμμειχότων ἡμῖν μετειληφέναι τὸ πρῶτον καὶ τὸ δεύτερον βιβλίον πρὶν ἡ διορθωθῇ, μὴ θαυμάσης, ἐὰν περιπίπτῃς αὐτοὺς ἐτέρως ἔχουσιν.

Ἀπὸ δὲ τῶν ὁκτὼ βιβλίων τὰ πρῶτα τέσσαρα πέπτυκεν εἰς ἀγωγὴν στοιχεῖωθη, περιέχει δὲ τὸ μὲν πρῶτον τὰς γενέσεις τῶν τριῶν τομῶν καὶ τῶν ἀντικειμένων καὶ τὰ ἐν αὐτῶι ἀρχικὰ συμπτώματα ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργασμένα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα, τὸ δὲ δεύτερον τὰ περὶ τὰς διαμέτρους καὶ τοὺς ἄξονας τῶν τομῶν συμβαίνοντα καὶ τὰς ἀσυμπτώτους καὶ ἄλλα γενικὴν καὶ ἀναγκαῖαν χρείαν παρεχόμενα πρὸς τοὺς διορισμοὺς· τῶν δὲ διαμέτρους καὶ τῶν ἄξονας καλῶ, εἰδῆςες ἐκ τούτου τοῦ βιβλίου. τὸ δὲ τρίτον πολλά καὶ παράδοξα δεινωμένα χρήσιμα πρὸς τὰς συνθέσεις τῶν στερεῶν τόπων καὶ τοὺς διορισμοὺς, ὥστε τὰ πλείστα καὶ κάλλιστα ξένα, ἃ καὶ κατανοήσαντες συνειδομένης ἡ συνθετέμενον ὑπὸ Εὐκλείδου τὸν ἐπὶ τρεῖς καὶ τέσσαρας γραμμὰς τόπον, ἀλλὰ μόριον τὸ τυχόν αὐτοῦ καὶ τοῦτο οὐκ εὐτυχῶς· οὐ γὰρ ἦν δυνατὸν ἄνευ τῶν προσευρημένων ἡμῖν τελειωθῆναι τὴν

*A necessary observation, because Archimedes had used the terms in a different sense.*

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to Alexandria and stayed with me, and that, when I had completed the investigation in eight books, I gave them to him at once, a little too hastily, because he was on the point of sailing, and so I was not able to correct them, but put down everything as it occurred to me, intending to make a revision at the end. Accordingly, as opportunity permits, I now publish on each occasion as much of the work as I have been able to correct. As certain other persons whom I have met have happened to get hold of the first and second books before they were corrected, do not be surprised if you come across them in a different form.

Of the eight books the first four form an elementary introduction. The first includes the methods of producing the three sections and the opposite branches [of the hyperbola] and their fundamental properties, which are investigated more fully and more generally than in the works of others. The second book includes the properties of the diameters and the axes of the sections as well as the asymptotes, with other things generally and necessarily used in determining limits of possibility; and what I call diameters and axes you will learn from this book. The third book includes many remarkable theorems useful for the syntheses of solid loci and for determining limits of possibility; most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me.¹

¹ For this locus, and Pappus's comments on Apollonius's claims, v. vol. i. pp. 486-489.
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σύνθεσιν. τὸ δὲ τέταρτον, ποσακῶς αἱ τῶν κώνων
tομαὶ ἀλλήλαις τε καὶ τῇ τοῦ κύκλου περιφέρειας
συμβάλλουσι, καὶ ἄλλα ἐκ περισσοῦ, ὅν οὐδέτερον
ὑπὸ τῶν πρὸ ἡμῶν γέγραπταί, κώνου τομῆς ἢ
κύκλου περιφέρεια κατὰ πόσα σημεῖα συμβάλ-
λουσί.

Τὰ δὲ λοιπὰ ἐστὶ περιουσιαστικώτερα: ἐστὶ
γάρ τὸ μὲν περὶ ἐλαχίστων καὶ μεγίστων ἐπὶ
πλέον, τὸ δὲ περὶ ἴσων καὶ ὁμοίων κώνων τομῶν,
tὸ δὲ περὶ διοριστικῶν θεωρημάτων, τὸ δὲ προ-
βλημάτων κωνικῶν διωρισμένων. οὐ μὴν ἄλλα
καὶ πάντων ἐκδοθέντων ἔξεστι τοῖς περιτυχάνουσι
κράνεων αὐτά, ᾧς ἂν αὐτῶν ἑκατοστὸς αἱρήται.
eὐτύχει.

(iii.) Definitions

Ibid., Defl., Apoll. Perg. ed. Heiberg i. 6. 2-8. 20

Ἐὰν ἀπὸ τινὸς σημείου πρὸς κύκλου περιφέρειαι,
ὅς οὐκ ἔστω ἐν τῷ αὐτῷ ἐπιπέδῳ τῷ σημείῳ,
eὐθεῖα ἐπιζευγνωσίᾳ ἐφ’ ἑκάτερα προσεκβληθῇ,
καὶ μένοντος τοῦ σημείου ἡ εὐθεία περιενεχθεῖσα
περὶ τὴν τοῦ κύκλου περιφέρειαν εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἦρξατο φέρεσθαι, τὴν γρα-
φεῖσαν ὑπὸ τῆς εὐθείας ἐπιφάνειαν, ἡ σύγκειται
ἐκ δύο ἐπιφάνειῶν κατὰ κορυφὴν ἀλλήλαις κε-
μένων, ὃν ἑκατέρα εἰς ἀνεπίρου αὐξεῖται τῆς

* Only the first four books survive in Greek. Books v.-vii. have survived in Arabic, but Book viii. is wholly lost. Halley (Oxford, 1710) edited the first seven books, and his edition is still the only source for Books vi. and vii. The first four books have since been edited by Heiberg (Leipzig, 1891-1893) and Book v. (up to Prop. 7) by L. Nix (Leipzig, 1889). The

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The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas].

The remaining books are thrown in by way of addition: one of them discusses fully minima and maxima, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.a

(iii.) Definitions

Ibid., Definitions, Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line a conical surface; it is composed of two surfaces lying vertically opposite to each other, of which each surviving books have been put into mathematical notation by T. L. Heath, Apollonius of Perga (Cambridge, 1896) and translated into French by Paul Ver Eecke, Les Coniques d'Apollonius de Perga (Bruges, 1923).

In ancient times Eutocius edited the first four books with a commentary which still survives and is published in Heiberg's edition. Serenus and Hypatia also wrote commentaries, and Pappus a number of lemmas.
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γραφούσης εὐθείας εἰς ἀπειρὸν προσεκβαλλομένης, καὶ ἐκ ἑαυτῆς τὸ μεμενηκὸς σημείου, ἀξονά δὲ τὴν διὰ τοῦ σημείου καὶ τοῦ κέντρου τοῦ κύκλου ἀγομένην εὐθείαν.

Κώνον δὲ τὸ περιεχόμενον σχῆμα ύπὸ τοῦ κύκλου καὶ τῆς μεταξὺ τῆς τε κυροφῆς καὶ τῆς τοῦ κύκλου περιφερείας κωνικῆς επιφανείας, κυρο-φῆν δὲ τοῦ κώνου τὸ σημείου, ὁ καὶ τῆς επιφανείας ἐστὶ κυροφῆ, ἀξονά δὲ τὴν ἀπὸ τῆς κυροφῆς ἐπὶ τὸ κέντρου τοῦ κύκλου ἀγομένην εὐθείαν, βάσιν δὲ τὸν κύκλον.

Τῶν δὲ κώνων ὀρθοὺς μὲν καὶ ἐκ τοῦ πρὸς ὀρθὰς ἔχοντας ταῖς βάσεσι τοῦς ἀξονάς, σκαληνοὺς δὲ τοὺς μὴ πρὸς ὀρθὰς ἔχοντας ταῖς βάσεσι τοὺς ἀξονάς.

Πάσης καμπύλης γραμμῆς, ἦτις ἐστὶν ἐν ἑνὶ ἐπίπεδῳ, διάμετρον μὲν καλῶν εὐθείαν, ἦτις ἰγμενὴ ἀπὸ τῆς καμπύλης γραμμῆς πάσας τὰς ἀγομένας ἐν τῇ γραμμῇ εὐθείᾳ τῳ παραλλήλῳ δίᾳ διαμεῖ, κυροφῆν δὲ τῆς γραμμῆς τὸ πέρα τῆς εὐθείᾳ τὸ πρὸς τῇ γραμμῇ, τεταγμένος δὲ ἐπὶ τὴν διάμετρον κατηχθαὶ ἐκάστην τῶν παραλ-λήλων.

'Ομοίως δὲ καὶ δύο καμπύλων γραμμῶν ἐν ἑνὶ ἐπίπεδῳ κειμένων διάμετρον καλῶν πλαγίων μὲν, ἦτις εὐθεία τείνουσα τὰς δύο γραμμὰς πάσας τὰς ἀγομένας ἐν ἐκατέρα τῶν γραμμῶν παρά τινα εὐθείαν δίᾳ τείνουσα, κυροφῆν δὲ τῶν γραμμῶν τὰ πρὸς τὰς γραμμάς πέρατα τῆς διάμετρον, ὀρθίαν δὲ, ἦτις κειμένη μεταξὺ τῶν δύο γραμμῶν πάσας τὰς ἀγομένας παραλλήλους εὐθείας εὐθεία τῳ καὶ ἀπολαμβανομένας μεταξὺ τῶν γραμμῶν δίᾳ

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extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the vertex, and the straight line drawn through this point and the centre of the circle I call the axis.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a cone, and by the vertex of the cone I mean the point which is the vertex of the surface, and by the axis I mean the straight line drawn from the vertex to the centre of the circle, and by the base I mean the circle.

Of cones, I term those right which have their axes at right angles to their bases, and scalene those which have their axes not at right angles to their bases.

In any plane curve I mean by a diameter a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the vertex of the curve I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn ordinate-wise to the diameter.

Similarly, in a pair of plane curves I mean by a transverse diameter a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the vertices of the curves I mean the extremities of the diameter on the curves; and by an erect diameter I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given
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tέμυνει, τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατήχθαι ἐκάστην τῶν παραλλήλων.

Συζυγεῖς καλῶ διαμέτρους [δύο] καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθείας, ὅν ἐκατέρα διάμετρος οὐσα τὰς τῇ ἐτέρα παραλλήλους δίχα διαιρεῖ.

"Ἀξονα δὲ καλῶ καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθείαν, ἦτις διάμετρος οὐσα τῆς γραμμῆς ἤ τῶν γραμμῶν πρὸς ὀρθὰς τέμυνε τὰς παραλλήλους.

Συζυγεῖς καλῶ ἄξονας καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν εὐθείας, αὐτῶν διάμετροι οὐσαί συζυγεῖς πρὸς ὀρθὰς τέμυνον τὰς ἀλλήλων παραλλήλους.

(iv.) Construction of the Sections

Ibid., Props. 7-9, Apoll. Perg. ed, Heiberg l. 22. 26-36. s

ζ'

'Εὰν κῶνος ἐπιπέδω τμηθῇ διὰ τοῦ ἄξονας, τμηθῇ δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμυνον τὸ ἐπίπεδον, ἐν ὧν ἐστὶν ἡ βάσις τοῦ κῶνου, κατ' εὐθείαν πρὸς ὀρθὰς οὐσὰν ἦτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονας τριγώνου ἢ τῇ ἐπ' εὐθείας αὐτῆ, αἱ ἀγόμεναι εὐθείαι ἀπὸ τῆς γεννηθείσης τομῆς ἐν τῇ τοῦ κῶνου ἐπιφανείᾳ, ἦν ἐποίησε τὸ τέμυνον ἐπίπεδον, παραλληλοὺ τῇ πρὸς ὀρθὰς τῇ βάσει τοῦ τριγώνου εὐθείᾳ ἐπὶ τὴν κοινὴν τομήν πεσοῦνται τοῦ τέμ-

1 δῦο om. Heiberg.

* This proposition defines a conic section in the most general way with reference to any diameter. It is only much
straight line; and I describe each of the parallels as drawn ordinate-wise to the diameter.

By conjugate diameters in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an axis of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By conjugate axes in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(iv.) Construction of the Sections

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

Prop. 7

If a cone be cut by a plane through the axis, and if it be also cut by another plane cutting the plane containing the base of the cone in a straight line perpendicular to the base of the axial triangle, or to the base produced, a section will be made on the surface of the cone by the cutting plane, and straight lines drawn in it parallel to the straight line perpendicular to the base of the axial triangle will meet the common section of the cutting plane and the axial later in the work (i. 52-58) that the principal axes are introduced as diameters at right angles to their ordinates. The proposition is an excellent example of the generality of Apollonius’s methods.

Apollonius followed rigorously the Euclidean form of proof. In consequence his general enunciations are extremely long and often can be made tolerable in an English rendering only by splitting them up; but, though Apollonius seems to have taken a malicious pleasure in their length, they are formed on a perfect logical pattern without a superfluous word.

Lit. "the triangle through the axis."

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νοντος ἐπὶ πέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου καὶ προσεκβαλλόμεναι ἓως τοῦ ἔτερου μέρους τῆς τομῆς διά τιμήθησονται ὑπ’ αὐτῆς, καὶ ἐὰν μὲν ὀρθὸς ἢ ὁ κώνος, ἢ ἐν τῇ βάσει εὐθεία πρὸς ὀρθᾶς ἐσται τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπὶ πέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου, ἐὰν δὲ σκαληνός, οὐκ αἰεὶ πρὸς ὀρθᾶς ἐσται, ἀλλ’ ὅταν τὸ διὰ τοῦ ἄξονος ἐπὶ πέδου πρὸς ὀρθᾶς ἢ τῇ βάσει τοῦ κώνου. Ἐστω κώνος, οὐ κορυφή μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπὶ πέδου διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τρίγωνον. τετμήσθω δὲ καὶ ἕτερο ἐπὶ πέδου τέμνοντι τὸ ἐπὶ πέδου, ἐν ζ ἐστὶν ὁ ΒΓ κύκλος, κατ’ εὐθείαν τὴν ΔΕ ἦτοι πρὸς ὀρθᾶς οὕσαν τῇ ΒΓ ἢ τῇ ἐπ’ εὐθείας αὐτῆς, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ· κοινῇ δὴ τομῇ τοῦ τέμνοντος ἐπὶ πέδου καὶ τοῦ ΑΒΓ τριγώνου ἢ ΖΗ. καὶ 290
triangle and, if produced to the other part of the section, will be bisected by it; if the cone be right, the straight line in the base will be perpendicular to the common section of the cutting plane and the axial triangle; but if it be scalene, it will not in general be perpendicular, but only when the plane through the axis is perpendicular to the base of the cone.

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BI'$, and let it be cut by a plane through the axis, and let the section so made be the triangle $ABI'$. Now let it be cut by another plane cutting the plane containing the circle $BI'$ in a straight line $DE$ which is either perpendicular to $BI'$ or to $BI'$ produced, and let the section made on the surface of the cone be $ΔZE$; then the common section of the cutting plane and of the triangle $ABI'$

* This applies only to the first two of the figures given in the ms.
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eἰλήφθω τι σημεῖον ἐπὶ τῆς ΔΖΕ τομῆς τὸ Ἐ, καὶ Ἰχθυς διὰ τοῦ Ἐ τῇ ΔΕ παράλληλος ἡ ΘΚ. λέγω, ὅτι Η ΘΚ συμβαλεῖ τῇ ΖΗ καὶ ἐκβαλλο-
μένη ἔως τοῦ ἔτερου μέρους τῆς ΔΖΕ τομῆς δίχα τμήθησαι ὑπὸ τῆς ΖΗ εὐθείας.

Ἐπεὶ γὰρ κῶνος, οὐ κορυφή μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, τέτμηται ἐπιπέδῳ διὰ τοῦ ἀξονός, καὶ ποιεῖ τομήν τὸ ΑΒΓ τριγώνου, εἰληφθεῖ δὲ τι σημεῖον ἐπὶ τῆς ἐπιφανείας, ὦ μὴ ἐστιν ἐπὶ πλευρᾶς τοῦ ΑΒΓ τριγώνου, τὸ Ἐ, καὶ ἐστι κάθετος ἡ ΔΗ ἐπὶ τὴν ΒΓ, ἡ ἁρά διὰ τοῦ Ἐ τῇ ΔΗ παράλληλος ἀγομένη, τούτου τίνι Η ἡ ἈΚ, συμβαλεῖ τῷ ΑΒΓ τριγώνῳ καὶ προσεκβαλλομένη ἕως τοῦ ἔτερου μέρους τῆς ἐπιφανείας δίχα τμήθη-
σαι ὑπὸ τοῦ τριγώνου. ἔπει οὖν ἡ διὰ τοῦ Ἐ τῇ ΔΕ παράλληλος ἀγομένη συμβάλλει τῷ ΑΒΓ τριγώνῳ καὶ ἐστιν ἐν τῷ διὰ τῆς ΔΖΕ τομῆς ἐπιπέδῳ, ἐπὶ τὴν κοινὴν ἁρά τομῆν πεσεῖται τοῦ τείμοντος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου. κοινῇ δὲ τομῆ ἐστὶ τῶν ἐπιπέδων ἡ ΖΗ. ἡ ἁρά διὰ τοῦ Ἐ τῇ ΔΕ παράλληλος ἀγομένη πεσεῖται ἐπὶ τὴν ΖΗ καὶ προσεκβαλλομένη ἕως τοῦ ἔτερου μέρους τῆς ΔΖΕ τομῆς δίχα τμήθησαι ὑπὸ τῆς ΖΗ εὐθείας.

Ἡτοι δὴ ὁ κῶνος ὀρθὸς ἐστιν, ἡ τὸ διὰ τοῦ ἀξονός τριγώνου τὸ ΑΒΓ ὀρθὸν ἐστὶ πρὸς τὸν ΒΓ κύκλον, ἡ οὐδέτερον.

Ἔστω πρότερον ὁ κῶνος ὀρθὸς· εἰὶ ἂν οὖν καὶ τὸ ΑΒΓ τριγώνον ὀρθὸν πρὸς τὸν ΒΓ κύκλον. ἐπεὶ οὖν ἐπιπέδου τὸ ΑΒΓ πρὸς ἐπιπέδου τὸ ΒΓ ὀρθὸν ἐστὶ, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΒΓ ἐν ἐν τῶν ἐπιπέδων τῷ ΒΓ πρὸς ὀρθὰς ἦκτα ἡ ΔΕ,
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is ZH. Let any point Θ be taken on ΔZE, and through Θ let ΘΚ be drawn parallel to ΔΕ. I say that ΘΚ intersects ZH and, if produced to the other part of the section ΔZE, it will be bisected by the straight line ZH.

For since the cone, whose vertex is the point A and base the circle BG, is cut by a plane through the axis and the section so made is the triangle ABG, and there has been taken any point Θ on the surface, not being on a side of the triangle ABG, and ΔH is perpendicular to BG, therefore the straight line drawn through Θ parallel to ΔH, that is ΘΚ, will meet the triangle ABG and, if produced to the other part of the surface, will be bisected by the triangle [Prop. 6]. Therefore, since the straight line drawn through Θ parallel to ΔE meets the triangle ABG and is in the plane containing the section ΔZE, it will fall upon the common section of the cutting plane and the triangle ABG. But the common section of those planes is ZH; therefore the straight line drawn through Θ parallel to ΔE will meet ZH; and if it be produced to the other part of the section ΔZE it will be bisected by the straight line ZH.

Now the cone is right, or the axial triangle ABG is perpendicular to the circle BG, or neither.

First, let the cone be right; then the triangle ABG will be perpendicular to the circle BG [Def. 3; Eucl. xi. 18]. Then since the plane ABG is perpendicular to the plane BG, and ΔE is drawn in one of the planes perpendicular to their common section BG, therefore
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η ΔΕ ἂρα τῶν ΑΒΓ τριγώνων ἐστὶ πρὸς ὀρθάς· καὶ πρὸς πάσας ἂρα τὰς ἀποτελέσματα αὐτῆς εὐθείας καὶ οὖσας ἐν τῷ ΑΒΓ τριγώνῳ ὀρθῇ ἐστιν. ῥωτε καὶ πρὸς τὴν ΖΗ ἐστὶ πρὸς ὀρθάς.

Μή ἔστω δὴ ὁ κώνος ὀρθός. εἰ μὲν οὖν τὸ διὰ τοῦ ἄξονος τριγώνων ὀρθὸν ἐστὶ πρὸς τὸν ΒΓ κύκλον, ὀμοίως δείξομεν, ὃτι καὶ η ΕΔ τῇ ΖΗ ἐστὶ πρὸς ὀρθάς. μὴ ἔστω δὴ τὸ διὰ τοῦ ἄξονος τριγώνου τὸ ΑΒΓ ὀρθὸν πρὸς τὸν ΒΓ κύκλον. λέγω, ὃτι οὖδε η ΕΔ τῇ ΖΗ ἐστὶ πρὸς ὀρθάς. εἰ γὰρ δυνατόν, ἔστω· ἔστι δὲ καὶ τῇ ΒΓ πρὸς ὀρθάς· η ἂρα ΕΔ ἐκατέρα τῶν ΒΓ, ΖΗ ἐστὶ πρὸς ὀρθάς. καὶ τῷ διὰ τῶν ΒΓ, ΖΗ ἐπιπέδῳ ἂρα πρὸς ὀρθάς ἐστι. τὸ δὲ διὰ τῶν ΒΓ, ΗΖ ἐπιπέδον ἐστὶ τὸ ΑΒΓ· καὶ η ΕΔ ἂρα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθάς. καὶ πάντα ἂρα τὰ δι’ αὐτῆς ἐπιπέδα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθάς. εἰν δὲ τὸν διὰ τῆς ΕΔ ἐπιπέδων ἐστὶν ὁ ΒΓ κύκλος· ο ΒΓ ἂρα κύκλος πρὸς ὀρθάς ἐστι τῷ ΑΒΓ τριγώνῳ. ῥωτε καὶ τὸ ΑΒΓ τριγώνου ὀρθὸν ἐσται πρὸς τὸν ΒΓ κύκλον· ὁπερ οὐχ ὑπόκειται, οὐκ ἂρα η ΕΔ τῇ ΖΗ ἐστὶ πρὸς ὀρθάς.

Πόρισμα

'Εκ δὴ τούτου φανερόν, ὅτι τῆς ΔΕΕ τομῆς διαμετρὸς ἐστὶν η ΖΗ, ἐπείπερ τὰς ἀγομένας παράλληλους εὐθεία των τῇ ΔΕ δίχα τέμνει, καὶ ὅτι δυνατὸν ἐστὶν ὑπὸ τῆς διαμέτρου τῆς ΖΗ παραλλήλους τινὰς δίχα τέμνεσθαι καὶ μὴ πρὸς ὀρθάς.

η'

'Εαν κώνος ἐπιπέδῳ τριήθη διὰ τοῦ ἄξονος,

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$\Delta E$ is perpendicular to the triangle $AB\Gamma$ [Eucl. xi. Def. 4]; and therefore it is perpendicular to all the straight lines in the triangle $AB\Gamma$ which meet it [Eucl. xi. Def. 3]. Therefore it is perpendicular to $ZH$.

Now let the cone be not right. Then, if the axial triangle is perpendicular to the circle $B\Gamma$, we may similarly show that $\Delta E$ is perpendicular to $ZH$. Now let the axial triangle $AB\Gamma$ be not perpendicular to the circle $B\Gamma$. I say that neither is $\Delta E$ perpendicular to $ZH$. For if it is possible, let it be; now it is also perpendicular to $B\Gamma$; therefore $\Delta E$ is perpendicular to both $B\Gamma$, $ZH$. And therefore it is perpendicular to the plane through $B\Gamma$, $ZH$ [Eucl. xi. 4]. But the plane through $B\Gamma$, $HZ$ is $AB\Gamma$; and therefore $\Delta E$ is perpendicular to the triangle $AB\Gamma$. Therefore all the planes through it are perpendicular to the triangle $AB\Gamma$ [Eucl. xi. 18]. But one of the planes through $\Delta E$ is the circle $B\Gamma$; therefore the circle $B\Gamma$ is perpendicular to the triangle $AB\Gamma$. Therefore the triangle $AB\Gamma$ is perpendicular to the circle $B\Gamma$; which is contrary to hypothesis. Therefore $\Delta E$ is not perpendicular to $ZH$.

**Corollary**

From this it is clear that $ZH$ is a diameter of the section $\Delta ZE$ [Def. 4], inasmuch as it bisects the straight lines drawn parallel to the given straight line $\Delta E$, and also that parallels can be bisected by the diameter $ZH$ without being perpendicular to it.

**Prop. 8**

*If a cone be cut by a plane through the axis, and it be*
τιμηθῇ δὲ καὶ ἕτερῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου καὶ εὐθείαν πρὸς ὅρθας οὕσαν τῇ βάσει τοῦ διὰ τοῦ ἀξονὸς τριγώνου, ἢ δὲ διάμετρος τῆς γυμνομένης ἐν τῇ ἐπιφανεία τομῆς ἤτοι παρὰ μίαν ἢ τῶν τοῦ τριγώνου πλευρῶν ἢ συμπίπτη αὐτήν εκτὸς τῆς κορυφῆς τοῦ κώνου, προσεκβάλλονται δὲ ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπιπέδον εἰς ἀπειρον, καὶ ἡ τομὴ εἰς ἀπειρον αὐξηθῆ-σεται, καὶ ἀπὸ τῆς διαμέτρου τῆς τομῆς πρὸς τῇ κορυφῇ πάση τῇ δοθείσῃ εὐθείᾳ ἢσθαν ἀπολήψεται τις εὐθεία ἀγομένη ἀπὸ τῆς τοῦ κώνου τομῆς παρὰ τὴν ἐν τῇ βάσει τοῦ κώνου εὐθείαν.

"Εστω κώνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμηθῶ ἐπιπέδῳ
dia τοῦ ἀξονὸς, καὶ ποιεῖτο τομὴν τὸ ΑΒΓ τρί-

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also cut by another plane cutting the base of the cone in a line perpendicular to the base of the axial triangle, and if the diameter of the section made on the surface be either parallel to one of the sides of the triangle or meet it beyond the vertex of the cone, and if the surface of the cone and the cutting plane be produced to infinity, the section will also increase to infinity, and a straight line can be drawn from the section of the cone parallel to the straight line in the base of the cone so as to cut off from the diameter of the section towards the vertex an intercept equal to any given straight line.

Let there be a cone whose vertex is the point A and base the circle BΓ', and let it be cut by a plane through the axis, and let the section so made be the triangle
γενοντες τετμησθω δε και έτερον επιπεδων τεμνοντι των ΒΓ κυκλον κατ' ευθειαν την ΔΕ προς ορθας 0υθαν τη ΒΓ, και ποιειτω τομην εν τη επιφανεια την ΔΖΕ γραμμην· η δε διαμετρος της ΔΖΕ τομης η ΖΗ ήτοι παραλληλος εστω τη ΑΓ η εκβαλλομενη συμπιπτετω αυτη εκτος του Α σημειου. λέγω, οτι και, εαν η τε του κωνου επιφανεια και το τεμνον επιπεδον εκβαλληται εις απειρον, και η ΔΖΕ τομη εις απειρον αυξηθησεται.

'Εκβεβλησθαι γιαρ η τε του κωνου επιφανεια και το τεμνον επιπεδον φανερον δη, οτι και αι ΑΒ, ΑΓ, ΖΗ συνεκβληθησονται. επει η ΖΗ τη ΑΓ ήτοι παραλληλος εστων η εκβαλλομενη συμπιπτει αυτη εκτος του Α σημειου, αι ΖΗ, ΑΓ αρα εκβαλλομεναι ως επι τα Γ, Η μερη ουδεποτε συμπεσουνται. εκβεβλησθωσαν ουν, και ειληφθων τι σημειου επι της ΖΗ τυχον το Θ, και δια του Θ σημειου τη μεν ΒΓ παραλληλος ηχθων η ΚΘΔ, τη δε ΔΕ παραλληλος η ΜΘΝ· το αρα δια των ΚΑ, ΜΝ επιπεδον παραλληλον εστι τω δια των ΒΓ, ΔΕ. κυκλος αρα εστι το ΚΛΜΝ επιπεδον. και επει τα Δ, Ε, Μ, Ν σημεια εν τω τεμνοντι εστων επιπεδων, εστι δε και εν τη επιφανεια του κωνου, επι της κοινης αρα τομης εστων· ηυξηται αρα η ΔΖΕ μεχρι των Μ, Ν σημειων. αυξηθεισης αρα της επιφανειας του κωνου και του τεμνοντος επιπεδου μεχρι τοι ΚΛΜΝ κυκλου ηυξηται και η ΔΖΕ τομη μεχρι των Μ, Ν σημειων. ομοιως δη δειξομεν, ότι και, εαν εις απειρον εκβαλληται η τε του κωνου επιφανεια και το τεμνον επιπεδον, και η ΜΔΖΕΝ τομη εις απειρον αυξηθησεται.

Και φανερον, ότι παση τη δοθειση ευθεια ισην
ABG; now let it be cut by another plane cutting the circle BG in the straight line DE perpendicular to BG, and let the section made on the surface be the curve ΔZE; let ZH, the diameter of the section ΔZE, be either parallel to AG or let it, when produced, meet AG beyond the point A. I say that if the surface of the cone and the cutting plane be produced to infinity, the section ΔZE will also increase to infinity.

For let the surface of the cone and the cutting plane be produced; it is clear that the straight lines, AB, AG, ZH are simultaneously produced. Since ZH is either parallel to AG or meets it, when produced, beyond the point A, therefore ZH, AG when produced in the directions H, G, will never meet. Let them be produced accordingly, and let there be taken any point Θ at random upon ZH, and through the point Θ let ΘA be drawn parallel to BG, and let ΘM be drawn parallel to DE; the plane through KA, MN is therefore parallel to the plane through BG, DE [Eucl. xi. 15]. Therefore the plane KAMN is a circle [Prop. 4]. And since the points Δ, E, M, N are in the cutting plane, and are also on the surface of the cone, they are therefore upon the common section; therefore ΔZE has increased to M, N. Therefore, when the surface of the cone and the cutting plane increase up to the circle KAMN, the section ΔZE increases up to the points M, N. Similarly we may prove that, if the surface of the cone and the cutting plane be produced to infinity, the section MΔZEN will increase to infinity.

And it is clear that there can be cut off from the
ἀπολήμεται τὸν ἀπὸ τῆς ΖΘ εὐθείας πρὸς τῷ Ζ σημείῳ. ἦν γὰρ τῆ δοθείσῃ ἱσθανθομεν τὴν ΖΞ καὶ διὰ τοῦ Ξ τῆς ΔΕ παράλληλον ἀγάγωμεν, συμπεσείται τῇ τομῇ, ὡσπερ καὶ ἡ διὰ τοῦ Θ ἀπεδείχθη συμπίπτουσα τῇ τομῇ κατὰ τὰ Μ, Ν σημεῖα· ὅστε ἄγεται τὴς εὐθείας συμπίπτουσα τῇ τομῇ παράλληλος ὅσα τῇ ΔΕ ἀπολαμβάνουσα ἀπὸ τῆς ΖΗ εὐθείαν ἵσθαν τῇ δοθείσῃ πρὸς τῷ Ζ σημείῳ.

θ′

'Εάν κάνος ἐπιπέδῳ τμῆθη συμπίπτοντι μὲν ἑκάτερα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν ἡγμένῳ μήτε ὑπεναντίως, ἡ τομὴ οὐκ ἔσται κύκλος.

'Εστω κάνος, οὗ κορυφῇ μὲν τὸ Α σημεῖον,

βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμημένῳ ἐπιπέδῳ των μήτε παραλλήλῳ ὄντι τῇ βάσει μήτε ὑπε-300
straight line \( Z\Theta \) in the direction of the point \( Z \) an intercept equal to any given straight line. For if we place \( Z\Xi \) equal to the given straight line and through \( \Xi \) draw a parallel to \( \Delta E \), it will meet the section, just as the parallel through \( \Theta \) was shown to meet the section at the points \( M, N \); therefore a straight line parallel to \( \Delta E \) has been drawn to meet the section so as to cut off from \( ZH \) in the direction of the point \( Z \) an intercept equal to the given straight line.

Prop. 9

*If a cone be cut by a plane meeting either side of the axial triangle, but neither parallel to the base nor subcontrary,\(^a\) the section will not be a circle.*

Let there be a cone whose vertex is the point \( A \) and base the circle \( \Gamma \), and let it be cut by a plane neither parallel to the base nor subcontrary, and let

\(^a\) In the figure of this theorem, the section of the cone by the plane \( \Delta E \) would be a *subcontrary section* (\( \nu\varepsilon\nu\varepsilon\alpha\tau\iota\alpha \tau\omicron\mu\iota \)) if the triangle \( \Delta \Gamma E \) were similar to the triangle \( \Delta \Gamma B \), but in a contrary sense, i.e., if angle \( \Delta \Gamma E \) = angle \( \Delta \Gamma B \). Apollonius proves in i. 5 that subcontrary sections of the cone are circles; it was proved in i. 4 that all sections parallel to the base are circles.
Εναντίως, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ τῆς ΔΚΕ γραμμῆς. λέγω, ὅτι ἡ ΔΚΕ γραμμὴ οὐκ ἔσται κύκλος.

Εἰ γὰρ δυνατὸν, ἔστω, καὶ συμπιπτέτω τὸ τέμνον ἐπίπεδον τῇ βάσει, καὶ ἔστω τῶν ἐπιπέδων κοινῆ τομῆ ἢ ΖΗ, τὸ δὲ κέντρον τοῦ ΒΓ κύκλου ἔστω τὸ Θ, καὶ ἀπ' αὐτοῦ κάθετος ἥχθω ἐπὶ τὴν ΖΗ ἢ ΘΗ, καὶ ἐκβεβληθὼ διὰ τῆς ΗΘ καὶ τοῦ ἀξονος ἐπίπεδον καὶ ποιεῖτω τομὰς ἐν τῇ κωνικῇ ἐπιφανείᾳ τὰς BA, ΑΓ εὐθείας. ἐπεὶ οὖν τὰ Δ, Ε, Η σημεῖα ἐν τῷ τῷ διὰ τῆς ΔΚΕ ἐπιπέδου ἔστω, ἔστω δὲ καὶ ἐν τῷ διὰ τῶν Α, Β, Γ, τὰ ἄρα Δ, Ε, Η σημεῖα ἐπὶ τῆς κοινῆς τομῆς τῶν ἐπιπέδων ἔστω εὐθεία ἀρα ἔστω ἡ ΗΕΔ. εἰλήφθω δὴ τι ἐπὶ τῆς ΔΚΕ γραμμῆς σημείων τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΖΗ παράλληλος ἥχθω ἢ ΚΛ· ἔσται δὴ ἢς ἡ ΚΜ τῇ ΜΔ. ἢ ἀρα ΔΕ διάμετρός ἔστω τοῦ ΔΚΛΕ κύκλου. ἥχθω δὴ διὰ τοῦ Μ τῇ ΒΓ παράλληλος ἢ ΝΜΕ· ἔστω δὲ καὶ ἢ ΚΛ τῇ ΖΗ παράλληλος· ἀφετέ τὸ διὰ τῶν ΝΕ, ΚΜ ἐπίπεδον παράλληλον ἔστω τῷ διὰ τῶν ΒΓ, ΖΗ, τούτου τῇ βάσει, καὶ ἔσται ἡ τομὴ κύκλοις. ἔστω οὖ ΝΚΕ. καὶ ἐπεὶ ἢ ΖΗ τῇ ΒΗ πρὸς ὀρθάς ἔστι, καὶ ἢ ΚΜ τῇ ΝΕ πρὸς ὀρθάς ἐστίν· ὅστε τὸ ὑπὸ τῶν ΝΜΕ ἢ ΚΜ ἢ ἔστι τῷ ἀπὸ τῆς ΚΜ. ἔστι δὲ τὸ ὑπὸ τῶν ΔΜΕ ἢ ΚΜ ἢ τῷ ἀπὸ τῆς ΚΜ· κύκλος γὰρ ὑπόκειται ἡ ΔΚΕ γραμμῆ, καὶ διάμετρος αὐτοῦ ἢ ΔΕ· τὸ ἀρα ὑπὸ τῶν ΝΜΕ ἢ ΚΜ ἢ ἔστι τῷ ὑπὸ ΔΜΕ· ἔστων ἢ ἔστω τῷ ὑπὸ ΔΜΕ. ἢ καὶ ἢ ΜΝ πρὸς ΜΔ, οὕτως ἢ ΕΜ πρὸς ΜΣ. ὅμοιον ἄρα ἔστι τὸ ΔΜΝ τρίγωνον τῷ ΕΜΕ τριγώνω, καὶ ἢ ὑπὸ ΔΝΜ γωνία ἢς ἔστι τῇ ὑπὸ ΜΕΣ. ἀλλὰ

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the section so made on the surface be the curve $\Delta KE$. I say that the curve $\Delta KE$ will not be a circle.

For, if possible, let it be, and let the cutting plane meet the base, and let the common section of the planes be $ZH$, and let the centre of the circle $\Gamma$ be $\Theta$, and from it let $\Theta H$ be drawn perpendicular to $ZH$, and let the plane through $H\Theta$ and the axis be produced, and let the sections made on the conical surface be the straight lines $BA, AT$. Then since the points $\Delta, E, H$ are in the plane through $\Delta KE$, and are also in the plane through $A, B, \Gamma$, therefore the points $\Delta, E, H$ are on the common section of the planes; therefore $HE\Delta$ is a straight line [Eucl. xi. 3]. Now let there be taken any point $K$ on the curve $\Delta KE$, and through $K$ let $KA$ be drawn parallel to $ZH$; then $KM$ will be equal to $MA$ [Prop. 7]. Therefore $\Delta E$ is a diameter of the circle $\Delta KEA$ [Prop. 7, coroll.]. Now let $N\Xi$ be drawn through $M$ parallel to $\Gamma$; but $KA$ is parallel to $ZH$; therefore the plane through $N\Xi$, $KM$ is parallel to the plane through $\Gamma$, $ZH$ [Eucl. xi. 15], that is to the base, and the section will be a circle [Prop. 4]. Let it be $NK\Xi$. And since $ZH$ is perpendicular to $BH$, $KM$ is also perpendicular to $N\Xi$ [Eucl. xi. 10]; therefore $NM \cdot \Xi = KM^2$. But $\Delta M \cdot ME = KM^2$; for the curve $\Delta KEA$ is by hypothesis a circle, and $\Delta E$ is a diameter in it. Therefore $NM \cdot \Xi = \Delta M \cdot ME$. Therefore $MN : M\Delta = EM : \Xi$. Therefore the triangle $\Delta MN$ is similar to the triangle $\Xi ME$, and the angle $\Delta NM$ is equal to the angle $ME\Xi$. 

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Εάν κώνος ἐπιπέδω τμηθῇ διὰ τοῦ ἄξονος, 
τμηθῇ δὲ καὶ ἐτέρω ἐπιπέδῳ τέμνοντι τὴν βάσιν 
τοῦ κώνου κατ' εὐθείαν πρὸς ὀρθὰς οὐσαν τῇ βάσει 
τοῦ διὰ τοῦ ἄξονος τριγώνου, ἐτι δὲ ἡ διάμετρος 
τῆς τομῆς παράλληλος ἢ μιᾷ πλευρᾷ τοῦ διὰ τοῦ 
ἄξονος τριγώνου, ἢτις ἂν ἀπὸ τῆς τομῆς τοῦ κώνου 
παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τοῦ τέμνοντος 
ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου μέχρι τῆς 
διαμέτρου τῆς τομῆς, δυνήσεται τὸ περιεχόμενον 
ὑπὸ τε τῆς ἀπολαμβανομένης ὑπ' αὐτῆς ἀπὸ τῆς 
διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς καὶ ἄλλης 
τινὸς εὐθείας, ἢ λόγου ἔχει πρὸς τὴν μεταξὺ τῆς 
τοῦ κώνου γωνίας καὶ τῆς κορυφῆς τῆς τομῆς, 
ὅν τὸ τετράγωνον τὸ ἀπὸ τῆς βάσεως τοῦ διὰ τοῦ 
ἄξονος τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν 
λοιπῶν τοῦ τριγώνου δύο πλευρῶν· καλείσθω δὲ 
ἡ τουατῇ τομῇ παραβολῆ.

"Εστώ κώνος, οὗ τὸ Α σημεῖον κορυφῆς, βάσις 
δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ διὰ τοῦ 
ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνου, 
tετμήσθω δὲ καὶ ἐτέρω ἐπιπέδῳ τέμνοντι τὴν 
βάσιν τοῦ κώνου κατ' εὐθείαν τὴν ΔΕ πρὸς ὀρθὰς
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But the angle $\Delta NM$ is equal to the angle $A\beta I'$; for $N\Xi$ is parallel to $B\Gamma$; and therefore the angle $A\beta I'$ is equal to the angle $M\Xi E$. Therefore the section is subcontrary [Prop. 5]; which is contrary to hypothesis. Therefore the curve $\Delta KE$ is not a circle.

(v.) Fundamental Properties

Ibid., Props. 11-14, Apoll. Perg. ed. Heiberg i. 36. 26-58. 7

Prop. 11

Let a cone be cut by a plane through the axis, and let it be also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and further let the diameter of the section be parallel to one side of the axial triangle; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the rectangle bounded by the intercept made by it on the diameter in the direction of the vertex of the section and a certain other straight line; this straight line will bear the same ratio to the intercept between the angle of the cone and the vertex of the segment as the square on the base of the axial triangle bears to the rectangle bounded by the remaining two sides of the triangle; and let such a section be called a parabola.

For let there be a cone whose vertex is the point $A$ and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $A\beta I'$, and let it be cut by another plane cutting the base of the cone in the straight line
οὗσαι τῇ ΒΓ, καὶ ποιεῖτω τομήν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τῆς ΔΖΕ, ἢ δὲ διάμετρος τῆς τομῆς ἡ ZH παράλληλος ἐστὶν μὲν πλευρὰ τοῦ διὰ τοῦ ἀξονὸς τριγώνου τῇ ΑΓ, καὶ ἀπὸ τοῦ Ζ σημείου τῇ ZH εἰθεῖα πρὸς ὁρθὰς ἡχθὸς ἡ ZΘ, καὶ πεποιήθω, ώσ τὸ ἀπὸ ΒΓ πρὸς τὸ ύπὸ ΒΑΓ, οὔτως ἡ ZΘ πρὸς ΖΑ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τυχόν τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΔΕ παράλληλος ἡ ΚΛ. λέγω, ὅτι τὸ ἀπὸ τῆς ΚΛ ἵσον ἐστὶ τῷ ύπὸ τῶν ΘΖΑ.

"Ἡχθὼ γὰρ διὰ τοῦ Λ τῇ ΒΓ παράλληλος ἡ ΜΝ· ἐστὶ δὲ καὶ ἡ ΚΛ τῇ ΔΕ παράλληλος· τὸ ἀρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον παράλληλον ἐστὶ τῷ διὰ τῶν ΒΓ, ΔΕ ἐπίπεδῳ, τούτῳ τῇ βάσει τοῦ κώνου. τὸ ἀρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον κύκλος ἐστίν, οὗ διάμετρος ἡ ΜΝ. καὶ ἐστὶ κάθετος ἐπὶ τῇ ΜΝ ἡ ΚΛ, ἐπεὶ καὶ ἡ ΔΕ ἐπὶ τῇ ΒΓ· τὸ ἀρα ύπὸ τῶν ΜΛΝ ἵσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ. καὶ ἐπεὶ ἐστίν, ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ύπὸ τῶν ΒΑΓ, οὔτως ἡ ΘΖ πρὸς ΖΑ, τὸ δὲ
ΔE perpendicular to BI, and let the section so made on the surface of the cone be ΔZE, and let ZH, the diameter of the section, be parallel to ΔΓ, one side of the axial triangle, and from the point Z let ZΘ be drawn perpendicular to ZH, and let BI : BA : ΔΓ = ZΘ : ZΑ, and let any point K be taken at random on the section, and through K let KA be drawn parallel to ΔE. I say that KA² = ΩZ : ZΑ.

For let MN be drawn through A parallel to BI; but KA is parallel to ΔE; therefore the plane through

\[ KA, MN \text{ is parallel to the plane through BI, } ΔE \] (Eucl. xi. 15), that is to the base of the cone. Therefore the plane through KA, MN is a circle, whose diameter is MN (Prop. 4). And KA is perpendicular to MN, since ΔE is perpendicular to BI (Eucl. xi. 10); therefore \[ MA \cdot AN = KA². \]

And since \[ BI : BA : ΔΓ = ΩZ : ZΑ, \]
ἀπὸ τῆς ΒΓ πρὸς τὸ ύπὸ τῶν ΒΑΓ λόγον ἔχει τὸν συγκείμενον ἐκ τε τοῦ, οὖν ἔχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ, ὁ ἀρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΒΓ πρὸς ΓΑ καὶ τοῦ τῆς ΓΒ πρὸς ΒΑ. ἀλλ' ὡς μὲν ἡ ΒΓ πρὸς ΓΑ, οὔτως ἡ ΜΝ πρὸς ΝΑ, τουτέστιν ἡ ΜΑ πρὸς ΛΖ, ως δὲ ἡ ΒΓ πρὸς ΒΑ, οὔτως ἡ ΜΝ πρὸς ΜΑ, τουτέστιν ἡ ΛΜ πρὸς ΜΖ, καὶ λοιπῇ ἡ ΝΑ πρὸς ΖΑ. ὁ ἀρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΜΑ πρὸς ΛΖ καὶ τοῦ τῆς ΝΑ πρὸς ΖΑ. ὁ δὲ συγκείμενος λόγος ἐκ τοῦ τῆς ΜΑ πρὸς ΛΖ καὶ τοῦ τῆς ΛΝ πρὸς ΖΑ ὁ τοῦ ύπὸ ΜΑΝ ἐστὶ πρὸς τὸ ύπὸ ΛΖΑ. ως ἀρα ἡ ΘΖ πρὸς ΖΑ, οὔτως τὸ ύπὸ ΜΑΝ πρὸς τὸ ύπὸ ΛΖΑ. ως δὲ ἡ ΘΖ πρὸς ΖΑ, τῆς ΖΛ κοινοῦ ύψους λαμβανομένης οὔτως τὸ ύπὸ ΘΖΑ πρὸς τὸ ύπὸ ΛΖΑ. ως ἀρα τὸ ύπὸ ΜΑΝ πρὸς τὸ ύπὸ ΛΖΑ, οὔτως τὸ ύπὸ ΘΖΑ πρὸς τὸ ύπὸ ΛΖΑ. ἵσον ἀρα ἐστὶ τὸ ύπὸ ΜΑΝ τῷ ύπὸ ΘΖΑ. τὸ δὲ ύπὸ ΜΑΝ ἵσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ. καὶ τὸ ἀπὸ τῆς ΚΛ ἁρὰ ἵσον ἐστὶ τῷ ύπὸ τῶν ΘΖΑ.

Καλεῖσθω δὲ ἡ μὲν τοιαύτη τομὴ παραβολῆ, ἡ δὲ ΘΖ παρ' ἑν δύνανται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν ΖΗ διάμετρον, καλεῖσθω δὲ καὶ ὀρθία.

"Εὰν κώνος ἐπιπέδῳ τιμηθῇ διὰ τοῦ ἄξονος, τιμηθῇ δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν

*a A parabola (παραβολή) because the square on the ordinate ΚΛ is applied (παραβαλέων) to the parameter ΘΖ in the form 808*
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while \( \mathbf{BI}^2 : \mathbf{BA} \cdot \mathbf{AI} = (\mathbf{BI} : \mathbf{GA})(\mathbf{BI} : \mathbf{BA}) \),
therefore \( \Theta Z : \mathbf{ZA} = (\mathbf{BI} : \mathbf{GA})(\mathbf{GB} : \mathbf{BA}) \).
But \( \mathbf{BI} : \mathbf{GA} = \mathbf{MN} : \mathbf{NA} = \mathbf{MA} : \mathbf{AZ} \), [Eucl. vi. 4
and \( \mathbf{BI} : \mathbf{BA} = \mathbf{MN} : \mathbf{MA} = \mathbf{AM} : \mathbf{MZ} \) [ibid.
\( = \mathbf{NA} : \mathbf{ZA} \). [Eucl. vi. 2
Therefore \( \Theta Z : \mathbf{ZA} = (\mathbf{MA} : \mathbf{AZ})(\mathbf{NA} : \mathbf{ZA}) \).
But \( (\mathbf{MA} : \mathbf{AZ})(\mathbf{AN} : \mathbf{ZA}) = \mathbf{MA} \cdot \mathbf{AN} : \mathbf{AZ} \cdot \mathbf{ZA} \).
Therefore \( \Theta Z : \mathbf{ZA} = \mathbf{MA} \cdot \mathbf{AN} : \mathbf{AZ} \cdot \mathbf{ZA} \).
But \( \Theta Z : \mathbf{ZA} = \Theta Z \cdot \mathbf{ZA} : \mathbf{AZ} \cdot \mathbf{ZA} \),
by taking a common height \( \mathbf{ZA} \);
therefore \( \mathbf{MA} \cdot \mathbf{AN} = \mathbf{AZ} \cdot \mathbf{ZA} = \Theta Z \cdot \mathbf{ZA} : \mathbf{AZ} \cdot \mathbf{ZA} \).
Therefore \( \mathbf{MA} \cdot \mathbf{AN} = \Theta Z \cdot \mathbf{ZA} \). [Eucl. v. 9
But \( \mathbf{MA} \cdot \mathbf{AN} = \mathbf{KA}^2 \);
and therefore \( \mathbf{KA}^2 = \Theta Z \cdot \mathbf{ZA} \).

Let such a section be called a parabola, and let \( \Theta Z \) be called the parameter of the ordinates to the diameter \( \mathbf{ZH} \), and let it also be called the erect side (latus rectum).\(^a\)

Prop. 12

Let a cone be cut by a plane through the axis, and let it be cut by another plane cutting the base of the cone in of the rectangle \( \Theta Z \cdot \mathbf{ZA} \), and is exactly equal to this rectangle. It was Apollonius's most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the application of areas (παραβολή τῶν χωρίων), for which v. vol. i. pp. 186-215. The explanation of the term latus rectum will become more obvious in the cases of the hyperbola and the ellipse; v. infra, p. 317 n. a.
GREEK MATHEMATICS

tou konou kat' eudeiaan prose orbas oudan tis basi
tou dia tou axonos trigwoun, kai h diametros tis
tomis ekballomevi symipte mia pleura tou
dia tou axonos trigwoun ekto tis tou konou
coruphis, hteis an apo tis tomis aikh parallilos
tis kouh tomis tou temvontos epipedou kai tis
basetous tou konou, eou tis diametrou tis tomis
dunhsetai ti xwriov parakeimevnov para tina
eudeiaan, prose hln logon eixe h ep' eudeia is men ouasa

tis diametrou tis tomis, upoteinouna de tis ekto

tou trigwoun gwnian, oin to tetragwou to apo

tis hgmenvis apo tis koruphous tou konou para

tou diametrou tis tomis eou tis basetous tou

tou trigwoun prose tou periechomevnous apo ton tou

basetous trigmata, on poi ei a axheisa, plato

exon tis apolamballonemevn h' auths apo tis
diametrou prose tis koruph tis tomis, uperballo

eidei oumio tis kai oumio keimeno tou periecho-

menov upo te tis upoteinoupos tis ekto

gwnian tou trigwoun kai tis par' hln dynahtai ai kat-

agomevai kalieswv de h toua unh tomis uperbolh.

'Eostw konov, oih koruphe men to A smieov,
basis de o BGI kiklos, kai tremphtov epipedo

dia tou axonov, kai poieitous tomhv to ABGI trig-
woun, tremphtov de kai eterev epipedov temvonti

tou basin tou konou kat' eudeiaan tis DE prose

orbas oudan tis BGI basi vou ABGI trigwoun,
kai poieitous tomhv en tis epifaneia tou konov tis

DZG gramh, h de diametros tis tomis h ZH

ekballomevi symiptetev mou plno of tou ABGI

trigwoun tis AI ekto tis tou konou korupheis

cat to TH, kai dia to A tis diametrou tis tomis

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a straight line perpendicular to the base of the axial triangle, and let the diameter of the section, when produced, meet one side of the axial triangle beyond the vertex of the cone; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the area applied to a certain straight line; this line is such that the straight line subtending the external angle of the triangle, lying in the same straight line with the diameter of the section, will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle bounded by the segments of the base made by the line so drawn; the breadth of the applied figure will be the intercept made by the ordinate on the diameter in the direction of the vertex of the section; and the applied figure will exceed by a figure similar and similarly situated to the rectangle bounded by the straight line subtending the external angle of the triangle and the parameter of the ordinates; and let such a section be called a hyperbola.

Let there be a cone whose vertex is the point \( A \) and whose base is the circle \( BG \), and let it be cut by a plane through the axis, and let the section so made be the triangle \( ABG \), and let it be cut by another plane cutting the base of the cone in the straight line \( DE \) perpendicular to \( BG \), the base of the triangle \( ABG \), and let the section so made on the surface of the cone be the curve \( DE \), and let \( ZH \), the diameter of the section, when produced, meet \( AT \), one side of the triangle \( ABG \), beyond the vertex of the cone at \( \Theta \), and through \( A \) let \( AK \) be drawn parallel to \( ZH \), the
ΓΕΡΣΚ ΜΑΤΗΜΑΤΙΚΑΣ

τὴ ᾽ΖΗ παράλληλος ᾽ηχθὼ ᾧ ΑΚ, καί τεμνέτω τὴν 
ΒΓ, καί ἀπὸ τοῦ Ζ τῇ ᾽ΖΗ πρὸς ὀρθὰς ᾽ηχθὼ ᾧ

ΖΑ, καί πεποίησθω, ὥσ τὸ ἀπὸ ΚΑ πρὸς τὸ ὑπὸ 
ΒΚΓ, οὕτως ᾧ ΖΘ πρὸς ΖΑ, καί εἰλήφθω τῖ 
σημείον ἐπὶ τῆς τομῆς τυχὸν τὸ Μ, καί διὰ τοῦ 
Μ τῇ ΔΕ παράλληλος ᾽ηχθὼ ᾧ ΜΝ, διὰ δὲ τοῦ Ν 
τῇ ΖΑ παράλληλος ᾧ ΝΟΞ, καί ἐπίζευξθησα ᾧ 
ΘΑ ἐκβεβλήσθω ἐπὶ τὸ Ξ, καὶ διὰ τῶν Δ, Ξ τῇ 
ΖΝ παράλληλοι ᾽ηχθωσαν αἱ ΛΟ, ΞΠ. λέγω, ὅτι 
ᾦ ΜΝ δύναται τὸ ΖΞ, ὁ παράκειται παρὰ τὴν 
ΖΑ, πλατὸς ἐχον τὴν ΖΝ, ύπερβάλλων εἰδεὶ τῷ 
ΔΞ ὅμοιῳ ὀντὶ τῷ ὑπὸ τῶν ΘΖΑ.

᾿Ηχθὼ γὰρ διὰ τοῦ Ν τῇ ΒΓ παράλληλος ᾧ 
PΝΣ. ἔστι δὲ καί ᾧ ΝΜ τῇ ΔΕ παράλληλος. τῷ
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diameter of the section, and let it cut $BI'$, and from $Z$ let $ZA$ be drawn perpendicular to $ZH$, and let $KA^2 : BK : KT = ZO : ZA$, and let there be taken at random any point $M$ on the section, and through $M$ let $MN$ be drawn parallel to $\Delta E$, and through $N$ let $NO\Xi$ be drawn parallel to $ZA$, and let $\Theta \Lambda$ be joined and produced to $\Xi$, and through $\Lambda, \Xi$, let $\Lambda O, \Xi II$ be drawn parallel to $ZN$. I say that the square on $MN$ is equal to $Z\Xi$, which is applied to the straight line $ZA$, having $ZN$ for its breadth, and exceeding by the figure $\Delta \Xi$ which is similar to the rectangle contained by $\Theta Z, Z\Lambda$.

For let $PN\Xi$ be drawn through $N$ parallel to $BI'$; but $NM$ is parallel to $\Delta E$; therefore the plane through
ἀρα διὰ τῶν MN, ΡΣ ἐπίπεδον παράλληλόν ἐστι
tῶ διὰ τῶν ΒΓ, ΔΕ, τούτεστι τῇ βάσει τοῦ κόνου.
ἐάν ἄρα ἐκβληθῇ τὸ διὰ τῶν MN, ΡΣ ἐπίπεδον,
ἡ τομὴ κύκλος ἐσται, οὔ διάμετρος ἡ ΡΝΣ. καὶ
ἐστιν ἐπ’ αὐτὴν κάθετος ἡ MN· τὸ ἄρα ὑπὸ τῶν
ΡΝΣ ἰσον ἐστὶ τῷ ἀπὸ τῆς MN. καὶ ἔτει ἐστιν,
ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΖΘ
πρὸς ΖΛ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ
λόγος σύγκειται ἐκ τε τοῦ, ὅν ἔχει ἡ ΑΚ πρὸς ΚΓ
καὶ ἡ ΑΚ πρὸς ΚΒ, καὶ ὁ τῆς ΖΘ ἄρα πρὸς τὴν
ΖΛ λόγος σύγκειται ἐκ τοῦ, ὅν ἔχει ἡ ΑΚ πρὸς
ΚΓ καὶ ἡ ΑΚ πρὸς ΚΒ. ἀλλ’ ὡς μὲν ἡ ΑΚ
πρὸς ΚΓ, οὕτως ἡ ΘΗ πρὸς ΗΓ, τούτεστιν ἡ ΘΟΝ
πρὸς ΝΣ, ὡς δὲ ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΖΗ πρὸς
ΗΒ, τούτεστιν ἡ ΖΝ πρὸς ΝΡ. ὁ ἄρα τῆς ΘΖ
πρὸς ΖΛ λόγος σύγκειται ἐκ τε τοῦ τῆς ΘΝ πρὸς
ΝΣ καὶ τοῦ τῆς ΖΝ πρὸς ΝΡ. ὁ δὲ συγκειμένος
λόγος ἐκ τοῦ τῆς ΘΝ πρὸς ΝΣ καὶ τοῦ τῆς ΖΝ
πρὸς ΝΡ ὁ τοῦ ὑπὸ τῶν ΘΝΖ ἐστὶ πρὸς τὸ ὑπὸ
tῶν ΣΝΡ· καὶ ὡς ἄρα τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ
ὑπὸ τῶν ΣΝΡ, οὕτως ἡ ΘΖ πρὸς ΖΛ, τούτεστιν
ἡ ΘΝ πρὸς ΝΞ. ἀλλ’ ὡς ἡ ΘΝ πρὸς ΝΞ, τῆς
ΖΝ κοινοῦ ὑψὸς λαμβανομένης οὕτως τὸ ὑπὸ
tῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΖΝΞ. καὶ ὡς ἄρα
τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΡ, οὕτως
τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΘΕΝΖ. τὸ
ἄρα ὑπὸ ΣΝΡ ἰσον ἐστὶ τῷ ὑπὸ ΘΕΝΖ. τὸ δὲ
ἀπὸ ΜΝ ἢ ἰσον ἐδείχθη τῷ ὑπὸ ΣΝΡ· καὶ τὸ ἀπὸ
tῆς ΜΝ ἄρα ἢ ἰσον ἐστὶ τῷ ὑπὸ τῶν ΘΕΝΖ. τὸ δὲ
ὑπὸ ΘΕΝΖ ἐστὶ τὸ ΞΖ παράλληλογράμμον. ἡ ἄρα

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MN, PΣ is parallel to the plane through BG, ΔΕ [Eucl. xi. 15], that is to the base of the cone. If, then, the plane through MN, PΣ be produced, the section will be a circle with diameter PNΣ [Prop. 4]. And MN is perpendicular to it; therefore

\[ PN \cdot NΣ = MN^2. \]

And since \( AK^2 : BK \cdot KΓ = ZΘ : ZΛ, \)
while \( AK^2 : BK \cdot KΓ = (AK : KΓ)(AK : KB), \)
therefore \( ZΘ : ZΛ = (AK : KΓ)(AK : KB). \)
But \( AK : KΓ = ΘH : HΓ, \)
\[ i.e., \]
\[ = ΘN : NΣ, \quad [Eucl. vi. 4 \]
and \( AK : KB = ZH : HB, \)
\[ i.e., \]
\[ = ZN : NP. \quad [ibid. \]
Therefore \( ΘZ : ZΛ = (ΘN : NΣ)(ZN : NP). \)
But \( (ΘN : NΣ)(ZN : NP) = ΘN \cdot NZ : ΣN \cdot NP; \)
and therefore
\[ ΘN \cdot NZ : ΣN \cdot NP = ΘZ : ZΛ = ΘN : NΣ. \quad [ibid. \]
But \( ΘN : NΣ = ΘN \cdot NZ : ZN \cdot NΣ, \)
by taking a common height ZN.
And therefore
\[ ΘN \cdot NZ : ΣN \cdot NP = ΘN \cdot NZ : ΕN \cdot NZ. \]
Therefore \( ΣN \cdot NP = ΕN \cdot NZ. \quad [Eucl. v. 9 \]
But \( MN^2 = ΣN \cdot NP, \)
as was proved;
and therefore \( MN^2 = ΕN \cdot NZ. \)
But the rectangle ΕN . NZ is the parallelogram ΕZ.
MN δύναται τὸ ΕΖ, ὃ παράκειται παρὰ τὴν ΖΛ, πλάτος ἔχον τὴν ΖΝ, ύπερβάλλον τῷ ΔΞ ὄμοιῷ ὄντι τῷ ὑπὸ τῶν ΘΖΛ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ύπερβολή, ἡ δὲ ΔΖ παρ' ἦν δύνανται αἱ ἐπὶ τὴν ΖΗ καταγόμεναι τεταγμένως: καλείσθω δὲ ἡ αὐτὴ καὶ ορθία, πλαγία δὲ ἡ ΖΘ.

'Εστω κώνος ἐπιπέδῳ τηθῇ διὰ τοῦ ἀξονος, τηθῇ δὲ καὶ ἐτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἐκατέρα πλευρά τοῦ διὰ τοῦ ἀξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν τοῦ κώνου ἡμεῖν μήτε ὑπεναντίως, τὸ δὲ ἐπίπεδον, ἐν ὦ ἐστιν ἡ βάσις τοῦ κώνου, καὶ τὸ τέμνον ἐπίπεδον συμπίπτῃ κατ' εὐθείαν πρὸς ὁρθὰς ὀδοὺς ἤτοι τῇ βάσει τοῦ διὰ τοῦ ἀξονος τριγώνου ἡ τῇ ἐπ' εὐθείας αὐτῆ, ἤτοι ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀκθῆ τῇ κοινῇ τομῇ τῶν ἐπίπεδων ἐως τῆς διαμέτρου τῆς τομῆς, δυνήσεται τι χωρίον παρακείμενον παρὰ τίνα εὐθείαν, πρὸς τὴν λόγου ἔχει ἡ διάμετρος τῆς τομῆς, ὅν τὸ τετράγωνον τὸ ἀπὸ τῆς ἡγμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διάμετρον τῆς τομῆς ἐως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν ἀπολαμβανομένων ὑπ' αὐτῆς πρὸς ταῖς τοῦ τριγώνου εὐθείας, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ἔλλειπον εἴδει ὁμοίῳ τε καὶ ὁμοίῳ κειμένῳ τῷ περιεχομένῳ ὑπὸ τε τῆς διαμέτρου καὶ τῆς παρ' ἦν δύνανται καλείσθω δὲ ἡ τοιαύτη τομὴ ἔλλειψις.

'Εστω κώνος, οὐ κορυφῆ μὲν τὸ Α σημεῖον,
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Therefore the square on MN is equal to ΕΖ, which is applied to ΖΛ, having ΖΝ for its breadth, and exceeding by ΛΕ similar to the rectangle contained by ΩΖ, ΖΛ. Let such a section be called a hyperbola, let ΛΖ be called the parameter to the ordinates to ΖΗ; and let this line be also called the erect side (latus rectum), and ΖΘ the transverse side.a

Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter; and let such a section be called an ellipse.

Let there be a cone, whose vertex is the point A

a The erect and transverse side, that is to say, of the figure (εἴδος) applied to the diameter. In the case of the parabola, the transverse side is infinite.
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βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἑπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ έτέρῳ ἑπιπέδῳ συμπίπτοντι μὲν ἐκατέρα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παραλλήλῳ τῇ βάσει τοῦ κῶνου μήτε ὑπεναντίως ἰγμένῳ, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κῶνου τῇ ΔΕ γραμμήν· κοινὴ

δὲ τομὴ τοῦ τέμνοντος ἑπιπέδου καὶ τοῦ, ἐν ὧν ἦν βάσις τοῦ κῶνου, ἐστώ ἡ ΖΗ πρὸς ὅρθας οὖσα τῇ ΒΓ, ἡ δὲ διάμετρος τῆς τομῆς ἐστώ ἡ ΕΔ, καὶ ἀπὸ τοῦ Ε τῇ ΕΔ πρὸς ὅρθας ἤχθω ἡ ΕΘ, καὶ διὰ τοῦ Α τῇ ΕΔ παράλληλος ἤχθω ἡ ΑΚ, καὶ πεποίησθω ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, καὶ εἰληφθεὶς τι σημεῖον ἐπὶ τῆς τομῆς τὸ Λ, καὶ διὰ τοῦ Λ τῇ ΖΗ παράλληλος ἤχθω ἡ ΛΜ. λέγω, ὅτι ἡ ΛΜ δύναται τι χωρίου, διὸ παράκειται παρὰ τὴν ΕΘ, πλάτος ἔχον τὴν ΕΜ, ἐλλειπὼν εἴδει ὑμοίῳ τῷ ὑπὸ τῶν ΔΕΘ.

Επεζεύχθω γὰρ ἡ ΔΘ, καὶ διὰ μὲν τοῦ Μ τῇ

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and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve $\Delta E$; let the common section of the cutting plane and of that containing the base of the cone be $ZH$, perpendicular to $B\Gamma$, and let the diameter of the section be $E\Delta$, and from $E$ let $E\Theta$ be drawn perpendicular to $E\Delta$, and through $A$ let $AK$ be drawn parallel to $E\Delta$, and let $AK^2 : BK \cdot K\Gamma = \Delta E : E\Theta$, and let any point $\Lambda$ be taken on the section, and through $\Lambda$ let $\Lambda M$ be drawn parallel to $ZH$. I say that the square on $\Lambda M$ is equal to an area applied to the straight line $E\Theta$, having $EM$ for its breadth, and being deficient by a figure similar to the rectangle contained by $\Delta E$, $E\Theta$.

For let $\Delta \Theta$ be joined, and through $M$ let $M\Xi N$ be
ΘΕ παράλληλος ἡχθων ἡ ΜΕΝ, διὰ δὲ τῶν Θ, Ξ
τῇ ΕΜ παράλληλοι ἡχθωσαν αἱ ΘΝ, ΞΟ, καὶ διὰ
τοῦ Μ τῇ ΒΓ παράλληλος ἡχθων ἡ ΠΜΡ. ἐπεῖ
οὖν ἡ ΠΡ τῇ ΒΓ παράλληλος ἐστιν, ἐστὶ δὲ καὶ
ἡ ΛΜ τῇ ΖΗ παράλληλος, τὸ ἀρα διὰ τῶν ΛΜ,
ΠΡ ἐπίπεδον παράλληλον ἐστὶ τῷ διὰ τῶν ΖΗ,
ΒΓ ἐπίπεδω, τούτεστι τῇ βάσει τοῦ κώνου.
ἐὰν ἀρα ἕκβληθῇ διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον, ἡ τομή
κύκλος ἔσται, οὐ διάμετρος ἡ ΠΡ. καὶ ἐστὶ
kάθετος ἐπ' αὐτὴν ἡ ΛΜ. τὸ ἀρα ὑπὸ τῶν ΠΜΡ
ἰσον ἐστὶ τῷ ἀπὸ τῆς ΛΜ. καὶ ἐπεῖ ἐστιν, ὡς τὸ
ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ, οὐτως ἡ ΕΔ
πρὸς τὴν ΕΘ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ
tῶν ΒΚΓ λόγος σύγκειται ἐκ τοῦ, ὃν ἔχει ἡ ΑΚ
πρὸς ΚΒ, καὶ ἡ ΑΚ πρὸς ΚΓ, ἀλλ' ὡς μὲν ἡ ΑΚ
πρὸς ΚΒ, οὐτως ἡ ΕΗ πρὸς ΗΒ, τούτεστιν ἡ
ΕΜ πρὸς ΜΠ, ὡς δὲ ἡ ΑΚ πρὸς ΚΓ, οὐτως ἡ
ΔΗ πρὸς ΗΓ, τούτεστιν ἡ ΔΜ πρὸς ΜΡ, ὁ ἀρα
τῆς ΔΕ πρὸς τὴν ΕΘ λόγος σύγκειται ἐκ τοῦ τῆς
ΕΜ πρὸς ΜΠ καὶ τοῦ τῆς ΔΜ πρὸς ΜΡ. ὁ
dε συγκείμενος λόγος ἐκ τοῦ, ὃν ἔχει ἡ ΕΜ
πρὸς ΜΠ, καὶ ἡ ΔΜ πρὸς ΜΡ, ὁ τοῦ ὑπὸ τῶν
ΕΜΔ ἐστὶ πρὸς τὸ ὑπὸ τῶν ΠΜΡ. ἐστὶν ἀρα ὡς
tὸ ὑπὸ τῶν ΕΜΔ πρὸς τὸ ὑπὸ τῶν ΠΜΡ, οὐτως
ἡ ΔΕ πρὸς τὴν ΕΘ, τούτεστιν ἡ ΔΜ πρὸς τὴν
ΜΕ. ὡς δὲ ἡ ΔΜ πρὸς ΜΕ, τῆς ΜΕ κοινοῦ
ὔμοις λαμβανομένης, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ
ὑπὸ ΞΕΜΕ. καὶ ὡς ἀρα τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ
ΠΜΡ, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΞΕΜΕ.
ἰσον ἀρα ἐστὶ τὸ ὑπὸ ΠΜΡ τῷ ὑπὸ ΞΕΜΕ. τὸ δὲ
ὑπὸ ΠΜΡ ἢςον ἐδείχθη τῷ ἀπὸ τῆς ΛΜ. καὶ τὸ
ὑπὸ ΞΕΜΕ ἀρα ἐστὶν ἢςον τῷ ἀπὸ τῆς ΛΜ. ἡ ΛΜ
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drawn parallel to ΘΕ, and through Θ, Ζ, let ON, ΖΟ be drawn parallel to EM, and through M let ΠM be drawn parallel to BG. Then since ΠP is parallel to BG, and AM is parallel to ZH, therefore the plane through AM, ΠP is parallel to the plane through ZH, BG [Eucl. xi. 15], that is to the base of the cone. If, therefore, the plane through AM, ΠP be produced, the section will be a circle with diameter ΠP [Prop. 4]. And AM is perpendicular to it; therefore

\[ ΠM \cdot MP = AM^2. \]

And since
\[ AK^2 : BK : KT = EΔ : EΘ, \]
and
\[ AK^2 : BK : KT = (AK : KB)(AK : KT), \]
while
\[ AK : KB = EH : HB = EM : ΜΠ, \] [Eucl. vi. 4]
and
\[ AK : KT = ΔH : HG = ΔM : MP, \] [ibid.]
therefore

\[ ΔE : EΘ = (EM : ΜΠ)(ΔM : MP). \]

But \( (EM : ΜΠ)(ΔM : MP) = EM \cdot MA : ΠM \cdot MP. \)

Therefore

\[ EM \cdot MA : ΠM \cdot MP = ΔE : EΘ = ΔM : ME. \] [ibid.]

But
\[ ΔM : ME = ΔM \cdot ME : ΞM \cdot ME, \]
by taking a common height ME.

Therefore \( ΔM \cdot ME : ΠM \cdot MP = ΔM \cdot ME : ΞM \cdot ME. \)

Therefore
\[ ΠM \cdot MP = ΞM \cdot ME. \] [Eucl. v. 9]
But
\[ ΠM \cdot MP = AM^2, \]
as was proved;
and therefore
\[ ΞM \cdot ME = AM^2. \]
ἀρα δύναται τὸ ΜΟ, δ' παράκειται παρὰ τὴν ΘΕ, πλάτος ἔχον τὴν ΕΜ, ἐλλείπον εἶδει τῷ ΟΝ ὁμοίω ὄντι τῷ ὑπὸ ΔΕΘ. καλείσθω δὲ ἢ μὲν τοιαύτη τομὴ ἐλλειψις, ἢ δὲ ΕΘ παρ' ἦν δύνανται αἱ κατ- αγόμεναι ἐπὶ τὴν ΔΕ τεταγμένως, ἢ δὲ αὐτῇ καὶ ὀρθία, πλαγίᾳ δὲ ἢ ΕΔ.

ιδ'

'Εαν αἱ κατὰ κορυφὴν ἐπιφάνειαι ἐπιπέδῳ τμῆ- θωσι μὴ διὰ τῆς κορυφῆς, ἔσται ἐν ἕκατέρα τῶν ἐπιφανειῶν τομῆ ἢ καλουμένη ὑπερβολή, καὶ τῶν δύο τομῶν ἢ τε διάμετρος ἢ αὐτῇ ἔσται, καὶ παρ' ἃς δύνανται αἱ ἐπὶ τὴν διάμετρον καταγόμεναι παράλληλαι τῇ ἐν τῇ βάσει τοῦ κώνου εὐθείᾳ ἢσι, καὶ τοῦ εἴδους ἢ πλαγία πλευρά κοινῇ ἢ μεταξὺ τῶν κορυφῶν τῶν τομῶν καλείσθωσαν δὲ αἱ τοιαύται τομαί ἀντικείμεναι.

"Εστῶσαν αἱ κατὰ κορυφῆν ἐπιφάνειαι, ὅν κορυφὴ τὸ Α σημεῖον, καὶ τετμήσθωσαν ἐπιπέδῳ μὴ διὰ τῆς κορυφῆς, καὶ ποιεῖτω ἐν τῇ ἐπιφανείᾳ τομᾶς ταῖς ΔΕΖ, ΗΘΚ. λέγω, ὅτι ἑκατέρα τῶν ΔΕΖ, ΗΘΚ τομῶν ἐστὶν ἢ καλουμένη ὑπερβολή.

* Let $p$ be the parameter of a conic section and $d$ the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of co-ordinates (in general oblique). Then Props. 11-13 are equivalent to the Cartesian equations.
Therefore the square on $\Delta M$ is equal to $MO$, which is applied to $OE$, having $EM$ for its breadth, and being deficient by the figure $ON$ similar to the rectangle $\Delta E \cdot EO$. Let such a section be called an eclipse, let $EO$ be called the parameter to the ordinates to $\Delta E$, and let this line be called the erect side (latus rectum), and $E\Delta$ the transverse side.\footnote{1}

Prop. 14

If the vertically opposite surfaces [of a double cone] be cut by a plane not through the vertex, there will be formed on each of the surfaces the section called a hyperbola, and the diameter of both sections will be the same, and the parameter to the ordinates drawn parallel to the straight line in the base of the cone will be equal, and the transverse side of the figure will be common, being the straight line between the vertices of the sections; and let such sections be called opposite.

Let there be vertically opposite surfaces having the point $A$ for vertex, and let them be cut by a plane not through the vertex, and let the sections so made on the surface be $\Delta EZ$, $H\Theta K$. I say that each of the sections $\Delta EZ$, $H\Theta K$ is the so-called hyperbola.

$$y^2 = px$$

(the parabola),

and

$$y^2 = px \pm \frac{p}{\alpha}x^2$$

(the hyperbola and ellipse respectively).

It is the essence of Apollonius’s treatment to express the fundamental properties of the conics as equations between areas, whereas Archimedes had given the fundamental properties of the central conics as proportions

$$y^2 : (a^2 \pm x^2) = a^2 : b^2.$$

This form is, however, equivalent to the Cartesian equations referred to axes through the centre.
"Εστώ γὰρ ὁ κύκλος, καθ' οὗ φέρεται ἡ τὴν ἐπιφάνειαν γράφουσα εὐθεία, ὁ ΒΔΓΖ, καὶ ἦχθω ἐν τῇ κατὰ κορυφὴν ἐπιφανεία παράλληλον αὐτῶ επίπεδον τὸ ΞΗΟΚ. κοινὰ δὲ τομαὶ τῶν ΗΘΚ, ΖΕΔ τομῶν καὶ τῶν κύκλων αἱ ΖΔ, ΗΚ. ἐσονται δὴ παράλληλοι. ἀξίων δὲ ἐστὶ τῆς κωνικῆς ἐπιφανείας ἡ ΛΑΥ εὐθεία, κέντρα δὲ τῶν κύκλων τὰ Λ, Υ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΖΔ κάθετος ἀχθεία ἐκβεβληθὼς ἐπὶ τὰ Β, Γ σημεία, καὶ διὰ τῆς ΒΓ καὶ τοῦ ἄξονος ἐπίπεδον ἐκβεβληθὼς συνῆσε δὴ τομάς ἐν μὲν τοῖς κύκλοις παράλληλους εὐθείας τὰς ΞΟ, ΒΓ, ἐν δὲ τῇ ἐπιφανείᾳ τὰς ΒΑΟ, ΓΑΞ.
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For let $B\Delta \Gamma Z$ be the circle round which revolves the straight line describing the surface, and in the vertically opposite surface let there be drawn parallel to it a plane $\Xi HOK$; the common sections of the sections $HOK$, $ZE\Delta$ and of the circles [Prop. 4] will be $Z\Delta$, $HK$; and they will be parallel [Eucl. xi. 16]. Let the axis of the conical surface be $\Lambda AY$, let the centres of the circles be $\Lambda$, $Y$, and from $\Lambda$ let a perpendicular be drawn to $Z\Delta$ and produced to the points $B$, $\Gamma$, and let the plane through $B\Gamma$ and the axis be produced; it will make in the circles the parallel straight lines $\Xi O$, $B\Gamma$, and on the surface $BAO$, $\Gamma A\Xi$;
ἐσται δὴ καὶ ἡ ἙΘ τῇ HK πρὸς ὅρθας, ἐπειδὴ καὶ ἡ ΒΓ τῇ ΖΔ ἐστὶ πρὸς ὅρθας, καὶ ἐστὶν ἑκατέρα παράλληλος. καὶ ἔπει τὸ διὰ τοῦ ἄξονος ἐπίπεδου ταῖς τομαῖς συμβάλλει κατὰ τὰ M, N σημεία ἔτος τῶν γραμμῶν, δὴλον, ὡς καὶ τὰς γραμμὰς τέμνει τὸ ἐπίπεδον. τεμνέτω κατὰ τὰ Θ, Ε τὰ ἄρα M, E, Θ, N σημεία ἐν τε τῷ διὰ τοῦ ἄξονος ἐστὶν ἐπίπεδω καὶ ἐν τῷ ἐπίπεδῳ, ἐν οἷς εἰσίν αἱ γραμμαί· εὐθεία ἁρᾷ ἐστὶν ἡ ΜΕΘΝ γραμμή· καὶ φανερῶν, ὅτι τὰ τῇ Ε, Θ, Α, Γ ἐπ’ εὐθείας ἐστι καὶ τὰ Β, Ε, Α, Θ· ἐν τε γὰρ τῇ κωνικῇ ἑπιφανείᾳ ἐστὶ καὶ ἐν τῷ διὰ τοῦ ἄξονος ἐπίπεδῳ. ἦχθωσαν δὴ ἀπὸ μὲν τῶν Θ, Ε τῇ ΘΕ πρὸς ὅρθας αἱ ΘΡ, ΕΠ, διὰ δὲ τοῦ Α τῇ ΜΕΘΝ παράλληλος ἤχθω ἡ ΣΑΤ, καὶ πεποίησαν, ὡς μὲν τὸ ἀπὸ τῆς ΑΣ πρὸς τὸ ύπὸ ΒΣΓ, οὕτως ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ τῆς ΑΤ πρὸς τὸ ύπὸ ΟΤΕ, οὕτως ἡ ΘΟ πρὸς ΘΡ· ἔπει οὖν κώνος, οὗ κυριφή μὲν τὸ Α σημεῖον, βάσις δὲ οἱ ΒΓ κύκλος, τέτμηται ἐπίπεδω διὰ τοῦ ἄξονος, καὶ πεποίηκε τομὴν τὸ ΑΒΓ τριγώνου, τέτμηται δὲ καὶ ἑτέρῳ ἐπίπεδῳ τέμνοντε τὴν βάσιν τοῦ κώνου κατ’ εὐθείαν τὴν ΔΜΖ πρὸς ὅρθας οὖσαν τῇ ΒΓ, καὶ πεποίηκε τομὴν ἐν τῇ ἑπιφανείᾳ τὴν ΔΕΖ, ἡ δὲ διάμετρος ἡ ME ἐκβαλλομένη συμπέπτουσα μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς κυριφῆς τοῦ κώνου, καὶ διὰ τοῦ Α σημείου τῇ διαμέτρῳ τῆς τομῆς τῇ EM παράλληλος ἢκται ἡ ΑΣ, καὶ ἀπὸ τοῦ E τῇ EM πρὸς ὅρθας ἢκται ἡ ΕΠ, καὶ ἐστὶν ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ύπὸ ΒΣΓ, οὕτως ἡ ΘΟ πρὸς ΕΠ, ἡ μὲν ΔΕΖ ἁρὰ τομὴ ύπερβολὴ ἐστὶν, ἡ δὲ ΕΠ παρ’ ἥν δύναται αἰ ἐπὶ τὴν EM καταγόμεναι τεταγμένως, πλαγία.
now \( OE \) will be perpendicular to HK, since \( BG \) is perpendicular to \( ZA \), and each is parallel [Eucl. xi. 10]. And since the plane through the axis meets the sections at the points \( M, N \) within the curves, it is clear that the plane cuts the curves. Let it cut them at the points \( \Theta, E \); then the points \( M, E, \Theta, N \) are both in the plane through the axis and in the plane containing the curves; therefore the line \( ME\Theta N \) is a straight line [Eucl. xi. 3]. And it is clear that \( E, \Theta, A, \Gamma \) are on a straight line, and also \( B, E, A, \Theta \); for they are both on the conical surface and in the plane through the axis. Now let \( \Theta P, EP \) be drawn from \( \Theta, E \) perpendicular to \( \Theta E \), and through \( A \) let \( \Sigma \Theta \) be drawn parallel to \( ME\Theta N \), and let

\[
\text{A} \Sigma^2 : B \Sigma \cdot \Sigma \Gamma = \Theta E : \Theta P
\]

and

\[
\text{A} \Sigma^2 : O \cdot T \Xi = E \Theta : \Theta P
\]

Then since the cone, whose vertex is the point \( A \) and whose base is the circle \( BG \), is cut by a plane through the axis, and the section so made is the triangle \( A\Theta \Gamma \), and it is cut by another plane cutting the base of the cone in the straight line \( \Delta MZ \) perpendicular to \( BG \), and the section so made on the surface is \( \Delta EZ \), and the diameter \( ME \) produced meets one side of the axial triangle beyond the vertex of the cone, and \( A\Sigma \) is drawn through the point \( A \) parallel to the diameter of the section \( EM \), and \( E\Pi \) is drawn from \( E \) perpendicular to \( EM \), and \( A\Sigma^2 : B \Sigma \cdot \Sigma \Gamma = E \Theta : E\Pi \), therefore the section \( \Delta EZ \) is a hyperbola, in which \( E\Pi \) is the parameter to the ordinates to \( EM \), and \( \Theta E \) is the
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dε τοῦ εἴδους πλευρὰ ἡ ΘΕ. ομοίως δὲ καὶ ἡ ΗΘΚ ύπερβολὴ ἐστὶν, ἦς διάμετρος μὲν ἡ ΘΝ, ἡ δὲ ΘΡ παρ’ ἦν δύνανται αἱ ἐπὶ τὴν ΘΝ καταγόμεναι τεταγμέναις, πλαγία δὲ τοῦ εἴδους πλευρὰ ἡ ΘΕ.

Λέγω, ὅτι ἵση ἐστὶν ἡ ΘΡ τῇ ΕΠ. επεὶ γὰρ παράλληλος ἐστὶν ἡ ΒΓ τῇ ΞΟ, ἐστὶν ὡς ἡ ΑΣ πρὸς ΣΓ, οὔτως ἡ ΑΤ πρὸς ΤΞ, καὶ ὡς ἡ ΑΣ πρὸς ΣΒ, οὔτως ἡ ΑΤ πρὸς ΤΟ. ἀλλ’ ὁ τῆς ΑΣ πρὸς ΣΓ λόγος μετὰ τοῦ τῆς ΑΣ πρὸς ΣΒ ὁ τοῦ ἀπὸ ΑΣ ἐστὶ πρὸς τὸ ὑπὸ ΒΣΓ, ὁ δὲ τῆς ΑΤ πρὸς ΤΞ μετὰ τοῦ τῆς ΑΤ πρὸς ΤΟ ὁ τοῦ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΕΣΤΟ. ἐστὶν ἥρα ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὔτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΣΤΟ. καὶ ἐστὶν ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΣΤΟ, ἡ ΘΕ πρὸς ΘΡ καὶ ὡς ἥρα ἡ ΘΕ πρὸς ΕΠ, ἡ ΕΘ πρὸς ΘΡ. ἵση ἥρα ἐστὶν ἡ ΕΠ τῇ ΘΡ.

(vi.) Transition to New Diameter

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

'Εάν ύπερβολής ἡ ἐλλείψεως ἡ κύκλου περιφερείας εὐθεία ἐπισφάλουσα συμπίπτῃ τῇ διάμετρῳ, καὶ διὰ τῆς ἁφῆς καὶ τοῦ κέντρου εὐθεία ἐκβληθῇ, ἀπὸ δὲ τῆς κορύφης ἀναχθεῖσα εὐθεία παρὰ τεταγμένως κατηγομένην συμπίπτῃ τῇ διὰ τῆς ἁφῆς καὶ

* Apollonius is the first person known to have recognized the opposite branches of a hyperbola as portions of the same 328
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transverse side of the figure [Prop. 12]. Similarly \(H\Theta K\) is a hyperbola, in which \(\Theta N\) is a diameter, \(\Theta P\) is the parameter to the ordinates to \(\Theta N\), and \(\Theta E\) is the transverse side of the figure.

I say that \(\Theta P = E\Pi\). For since \(B\Gamma\) is parallel to \(\Xi O\),

\[
\frac{A\Sigma}{\Sigma \Gamma} = \frac{AT}{TE},
\]

and

\[
\frac{A\Sigma}{\Sigma B} = \frac{AT}{TO}.
\]

But

\[
\left(\frac{A\Sigma}{\Sigma \Gamma}\right) \left(\frac{A\Sigma}{\Sigma B}\right) = \frac{A\Sigma^2}{B\Sigma \cdot \Sigma \Gamma},
\]

and

\[
\left(\frac{AT}{TE}\right) \left(\frac{AT}{TO}\right) = \frac{AT^2}{\Xi T \cdot TO}.
\]

Therefore

\[
\frac{A\Sigma^2}{B\Sigma \cdot \Sigma \Gamma} = \frac{AT^2}{\Xi T \cdot TO}.
\]

But

\[
\frac{A\Sigma^2}{B\Sigma \cdot \Sigma \Gamma} = \frac{\Theta E}{\Theta P},
\]

while

\[
\frac{AT^2}{\Xi T \cdot TO} = \frac{\Theta E}{\Theta P};
\]

therefore

\[
\frac{\Theta E}{\Theta P} = \frac{E\Pi}{\Xi E} = \frac{E\Theta}{\Theta P}.
\]

Therefore

\[
E\Pi = \Theta P.\]

[Eucl. v. 9]

(vi.) Transition to New Diameter

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

Prop. 50

In a hyperbola, ellipse or circumference of a circle let a straight line be drawn to touch [the curve] and meet the diameter, and let the straight line through the point of contact and the centre be produced, and from the vertex let a straight line be drawn parallel to a straight line drawn ordinate-wise so as to meet the straight line drawn curve. It is his practice, however, where possible to discuss the single-branch hyperbola (or the hyperbola simpliciter as he would call it) together with the ellipse and circle, and to deal with the opposite branches separately. But occasionally, as in i. 30, the double-branch hyperbola and the ellipse are included in one enunciation.
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tού κέντρου ἡμένη εὐθεία, καὶ ποιηθῇ, ωσ τὸ τμῆμα τῆς ἐφαπτομένης τὸ μεταξὺ τῆς ἄφης καὶ τῆς ἀνγγεμένης πρὸς τὸ τμῆμα τῆς ἡμένης διὰ τῆς ἄφης καὶ τοῦ κέντρου τὸ μεταξὺ τῆς ἄφης καὶ τῆς ἀνγγεμένης, εὐθεία τις πρὸς τὴν διπλασίαν τῆς ἐφαπτομένης, ἦτις ἀν ἀπὸ τῆς τομῆς ἀχθῇ ἐπὶ τὴν διὰ τῆς ἄφης καὶ τοῦ κέντρου ἡμένην εὐθείαν παράλληλος τῇ ἐφαπτομένῃ, δυνήσεται τι χωρίον ὀρθογώνιον παρακείμενον παρὰ τὴν πορισθείσαν, πλάτος ἐξοῦ τὴν ἀπολαμβανομένην ὑπ’ αὐτῆς πρὸς τῇ ἄφη, ἐπὶ μὲν τῆς ὑπερβολῆς ὑπερβάλλον εἰδεί ὁμοίω τῷ περιεχομένῳ ὑπὸ τῆς διπλασίας τῆς μεταξὺ τοῦ κέντρου καὶ τῆς ἄφης καὶ τῆς πορισθείσης εὐθείας, ἐπὶ δὲ τῆς ἐλλειψεως καὶ τοῦ κύκλου ἐλλείπον.

"Εστω ὑπερβολή ἡ ἐλλειψις ἡ κύκλου περιφέρεια, ἡς διάμετρος ἡ ΑΒ, κέντρον δὲ τὸ Γ, ἐφαπτομένη δὲ ἡ ΔΕ, καὶ ἐπίζευχθείσα ἡ ΓΕ ἐκβεβληθὼς ἐφ’ ἐκάτερα, καὶ κείσθω τῇ ΕΓ ἵση ἡ ΓΚ, καὶ διὰ τοῦ Β τεταγμένου ἀνήχθου ἡ ΒΖΗ, διὰ δὲ τοῦ Ε τῇ ΕΓ πρὸς ὀρθάς ἡχθω ἡ ΕΘ, καὶ γνέσθω, ὡς ἡ ΖΕ πρὸς ΕΗ, οὕτως ἡ ΕΘ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐπίζευχθείσα ἡ ΘΚ ἐκβεβλήθω, καὶ εἰλήφθω τι ἐπὶ τῆς τομῆς σημείου τὸ Δ, καὶ δ’ αὐτοῦ τῇ ΕΔ παράλληλος ἡχθω ἡ ΛΜΞ, τῇ δὲ

* To save space, the figure is here given for the hyperbola only; in the ms. there are figures for the ellipse and circle as well.

The general enunciation is not easy to follow, but the particular enunciation will make it easier to understand. The
through the point of contact and the centre, and let the segment of the tangent between the point of contact and the line drawn ordinate-wise bear to the segment of the line drawn through the point of contact and the centre between the point of contact and the line drawn ordinate-wise the same ratio as a certain straight line bears to double the tangent; then if any straight line be drawn from the section parallel to the tangent so as to meet the straight line drawn through the point of contact and the centre, its square will be equal to a certain rectilineal area applied to the postulated straight line, having for its breadth the intercept between it and the point of contact, in the case of the hyperbola exceeding by a figure similar to the rectangle bounded by double the straight line between the centre and the point of contact and the postulated straight line, in the case of the ellipse and circle falling short.\(^a\)

In a hyperbola, ellipse or circumference of a circle, with diameter \(AB\) and centre \(\Gamma\), let \(\Delta E\) be a tangent, and let \(\Gamma E\) be joined and produced in either direction, and let \(\Gamma K\) be placed equal to \(E\Gamma\), and through \(B\) let \(BZH\) be drawn ordinate-wise, and through \(E\) let \(E\Theta\) be drawn perpendicular to \(E\Gamma\), and let \(ZE : EH = E\Theta : 2E\Delta\), and let \(\Theta K\) be joined and produced; and let any point \(\Lambda\) be taken on the section, and through it let \(\Lambda M\Xi\) be drawn parallel to \(E\Delta\) and purpose of this important proposition is to show that, if any other diameter be taken, the ordinate-property of the conic with reference to this diameter has the same form as the ordinate-property with reference to the original diameter. The theorem amounts to a transformation of co-ordinates from the original diameter and the tangent at its extremity to any diameter and the tangent at its extremity. In succeeding propositions, showing how to construct conics from certain data, Apollonius introduces the axes for the first time as special cases of diameters.
ΕΓ πρὸς ΚΓ, ή ΕΣ πρὸς ΣΘ, ἢ σῆ ἄρα καὶ ή ΕΣ
tῆ ΣΘ. καὶ ἐπεῖ ἐστὶν, ώσ ἡ ΖΕ πρὸς ΕΗ, ή
ΘΕ πρὸς τῆν διπλασίαν τῆς ΕΔ, καὶ ἐστὶ τῆς
ἡμίσεια ἡ ΕΣ, ἐστὶν ἄρα, ώσ ἡ ΖΕ πρὸς ΕΗ, ή
ΣΕ πρὸς ΕΔ. ώσ δὲ ἡ ΖΕ πρὸς ΕΗ, ή ΛΜ πρὸς
ΜΡ. ώσ ἄρα ή ΛΜ πρὸς ΜΡ, ή ΣΕ πρὸς ΕΔ.
καὶ ἐπεῖ τὸ ΡΝΓ τρίγωνον τοῦ ἩΒΓ τριγώνου,
tουτεστὶ τοῦ ΓΔΕ, ἐπὶ μὲν τῆς ὑπερβολῆς µείζον
ἐδείχθη, ἐπὶ δὲ τῆς ἐλλειψεως καὶ τοῦ κύκλου
ἐλασσον τῶ ΛΝΞ, κοινῶν ἀφαιρεθέντων ἐπὶ µὲν
τῆς ὑπερβολῆς τοῦ τε ΕΓΔ τριγώνου καὶ τοῦ
ΝΡΜΞ τετραπλεύρου, ἐπὶ δὲ τῆς ἐλλειψεως καὶ
tοῦ κύκλου τοῦ ΜΞΓ τριγώνου, τὸ ΛΜΡ τρίγωνον
tῶ ΜΕΔΞ τετραπλεύρῳ ἐστὶν ἴσον. καὶ ἐστὶ
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ΔPN parallel to BH, and let ΜΠ be drawn parallel to EΘ. I say that ΔM² = EM · ΜΠ.

For through Γ let ΓΣΘ be drawn parallel to ΚΠ. Then since

$EG = GI$ $K$ and $EG : GI = ES : ΣΘ, \quad [Euc. \, vi. \, 2$

therefore $ES = ΣΘ.$

And since $ZE : EH = ΘE : 2ΣΔ,$

and $ES = 1/2 EΘ,$

therefore $ZE : EH = ΣE : EΔ.$

But $ZE : EH = ΔM : MP; \quad [Euc. \, vi. \, 4$

therefore $ΔM : MP = ΣE : EΔ.$

And since it has been proved [Prop. 43] that in the hyperbola

triangle PΝΓ = triangle HΒΓ + triangle ΔΝΞ,

i.e., triangle PΝΓ = triangle ΓΔΕ + triangle ΔΝΞ,\(^a\)

while in the ellipse and the circle

triangle PΝΓ = triangle HΒΓ −

triangle ΔΝΞ,

i.e., triangle PΝΓ + triangle ΔΝΞ = triangle ΓΔΕ,\(^b\)

therefore by taking away the common elements—in the hyperbola the triangle ΕΓΔ and the quadrilateral ΝΠΜΕΞ, in the ellipse and the circle the triangle ΜΞΓ,'

triangle ΔΜΡ = quadrilateral ΜΕΔΞ.

\(^a\) For this step v. Eutocius's comment on Prop. 43.

\(^b\) See Eutocius.
παράλληλος ἡ ΜΕ τῇ ΔΕ, ἡ δὲ ὑπὸ ΛΜΡ τῇ ὑπὸ ΕΜΕ ἐστὶν ἴση· ἴσον ἀρα ἐστὶ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΕΜ καὶ συναμφότερον τῆς ΕΔ, ΜΕ. καὶ ἐπεὶ ἐστὶν, ὡς ἡ ΜΓ πρὸς ΓΕ, ἤ τε ΜΕ πρὸς ΕΔ καὶ ἡ ΜΟ πρὸς ΕΣ, ὡς ἀρα ἡ ΜΟ πρὸς ΕΣ, ἡ ΜΕ πρὸς ΔΕ. καὶ συνθέντι, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς ΕΣ, οὕτως συναμφότερος ἡ ΜΕ, ΕΔ πρὸς ΕΔ· ἐναλλάξ, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς συναμφότερον τὴν ΞΜ, ΕΔ ἢ ΣΕ πρὸς ΕΔ. ἀλλ' ὡς μὲν συναμφότερος ἡ ΜΟ, ΕΣ πρὸς συναμφότερον τὴν ΜΕ, ΔΕ, τὸ ὑπὸ συναμφότερον τῆς ΜΟ, ΕΣ καὶ τῆς ΕΜ πρὸς τὸ ὑπὸ συναμφότερον τῆς ΜΕ, ΕΔ καὶ τῆς ΕΜ, ὡς δὲ ἡ ΣΕ πρὸς ΕΔ, ἡ ΖΕ πρὸς ΕΗ, τούτεστιν ἡ ΛΜ πρὸς ΜΡ, τούτεστι τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ· ὡς ἀρα τὸ ὑπὸ συναμφότερον τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ συναμφότερον τῆς ΜΕ, ΕΔ καὶ τῆς ΕΜ, τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ. καὶ ἐναλλάξ, ὡς τὸ ὑπὸ συναμφότερον τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ἀπὸ ΜΑ, οὕτως τὸ ὑπὸ συναμφότερον τῆς ΜΕ, ΕΔ καὶ τῆς ΜΕ πρὸς τὸ ὑπὸ ΛΜΡ· ἴσον δὲ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΜΕ καὶ συναμφότερον τῆς ΜΕ, ΕΔ· ἴσον ἀρα καὶ τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜ καὶ συναμφότερον τῆς ΜΟ, ΕΣ. καὶ ἐστὶν ἡ μὲν ΣΕ τῇ ΣΘ ἴση, ἡ δὲ ΣΘ τῇ ΟΠ· ἴσον ἀρα τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜΠ.

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But $ME$ is parallel to $DE$ and angle $AMP = \text{angle } EM\Xi$ (Eucl. i. 15);

therefore

$$\lambda M \cdot MP = EM \cdot (ED + ME).$$

And since

$$MG : GE = ME : ED,$$

and

$$MG : GE = MO : ES,$$

therefore

$$MO : ES = ME : DE.$$

*Componendo,*

$$MO + SE : ES = ME + ED : ED;$$

and *permutando*

$$MO + SE : EM + ED = SE : ED.$$

But

$$MO + SE : EM + ED = (MO + ES) \cdot EM : (ME + ED) \cdot EM,$$

and

$$SE : ED = ZE : EH = \lambda M : MP$$

$$= \lambda M^2 : \lambda M \cdot MP;$$

therefore

$$(MO + ES) \cdot ME : (ME + ED) \cdot EM = \lambda M^2 : \lambda M \cdot MP.$$

*And permutando*

$$(MO + ES) \cdot ME : MA^2 = (ME + ED) \cdot ME : \lambda M \cdot MP.$$

But

$$\lambda M \cdot MP = ME \cdot (ME + ED);$$

therefore

$$\lambda M^2 = EM \cdot (MO + ES).$$

*And $SE = \Sigma \Theta,$ while $\Sigma \Theta = O\Pi$ [Eucl. i. 34];*

therefore

$$\lambda M^2 = EM \cdot M\Pi.$$
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(b) Other Works

(i.) General

Papp. Coll. vii. 3, ed. Hultsch 636.18-23

Τῶν δὲ προειρημένων τοῦ 'Αναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη: Εὐκλείδου Δεδομένων βιβλίων ἀ', 'Απολλωνίου Λόγου ἀποτομῆς β', Ἑωρίου ἀποτομῆς β', Διωμόμενης τομῆς δύο, 'Ἐπαφών δύο, Εὐκλείδου Πορισμάτων τρία, 'Απολλωνίου Νεώσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο, Κωνικῶν ἦ.

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

Τῆς δ' Ἀποτομῆς τοῦ λόγου βιβλίων ὃντων β' πρότασις ἐστὶν μία ὑποδιηρμένη, διὸ καὶ μίαν πρότασιν οὕτως γράφω: διὰ τοῦ δοθέντος σημείου εὐθείαν γραμμήν ἀγαγεῖν τέμνονσαν ἀπὸ τῶν τῆς θέσει δοθεισῶν δύο εὐθείων πρὸς τοὺς ἐπ' αὐτῶν δοθείσι σημείοις λόγου ἐγκυύσας τῶν αὐτῶν τῷ δοθέντι. τὰς δὲ γραφὰς διαφόρους γενέσθαι καὶ πλῆθος λαβεῖν συμβεβήκεν ὑποδιαιρέσεως γενομένης ἑνὲκα τῆς τε πρὸς ἄλληλας θέσεως τῶν διομένων εὐθείων καὶ τῶν διαφόρων πτώσεων τοῦ διομένου σημείου καὶ διὰ τὰς ἀναλύσεις καὶ συνθέσεις αὐτῶν τε καὶ τῶν διορισμῶν. ἔχει γάρ τὸ μὲν πρῶτον βιβλίον τῶν Λόγου ἀποτομῆς

* Unhappily the only work by Apollonius which has survived, in addition to the Conics, is On the Cutting-off of a
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(b) Other Works

(i.) General

Pappus, Collection vii. 3, ed. Hultsch 636. 18-23

The order of the aforesaid books in the Treasury of Analysis is as follows: the one book of Euclid’s Data, the two books of Apollonius’s On the Cutting-off of a Ratio, his two books On the Cutting-off of an Area, his two books On Determinate Section, his two books On Tangencies, the three books of Euclid’s Porisms, the two books of Apollonius’s On Vergings, the two books of the same writer On Plane Loci, his eight books of Conics.¹

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

In the two books On the Cutting-off of a Ratio there is one enunciation which is subdivided, for which reason I state one enunciation thus: Through a given point to draw a straight line cutting off from two straight lines given in position intercepts, measured from two given points on them, which shall have a given ratio. When the subdivision is made, this leads to many different figures according to the position of the given straight lines in relation one to another and according to the different cases of the given point, and owing to the analysis and the synthesis both of these cases and of the propositions determining the limits of possibility. The first book of those On the Cutting-off of a Ratio, and that only in Arabic. Halley published a Latin translation in 1706. But the contents of the other works are indicated fairly closely by Pappus’s references.
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tópous ἧς, πτώσεις καὶ διορισμοῦς δὲ ἐς, ὅν τρεῖς μὲν εἰσὶν μέγιστοι, δύο δὲ ἐλάχιστοι. . . . τὸ δὲ δεύτερον βιβλίον Λόγου ἀποτομὴς ἔχει τόπους ὧς, πτώσεις δὲ ἤγγο, διορισμοῦς δὲ τοὺς ἐκ τοῦ πρῶτου ἀπάγεται γὰρ ὅλον εἰς τὸ πρῶτον.

(iii.) On the Cutting-off of an Area
Ibid. vii. 7, ed. Hultsch 640. 26–642. 5

Τῆς δ’ Ἀποτομῆς τοῦ χωρίου βιβλία μὲν ἐστὶν δύο, πρόβλημα δὲ καὶ τούτος ἐν ὑποδιαιρούμενον δία, καὶ τούτων μία πρότασις ἐστὶν τὰ μὲν ἄλλα ὁμοίως ἔχουσα τῇ προτέρᾳ, μόνη δὲ τούτῳ διαφέρουσα τῷ δεῖν τὰς ἀποτεμνομένας δύο εὐθείας ἐν ἐκείνῃ μὲν λόγον ἔχουσας δοθέντα ποιεῖν, ἐν δὲ ταύτῃ χωρίου περιεχοῦσας δοθέν.

(iv.) On Determinate Section
Ibid. vii. 9, ed. Hultsch 642. 19–644. 16

'Εξής τούτων ἀναδεδονται τῆς Διωρισμένης τομῆς βιβλία β', ὅν ὁμοίως τοῖς πρότερον μίαν πρότασιν πάρεστιν λέγειν, διεζευγμένην δὲ ταύτην.

*The Arabic text shows that Apollonius first discussed the cases in which the lines are parallel, then the cases in which the lines intersect but one of the given points is at the point of intersection; in the second book he proceeds to the general case, but shows that it can be reduced to the case where one
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Ratio contains seven loci, twenty-four cases and five determinations of the limits of possibility, of which three are maxima and two are minima. . . . The second book On the Cutting-off of a Ratio contains fourteen loci, sixty-three cases and the same determinations of the limits of possibility as the first; for they are all reduced to those in the first book.ª

(iii.) On the Cutting-off of an Area

Ibid. vii. 7, ed. Hultsch 640. 26–642. 5

In the work On the Cutting-off of an Area there are two books, but in them there is only one problem, twice subdivided, and the one enunciation is similar in other respects to the preceding, differing only in this, that in the former work the intercepts on the two given lines were required to have a given ratio, in this to comprehend a given area.ª

(iv.) On Determinate Section

Ibid. vii. 9, ed. Hultsch 642. 19–644. 16

Next in order after these are published the two books On Determinate Section, of which, as in the previous cases, it is possible to state one comprehen-
of the given points is at the intersection of the two lines. By this means the problem is reduced to the application of a rectangle. In all cases Apollonius works by analysis and synthesis.ª

ª Halley attempted to restore this work in his edition of the De sections rationis. As in that treatise, the general case can be reduced to the case where one of the given points is at the intersection of the two lines, and the problem is reduced to the application of a certain rectangle.
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τὴν δοθείσαν ἀπειρον εὐθείαν ἕνι σημεῖῳ τεμεῖν, ὡστε τῶν ἀπολαμβανομένων εὐθειῶν πρὸς τοῖς ἐπ’ αὐτῆς δοθείσι σημείωσι ήτοι τὸ ἀπὸ μίας τετράγωνον η τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθογώνιον δοθέντα λόγον ἔχειν ἦτοι πρὸς τὸ ἀπὸ μίας τετράγωνον η πρὸς τὸ ὑπὸ μίας ἀπολαμβανομένης καὶ τῆς ἔξω δοθεῖσι η πρὸς τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθο-

(v.) On Tangencies

Ibid. vii. 11, ed. Hultsch 644. 23-646. 19

Ἐξῆς δὲ τούτων τῶν Ἐπαφῶν ἔστιν βιβλία δύο. προτάσεις δὲ ἐν αὐτοῖς δοκούσιν εἶναι πλεῖόνες, ἀλλὰ καὶ τούτων μίαν τιθεμέν οὕτως ἔχουσαν ἐξῆς—

σημείων καὶ εὐθειῶν καὶ κύκλων τριῶν ὀποιονδήποτε ἔστει δοθέντων κύκλων ἁγαγεῖν δὲ ἐκάστου τῶν
doθέντων σημείων, εἰ δοθείη, ἡ ἐφαπτόμενον
ekásthēs tōn dotheiōn γραμμῶν. ταύτης διὰ

As the Greeks never grasped the conception of one point being two coincident points, it was not possible to enunciate this problem so concisely as we can do: Given four points A, B, C, D on a straight line, of which A may coincide with C and B with D, to find another point P on the same straight line such that AP : CP : BP : DP has a given value. If AP : CP = λ. BP : DP, where A, B, C, D, λ are given, the determination of P is equivalent to the solution of a quadratic equation, which the Greeks could achieve by means of the
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sive enunciation thus: To cut a given infinite straight line in a point so that the intercepts between this point and given points on the line shall furnish a given ratio, the ratio being that of the square on one intercept, or the rectangle contained by two, towards the square on the remaining intercept, or the rectangle contained by the remaining intercept and a given independent straight line, or the rectangle contained by two remaining intercepts, whichever way the given points [are situated]. . . . The first book contains six problems, sixteen subdivisions and five limits of possibility, of which four are maxima and one is a minimum. . . . The second book On Determinate Section contains three problems, nine subdivisions, and three limits of possibility.\(^a\)

(v.) On Tangencies


Next in order are the two books On Tangencies. Their enunciations are more numerous, but we may bring these also under one enunciation thus stated: Given three entities, of which any one may be a point or a straight line or a circle, to draw a circle which shall pass through each of the given points, so far as it is points which are given, or to touch each of the given lines.\(^b\) In application of areas. But the fact that limits of possibility, and maxima and minima were discussed leads Heath (*H.G.M.* ii. 180-181) to conjecture that Apollonius investigated the series of point-pairs determined by the equation for different values of \(\lambda\), and that “the treatise contained what amounts to a complete Theory of Involution.” The importance of the work is shown by the large number of lemmas which Pappus collected.

\(^a\) The word “lines” here covers both the straight lines and the circles.
πλήθη τῶν ἐν ταῖς υποθέσεις δεδομένων ὄμοιων ἢ ἀνομοίων κατὰ μέρος διαφόρους προτάσεις ἀναγκαίων γίνονται δέκα: ἐκ τῶν τριῶν γάρ ἀνομοίων γενῶν τριάδες διάφοροι ἀτακτοὶ γίνονται ἦ. ἦτοι γὰρ τὰ διδόμενα τρία σημεῖα ἢ τρεῖς εὐθείαι ἢ δύο σημεῖα καὶ εὐθεία ἢ δύο εὐθείαι καὶ σημεῖον ἢ δύο σημεῖα καὶ κύκλος ἢ δύο κύκλοι καὶ σημεῖον ἢ δύο εὐθείαι καὶ κύκλος ἢ δύο κύκλοι καὶ εὐθεία ἢ σημεῖον καὶ εὐθεία καὶ κύκλος ἢ τρεῖς κύκλοι. τούτων δύο μὲν τὰ πρῶτα δέδεκται ἐν τῷ δ’ βιβλίῳ τῶν πρώτων Στοιχείων, διὸ παρίει μὴ γράφων· τὸ μὲν γὰρ τριῶν δοθέντων σημείων μὴ ἐπ’ εὐθείας ὀντων τὸ αὐτὸ ἐστὶν τῷ περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι, τὸ δὲ γ’ δοθεισῶν εὐθείων μὴ παραλλήλων οὐσῶν, ἀλλὰ τῶν τριῶν συμπληροῦσαν, τὸ αὐτὸ ἐστὶν τῷ εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι· τὸ δὲ δύο παραλλήλων οὐσῶν καὶ μιᾶς ἐμπιπτούσης ὡς μέρος οὖν τῆς β’ υποδιαιρέσεως προγράφεται ἐν τούτωι πάντων. καὶ τὰ ἔξης ἡ ἐν τῷ πρώτῳ βιβλίῳ τὰ δὲ λειτούρμενα δύο, τὸ δύο δοθεισῶν εὐθείων καὶ κύκλου ἢ τριῶν δοθέντων κύκλων μόνον ἐν τῷ δευτέρῳ βιβλίῳ διὰ τὰς πρὸς ἀλλήλους θέσεις τῶν κύκλων τε καὶ εὐθειῶν πλειονάς οὐσῶς καὶ πλειονῶν διορισμῶν δεομένας.

* Eucl. iv. 5 and 4.
* The last problem, to describe a circle touching three
this problem, according to the number of like or unlike entities in the hypotheses, there are bound to be, when the problem is subdivided, ten enunciations. For the number of different ways in which three entities can be taken out of the three unlike sets is ten. For the given entities must be (1) three points or (2) three straight lines or (3) two points and a straight line or (4) two straight lines and a point or (5) two points and a circle or (6) two circles and a point or (7) two straight lines and a circle or (8) two circles and a straight line or (9) a point and a straight line and a circle or (10) three circles. Of these, the first two cases are proved in the fourth book of the first Elements,¹ for which reason they will not be described; for to describe a circle through three points, not being in a straight line, is the same thing as to circumscribe a given triangle, and to describe a circle to touch three given straight lines, not being parallel but meeting each other, is the same thing as to inscribe a circle in a given triangle; the case where two of the three lines are parallel and one meets them is a subdivision of the second problem but is here given first place. The next six problems in order are investigated in the first book, while the remaining two, the case of two given straight lines and a circle and the case of three circles, are the sole subjects of the second book on account of the manifold positions of the circles and straight lines with respect one to another and the need for numerous investigations of the limits of possibility.²
given circles, has been investigated by many famous geometers, including Newton (Arithmetica Universalis, Prob. 47). The lemmas given by Pappus enable Heath (H.G.M. ii. 182-185) to restore Apollonius’s solution—a “plane” solution depending only on the straight line and circle.
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(vi.) On Plane Loci

Ibid. vii. 23, ed. Hultsch 662. 19–664. 7

Οἱ μὲν οὖν ἀρχαῖοι εἰς τὴν τῶν ἐπιπέδων τούτων τόπων τάξειν ἀποβλέποντες ἐστοιχείωσαν ἃς ἀμελήσαντες οἱ μετά αὐτοὺς προσέθηκαν ἐτέρους, ὡς οὐκ ἀπειρων τὸ πλῆθος ὅντων, εἴ θέλοι τις προσγράφειν ὑπὸ τῆς τάξεως ἐκείνης ἐχόμενα. θὴσον οὖν τὰ μὲν προσκείμενα υστερα, τὰ δ' ἐκ τῆς τάξεως πρότερα μιᾷ περιλαβῶν προτάσσει ταῦτη.

Εὰν δύο εὐθείαι ἀχθῶσιν ήτοι ἀπὸ ἕνος δεδομένου σημείου ἡ ἀπὸ δύο καὶ ήτοι ἐπ' εὐθείας ἡ παράλληλοι ἡ δεδομένην περιέχουσα γωνίαν καὶ ήτοι λόγον ἔχουσαι πρὸς ἀλλήλας ἡ χωρίον περιέχουσαι δεδομένον, ἀπτηται δὲ τὸ τῆς μιᾶς πέρας ἐπιπέδου τόπον θέσει δεδομένου, ἀφεται καὶ τὸ τῆς ἑτέρας πέρας ἐπιπέδου τόπον θέσει δεδομένου ὅτε μὲν τοῦ ὀμογενοῦς, ὅτε δὲ τοῦ ἑτέρου, καὶ ὅτε μὲν ὀμοίως κειμένου πρὸς τὴν εὐθείαν, ὅτε δὲ ἑναντίως. ταῦτα δὲ γίνεται παρὰ τὰς διαφόρας τῶν ὑποκειμένων.

(vii.) On Vergings

Ibid. vii. 27-28, ed. Hultsch 670. 4-672. 3

Νεύειν λέγεται γραμμὴ ἐπὶ σημείου, εἰν ἐπεκβαλλομένη ἐπ' αὐτὸ παραγώνηται [ . . . ]

1 τούτων is attributed by Hultsch to dittography.

* These words follow the passage (quoted supra, pp. 262-265) wherein Pappus divides loci into ἐφεκτικοῖ, διεξοδικοῖ and ἀναστροφικοῖ.

* It is not clear what straight line is meant—probably the most obvious straight line in each figure.

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(vi.) On Plane Loci

Ibid. vii. 23, ed. Hultsch 662. 19-664. 7

The ancients had regard to the arrangement of these plane loci with a view to instruction in the elements; heedless of this consideration, their successors have added others, as though the number could not be infinitely increased if one were to make additions from outside that arrangement. Accordingly I shall set out the additions later, giving first those in the arrangement, and including them in this single enunciation:

If two straight lines be drawn, from one given point or from two, which are in a straight line or parallel or include a given angle, and either bear a given ratio one towards the other or contain a given rectangle, then, if the locus of the extremity of one of the lines be a plane locus given in position, the locus of the extremity of the other will also be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of a different kind, and will sometimes be similarly situated with respect to the straight line, sometimes contrariwise. These different cases arise according to the differences in the suppositions.

(vii.) On Vergings

Ibid. vii. 27-28, ed. Hultsch 670. 4-672. 3

A line is said to verge to a point if, when produced, it passes through the point. [ . . . ] The general

* Pappus proceeds to give seven other enunciations from the first book and eight from the second book. These have enabled reconstructions of the work to be made by Fermat, van Schooten and Robert Simson.

* Examples of vergings have already been encountered several times; v. pp. 186-189 and vol. i. p. 244 n. a.
προβλήματος δὲ ὄντος καθολικοῦ τούτου· δύο
dοθεισῶν γραμμῶν θέσει θείναι μεταξὺ τούτων
eυθείαν τῷ μεγέθει δεδομένην νεύονσαν ἐπὶ δοθέν
σημεῖον, ἐπὶ τούτου τῶν ἐπὶ μέρους διάφορα τὰ
ὑποκείμενα ἐξόντων, ἀ μὲν ἢ ἐπίπεδα, ἀ δὲ
στερεά, ἀ δὲ γραμμικά, τῶν δ' ἐπιπέδων ἀποκλη-
ρώσαντες τὰ πρὸς πολλὰ χρησιμότερα ἔδειξαν τὰ
προβλήματα ταῦτα.
Θέσει δεδομένων ἡμικυκλίου τε καὶ εὐθείας πρὸς
ὅρθας τῇ βάσει ἡ δύο ἡμικυκλίων ἐπὶ εὐθείας
ἐχόντων τὰς βάσεις θείναι δοθέσαν τῷ μεγέθει
eυθείαν μεταξὺ τῶν δύο γραμμῶν νεύονσαν ἐπὶ
gωνίαν ἡμικυκλίου.
Καὶ ρόμβου δοθέντος καὶ ἐπεκβεβλημένης μιᾶς
πλευράς ἀρμόσαι ὑπὸ τὴν ἐκτὸς γωνίαν δεδομένην
tῷ μεγέθει εὐθείαν νεύονσαν ἐπὶ τὴν ἀντικρος
γωνίαν.
Καὶ θέσει δοθέντος κύκλου ἐναρμόσαι εὐθείαν
μεγέθει δεδομένην νεύονσαν ἐπὶ δοθέν.
Τοῦτων δὲ ἐν μὲν τῷ πρῶτῳ τεύχει δέδεκται τὸ
ἐπὶ τοῦ ἑνὸς ἡμικυκλίου καὶ εὐθείας ἔχον πτώσεις
δ καὶ τὸ ἐπὶ τοῦ κύκλου ἔχον πτώσεις δύο καὶ τὸ
ἐπὶ τοῦ ρόμβου πτώσεις ἔχον β, ἐν δὲ τῷ δευτέρῳ
τεύχει τὸ ἐπὶ τῶν δύω ἡμικυκλίων τῆς ὑποθέσεως
πτώσεις ἐχούσης ἰ, ἐν δὲ ταῦται ὑποδιαρέσεις
πλείονες διοριστικαὶ ἕνεκα τοῦ δεδομένου μεγέθους
tῆς εὐθείας.
problem is: *Two straight lines being given in position, to place between them a straight line of given length so as to verge to a given point.* When it is subdivided the subordinate problems are, according to differences in the suppositions, sometimes plane, sometimes solid, sometimes linear. Among the plane problems, a selection was made of those more generally useful, and these problems have been proved:

*Given a semicircle and a straight line perpendicular to the base, or two semicircles with their bases in a straight line, to place a straight line of given length between the two lines and verging to an angle of the semicircle [or of one of the semicircles];*

*Given a rhombus with one side produced, to insert a straight line of given length in the external angle so that it verges to the opposite angle;*

*Given a circle, to insert a chord of given length verging to a given point.*

Of these, there are proved in the first book four cases of the problem of one semicircle and a straight line, two cases of the circle, and two cases of the rhombus; in the second book there are proved ten cases of the problem in which two semicircles are assumed, and in these there are numerous subdivisions concerned with limits of possibility according to the given length of the straight line.a

a A restoration of Apollonius's work *On Vergings* has been attempted by several writers, most completely by Samuel Horsley (Oxford, 1770). A lemma by Pappus enables Apollonius's construction in the case of the rhombus to be restored with certainty; v. Heath, *H.G.M.* ii. 190-192.
(viii.) On the Dodecahedron and the Icosahedron

Hypsicl. [Eucl. Elem. xiv.], Eucl. ed. Heiberg
v. 6. 19-8. 5

"Ό αὐτὸς κύκλος περιλαμβάνει τὸ τε τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἴκοσαέδρου τρίγωνον τῶν εἰς τὴν αὐτὴν σφαῖραν ἐγγραφομένων. τούτῳ δὲ γράφεται ὑπὸ μὲν 'Αρισταίου ἐν τῷ ἐπιγραφομένῳ Τῶν ἐ σχημάτων συγκρίσει, ὑπὸ δὲ Ἀπολλωνίου ἐν τῇ δευτέρᾳ ἐκδόσει τῆς Συγκρίσεως τοῦ δωδεκαέδρου πρὸς τὸ εἴκοσαέδρον, διὸ ἐστὶν, ὡς ἡ τοῦ δωδεκαέδρου ἑπιφάνεια πρὸς τὴν τοῦ εἴκοσαέδρου ἑπιφάνειαν, οὕτως καὶ αὐτὸ τὸ δωδεκάεδρον πρὸς τὸ εἴκοσαέδρον διὰ τὸ τὴν αὐτὴν εἶναι κάθετον ἀπὸ τοῦ κέντρου τῆς σφαίρας ἐπὶ τὸ τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἴκοσαέδρου τρίγωνον.

(ix.) Principles of Mathematics

Marin. in Eucl. Dat., Eucl. ed. Heiberg vi. 234. 13-17

Διὸ τῶν ἀπλούστερον¹ καὶ μιὰ τινὶ διαφορὰ περιγράφειν τὸ δεδομένον προθεμένων οἱ μὲν τεταγμένων, ὡς 'Απολλώνιος ἐν τῇ Περὶ νεύσεων καὶ

¹ ἀπλούστερον Heiberg, ἀπλούστερων cod.
APOLLONIUS OF PERGA

(viii.) On the Dodecahedron and the Icosahedron

Hypsicles [Euclid, Elements xiv.],* Eucl. ed. Heiberg v. 6. 19-8. 5

The pentagon of the dodecahedron and the triangle of the icosahedron b inscribed in the same sphere can be included in the same circle. For this is proved by Aristaeus in the work which he wrote On the Comparison of the Five Figures, c and it is proved by Apollonius in the second edition of his work On the Comparison of the Dodecahedron and the Icosahedron that the surface of the dodecahedron bears to the surface of the icosahedron the same ratio as the volume of the dodecahedron bears to the volume of the icosahedron, by reason of there being a common perpendicular from the centre of the sphere to the pentagon of the dodecahedron and the triangle of the icosahedron.

(ix.) Principles of Mathematics

Marinus, Commentary on Euclid's Data, Eucl. ed.
Heiberg vi. 234. 13-17

Therefore, among those who made it their aim to define the datum more simply and with a single differentia, some called it the assigned, such as Apollonius in his book On Vergings and in his

* The so-called fourteenth book of Euclid's Elements is really the work of Hypsicles, for whom v. infra, pp. 394-397.

b For the regular solids v. vol. i. pp. 216-225. The face of the dodecahedron is a pentagon and the face of the icosahedron a triangle.

c A proof is given by Hypsicles as Prop. 2 of his book. Whether the Aristaeus is the same person as the author of the Solid Loci is not known.
GREEK MATHEMATICS

ἐν τῇ Καθόλου πραγματεία, οἱ δὲ γνώριμοι, ὡς Διόδορος.

(x.) On the Cochlias

Procl. in Eucl. i., ed. Friedlein 105. 1-6

Τῇν περὶ τὸν κύλινδρον ἐλικα γραφομένην, ὅταν εὐθεῖας κινούμενης περὶ τὴν ἐπιφάνειαν τοῦ κύλινδρου σημεῖον ὀμοταχῶς ἐπὶ αὐτῆς κινήται. γίνεται γὰρ ἐπὶ, ἤς ὀμοιομερῶς πάντα τὰ μέρη πάσιν ἐφαρμόζει, καθάπερ 'Απολλώνιος ἐν τῷ Περὶ τοῦ κοχλίου γράμματι δείκνυσιν.

(xi.) On Unordered Irrationals

Procl. in Eucl. i., ed. Friedlein 74. 23-24

Τὰ Περὶ τῶν ἀτάκτων ἀλόγων, ἀ' Ἀπολλώνιος ἐπὶ πλέον ἐξειργάσατο.

Schol. i. in Eucl. Elem. x., Eucl. ed. Heiberg v. 414. 10-16

Ἐν μὲν οὖν τοῖς πρῶτοι περὶ συμμέτρων καὶ ἀσυμμέτρων διαλαμβάνει πρὸς τὴν φύσιν αὐτῶν αὐτὰ ἐξετάζων, ἐν δὲ τοῖς ἐξῆς περὶ ῥητῶν καὶ ἀλόγων οὐ πασῶν· τινὲς γὰρ αὐτῶ ὡς ἐνιστάμενοι ἐγκαλοῦσιν· ἄλλα τῶν ἀπλοῦστάτων εἰδῶν, ὅν

* Heath (H.G.M. ii. 192-193) conjectures that this work must have dealt with the fundamental principles of mathematics, and to it he assigns various remarks on such subjects attributed to Apollonius by Proclus, and in particular his attempts to prove the axioms. The different ways in which entities are said to be given are stated in the definitions quoted from Euclid's Data in vol. i. pp. 478-479.
APOLLONIUS OF PERGA

General Treatise, others the known, such as Diodorus.

(x.) On the Cochlias

Proclus, On Euclid i., ed. Friedlein 105. 1-6

The cylindrical helix is described when a point moves uniformly along a straight line which itself moves round the surface of a cylinder. For in this way there is generated a helix which is homoeomeris, any part being such that it will coincide with any other part, as is shown by Apollonius in his work On the Cochlias.

(xi.) On Unordered Irrationals

Proclus, On Euclid i., ed. Friedlein 74. 23-24

The theory of unordered irrationals, which Apollonius fully investigated.

Euclid, Elements x., Scholium i., ed. Heiberg v. 414. 10-16

Therefore in the first [theorems of the tenth book] he treats of symmetrical and asymmetrical magnitudes, investigating them according to their nature, and in the succeeding theorems he deals with rational and irrational quantities, but not all, which is held up against him by certain detractors; for he dealt only with the simplest kinds, by the combination of which

* Possibly Diodorus of Alexandria, for whom v. vol. i. p. 300 and p. 301 n. b.

* In Studien über Euklid, p. 170, Heiberg conjectured that this scholium was extracted from Pappus’s commentary, and he has established his conjecture in Videnskabernes Selskabs Skrifter, 6 Raekke, hist.-philos. Afd. ii. p. 236 seq. (1888).
GREEK MATHEMATICS

συντιθεμένων γίνονται ἀπειροῦ ἁλογοι, ὡν τινας καὶ ὁ Ἀπολλώνιος ἀναγράφει.

(xii.) Measurement of a Circle


Ἰστέον δέ, ὅτι καὶ Ἀπολλώνιος ὁ Περγαῖος ἐν τῷ Ὑκτοκίω ἀπέδειξεν αὐτό δι' ἀριθμῶν ἑτέρων ἐπὶ τὸ σύνεγγυς μάλλον ἁγαγόν. τούτῳ δὲ ἀκριβέστερον μὲν εἶναι δοκεῖ, οὐ χρήσιμον δὲ πρὸς τὸν Ἀρχιμήδους σκοπὸν, ἐφάμεν γὰρ αὐτὸν σκοπὸν ἔχειν ἐν τῶδε τῷ βιβλίῳ τὸ σύνεγγυς εὑρεῖν διὰ τὰς ἐν τῷ βιώ χρείας.

(xiii.) Continued Multiplications


Τούτου δὴ προτεθεωρημένου πρόδηλου, πῶς ἔστω τὸν δοθέντα στὶχον πολλαπλασιάσαι καὶ εἰπεῖν τὸν γενόμενον ἀριθμὸν ἐκ τοῦ τὸν πρῶτον ἀριθμὸν ὅν εἰληφε τὸ πρῶτον τῶν γραμμάτων ἐπὶ τὸν δεύτερον ἀριθμὸν ὅν εἰληφε τὸ δεύτερον τῶν γραμμάτων πολλαπλασιασθῆναι καὶ τὸν γενόμενον ἐπὶ τὸν τρίτον ἀριθμὸν ὅν εἰληφε τὸ τρίτον γράμμα

1 The extensive interpolations are omitted.

* Pappus's commentary on Eucl. Elem. x. was discovered in an Arabic translation by Woepcke (Mémoires présentées par divers savants à l'Académie des sciences, 1856, xiv.). It contains several references to Apollonius's work, of which one is thus translated by Woepcke (p. 693): "Enfin, Apollonius distingua les espèces des irrationnelles ordonnées, et 352.
an infinite number of irrationals are formed, of which latter Apollonius also describes some.\footnote{We do not know what the approximation was.}

(xii.) *Measurement of a Circle*


It should be noticed, however, that Apollonius of Perga proved the same thing (sc. the ratio of the circumference of a circle to the diameter) in the *Quick-deliverer* by a different calculation leading to a closer approximation. This appears to be more accurate, but it is of no use for Archimedes' purpose; for we have stated that his purpose in this book was to find an approximation suitable for the everyday needs of life.\footnote{Heiberg (Apollon. Perg. ed. Heiberg ii. 124, n. 1) suggests that these calculations were contained in the Όκυτόκον, but there is no definite evidence.}

(xiii.) *Continued Multiplications*\footnote{The passages, chiefly detailed calculations, adjudged by Hultsch to be interpolations are omitted.}

Pappus, *Collection* ii. 17-21, ed. Hultsch 18. 23-24. 20d

This theorem having first been proved, it is clear how to multiply together a given verse and to tell the number which results when the number represented by the first letter is multiplied into the number represented by the second letter and the product is multiplied into the number represented by the third
découvrit la science des quantités appelées (irrationnelles) inordonnées, dont il produisit un très-grand nombre par des méthodes exactes."
καὶ κατὰ τὸ ἔξοχον ἔφη ἴνα κούραι
(τὸ δὲ κλεῖτε φησιν ἀντὶ τοῦ ὑπομνήσατε).

Ἐπεὶ οὖν γράμματα ἐστὶν λή τοῦ στίχου, ταῦτα
dὲ περιέχει ἀριθμοῦς δέκα τοὺς ρ ῃ ὅ ῃ ρ ῃ χ ϵ ρ, ὅν ἔκαστος ἐλάσσων μὲν ἐστὶν χιλιάδος μετρεῖται
dὲ ὑπὸ ἐκατοντάδος, καὶ ἀριθμοὺς ἵτι τοὺς μ i o k
λ ῃ κ ῃ ξ ῃ ξ ἵν ξ ἵ n κ ϰ i, ὅν ἔκαστος ἐλάσσων μὲν ἐστιν ἐκατοντάδος μετρεῖται δὲ ὑπὸ δεκάδος,
καὶ τοὺς λοιποὺς τοὺς ἄ ἓ δ ἓ ἓ ἓ ἓ ἓ ἓ ἓ ἓ ἓ ἓ ἓ, ὅν ἔκαστος ἐλάσσων δεκάδος, εἰς ἄρα τοῖς μὲν δέκα
ἀριθμοῖς ὑποτάξωμεν ἰσάριθμους δέκα κατὰ τάξιν ἐκατοντάδος, τοῖς δὲ ἵτι ὁμοίως ὑποτάξωμεν
δεκάδας ἵτι, φανεροῖς ἐκ τοῦ ἀνώτερου λογιστικοῦ
θεωρήματος ἵτι ὁτα δέκα ἐκατοντάδες μετὰ τῶν
ἵτι δεκάδων ποιοῦσι μυριάδας ἐναπλάς δέκα.

Ἐπεὶ δὲ καὶ πυθμένες ὁμοί τῶν μετρουμένων
ἀριθμῶν ὑπὸ ἐκατοντάδος καὶ τῶν μετρουμένων
ὑπὸ δεκάδος εἰσὶν οἱ ὑποκείμενοι ἵτι

\[ \text{ἀ γ θ γ ἃ γ θ ἃ δ ἃ} \]
\[ \text{δ ἃ ξ β γ ἃ ξ β ξ = ξ ξ = ε ε ε ξ ξ}, \]

*Apollonius, it is clear from Pappus, had a system of
tetrad for calculations involving big numbers, the unit being
the myriad or fourth power of 10. The tetrad are called
μυριάδες ἀπλαί, μυριάδες διπλαί, μυριάδες τριπλαί, simple
myriads, double myriads, triple myriads and so on, by which
are meant 10000, 10000, 10000 and so on. In the text of
letter and so on in order until the end of the verse which Apollonius gave in the beginning, that is

'Ἀρτέμιδος κλείτε κράτος ἐξοχον ἐννέα κοῦραι

(where he says κλείτε for ὑπομνήσατε, recall to mind).

Since there are thirty-eight letters in the verse, of which ten, namely ῥ ῦ ὚ ῦ ῦ ῦ ῦ ῦ ῦ ῦ ( = 100, 300, 200 300, 100, 300, 200, 600, 400, 100), represent numbers less than 1000 and divisible by 100, and seventeen, namely μ ἢ οκ λ ἴ κ ἴ ξ ὒ ὒ ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν ν
άλλα καὶ τῶν ἑλασσόνων δεκάδος εἶσιν Ἴᾳ, τοὐτέστιν ἀριθμοὶ οἱ

\[\text{\textalpha\textepsilon\textdelta\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon}\]

, εἰών τὸν ἐκ τοῦτων τῶν Ἴᾳ καὶ τὸν ἐκ τῶν \(\overline{\kappa}\) πυθμένων στερεὸν δι’ ἄλληλων πολλαπλασιάσωμεν, ἐσται ὁ στερεὸς μυριάδων τετραπλῶν ἦθ καὶ τριπλῶν \(\overline{5\lambda}\) καὶ διπλῶν \(\etaτ\).

Αὕται δὴ συμπολλαπλασιαζόμεναι ἐπὶ τὸν ἐκ τῶν ἐκατοντάδων καὶ δεκάδων στερεῶν, τοὐτέστι τὰς προκειμένας μυριάδας ἐνναπλᾶς δέκα, ποιοῦσιν μυριάδας τρισκαίδεκαπλᾶς \(\overline{ρ\xi\zeta}\), δωδεκαπλᾶς \(\overline{τ\varepsilon\η}\), ἐνδεκαπλᾶς \(\delta\omega\).

(xiv.) On the Burning Mirror

Fragmentum mathematicum Bobiense 113. 98-33, ed.
Belger, Hermes, xvi., 1881, 279-280

Οἱ μὲν οὖν παλαιοὶ ὑπέλαβον τὴν ἔξαφνα ποιεῖσθαι περὶ τὸ κέντρον τοῦ κατόπτρου, τοῦτο δὲ ψεύδος Ἀπολλώνιος μάλα δεόντως . . . (ἐν τῷ) πρὸς τοὺς κατοπτρικοὺς ἔδειξεν, καὶ περὶ τίνα δὲ τόπον ἡ ἐκπύρωσις ἐσται, διασεσάφηκεν ἐν τῷ Περὶ τοῦ πυρίου.

1 As amended by Heiberg, Zeitschrift für Mathematik und Physik, xxviii., 1883, hist. Ἀβθ. 124-125.
while there are eleven less than ten, that is the numbers

1, 5, 4, 5, 5, 1, 5, 5, 5, 1, 1,

if we multiply together the solid number formed by these eleven with the solid number formed by the twenty-seven the result will be the solid number

$19 \cdot 10000^4 + 6036 \cdot 10000^3 + 8480 \cdot 10000^2$.

When these numbers are multiplied into the solid number formed by the hundreds and the tens, that is with $10 \cdot 10000^9$ as calculated above, the result is

$196 \cdot 10000^{12} + 368 \cdot 10000^{12} + 4800 \cdot 10000^{11}$.

(xiv.) On the Burning Mirror

*Fragmentum mathematicum Bobiense 113. 28-33,* ed.
Belger, *Hermes,* xvi., 1881, 279-280

The older geometers thought that the burning took place at the centre of the mirror, but Apollonius very suitably showed this to be false ... in his work on mirrors, and he explained clearly where the kindling takes place in his works *On the Burning Mirror.*

*a* This fragment is attributed to Anthemius by Heiberg, but its antiquated terminology leads Heath (*H.G.M.* ii. 194) to suppose that it is much earlier.

*b* Of Apollonius's other achievements, his solution of the problem of finding two mean proportionals has already been mentioned (vol. i. p. 267 n. 6) and sufficiently indicated; for his astronomical work the reader is referred to Heath, *H.G.M.* ii. 195-196.
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The text appears to be a continuous block of paragraphs, possibly discussing a technical or scientific subject, but the details are not discernible.
XX. LATER DEVELOPMENTS IN GEOMETRY
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) CLASSIFICATION OF CURVES

Procl. in Eucl. i., ed. Friedlein 111. 1-112. 11

Διανικῇ δ’ αὖ τὴν γραμμήν ὁ Γέμνος¹ πρῶτον μὲν εἰς τὴν ἄσυνθετον καὶ τὴν σύνθετον—καλεῖ δὲ σύνθετον τὴν κεκλασμένην καὶ γωνίαν ποιοῦσαν—ἐπειτα τὴν ἄσυνθετον² εἰς τῇ σχηματοποιοῦσαν καὶ τὴν ἐπ’ ἀπειρον ἐκβαλλομένην, σχήμα λέγων ποιεῖν τὴν κυκλικὴν, τὴν τοῦ θυρεοῦ, τὴν κυκτοειδή, μὴ ποιεῖν δὲ τὴν τοῦ ὀρθογωνίου κόνου τομήν, τὴν τοῦ ἀμβλυγωνίου, τὴν κογχοειδῆ, τὴν εὐθείαν, πᾶσας τὰς τοιαύτας. καὶ πάλιν κατ’ ἄλλον τρόπον τῆς ἄσυνθετον γραμμῆς τὴν μὲν ἀπλὴν εἶναι, τὴν δὲ μικτὴν, καὶ τῆς ἀπλῆς τὴν μὲν σχήμα ποιεῖν ὡς τὴν κυκλικὴν, τὴν δὲ ἀόριστον εἶναι ὡς τὴν εὐθείαν, τῆς δὲ μικτῆς τὴν μὲν ἐν τοῖς ἐπιπέδοις εἶναι, τὴν δὲ ἐν τοῖς στερεοῖς, καὶ τῆς ἐν ἐπιπέδοις τὴν μὲν ἐν αὐτῇ συμπίπτειν ὡς τὴν κυκτοειδὴ, τὴν δ’ ἐπ’ ἀπειρον ἐκβάλλεσθαι, τῆς δὲ ἐν στερεοῖς

¹ Γέμνος Tittel, Γέμνος Friedlein.
² σύνθετον codd., correxii.

* No great new developments in geometry were made by the Greeks after the death of Apollonius, probably through 360
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) Classification of Curves

Proclus, *On Euclid i.*, ed. Friedlein 111. 1–112. 11

Geminus first divides lines into the incomposite and the composite, meaning by composite the broken line forming an angle; and then he divides the incomposite into those forming a figure and those extending without limit, including among those forming a figure the circle, the ellipse and the cissoid, and among those not forming a figure the parabola, the hyperbola, the conchoid, the straight line, and all such lines. Again, in another manner he says that some incomposite lines are simple, others mixed, and among the simple are some forming a figure, such as the circle, and others indeterminate, such as the straight line, while the mixed include both lines on planes and lines on solids, and among the lines on planes are lines meeting themselves, such as the cissoid, and others extending without limit, and among lines on solids are the limits imposed by their methods, and the recorded additions to the corpus of Greek mathematics may be described as reflections upon existing work or "stock-taking." On the basis of geometry, however, the new sciences of trigonometry and mensuration were founded, as will be described, and the revival of geometry by Pappus will also be reserved for separate treatment.

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τὴν μὲν κατὰ τὰς τομὰς ἐπινοεῖσθαι τῶν στερεῶν, τὴν δὲ περὶ τὰ στερεὰ ύφιστασθαι. τὴν μὲν γὰρ ἔλικα τὴν περὶ σφαῖραν ἢ κῶνον περὶ τὰ στερεὰ ύφεστάναι, τὰς δὲ κωνικὰς τομὰς ἢ τὰς σπειρικὰς ἀπὸ τοιάδε τοµῆς γεννάσθαι τῶν στερεῶν. ἐπινοοῦσθαι δὲ ταύτας τὰς τομὰς τὰς μὲν ὑπὸ Μεναίχμου τὰς κωνικὰς, ὁ καὶ Ἑρατοσθένης ἱστορῶν λέγει: "μὴ δὲ Μεναίχμους κωνοτομεῖν τριάδας" τὰς δὲ ὑπὸ Περσέως, ὅσ καὶ τὸ ἐπίγραμμα ἐποίησεν ἐπὶ τῇ εὐρέσει—

Τρεῖς γραμμάς ἐπὶ πέντε τομαῖς εὐρών ἐλκώδεις
Pερσέως τῶν δὲ ἐνεκεν δαίμονας ἡλάστατο.

αἱ μὲν δὴ τρεῖς τομαὶ τῶν κώνων εἰσὶν παραβολὴ καὶ ὑπερβολὴ καὶ ἐλλειψις, τῶν δὲ σπειρικῶν τομῶν ἡ μὲν ἐστὶν ἐμπεπλεγμένη, ἐνκυκία τῇ τοῦ ἱπποῦ πέδη, ἡ δὲ κατὰ τὰ μέσα πλατύνεται, ἐξ ἐκατέρου δὲ ἀπολύγει μέρους, ἡ δὲ παραμήκης οὔσα τῷ μὲν μέσῳ διαστήματι ἐλάττων χρῆται, εὑρόνεται δὲ ἐφ᾽ ἐκάτερα. τῶν δὲ ἄλλων μίξεων τὸ πλῆθος ἀπέραντον ἐστὶν καὶ γὰρ στερεῶν σχημάτων πλῆθος ἐστὶν ἀπειρον καὶ τομαὶ αὐτῶν συνίστανται πολυειδεῖς.

Ibid., ed. Friedlein 356. 8-12

Καὶ γὰρ Ἀπολλώνιος ἐφ᾽ ἐκάστης τῶν κωνικῶν γραμμῶν τί τὸ σύμπτωμα δείκνυαι, καὶ ὁ Νικομήδης ἐπὶ τῶν κοχχοειδῶν, καὶ ὁ Ἰππίας ἐπὶ

1 ἐλκώδεις Knoche, εὐρών τὰς σπειρικὰς λέγων codd.

* v. vol. i. pp. 296-297.
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lines conceived as *formed by sections* of the solids and lines *formed round the solids*. The helix round the sphere or cone is an example of the lines formed round solids, and the conic sections or the spiric curves are generated by various sections of solids. Of these sections, the conic sections were discovered by Menaechmus, and Eratosthenes in his account says: "Cut not the cone in the triads of Menaechmus"; and the others were discovered by Perseus, who wrote an epigram on the discovery—

Three spiric lines upon five sections finding,
Perseus thanked the gods therefor.

Now the three conic sections are the parabola, the hyperbola and the ellipse, while of the spiric sections one is *interlaced*, resembling the horse-fetter, another is widened out in the middle and contracts on each side, a third is elongated and is narrower in the middle, broadening out on either side. The number of the other mixed lines is unlimited; for the number of solid figures is infinite and there are many different kinds of section of them.


For Apollonius shows for each of the conic curves what is its property, as does Nicomedes for the
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τῶν τετραγωνιζουσῶν, καὶ ὁ Περσεύς ἐπὶ τῶν σπειρικῶν.

Ibid., ed. Friedlein 119. 8-17

"Ὁ δὲ συμβαίνειν φαμέν κατὰ τὴν σπειρικὴν ἐπι-
φάνειαν· κατὰ γὰρ κύκλου νοεῖται στροφὴν όρθον
διαμένοντος καὶ στρεφομένου περὶ τὸ αὐτὸ σημεῖον,
ὁ μὴ ἐστὶ κέντρον τοῦ κύκλου, διὸ καὶ τριχώς ἡ
σπείρα γίνεται, ἡ γὰρ ἐπὶ τῆς περιφερείας ἐστὶ τὸ
κέντρον ἡ ἐντὸς ἡ ἐκτὸς. καὶ εἴ μὲν ἐπὶ τῆς περι-
φερείας ἐστὶ τὸ κέντρον, γίνεται σπείρα συνεχῆς,
εἰ δὲ ἐντός, ἡ ἐμπεπλεγμένη, εἰ δὲ ἐκτός, ἡ διεχθής.
καὶ τρεῖς αἱ σπειρικαὶ τομαὶ κατὰ τὰς τρεῖς ταύτας
diaforάς.

Obviously the work of Perseus was on a substantial
scale to be associated with these names, but nothing is known
of him beyond these two references. He presumably
flourished after Euclid (since the conic sections were probably
well developed before the spiric sections were tackled) and
before Geminus (since Proclus relies on Geminus for his
knowledge of the spiric curves). He may therefore be
placed between 300 and 75 a.c.

Nicomedes appears to have flourished between Eratosthenes
and Apollonius. He is known only as the inventor of the
conchoid, which has already been fully described (vol. i.
pp. 298-309).

It is convenient to recall here that about a century later
flourished Diocles, whose discovery of the cissoid has already
been sufficiently noted (vol. i. pp. 270-279). He has also
been referred to as the author of a brilliant solution of the
problem of dividing a cone in a given ratio, which is equi-

The Dionysodorus who solved the same problem (ibid.) may
have been the Dionysodorus of Caunus-mentioned in the
Herculaneum Roll, No. 1044 (so W. Schmidt in Bibliotheca
mathematica, iv. pp. 321-325), a younger contemporary of
Apollonius; he is presumably the same person as the
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conchoid and Hippias for the quadratiees and Perseus for the spirie curves.\(^a\)

*Ibid.*, ed. Friedlein 119. 8-17

We say that this is the case with the spirie surface; for it is conceived as generated by the revolution of a circle remaining perpendicular [to a given plane] and turning about a fixed point which is not its centre. Hence there are three forms of spirie according as the centre is on the circumference, or within it, or without. If the centre is on the circumference, the spirie generated is said to be *continuous*, if within *interlaced*, and if without *open*. And there are three spirie sections according to these three differences.\(^b\)

Dionysodorus mentioned by Heron, *Metrica* ii. 13 (cited *infra*, p. 481), as the author of a book *On the Spire.*

\(^b\) This last sentence is believed to be a slip, perhaps due to too hurried transcription from Geminus. At any rate, no satisfactory meaning can be obtained from the sentence as it stands. Tannery (*Mémoires scientifiques* ii. pp. 24-28) interprets Perseus' epigram as meaning "three curves in addition to five sections." He explains the passages thus: Let \(a\) be the radius of the generating circle, \(e\) the distance of the centre of the generating circle from the axis of revolution, \(d\) the perpendicular distance of the plane of section (assumed to be parallel to the axis of revolution) from the axis of revolution. Then in the *open* spirie, in which \(e > a\), there are five different cases:

1. \(e + a > d > e\). The curve is an oval.
2. \(d = e\). Transition to (3).
3. \(e > d > e - a\). The curve is a closed curve narrowest in the middle.
4. \(d = e - a\). The curve is the *hippopede* (horse-fetter), which is shaped like the figure of 8 (v. vol. i. pp. 414-415 for the use of this curve by Eudoxus).
5. \(e - a > d > 0\). The section consists of two symmetrical ovals.

Tannery identifies the "five sections" of Perseus with these five types of section of the open spirie; the three curves
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(b) ATTEMPTS TO PROVE THE PARALLEL POSTULATE

(i.) General

Procl. in Eucl. i., ed. Friedlein 191. 16–193. 9

"Καὶ ἐὰν εἰς δύο εὐθεῖας εὐθεία ἐμπίπτουσα τὰς ἑντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὅρθων ἐλάττων ποιῆ, ἐκβαλλομένας τὰς εὐθείας ἐπ' ἀπειρον συμπίπτειν, ἐφ' ἀ μέρη εἰσὶν αἱ τῶν δύο ὅρθων ἐλάττωνς."

Τούτῳ καὶ παντελῶς διαγράφειν χρῆ τῶν αἰτημάτων θεώρημα γάρ ἐστι, πολλὰς μὲν ἀπόριας ἐπιδεχόμενον, ἂς καὶ ὁ Πτολεμαῖος ἐν τινὶ βιβλίῳ διαλύσαι προῦθετο, πολλών δὲ εἰς ἀπώδειξιν δεόμενον καὶ ὅρων καὶ θεωρημάτων. καὶ τό γε ἀντιστρέφον καὶ ὁ Εὐκλείδης ὡς θεώρημα δείκνυσιν. ἵππος δὲ ἀν τινὲς ἀπατώμενοι καὶ τούτῳ τάττειν ἐν τοῖς αἰτήμασιν ἀξιώσειαν, ὡς διὰ τὴν ἐλάττωσιν τῶν δύο ὅρθων αὐτόθεν τὴν πίστιν παρεχόμενον

described by Proclus are (1), (3) and (4). When the spire is continuous or closed, \( c = a \) and there are only three sections corresponding to (1), (2) and (3) ; (4) and (5) reduce to two equal circles touching one another. But the interlaced spire, in which \( c < a \), gives three new types of section, and in these Tannery sees his "three curves in addition to five sections." There are difficulties in the way of accepting this interpretation, but no better has been proposed.

Further passages on the spire by Heron, including a formula for its volume, are given infra, pp. 476-483.

a Eucl. i. Post. 5, for which v. vol. i. pp. 442-443, especially n. c.

Aristotle (Anal. Prior. ii. 16, 65 a 4) alludes to a pettitio principii current in his day among those who "think they establish the theory of parallels"—\( τὰς παραλληλοὺς γράφειν. \) As Heath notes (The Thirteen Books of Euclid's Elements, 366
LATER DEVELOPMENTS IN GEOMETRY

(b) Attempts to Prove the Parallel Postulate

(i.) General

Proclus, On Euclid i., ed. Friedlein 191. 16-193. 9

"If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles." ¹

This ought to be struck right out of the Postulates; for it is a theorem, and one involving many difficulties, which Ptolemy set himself to resolve in one of his books, and for its proof it needs a number of definitions as well as theorems. Euclid actually proves its converse as a theorem. Possibly some would erroneously consider it right to place this assumption among the Postulates, arguing that, as the angles are less than two right angles, there is

vol. i. pp. 191-192), Philoponus’s comment on this passage suggests that the pettio principii lay in a direction theory of parallels. Euclid appears to have admitted the validity of the criticism and, by assuming his famous postulate once and for all, to have countered any logical objections.

Nevertheless, as the extracts here given will show, ancient geometers were not prepared to accept the undemonstrable character of the postulate. Attempts to prove it continued to be made until recent times, and are summarized by R. Bonola, "Sulla teoria delle parallele e sulle geometrie non-euclidee" in Questioni riguardanti la geometria elementare, and by Heath, loc. cit., pp. 204-219. The chapter on the subject in W. Rouse Ball’s Mathematical Essays and Recreations, pp. 307-326, may also be read with profit. Attempts to prove the postulate were abandoned only when it was shown that, by not conceding it, alternative geometries could be built.

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tῆς τῶν εὐθειῶν συνεύςεως καὶ συμπτώσεως. πρὸς οὖς ὁ Γεμῖνος ὅρθως ἀπήντησε λέγων ὅτι παρ᾽ αὐτῶν ἐμάδομεν τῶν τῆς ἐπιστήμης ταύτης ἡγεμόνων μὴ πάνυ προσέχειν τῶν νοῦν ταῖς πιθαναῖς φαντασίαις εἰς τὴν τῶν λόγων τῶν ἐν γεωμετρία παραδοχῆς. ὁμοίων γὰρ φησὶ καὶ Ἀριστοτέλης ῥητορικὸν ἀποδείξεις ἀπαίτει καὶ γεωμέτρου πιθανολογοῦντος ἀνέχεσθαι, καὶ ὁ παρὰ τῶν Πλάτωνι Σιμμίας, ὅτι "τοῖς ἐκ τῶν εἰκότων ταῖς ἀποδείξεις ποιομένωι σύνοιδα σύνισιν ἀλαζοῦσι." κάνταῦθα τοῖς τούτων τὸ μὲν ἡλαττωμένων τῶν ὀρθῶν συνεύςει τὰς εὐθείας ἀληθὲς καὶ ἀναγκαῖον, τὸ δὲ συνενοῦσα ἐπὶ πλέον ἐν τῷ ἐκβάλλεσθαι συμπεσεῖται ποτε πιθανον, ἀλλ' οὐκ ἀναγκαῖον, εἰ μή τις ἀποδείξειν λόγος, ὅτι ἐπὶ τῶν εὐθειῶν τοῦτο ἀληθὲς. τὸ γὰρ εἶναι τινας γραμμὰς συνιούσας μὲν ἐπὶ ἀπειρον, ἀσυμπτώτους δὲ ύπαρχούσας, καίτω δοκοῦν ἀπίθανον εἶναι καὶ παραδοξοῦν, ὅμως ἀληθὲς ἔστι καὶ πεφώραται ἐπὶ ἄλλων εἰδώλ τῆς γραμμῆς. μῆποτε οὖν τοῦτο καὶ ἐπὶ τῶν εὐθειῶν δυνατόν, ὅπερ ἐπὶ ἐκείνων τῶν γραμμῶν; εἰς γὰρ ἄν δὲ ἀποδειξως αὐτὸ καταδεικτεῖθα, περισσὰ τῆς φαντασίας τὰ ἐπὶ ἄλλων δεικτείται γραμμῶν. εἰ δὲ καὶ οἱ διαμφισβητοῦντες λόγοι πρὸς τὴν σύμπτωσιν πολὺ τὸ πληκτικὸν ἔχων, πῶς οὐχὶ πολλῷ πλέον ἢν τὸ πιθανὸν τοῦτο καὶ τὸ ἄλογον ἑκβάλλουμεν τῆς ἠμετέρας παραδοχῆς;

'Αλλ' ὅτι μὲν ἀποδείξειν χρῆ ἔχειν τῶν προκειμένου θεωρήματος δήλον ἐκ τούτων, καὶ ὃτι

* For Geminus, v. infra, p. 370 n. e.
immediate reason for believing that the straight lines converge and meet. To such, Geminus\(^a\) rightly rejoined that we have learnt from the pioneers of this science not to incline our mind to mere plausible imaginings when it is a question of the arguments to be used in geometry. For Aristotle\(^b\) says it is as reasonable to demand scientific proof from a rhetorician as to accept mere plausibilities from a geometer, and Simmias is made to say by Plato\(^c\) that he "recognizes as quacks those who base their proofs on probabilities." In this case the convergence of the straight lines by reason of the lessening of the right angles is true and necessary, but the statement that, since they converge more and more as they are produced, they will some time meet is plausible but not necessary, unless some argument is produced to show that this is true in the case of straight lines. For the fact that there are certain lines which converge indefinitely but remain non-secant, although it seems improbable and paradoxical, is nevertheless true and well-established in the case of other species of lines. May not this same thing be possible in the case of straight lines as happens in the case of those other lines? For until it is established by rigid proof, the facts shown in the case of other lines may turn our minds the other way. And though the controversial arguments against the meeting of the two lines should contain much that is surprising, is that not all the more reason for expelling this merely plausible and irrational assumption from our accepted teaching?

It is clear that a proof of the theorem in question must be sought, and that it is alien to the special

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\(^a\) Eth. Nic. i. 3. 4, 1094 b 25-27.  
\(^b\) Phaedo 92 d.
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tῆς τῶν αἴτημάτων ἐστὶν ἀλλότριον ἰδιότητος, πώς δὲ ἀποδεικτέον αὐτῷ καὶ διὰ ποιῶν λόγων ἀναφετέον τάς πρός αὐτὸ φερομένας ἐνστάσεις, τηνικά ἣν καὶ ὁ στοιχειώτης αὐτοῦ μέλη ποιεῖσθαι μνήμην ὡς ἐναργεῖ προσ-χρώμενος. τότε γὰρ ἀναγκαῖον αὐτοῦ δεῖξαι τὴν ἐνάργειαν οὐκ ἀναποδείκτως προφαινομένην ἀλλὰ δὲ ἀποδείξεως γνώριμον γιγνομένην.

(ii.) Posidonius and Geminus

Ibid., ed. Friedlein 176. 5-10

Καὶ ὁ μὲν Ἐὐκλείδης τούτον ὁρίζει καὶ τὸν πρόπον τὰς παραλλήλους εὐθείας, ὁ δὲ Ποσειδώνιος, παράλληλου, φησίν, εἰσὶν αἱ μὴ συνεύονσαι μὴ τῇ ἀπονένονσαι ἐν ἐνὶ ἐπιπέδῳ, ἀλλ' ἵσας ἔχουσαι

* i.e., Eucl. i. 28.

Posidonius was a Stoic and the teacher of Cicero; he was born at Apamea and taught at Rhodes, flourishing 151–135 B.C. He contributed a number of definitions to elementary geometry, as we know from Proclus, but is more famous for a geographical work On the Ocean (lost but copiously quoted by Strabo) and for an astronomical work Περὶ μετέωρῶν. In this he estimated the circumference of the earth (κ. supra, p. 267) and he also wrote a separate work on the size of the sun.

* As with so many of the great mathematicians of antiquity, we know practically nothing about Geminus's life, not even his date, birthplace or the correct spelling of his name. As he wrote a commentary on Posidonius's Περὶ μετέωρῶν, we have an upper limit for his date, and "the view most generally accepted is that he was a Stoic philosopher, born probably in the island of Rhodes, and a pupil of Posidonius, and that he wrote about 73–67 B.C." (Heath, H.G.M. ii. 223). Further details may be found in Manilius's edition of the so-called Geminii elementa astronomiae.

Geminus wrote an encyclopaedic work on mathematics
character of the Postulates. But how it should be proved, and by what sort of arguments the objections made against it may be removed, must be stated at the point where the writer of the Elements is about to recall it and to use it as obvious. Then it will be necessary to prove that its obvious character does not appear independently of proof, but by proof is made a matter of knowledge.

(ii.) Posidonius b and Geminus

Ibid., ed. Friedlein 176. 5-10

Such is the manner in which Euclid defines parallel straight lines, but Posidonius says that parallels are lines in one plane which neither converge nor diverge which is referred to by ancient writers under various names, but that used by Eutocius (Τῶν μαθημάτων θεωρία, ε. supra, pp. 280-281) was most probably the actual title. It is unfortunately no longer extant, but frequent references are made to it by Proclus, and long extracts are preserved in an Arabic commentary by an-Nairizi.

It is from this commentary that Geminus is known to have attempted to prove the parallel-postulate by a definition of parallels similar to that of Posidonius. The method is reproduced in Heath, H.G.M. ii. 228-230. It tacitly assumes "Playfair's axiom," that through a given point only one parallel can be drawn to a given straight line; this axiom—which was explicitly stated by Proclus in his commentary on Eucl. i. 30 (Procl. in Eucl. i., ed. Friedlein 374. 18-375. 3)—is, in fact, equivalent to Euclid's Postulate 5. Saccheri noted an even more fundamental objection, that, before Geminus's definition of parallels can be used, it has to be proved that the locus of points equidistant from a straight line is a straight line; and this cannot be done without some equivalent postulate. Nevertheless, Geminus deserves to be held in honour as the author of the first known attempt to prove the parallel-postulate, a worthy predecessor to Lobachewsky and Riemann.
πάσας τὰς καθέτους τὰς ἀγομένας ἀπὸ τῶν τῆς ἕτερας σημείων ἐπὶ τὴν λυσὶν.

(iii.) Ptolemy

Ibid., ed. Friedlein 362, 12–363. 18

'Αλλ' ὡπως μὲν ὁ Στοιχειωτῆς δείκνυσιν ὅτι δύο ὀρθαῖς ἵσων οὖσῶν τῶν ἐντὸς αἱ εὐθεῖαι παράλληλοι εἰσί, φανερὸν ἐκ τῶν γεγραμμένων. Πτολεμαῖος δὲ ἐν οἷς ἀποδείξει προεθετο τὰς ἀπ' ἐλαττῶν ἡ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, ἐφ' ἀ μέρη εἰσίν αἱ τῶν δύο ὀρθῶν ἔλασσονες, τούτο πρὸ πάντων δεικνὺς τὸ θεώρημα τὸ δυνώ ὀρθαίς ἵσων ὑπαρχοῦσῶν τῶν ἐντὸς παράλληλος εἰσὶ τὰς εὐθείας οὕτω πως δείκνυσιν.

'Εστσαν δύο εὐθείαι αἱ AB, ΓΔ, καὶ τεμνέτω τις αὐτὰς εὐθεία ἡ EZΗΘ, ὡστε τὰς ὑπὸ BZH καὶ ὑπὸ ZΗΔ γωνίας δύο ὀρθαίς ἵσως ποιεῖν. λέγω ὅτι παράλληλοι εἰσίν αἱ εὐθεῖαι, τοποθετοῦν 372.
but the perpendiculars drawn from points on one of the lines to the other are all equal.

(iii.) Ptolemy

Ibid., ed. Friedlein 362. 12-363. 18

How the writer of the Elements proves that, if the interior angles be equal to two right angles, the straight lines are parallel is clear from what has been written. But Ptolemy, in the work in which he attempted to prove that straight lines produced from angles less than two right angles will meet on the side on which the angles are less than two right angles, first proved this theorem, that if the interior angles be equal to two right angles the lines are parallel, and he proves it somewhat after this fashion.

Let the two straight lines be AB, ΓΔ, and let any straight line EZΗΘ cut them so as to make the angles BZH and ΖΗΔ equal to two right angles. I say that the straight lines are parallel, that is they are non-

* For the few details known about Ptolemy, v. infra, p. 408 and n. b.
* This work is not otherwise known.
ἐσῳμπτωτοί εἰςιν. εἴ γὰρ δυνατόν, συμπιπτέτωσαν ἐκβαλλόμεναι αἱ ΒΣ, ΗΔ κατὰ τὸ Κ. ἐπεὶ οὖν εὐθεία η ἩΖ εὗφεστηκεν ἐπὶ τὴν ΑΒ, δύο ὀρθαῖς ἴσας ποιεῖ τὰς ὑπὸ ΑΖΗ, ΒΖΗ γωνίας. ὁμοίως δὲ, ἐπεὶ η ἩΖ εὗφεστηκεν ἐπὶ τὴν ΓΔ, δύο ὀρθαῖς ἴσας ποιεῖ τὰς ὑπὸ ΓΗΖ, ΔΗΖ γωνίας. αἱ τέσσαρες ἀρὰ αἱ ὑπὸ ΑΖΗ, ΒΖΗ, ΓΗΖ, ΔΗΖ τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ὅν αἱ δύο αἱ ὑπὸ ΒΖΗ, ΖΗΔ δύο ὀρθαῖς ὑπόκεινται ἴσαι. λοιποὶ ἀρὰ αἱ ὑπὸ ΑΖΗ, ΓΗΖ καὶ αὐταὶ δύο ὀρθαῖς ἴσαι. εἰ οὖν αἱ ΖΒ, ΗΔ δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς ἐκβαλλόμεναι συνέπεσον κατὰ τὸ Κ, καὶ αἱ ΖΑ, ΗΓ ἐκβαλλόμεναι συμπεσούνται. δύο γὰρ ὀρθαῖς καὶ αἱ ὑπὸ ΑΖΗ, ΓΗΖ ἴσαι εἰσίν. ἡ γὰρ κατ’ ἀμφότερα συμπεσοῦνται αἱ εὐθείαι, ἡ κατ’ οὐδέτερα, εἰπέρ καὶ αὐταὶ κάκεινα δύο ὀρθαῖς ἴσων ἴσαι. συμπιπτέτωσαν οὖν αἱ ΖΑ, ΗΓ κατὰ τὸ Λ. αἱ ἀρὰ ΛΑΒΚ, ΛΔΚ εὐθεῖαι χωρίον περιέχουσιν, ὅπερ ἄδυνατον. οὖν ἀρὰ δυνατόν ἐστιν δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς συμπίπτειν τὰς εὐθείας. παράλληλοι ἄρα εἰσίν.

Ibid., ed. Friedlein 365. 5-367. 27

*Ἡδη μὲν οὖν καὶ ἄλλοι τινὲς ὑπὸ θεώρημα προτάζοντες τούτο ἀιτήμα παρὰ τῷ Στοιχεωτῇ ληφθέν ἀποδείξεως ἥξιωσαν. δοκεῖ δὲ καὶ ὁ Πτολεμαῖος

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* There is a Common Notion to this effect interpolated in the text of Euclid: v. vol. i. pp. 444 and 445 n. a.

* The argument would have been clearer if it had been proved that the two interior angles on one side of ZH were severally equal to the two interior angles on the other side, that is BZH = ΓΗΖ and ΔΗΖ = ΑΖΗ; whence, if ZA, ΗΓ meet at Λ, the triangle ZΗΔ can be rotated about the mid-
secant. For, if it be possible, let BZ, HΔ, when produced, meet at K. Then since the straight line HZ stands on AB, it makes the angles AZH, BZH equal to two right angles [Eucl. i. 13]. Similarly, since HZ stands on ΓΔ, it makes the angles ΓHZ, ΔHZ equal to two right angles [*ibid.*]. Therefore the four angles AZH, BZH, ΓHZ, ΔHZ are equal to four right angles, and of them two, BZH, ZHΔ, are by hypothesis equal to two right angles. Therefore the remaining angles AZH, ΓHZ are also themselves equal to two right angles. If then, the interior angles being equal to two right angles, ZB, HΔ meet at K when produced, ZA, HΓ will also meet when produced. For the angles AZH, ΓHZ are also equal to two right angles. Therefore the straight lines will either meet on both sides or on neither, since these angles also are equal to two right angles. Let ZA, HΓ meet, then, at Λ. Then the straight lines ΔABK, ΔΓΔK enclose a space, which is impossible. Therefore it is not possible that, if the interior angles be equal to two right angles, the straight lines should meet. Therefore they are parallel.

*Ibid.*, ed. Friedlein 365. 5-367. 27

Therefore certain others already classed as a theorem this postulate assumed by the writer of the *Elements* and demanded a proof. Ptolemy appears point of ZH so that ZH lies where HZ is in the figure, while ZK, HΚ lie along the sides HΓ, ZA respectively; and therefore HΓ, ZΑ must meet at the point where K falls. The proof is based on the assumption that two straight lines cannot enclose a space. But Riemann devised a geometry in which this assumption does not hold good, for all straight lines having a common point have another point common also.
αὐτὸ δεικνύναι ἐν τῷ περὶ τοῦ τὰς ἀπ᾿ ἑλαττόνων ἡ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, καὶ δείκνυσι πολλὰ προλαβῶν τῶν μέχρι τοῦθὲ τοῦ θεωρήματος ὑπὸ τοῦ Στοιχειωτοῦ προαποδειγμένων. καὶ ὑποκείσθω πάντα εἶναι ἀληθῆ, ἢν μὴ καὶ ἡμεῖς ὄχλον ἑπεισάγωμεν ἄλλον, καὶ ὡς λημμάτιον τοῦτο δείκνυσθαι διὰ τῶν προειρημένων. ἐν δὲ καὶ τούτῳ τῶν προδειγμένων τὸ τὰς ἀπὸ δυνεῖν ὀρθῶς ἴσων ἐκβαλλομένας μηδεμῶς συμπίπτειν. λέγω τούτων ὅτι καὶ τὸ ἀνάπαλων ἄλλησι, καὶ τὸ παράλληλων ὑσῶν τῶν εὐθειῶν καὶ τεμνομένων ὑπὸ μιᾶς εὐθείας τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἴσας εἶναι. ἀνάγκη γὰρ τὴν τέμνουσαν τὰς παράλληλους ἡ δύο ὀρθῶς ἴσας ποιεῖ τὰς ἐντός καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας ἡ δύο ὀρθῶν ἐλάσσους ἡ μείζους. ἔστωσαν οὖν παράλληλοι αἱ ΔΒ, ΓΔ, καὶ ἐμπιπτέτω εἰς αὐτὰς ἡ ΖΖ; λέγω ὅτι οὐ ποιεῖ δύο ὀρθῶν μείζους τὰς ἐντός καὶ ἐπὶ τὰ αὐτὰ. εἰ γὰρ αἱ ὑπὸ ΑΖΖ, ΓΖΖ δύο ὀρθῶν μείζους, αἱ λοιπαι αἱ ὑπὸ ΒΖΖ, ΔΖΖ δύο ὀρθῶν ἐλάσσους. ἀλλὰ καὶ δύο ὀρθῶν μείζους αἱ αὐταί. οὐδὲν γὰρ μᾶλλον αἱ ΑΖ, ΓΗ παράλληλης ἡ ΖΖ, ΓΔ, ὡστε εἰ ἡ ἐμπεσοῦσα εἰς τὰς ΑΖ, ΓΗ δύο ὀρθῶν μείζους ποιεῖ τὰς ἐντός, καὶ 376.
to have proved it in his book on the proposition that *straight lines drawn from angles less than two right angles meet if produced*, and he uses in the proof many of the propositions proved by the writer of the *Elements* before this theorem. Let all these be taken as true, in order that we may not introduce another mass of propositions, and by means of the aforesaid propositions this theorem is proved as a lemma, that *straight lines drawn from two angles together equal to two right angles do not meet when produced*—for this is common to both sets of preparatory theorems. I say then that the converse is also true, that if parallel straight lines be cut by one straight line the interior angles on the same side are equal to two right angles. For the straight line cutting the parallel straight lines must make the interior angles on the same side equal to two right angles or less or greater. Let $AB, \Gamma\Delta$ be parallel straight lines, and let $HZ$ cut them; I say that it does not make the interior angles on the same side greater than two right angles. For if the angles $AZH, \GammaHZ$ are greater than two right angles, the remaining angles $BZH, \DeltaHZ$ are less than two right angles. But these same angles are greater than two right angles; for $AZ, \GammaH$ are not more parallel than $ZB, \H\Delta$, so that if the straight line falling on $AZ, \GammaH$ make the interior angles greater than two right angles, the same straight line falling

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* This is equivalent to Eucl. i. 28.
* This is equivalent to Eucl. i. 29.
* By Eucl. i. 13, for the angles $AZH, BZH$ are together equal to two right angles and so are the angles $\GammaHZ, \DeltaHZ$. 377
η εἰς τὰς ΖΒ, ΗΔ ἐμπίπτουσα δύο ὀρθῶν ποιήσει μεῖζον τὸς ἐντὸς. ἂλλ' αἱ αὐταὶ καὶ δύο ὀρθῶν ἐλάσσουσι· αἱ γὰρ τέσσαρες αἱ ὑπὸ AZH, ΓΗΖ, ΒΖΗ, ΔΗΖ τετραγων όρθαῖς ἵσαν· οπερ ἄδυνατον. ομοίως δὴ δεῖξομεν ὅτι εἰς τὰς παραλληλοὺς ἐμπίπτουσα οὐ ποιεῖ δύο ὀρθῶν ἐλάσσους τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας. εἰ δὲ μὴτε μεῖζον μήτε ἐλάσσους ποιεῖ τῶν δύο ὀρθῶν, λείπεται τὴν ἐμπίπτουσαν δύο ὀρθαῖς ἵσας ποιεῖν τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας.

Τούτου δὴ οὖν προδεδευγμένου τὸ προκείμενον ἀναμφισβήτητως ἀποδείκνυται. λέγω γὰρ ὅτι εἰναὶ δύο εὐθείας εὐθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσους ποιῆσαν καὶ συμπεσοῦνται αἱ εὐθείαι ἐκβαλλόμεναι, ἐφ' ἀ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσους. μὴ γὰρ συμπεπτέτωσαν. ἀλλ' εἰ ἀσύμπτωτοι εἰσίν, ἐφ' ἀ μέρη αἱ τῶν δύο ὀρθῶν ἐλάσσους, πολλῷ μᾶλλον ἐσονται ἀσύμπτωτοι ἑπὶ βάτερα, ἐφ' ἀ τῶν δύο εἰσὶν ὀρθῶν αἱ μεῖζονες, ὅστε ἐφ' ἐκάτερα ἀν εἰεν ἀσύμπτωτοι αἱ εὐθείαι. εἰ δὲ τούτο, παράλληλοι εἰσίν. ἀλλ' δεδεικται ὅτι η εἰς τὰς παραλληλοὺς ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθαῖς ἵσαν ποιήσει γωνίας. αἱ αὐταὶ ἄρα καὶ δύο ὀρθαῖς ἵσαι καὶ δύο ὀρθῶν ἐλάσσους, ὀπερ ἄδυνατον.

Ταύτα προδεειχόνος ὁ Πτολεμαῖος καὶ καταγ-
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on ZB, HΔ also makes the interior angles greater than two right angles; but these same angles are less than two right angles, for the four angles AZH, ΓHZ, ΒZH, ΔHZ are equal to four right angles; which is impossible. Similarly we may prove that a straight line falling on parallel straight lines does not make the interior angles on the same side less than two right angles. But if it make them neither greater nor less than two right angles, the only conclusion left is that the transversal makes the interior angles on the same side equal to two right angles.

With this preliminary proof, the theorem in question is proved beyond dispute. I mean that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced, will meet on that side on which are the angles less than two right angles. For [, if possible,] let them not meet. But if they are non-secant on the side on which are the angles less than two right angles, by much more will they be non-secant on the other side, on which are the angles greater than two right angles, so that the straight lines would be non-secant on both sides. Now if this should be so, they are parallel. But it has been proved that a straight line falling on parallel straight lines makes the interior angles on the same side equal to two right angles. Therefore the same angles are both equal to and less than two right angles, which is impossible.

Having first proved these things and squarely faced tion is equivalent to the hypothesis that through a given point only one parallel can be drawn to a given straight line; but this hypothesis can be proved equivalent to Euclid's postulate. It is known as "Playfair's Axiom," but is, in fact, stated by Proclus in his note on Eucl. i. 31.
τήσας εἰς τὸ προκείμενον ἀκριβέστερόν τι προσθείναι βούλεται καὶ δεῖξαι ὅτι, ἐὰν εἰς δύο ἐνθείας ἐνθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθῶν ποὺ ἐλάσσονας, οὐ μόνον οὐκ εἰσὶν ἴσους μείονοι αἱ ἐνθείαι, ὡς ἐδείκται, ἀλλὰ καὶ ἡ σύμπτωσις αὐτῶν κατ' ἐκεῖνα γίνεται τὰ μέρη, ἐφ' ἂν τῶν δύο ὀρθῶν ἐλάσσονες, οὐκ ἐφ' ἂν μείζονες. ἔστωσαν γὰρ δύο ἐνθείαι αἱ AB, ΘΔ καὶ ἐμπίπτουσα εἰς αὐτὰς ἡ EZΗΘ ποιεῖτω τὰς ὑπὸ AZΗ καὶ ὑπὸ ΓΗΖ δύο ὀρθῶν ἐλάσσους.

αἱ λοιπαὶ ἀρα μείζονες δύο ὀρθῶν. ὅτι μὲν [οὖν] οὐκ ἴσους μείονοι αἱ ἐνθείαι ἐδείκται. εἴ δὲ συμπιπτοῦσαν, ἡ ἐπὶ τὰ Α, Γ συμπεσοῦνται, ἡ ἐπὶ τὰ Β, Δ. συμπιπτέτωσαν ἐπὶ τὰ Β, Δ κατὰ τὸ K. ἐπεὶ οὖν αἱ μὲν ὑπὸ AZΗ καὶ ΓΗΖ δύο ὀρθῶν εἰσὶν ἐλάσσους, αἱ δὲ ὑπὸ AZΗ, ΒΖΗ δύο ὀρθαῖς ἴσαι, κοινῆς ἀφαιρεθεῖσης τῆς ὑπὸ AZΗ, 380
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the theorem in question, Ptolemy tries to make a more precise addition and to prove that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, not only are the straight lines not non-secant, as has been proved, but their meeting takes place on that side on which the angles are less than two right angles, and not on the side on which they are greater. For let AB, ΓΔ be two straight lines and let EZΗΘ fall on them and make the angles AZH, ΓHZ less than two right angles. Then the remaining angles are greater than two right angles [Eucl. i. 13]. Now it has been proved that the straight lines are not non-secant. If they meet, they will meet either on the side of A, Γ or on the side of B, Δ. Let them meet on the side of B, Δ at K. Then since the angles AZH, ΓHZ are less than two right angles, while the angles AZH, BZH are equal to two right angles, when the common angle AZH is taken away, the angle ΓHZ will be less

*obv* is clearly out of place.
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ἡ ὑπὸ ΓΖΗ ἐλάσσων ἔσται τῆς ὑπὸ ΒΖΗ, τριγώνου ἀρα τοῦ ΚΖΗ ἡ ἐκτὸς τῆς ἐντὸς καὶ ἀπεναντίον ἐλάσσων, ὅπερ ἀδύνατον. οὐκ ἀρα κατὰ ταῦτα συμπίπτουσιν. ἄλλα μὴν συμπίπτουσιν. κατὰ θάτερα ἀρα ἡ σύμπτωσις αὐτῶν ἔσται, καθ’ ἀ αἱ τῶν δύο ὀρθῶν εἰσιν ἐλάσσονες.

(iv.) Proclus

Ibid., ed. Friedlein 371. 23-373. 2

Τούτου δὴ προσποτεθέντος λέγω ὅτι, ἐὰν παράλληλων εὐθείων τὴν ἐτέραν τέμνει τις εὐθεία, τεμεῖ καὶ τῇ λοιπῇ.

"Εστώσαν γὰρ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ τεμνόντο τῆν ΑΒ ἡ ΕΖΗ. λέγω ὅτι τῆν ΓΔ τεμεῖ.

\[
A \quad E \quad H \quad Z \quad B
\]

\[
Γ \quad \quad Δ
\]

'Εστι γὰρ δύο εὐθείαι εἰσιν ἀφ’ ἐνὸς σημείου τοῦ Ζ, εἰς ἀπειρὸν ἐκβαλλόμεναι αἱ ΒΖ, ΖΗ, παντὸς μεγέθους μείζονα ἔχουσι διάστασιν, ὡστέ καὶ τούτου, ὅσον ἔστι τὸ μεταξὺ τῶν παραλληλῶν.
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than the angle BZH. Therefore the exterior angle of the triangle KZH will be less than the interior and opposite angle, which is impossible [Eucl. i. 16]. Therefore they will not meet on this side. But they do meet. Therefore their meeting will be on the other side, on which the angles are less than two right angles.

(iv.) Proclus

Ibid., ed. Friedlein 371. 23-373. 2

This having first been assumed, I say that, if any straight line cut one of parallel straight lines, it will cut the other also.

For let AB, ΓΔ be parallel straight lines, and let EZH cut AB. I say that it will cut ΓΔ.

For since BZ, ZH are two straight lines drawn from one point Z, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than that between the parallels.

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οταν οὖν μείζον ἀλλήλων διαστάσεως τῆς τούτων διαστάσεως τεμεῖ η ὉΗ τῆν ΓΔ. εάν άρα παραλλήλων τὴν ἐτέραν τέμνῃ τις εὐθεία, τεμεῖ καὶ τῆν λοιπήν.

Τούτου προαποδείχθεντος ἀκολούθως δείξομεν τὸ προκείμενον. έστωσαν γὰρ δύο εὐθείαι αἱ ΑΒ, ΓΔ, καὶ ἐμπιπτέτω εἰς αὐτὰς ἡ EZ ἐλάσσονας δύο ὀρθῶν ποιοῦσα τὰς ὑπὸ BEZ, ΔΖΕ. λέγω ὅτι συμπεσοῦνται αἱ εὐθείαι κατὰ ταῦτα τὰ μέρη, ἐφ' ἂν αἱ τῶν δύο ὀρθῶν εἰσιν ἐλάσσον.

'Επειδὴ γὰρ αἱ ὑπὸ BEZ, ΔΖΕ ἐλάσσονες εἰσὶν δύο ὀρθῶν, τῇ ὑπεροχῇ τῶν δύο ὀρθῶν ἔστω ἤση ἡ ὑπὸ ΘΕΒ. καὶ ἐκβεβλήσω ἡ ΘΕ ἐπὶ τὸ K. ἔπει οὖν εἰς τὰς ΚΘ, ΓΔ ἐμπέπτωκεν ἡ EZ καὶ ποιεῖ τὰς ἐντὸς δύο ὀρθῶν ἕσσας τὰς ὑπὸ ΘΕZ, ΔΖΕ, παράλληλοι εἰσὶν αἱ ΘΚ, ΓΔ εὐθείαι. καὶ τείμει τὴν ΚΘ ἡ ΑΒ. τεμεῖ ἀρα καὶ τὴν ΓΔ διὰ τὸ προδειγμένον. συμπεσοῦνται ἀρα αἱ ΑΒ, ΓΔ κατὰ τὰ μέρη ἑκείνα, ἐφ' ἂν αἱ τῶν δύο ὀρθῶν ἐλάσσονες, ὡστε δεδεικταὶ τὸ προκείμενον.

1 ΔΕΖ codd., correxī.

* The method is ingenious, but Clavius detected the flaw, which lies in the initial assumption, taken from Aristotle, that two divergent straight lines will eventually be so far apart that a perpendicular drawn from a point on one to the other will be greater than any assigned distance; Clavius draws attention to the conchoid of Nicomedes (v. vol. i. pp. 298-301), which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; but the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote.
Whenever, therefore, they are at a distance from one another greater than the distance between the parallels, ZH will cut ΓΔ. If, therefore, any straight line cuts one of parallels, it will cut the other also.

This having first been established, we shall prove in turn the theorem in question. For let AB, ΓΔ be two straight lines, and let EZ fall on them so as to make the angles BEZ, ΔZE less than two right angles. I say that the straight lines will meet on that side on which are the angles less than two right angles.

For since the angles BEZ, ΔZE are less than two right angles, let the angle ΘEB be equal to the excess of the two right angles. And let ΘE be produced to K. Then since EZ falls on KO, ΓΔ and makes the interior angles ΘEZ, ΔZE equal to two right angles, the straight lines ΘK, ΓΔ are parallel. And AB cuts KO; therefore, by what was before shown, it will also cut ΓΔ. Therefore AB, ΓΔ will meet on that side on which are the angles less than two right angles, so that the theorem in question is proved.
(c) ISOPERIMETRIC FIGURES


"'Ωσαύτως δ' ὅτι, τῶν ἴσην περίμετρον ἐχόντων σχημάτων διαφόρων, ἐπειδή μείζονα ἐστίν τὰ πολυγωνότερα, τῶν μὲν ἐπιπέδων ὁ κύκλος γίνεται μείζων, τῶν δὲ στερεῶν ἡ σφαῖρα."

Ποιησόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἐπιτομῇ ἐκ τῶν Ζηνοδότου δεδειγμένων ἐν τῷ Περὶ ἴσοπεριμέτρων σχημάτων.

Τῶν ἴσην περίμετρον ἐχόντων τεταγμένων εὐ-

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* Ptolemy, Math. Syn. i. 3, ed. Heiberg i. pars i. 13. 16-19.
* Zenodorus, as will shortly be seen, cites a proposition by Archimedes, and therefore must be later in date than Archimedes: as he follows the style of Archimedes closely, he is generally put not much later. Zenodorus's work is not
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(c) ISOPERIMETRIC FIGURES


"In the same way, since the greatest of the various figures having an equal perimeter is that which has most angles, the circle is the greatest among plane figures and the sphere among solid." "

We shall give the proof of these propositions in a summary taken from the proofs by Zenodorus in his book *On Isoperimetric Figures*.

*Of all rectilinear figures having an equal perimeter—*

extant, but Pappus also quotes from it extensively (*Coll. v.* ad *init.*), and so does the passage edited by Hultsch (*Papp. Coll.*., ed. Hultsch 1138-1165) which is extracted from an introduction to Ptolemy's *Syntaxis* of uncertain authorship (*v*. Rome, *Studi e Testi*, liv., 1931, pp. xiii-xvii). It is disputed which of these versions is the most faithful.
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θυγράμμων σχημάτων, λέγω δή ἰσοπλεύρων τε καὶ ἰσογωνίων, τὸ πολυγωνότερον μεῖζὸν ἐστιν.

"Εστιν γὰρ ἰσοπερίμετρα ἰσόπλευρα τε καὶ ἰσογώνια τὰ ἈΒΓ, ΔΕΖ, πολυγωνότερον δὲ ἐστὶν τὸ ἈΒΓ. λέγω, ὅτι μεῖζὸν ἐστὶν τὸ ἈΒΓ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν περὶ τὰ ἈΒΓ, ΔΕΖ πολύγωνα περιγραφομένων κύκλων τὰ Ἡ, Θ, καὶ ἐπεξεύχθωσαν αἵ ἩΒ, ἩΓ, ἩΕ, ΘΖ. καὶ ἔτι ἀπὸ τῶν Ἡ, Θ ἐπὶ τὰς ΒΓ, ΕΖ κάθετοι ἤχθωσαν αἵ ὩΚ, ὩΛ. ἐπεὶ οὖν πολυγωνότερον ἐστὶν τὸ ἈΒΓ τοῦ ΔΕΖ, πλεονάκις ἡ ΒΓ τὴν τοῦ ἈΒΓ περίμετρον καταμετρεῖ ἤπερ ἡ ΕΖ τὴν τοῦ ΔΕΖ. καὶ εἶσαι ἰσοὶ αἱ περίμετροι. μεῖζων ἀρα ἡ ΕΖ τῆς ΒΓ. ὡστε καὶ ἡ ΕΛ τῆς ΒΚ. κείσθω τῇ ΒΚ ἢ ἡ ΛΜ, καὶ ἐπεξεύχθω ἡ ὩΜ. καὶ ἔπει ἑστὶν ὡς ἡ ΕΖ εὐθείᾳ πρὸς τὴν τοῦ ΔΕΖ πολυγώνου περίμετρον οὕτως ἡ ὑπὸ ΘΖ πρὸς δ ὀρθάς, διὰ τὸ ἰσοπλεύρων εἶναι τὸ πολύγωνον καὶ ἵσας ἀπολαμβάνειν περιφερείας τοῦ περιγραφομένου κύκλου καὶ τὰς πρὸς τῷ κέντρῳ γωνίας τὸν αὐτὸν ἐχειν λόγον ταῖς περιφερείαις ἐφ' ὧν βεβήκασιν, ὡς δὲ ἡ τοῦ ΔΕΖ περίμετρος, τοῦτεστιν ἡ τοῦ ἈΒΓ, πρὸς τὴν ΒΓ οὕτως αἵ δ ὀρθαι πρὸς τὴν ὑπὸ ΒΗΓ, δι' ἵσου ἀρα ὡς ἡ ΕΖ πρὸς ΒΓ, τοῦτεστιν ἡ ΕΛ πρὸς ΛΜ, οὕτως καὶ ἡ ὑπὸ ΘΖ γωνία πρὸς τὴν ὑπὸ ΒΗΓ, τοῦτεστιν ἡ ὑπὸ ΘΛ πρὸς τὴν ὑπὸ ΒΗΚ. καὶ ἔπει ἡ ΕΛ πρὸς ΛΜ μεῖζων λόγον ἐχει ἤπερ ἡ ὑπὸ ΘΛ γωνία πρὸς τὴν ὑπὸ ΜΘΛ, ὡς ἐξῆς δεῖξομεν, ὡς δὲ ἡ ΕΛ.
I mean equilateral and equiangular figures—the greatest is that which has most angles.

For let $\mathrm{ABG}$, $\Delta EZ$ be equilateral and equiangular figures having equal perimeters, and let $\mathrm{ABG}$ have the more angles. I say that $\mathrm{ABG}$ is the greater.

For let $H$, $\Theta$ be the centres of the circles circumscribed about the polygons $\mathrm{ABG}$, $\Delta EZ$, and let $\mathrm{HB}$, $\mathrm{HG}$, $\Theta E$, $\Theta Z$ be joined. And from $H$, $\Theta$ let $\mathrm{HK}$, $\Theta \Lambda$ be drawn perpendicular to $\mathrm{BG}$, $\mathrm{EZ}$. Then since $\mathrm{ABG}$ has more angles than $\Delta EZ$, $\mathrm{BG}$ is contained more often in the perimeter of $\mathrm{ABG}$ than $\mathrm{EZ}$ is contained in the perimeter of $\Delta EZ$. And the perimeters are equal. Therefore $\mathrm{EZ} > \mathrm{BG}$; and therefore $\mathrm{EA} > \mathrm{BK}$. Let $\Lambda M$ be placed equal to $\mathrm{BK}$, and let $\Theta M$ be joined. Then since the straight line $\mathrm{EZ}$ bears to the perimeter of the polygon $\Delta EZ$ the same ratio as the angle $\Theta OZ$ bears to four right angles—owing to the fact that the polygon is equilateral and the sides cut off equal arcs from the circumscribing circle, while the angles at the centre are in the same ratio as the arcs on which they stand [Eucl. iii. 26]—and the perimeter of $\Delta EZ$, that is the perimeter of $\mathrm{ABG}$, bears to $\mathrm{BG}$ the same ratio as four right angles bears to the angle $\mathrm{BHG}$, therefore ex aequali [Eucl. v. 17]

\[
\frac{\mathrm{EZ}}{\mathrm{BG}} = \frac{\angle \Theta \mathrm{OZ}}{\angle \mathrm{BHG}},
\]

\[
i.e.,
\frac{\mathrm{EA}}{\Lambda \mathrm{M}} = \frac{\angle \Theta \mathrm{OZ}}{\angle \mathrm{BHG}},
\]

\[
i.e.,
\frac{\mathrm{EA}}{\Lambda \mathrm{M}} = \frac{\angle \Theta \Lambda}{\angle \mathrm{BHK}}.
\]

And since \[
\frac{\mathrm{EA}}{\Lambda \mathrm{M}} > \frac{\angle \Theta \Lambda}{\angle \Theta \mathrm{OZ}},
\]
as we shall prove in due course,\(^b\)

and $\Theta M$ as radius cutting $\Theta E$ and $\Theta \Lambda$ produced, as in Eucl. Optic. 8 (v. vol. i. pp. 502-505); the proposition is equivalent to the formula

\[
\tan a : \tan \beta > a : \beta \text{ if } \frac{\pi}{2} > a > \beta.
\]

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πρὸς ΔΜ ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ, ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ μεῖζονα λόγου ἔχει ῥηπερ πρὸς τὴν ὑπὸ ΜΘΛ. μείζων ἀρα ἡ ὑπὸ ΜΘΛ γωνία τῆς ὑπὸ ΒΗΚ. ἦστιν δὲ καὶ ὀρθὴ ἡ πρὸς τῷ Λ ὀρθὴ τῇ πρὸς τῷ Κ ἑσθ. λοιπὴ ἀρα ἡ ὑπὸ ΗΒΚ μεῖζων ἦσται τῆς ὑπὸ ΘΜΛ. κείσθω τῇ ὑπὸ ΗΒΚ ἑσθ ἡ ὑπὸ ΛΜΝ καὶ διήκθω ἡ ΔΘ ἐπὶ τῷ Ν. καὶ ἐπεὶ ἑσθ ἦστιν ἡ ὑπὸ ΗΒΚ τῇ ὑπὸ ΝΜΛ, ἄλλα καὶ ἡ πρὸς τῷ Λ ἑσθ τῇ πρὸς τῷ Κ, ἦστι δὲ καὶ ἡ ΒΚ πλευρά τῇ ΜΛ ἑσθ, ἑσθ ἀρα καὶ ἡ ΗΚ τῇ ΝΛ. μεῖζων ἀρα ἡ ΗΚ τῆς ΘΔ. μείζων ἀρα καὶ τῷ ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ τοῦ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΔ. καὶ ἦστιν τῷ μὲν ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ διπλάσιον τοῦ ΑΒΓ πολυγώνου, ἐπεὶ καὶ τῷ ὑπὸ τῆς ΒΓ καὶ τῆς ΗΚ διπλάσιον ἦστιν τοῦ ΗΒΓ τριγώνον. τὸ δὲ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΔ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μείζων ἀρα τῷ ΑΒΓ πολυγώνου τοῦ ΔΕΖ.

Ibid. 358. 12-360. 3

Τοῦτον δεδειγμένου λέγω, ὅτι ἐὰν κύκλος εὐθυγράμμω ἱσοπλεύρῳ τε καὶ ἱσογωνίῳ ἱσοπερίμετρῳ ἦ, μείζων ἦσται ὁ κύκλος.

Κύκλος γὰρ ὁ ΑΒΓ ἱσοπλεύρῳ τε καὶ ἱσογωνίῳ τῷ ΔΕΖ εὐθυγράμμῳ ἱσοπερίμετρῳ ἦστιν λέγω, ὅτι μείζων ἦστιν ὁ κύκλος.

Εἰλήφθω τοῦ μὲν ΑΒΓ κύκλου κέντρον τὸ H, τοῦ δὲ περι τὸ ΔΕΖ πολυγώνου περιγραφομένου τὸ Θ, καὶ περιγεγράφθω περι τὸν ΑΒΓ κύκλον 890
and \( \angle \Lambda \Delta \angle \Theta \Lambda \angle \angle \Delta \Theta \) \( \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle \angle 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πολύγωνον ὁμοιον τῷ ΔΕΖ τῷ ΚΛΜ, καὶ ἐπεζεύχθω ἢ ΗΒ, καὶ κάθετος ἀπὸ τοῦ Θ ἐπὶ τὴν ΕΖ ἤχθω ἢ ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΛ, ΘΕ.

ἐπεὶ οὖν ἢ τοῦ ΚΛΜ πολυγώνου περίμετρος μεῖζων ἐστὶν τῆς τοῦ ΑΒΓ κύκλου περίμετρον ὡς ἐν τῷ Περὶ σφαῖρας καὶ κυλίνδρου Ἀρχιμήδης, ἵστη δὲ ἢ τοῦ ΑΒΓ κύκλου περίμετρος τῆς τοῦ ΔΕΖ πολυγώνου περίμετρω, μεῖζων ἀρα καὶ ἢ τοῦ ΚΛΜ πολυγώνου περίμετρος τῆς τοῦ ΔΕΖ πολυγώνου περίμετρου. καὶ εἰσὶν ὁμοία τὰ πολυγώνα. μεῖζων ἀρα ἢ ΒΛ τῆς ΝΕ. καὶ ὁμοιον τὸ ΗΛΒ τρίγωνον τῷ ΘΕΝ τρυγώνῳ, ἐπεὶ καὶ τὰ 392
polygon $\triangle KAM$ similar to $\triangle EZ$, and let $HB$ be joined, and from $\Theta$ let $\Theta N$ be drawn perpendicular to $EZ$, and let $HA, \Theta E$ be joined. Then since the perimeter of the polygon $\triangle KAM$ is greater than the perimeter of the circle $\triangle ABG$, as Archimedes proves in his work *On the Sphere and Cylinder*, while the perimeter of the circle $\triangle ABG$ is equal to the perimeter of the polygon $\triangle EZ$, therefore the perimeter of the polygon $\triangle KAM$ is greater than the perimeter of the polygon $\triangle EZ$. And the polygons are similar; therefore $BA > NE$. And the triangle $HAB$ is similar to the triangle $\Theta EN$.

*Prop. 1, c. supra, pp. 48-49.*
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ολα πολύγωνα. μείζων ἄρα καὶ ἡ ΗΒ τῆς ΘΝ. καὶ ἔστω ἢ ὑπὸ τοῦ ΑΒΓ κύκλου περίμετρος τῇ τοῦ ΔΕΖ πολυγώνου περιμέτρῳ. τὸ ἄρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ΗΒ μείζὸν ἐστὶν τοῦ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ. ἀλλὰ τὸ μὲν ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ΗΒ διπλάσιον τοῦ ΑΒΓ κύκλου Ἀρχιμήδης ἔδειξεν, οὗ καὶ τὴν δείξεων ἐξῆς ἐκθεθήκεθα· τὸ δὲ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μείζων ἄρα ὁ ΑΒΓ κύκλος τοῦ ΔΕΖ πολυγώνου, ὅπερ ἔδει δεῖξαι.

Ibid. 364. 12-14

Λέγω δὴ καὶ ὅτι τῶν ἴσοπεριμέτρων εὐθυγράμμων σχημάτων καὶ τὰς πλευρὰς ἴσοπληθεῖς ἑχόντων τὸ μέγιστον ἴσοπλευρὸν τέ ἐστιν καὶ ἴσογώνιον.

Ibid. 374. 12-14

Λέγω δὴ ὅτι καὶ ἡ σφαίρα μείζων ἐστὶν πάντων τῶν ἴσην ἐπιφάνειαν ἑχόντων στερεῶν σχημάτων, προσχεθημένος τοῖς ὑπὸ Ἀρχιμήδους δεδειγμένοις ἐν τῷ Περὶ σφαιρας καὶ κυλίνδρου.

(d) DIVISION OF ZODIAC CIRCLE INTO 360 PARTS: HYPSCIcles

Hypsicl. Anaph., ed. Manitius 5. 25-31

Τοῦ τῶν Ζωδίων κύκλου εἰς τῇ περιφερείας ἴσας

* The proofs of these two last propositions are worked out by similar methods.

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since the whole polygons are similar; therefore HB > ΘN. And the perimeter of the circle AΒΓ is equal to the perimeter of the polygon ΔΕΖ. Therefore the rectangle contained by the perimeter of the circle AΒΓ and HB is greater than the rectangle contained by the perimeter of the polygon ΔΕΖ and ΘN. But the rectangle contained by the perimeter of the circle AΒΓ and HB is double of the circle AΒΓ as was proved by Archimedes, whose proof we shall set out next; and the rectangle contained by the perimeter of the polygon ΔΕΖ and ΘN is double of the polygon ΔΕΖ [by Eucl. i. 41]. Therefore the circle AΒΓ is greater than the polygon ΔΕΖ, which was to be proved.

Ibid. 364. 12-14

Now I say that, of all rectilineal figures having an equal number of sides and equal perimeter, the greatest is that which is equilateral and equiangular.

Ibid. 374. 12-14

Now I say that, of all solid figures having an equal surface, the sphere is the greatest; and I shall use the theorems proved by Archimedes in his work On the Sphere and Cylinder.

(d) Division of Zodiac Circle into 360 Parts:

Hypsicles

Hypsicles, On Risings, ed. Manitius * 5. 25-31

The circumference of the zodiac circle having been

* Des Hypsikles Schrift Anaphorikos nach Überlieferung und Inhalt kritisch behandelt, in Programm des Gymnasiums zum Heiligen Kreuz in Dresden (Dresden, 1888), I Abt. 395
διηρημένου, ἐκάστη τῶν περιφερειῶν μοῖρα τοπικὴ καλείσθω. ὁμοίως δὴ καὶ τοῦ χρόνου, ἐν ὧν ὁ ἥγιος ἀρ' ὠδ' ἐτύχε σημείου ἐπὶ τὸ αὐτὸ σημείον παραγίγνεται, εἰς τὲ χρόνους ἰσοὺς διηρημένου, ἑκαστὸς τῶν χρόνων μοῖρα χρονικὴ καλείσθω.

(e) Handbooks

(i.) Cleomedes

Cleom. De motu circ. ii. 6, ed. Ziegler 218. 8-224. 8

Τοιούτων δὲ τῶν περὶ τὴν ἐκλειψιν τῆς σελήνης εἶναι ἐπιδειγμένων δοκεῖ ἐναντιοῦσαί τῷ λόγῳ τῶν κατασκευασμένη ἐκλείπειν τὴν σελήνην εἰς τὴν σκιὰν ἐμπίπτουσαν τῆς γῆς τὰ λεγόμενα κατὰ τὰς παραδόξους τῶν ἐκλείπεων. φασὶ γὰρ τὶνες, ὅτι γίνεται σελήνης ἐκλειψις καὶ ἀμφότερον τῶν φωτῶν ὑπὲρ τῶν ὀρίζοντα θεωρομένων. τούτον δὲ δὴλον ποιεῖ, διότι μὴ ἐκλείπει η ἑκλήνη τῇ σκίᾳ

* Hypsicles, who flourished in the second half of the second century B.C., is the author of the continuation of Euclid’s Elements known as Book xiv. Diophantus attributed to him a definition of a polygonal number which is equivalent to the formula \( \frac{1}{2} n(2 + (n - 1)(a - 2)) \) for the \( n \)th \( a \)-gonal number.

The passage here cited is the earliest known reference in Greek to the division of the ecliptic into 360 degrees. This number appears to have been adopted by the Greeks from the Chaldaeans, among whom the zodiac was divided into twelve signs and each sign into thirty parts according to one system, sixty according to another (v. Tannery, Mémoires scientifiques, ii. pp. 256-268). The Chaldaeans do not, however, seem to have applied this system to other circles; Hipparchus is believed to have been the first to divide the
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divided into 360 equal arcs, let each of the arcs be called a degree in space, and similarly, if the time in which the zodiac circle returns to any position it has left be divided into 360 equal times, let each of the times be called a degree in time.\(^a\)

\[(c)\] **Handbooks**

\[(i.)\] **Cleomedes**\(^b\)

Cleomedes, *On the Circular Motion of the Heavenly Bodies* ii. 6, ed. Ziegler 218. 8–224. 8

Although these facts have been proved with regard to the eclipse of the moon, the argument that the moon suffers eclipse by falling into the shadow of the earth seems to be refuted by the stories told about paradoxical eclipses. For some say that an eclipse of the moon may take place even when both luminaries are seen above the horizon. This should make it clear that the moon does not suffer eclipse by circle in general into 360 degrees. The problem which Hypsicles sets himself in his book is: *Given the ratio between the length of the longest day and the length of the shortest day at any given place, to find how many time-degrees it takes any given sign to rise.* A number of arithmetical lemmas are proved.

\(^b\) Cleomedes is known only as the author of the two books Κυκλικὴ θεωρία μετεώρων. This work is almost wholly based on Posidonius. He must therefore have lived after Posidonius and presumably before Ptolemy, as he appears to know nothing of Ptolemy's works. In default of better evidence, he is generally assigned to the middle of the first century B.C.

The passage explaining the measurement of the earth by Eratosthenes has already been cited (*supra*, pp. 266-273). This is the only other passage calling for notice.
τῆς γῆς περιπτώσεως, ἀλλ' ἔτερον τρόπον. . . . οἱ παλαιότεροι τῶν μαθηματικῶν οὕτως ἐπεχείρον λυέιν τὴν ἀπορίαν ταύτην. ἔφασαν γὰρ, ὅτι . . . οἱ δ' ἔπι γῆς ἔστώτες οὐδὲν ἂν κωλύσωτο ὅραν ἀμφοτέρους αὐτοὺς ἐπὶ τοῖς κυρτῷμασι τῆς γῆς ἔστωτε. . . . τοιαύτην μὲν οὖν οἱ παλαιότεροι τῶν μαθηματικῶν τῆς τῆς προσαγομένης ἀπορίας λύσιν ἐποιήσαντο. μὴ ποτε δ' οὖχ ὑγιῶς εἰσώ ἐννεγμένου. ἐφ' ὑψὸς μὲν γὰρ ἡ ὁψὶς ἡμῶν γενομένη δύνατ' ἂν τοῦτο παθεῖν, κωνοειδοὺς τοῦ ὀρίζοντος γενομένου πολὺ ἀπὸ τῆς γῆς ἐκ τὸν ἁέρα ἡμῶν ἐξαρθέντων, ἐπὶ δὲ τῆς γῆς ἔστωτων οὐδαμῶς. εἰ γὰρ καὶ κύρτωμά ἐστιν, ἐφ' οὖ βεβήκαμεν, ἀφανίζεται ἡμῶν ἡ ὁψὶς ὑπὸ τοῦ μεγέθους τῆς γῆς. . . . ἀλλὰ πρῶτον μὲν ἀπαντητέον λέγοντας, οτι πέπλασαι ὁ λόγος οὕτως ὑπὸ τῶν ἀπορίαν βουλομένων ἐμποίησαι τοῖς περὶ ταύτα καταγινομένοις τῶν ἀστρολόγων καὶ φιλοσόφων. . . . πολλῶν δὲ καὶ παντοδαπῶν περὶ τὸν ἁέρα παθῶν συνιστασθαι πεφυκότων οὐκ ἂν εἶν ἀδύνατον, ἡδη καταδεδυκότος τοῦ ἥλιου καὶ ὑπὸ τὸν ὀρίζοντα ὄντος φαντασίαν ἡμῶν προσπεσεῖν ὡς μηδέπω καταδεδυκότος αὐτού, ἡ νέφους παχυτέρου πρὸς τῇ δύσει ὄντος καὶ λαμπρομένου ὑπὸ τῶν ἥλιακῶν ἀκτίνων καὶ ἥλιου ἡμῶν φαντασίαν ἀποστέπῳ τοῦ ἀνθρώπου γενομένου. καὶ γὰρ

* i.e., the horizon would form the base of a cone whose vertex would be at the eye of the observer. He could thus look down on both the sun and moon as along the generators of a cone, even though they were diametrically opposite each other.
falling into the shadow of the earth, but in some other way. . . . The more ancient of the mathematicians tried to explain this difficulty after this fashion. They said that persons standing on the earth would not be prevented from seeing them both because they would be standing on the convexities of the earth. . . . Such is the solution of the alleged difficulty given by the more ancient of the mathematicians. But its soundness may be doubted. For, if our eye were situated on a height, the phenomenon in question might take place, the horizon becoming conical if we were raised sufficiently far above the earth, but it could in no wise happen if we stood on the earth. For though there might be some convexity where we stood, our sight itself becomes evanescent owing to the size of the earth. . . . The fundamental objection must first be made, that this story has been invented by certain persons wishing to make difficulty for the astronomers and philosophers who busy themselves with such matters. . . . Nevertheless, as the conditions which naturally arise in the air are many and various, it would not be impossible that, when the sun has just set and is below the horizon, we should receive the impression of its not having yet set, if there were a cloud of considerable density at the place of setting and if it were illumined by the solar rays and transmitted to us an image of the sun, or if there were a mock sun. For such images are often

Lit. "anthelion," defined in the Oxford English Dictionary as "a luminous ring or nimbus seen (chiefly in alpine or polar regions) surrounding the shadow of the observer's head projected on a cloud or fog bank opposite the sun." The explanation here tentatively put forward by Cleomedes is, of course, the true one.
τοιαῦτα πολλὰ φαντάζεται ἐν τῷ ἀέρι, καὶ μᾶλλον περὶ τὸν Πόντον.

(ii.) Theon of Smyrna

Ptol. Math. Syn. x. 1, ed. Heiberg l. pars ii. 296. 14-16

'Ἐν μὲν γὰρ ταῖς παρὰ Θέωνος τοῦ μαθηματικοῦ δοθεῖσαι ἡμῖν εὐρομεν ἀναγεγραμμενὴν τίησιν τῶν ἰ5' ἦτεὶ Ἀδριανοῦ.

Theon Smyr., ed. Hiller 1. 1-2. 2

"Ὅτι μὲν οὐχ οἶδον τε συνεῖναι τῶν μαθηματικῶν λεγομένων παρὰ Πλάτωνι μὴ καὶ αὐτὸν ἑσκημένου ἐν τῇ θεωρίᾳ ταῦτη, πᾶς ἂν που ὀμολογήσειν ὅσο δὲ οὐδὲ τὰ ἀλλὰ ἀνωφελῆς οὐδὲ ἀνόητος ἐπὶ ταῦτα ἐμπείρια, διὰ πολλῶν αὐτὸς ἐμφανίζειν έοικε. τὸ μὲν οὖν συμπάσχης γεωμετρίας καὶ συμπάσχης μονακής καὶ ἀστρονομίας ἐμπείρων γενόμενον τοῖς Πλάτωνοις συγγράμμασιν ἐνυγχάνεις μακαριστὸν μὲν εἴ τι πέντε, οὐ μὴν εὔπορον οὐδὲ βάδιον ἀλλὰ πάνιν πολλοῦ τοῦ ἐκ παιδῶν πόνου δεόμενον. ὡστε δὲ τοὺς διημαρτηκότας τοῦ ἐν τοῖς μαθήμασι ἀσκητήμαι, ὄρεγομένους δὲ τῆς γνώσεως τῶν συγγραμμάτων αὐτοῦ μὴ παντάπασιν ὃν ποθοῦσι διαμαρτεῖν, κεφαλαίδη καὶ σύντομον ποιησόμεθα τῶν ἀναγκαίων καὶ ὃν δὲι μάλιστα τοῖς ἐνεπεξεργαζομενοῖς Πλάτωνι μαθηματικῶν θεωρημάτων παράδοσιν, ἀριθμητικῶν τε καὶ μονακῶν καὶ γεωμετρικῶν τῶν τε κατὰ στερεομετρίαν καὶ ἀστρονομίαν, ὃν χωρίς οὕχ 400
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seen in the air, and especially in the neighbourhood of Pontus.

(ii.) Theon of Smyrna

Ptolemy, Syntaxis x. 1, ed. Heiberg i. pars ii. 296. 14-16

For in the account given to us by Theon the mathematician we find recorded an observation made in the sixteenth year of Hadrian.4

Theon of Smyrna, ed. Hiller 1. 1-2. 2

Everyone would agree that he could not understand the mathematical arguments used by Plato unless he were practised in this science; and that the study of these matters is neither unintelligent nor unprofitable in other respects Plato himself would seem to make plain in many ways. One who had become skilled in all geometry and all music and astronomy would be reckoned most happy on making acquaintance with the writings of Plato, but this cannot be come by easily or readily, for it calls for a very great deal of application from youth upwards. In order that those who have failed to become practised in these studies, but aim at a knowledge of his writings, should not wholly fail in their desires, I shall make a summary and concise sketch of the mathematical theorems which are specially necessary for readers of Plato, covering not only arithmetic and music and geometry, but also their application to stereometry and astronomy, for

By way of example, Theon proceeds to relate Plato's reply to the craftsmen about the doubling of the cube (v.
LATER DEVELOPMENTS IN GEOMETRY

without these studies, as he says, it is not possible to attain the best life, and in many ways he makes clear that mathematics should not be ignored.\(^a\)

vol. i. p. 257), and also the Epinomis. Theon’s work, which has often been cited in these volumes, is a curious hotchpotch, containing little of real value to the study of Plato and no original work.
XXI. TRIGONOMETRY
XXI. TRIGONOMETRY

1. HIPPARCHUS AND MENELAUS

Theon Alex. in Ptol. Math. Syn. Comm. i. 10, ed. Rome, 
Studia e Testi, lxxii. (1936), 451: 4-5

Δέδεικται μὲν οὖν καὶ Ἰππάρχῳ πραγματεία 
tῶν ἐν κύκλῳ εὐθείων ἐν ἰβ βιβλίοις, έτι τε καὶ 
Μενελάῳ ἐν ἂ.

Heron, Metr. i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

*Εστω ἐννάγωνον ἴσόπλευρον καὶ ἴσογώνιον τὸ 
ΑΒΓΔΕΖΗΘΩΚ, οὗ ἕκαστη τῶν πλευρῶν μονάδων 
ι. εὑρεῖν αὐτοῦ τὸ ἐμβαδόν. περιγεγράφθη περὶ 
αὐτὸ κύκλος, οὗ κέντρον ἐστὶ τὸ Λ, καὶ ἐπε-

—

* The beginnings of Greek trigonometry may be found in the science of sphaeric, the geometry of the sphere, for which v. vol. i. p. 5 n. b. It reached its culminating point in the Sphaerica of Theodosius.

Trigonometry in the strict sense was founded, so far as we know, by Hipparchus, the great astronomer, who was born at Nicaea in Bithynia and is recorded by Ptolemy to have made observations between 161 and 126 B.C., the most important of them at Rhodes. His greatest achievement was the discovery of the precession of the equinoxes, and he made a calculation of the mean lunar month which differs by less than a second from the present accepted figure. Unfortunately the only work of his which has survived is his early Commentary on the Phenomena of Eudoxus and Aratus. It

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XXI. TRIGONOMETRY

1. HIPPARCUS AND MENELAUS

Theon of Alexandria, Commentary on Ptolemy's Syntaxis i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 451. 4-5

An investigation of the chords in a circle is made by Hipparchus in twelve books and again by Menelaus in six.a

Heron, Metrics i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

Let AΓΔΕΖΗΘΚ be an equilateral and equiangular enneagon,b whose sides are each equal to 10. *To find its area.* Let there be described about it a circle with centre Λ, and let ΕΛ be joined and pro-
is clear, however, from the passage here cited, that he drew up, as did Ptolemy, a table of chords, or, as we should say, a table of sines; and Heron may have used this table (e. the next passage cited and the accompanying note).

Menelaus, who also drew up a table of chords, is recorded by Ptolemy to have made an observation in the first year of Trajan's reign (A.D. 98). He has already been encountered (vol. i. pp. 348-349 and n. ε) as the discoverer of a curve called "paradoxical." His trigonometrical work Sphaerica has fortunately been preserved, but only in Arabic, which will prevent citation here. A proof of the famous theorem in spherical trigonometry bearing his name can, however, be given in the Greek of Ptolemy (infra, pp. 458-463); and a summary from the Arabic is provided by Heath, H.G.M. ii. 262-273.

b i.e., a figure of nine sides.
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ζεύχων ἡ ΕΛ καὶ ἐκβεβλῆσθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχων ἡ ΜΖ. τὸ ἀρὰ ΕΖΜ τρίγωνον δοθέν ἐστιν τοῦ ἐνναγώνου. δεδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν, ὅτι ἡ ΖΕ τῆς ΕΜ τρίτου μέρος ἐστὶν ὡς ἐγγυστα.

2. PTOLEMY

(a) General

Suidas, s.v. Πτολεμαῖος

Πτολεμαῖος, ὁ Κλαύδιος χρηματίσας, Ἀλέξανδρεύς, φιλόσοφος, γεγονὼς ἐπὶ τῶν χρόνων Μάρκου τοῦ βασιλέως. οὗτος ἔγραψε Μηχανικὰ βιβλία γ', Περὶ φάσεων καὶ ἐπισημασιῶν ἀστερόν ἀπλανῶν βιβλία β', "Ἀπλωσιν ἐπιφανείας σφαίρας, Κανώνα πρόχειρον, τὸν Μέγαν ἀστρονόμον ἦτοι Σύνταξιν καὶ ἄλλα.

* A similar passage (i. 24, ed. H. Schöne 62, 11-20) asserts that the ratio of the side of a regular hendecagon to the diameter of the circumscribing circle is approximately \( \frac{7}{23} \); and of this assertion also it is said δεδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν. These are presumably the works of Hipparchus and Menelaus, though this opinion is controverted by A. Rome, "Premiers essais de trigonométrie rectiligne chez les Grecs" in L'Antiquité classique, t. 2 (1933), pp. 177-192. The assertions are equivalent to saying that sin 20° is approximately 0.333... and sin 16° 21' 49" is approximately 0.28.

* Nothing else is certainly known of the life of Ptolemy except, as can be gleaned from his own works, that he made observations between A.D. 125 and 141 (or perhaps 151). Arabian traditions add details on which too much reliance should not be placed. Suidas's statement that he was born
duced to M, and let MZ be joined. Then the triangle EZM is given in the enneagon. But it has been proved in the works on chords in a circle that ZE : EM is approximately $\frac{1}{2}$.\textsuperscript{a}

2. PTOLEMY

(a) General

Suidas, s.v. Ptolemaeus

Ptolemy, called Claudius, an Alexandrian, a philosopher, born in the time of the Emperor Marcus. He wrote Mechanics, three books, On the Phases and Seasons of the Fixed Stars, two books, Explanation of the Surface of a Sphere, A Ready Reckoner, the Great Astronomy or Syntaxis; and others.\textsuperscript{b}

in the time of the Emperor Marcus [Aurelius] is not accurate as Marcus reigned from A.D. 161 to 180.

Ptolemy’s Mechanics has not survived in any form; but the books On Balancing and On the Elements mentioned by Simplicius may have been contained in it. The lesser astronomical works of Ptolemy published in the second volume of Heiberg’s edition of Ptolemy include, in Greek, Φάσεως ἀπλανῶν ἀστέρων καὶ συναγωγῆς ἐπισημασίων and Προχείρων κανόνων διάταξις καὶ ψηφοφορία, which can be identified with two titles in Suidas’s notice. In the same edition is the Planisphaerium, a Latin translation from the Arabic, which can be identified with the “Ἀπλωσις ἐπιφάνειας σφαίρας of Suidas; it is an explanation of the stereographic system of projection by which points on the heavenly sphere are represented on the equatorial plane by projection from a pole—circles are projected into circles, as Ptolemy notes, except great circles through the poles, which are projected into straight lines.

Allied to this, but not mentioned by Suidas, is Ptolemy’s Analemma, which explains how points on the heavenly sphere can be represented as points on a plane by means of orthogonal projection upon three planes mutually at right angles—
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Simpl. in De caelo iv. 4 (Aristot. 311 b 1), ed. Heiberg 710. 14-19

Πτολεμαῖος δὲ ὁ μαθηματικὸς ἐν τῷ Περὶ ῥοπῶν τὴν ἐναυτίαν ἐχὼν τῷ Ἀριστοτέλει δόξαν πειράται κατασκευάζειν καὶ αὐτὸς, ὅτι ἐν τῇ ἐαυτῶν χώρᾳ οὔτε τὸ ὕδωρ οὔτε ὁ ἄλρ ἔχει βάρος. καὶ ὅτι μὲν τὸ ὕδωρ οὐκ ἔχει, δείκνυσιν ἐκ τοῦ τούς κατα-

dύσοντας μὴ αἰσθάνεσθαι βάρους τοῦ ἑπικειμένου ὕδατος, καὶ τοι τινὰς εἰς πολὺ καταδύσοντας βάθος.

Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11

Πτολεμαῖος ἐν τῷ Περὶ τῶν στοιχείων βιβλίων καὶ ἐν τοῖς Ὀπτικοῖς . . .

Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27

'Ὁ δὲ βασιλεὺς Πτολεμαῖος ἐν τῷ Περὶ διαστάσεως μονοβιβλίῳ ἀπεδείξεν, ὅτι οὐκ εἰσὶ

the meridian, the horizontal and the "prime vertical." Only fragments of the Greek and a Latin version from the Arabic have survived; they are given in Heiberg's second volume.

Among the "other works" mentioned by Suidas are presumably the Inscription in Canobus (a record of some of Ptolemy's discoveries), which exists in Greek; the Τοποθεσία τῶν πλανώμενων, of which the first book is extant in Greek and the second in Arabic; and the Optics and the book On Dimension mentioned by Simplicius.

But Ptolemy's fame rests most securely on his Great Astronomy or Syntaxis as it is called by Suidas. Ptolemy himself called this majestic astronomical work in thirteen books the Μαθηματικὴ σύνταξις or Mathematical Collection. In due course the lesser astronomical works came to be called the Μικρὸς ἀστρονομοῦμενος (τόπος), the Little Astronomy, and the Syntaxis came to be called the Μεγάλη σύνταξις, or Great Collection. Later still the Arabs, combining their article Al
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Simplicius, *Commentary on Aristotle's De caelo* iv. 4 (311 b 1), ed. Heiberg 710. 14-19

Ptolemy the mathematician in his work *On Balancing* maintains an opinion contrary to that of Aristotle and tries to show that in its own place neither water nor air has weight. And he proves that water has not weight from the fact that divers do not feel the weight of the water above them, even though some of them dive into considerable depths.

*Ibid.* i. 2, 269 a 9, ed. Heiberg 20. 11

Ptolemy in his book *On the Elements* and in his *Optics* . . .


The gifted Ptolemy in his book *On Dimension* showed that there are not more than three dimension with the Greek superlative μέγιστος, called it Al-majisti; corrupted into *Almagest*, this has since been the favourite name for the work.

The *Syntaxis* was the subject of commentaries by Pappus and Theon of Alexandria. The trigonometry in it appears to have been abstracted from earlier treatises, but condensed and arranged more systematically.

Ptolemy's attempt to prove the parallel-postulate has already been noticed (*supra*, pp. 372-383).

* Ptolemy's *Optics* exists in an Arabic version, which was translated into Latin in the twelfth century by Admiral Eugenius Siculus (v. G. Govi, *L'ottica di Claudio Tolomeo di Eugenio Ammiraglio di Sicilia*); but of the five books the first and the end of the last are missing. Until the Arabic text was discovered, Ptolemy's *Optics* was commonly supposed to be identical with the Latin work known as *De Speculis*; but this is now thought to be a translation of Heron's *Catoptrica* by William of Moerbeke (v. *infra*, p. 502 n. a).
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πλείονες τῶν τριῶν διαστάσεις, ἐκ τοῦ δεῖν μὲν τὰς διαστάσεις ύψιστάς εἶναι, τὰς δὲ ύψιστάς διαστάσεις κατ᾽ εὐθείας λαμβάνεσθαι καθέτους, τρεῖς δὲ μόνας πρὸς ὀρθὸς ἀλλήλας εὐθείας δύνατον εἶναι λαβεῖν, δύο μὲν καθ’ ὁς τὸ ἐπὶ πέδουν ὀρίζεται, τρίτην δὲ τῆς τὸ βάθος μετροῦσαν ὡστε, εἰ τις εἰς μετὰ τὴν τριχή διάστασιν ἀλλή, ἀμέτρος ἀν εἰς παντελῶς καὶ ἀόριστος.

(b) Table of Sines

(i.) Introduction

Ptol. Math. Syn. i. 10, ed. Heiberg i. pars i. 31. 7-32. 9

τ. Περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθείων

Πρὸς μὲν οὖν τὴν ἐξ ἑτοῖμον χρήσιν κανονικῆς τοια μετὰ ταῦτα ἔκθεσιν ποιησόμεθα τῆς πηλικότητος αὐτῶν τὴν μὲν περὶμετρον εἰς τῇ τμῆμα τα θελόντες, παρατιθέμενες δὲ τὰς ὑπὸ τὰς καθ’ ἡμιμοίρους παραμετρίες τῶν περιφερειῶν ύποτευνομένας εὐθείας, τούτους πόσων εἰσὶ τὰς τμήματος ἀνός τῆς διαμέτρου διὰ τὸ ἐξ αὐτῶν τῶν ἐπιλογισμῶν φανερόμενον ἐν τοῖς ἀρίθμοις εὐχρηστον εἰς ἕκ τοὺς τμήματα διηρθημένης. προτερον δὲ δειδομεν, πῶς ἄν ὡς εἰν μάλιστα δι᾽ ὀλίγων καὶ τῶν αὐτῶν θεωρημάτων εὐμεθοδευτον καὶ ταξεῖν τὴν ἐπιβολῆν τὴν πρὸς τὰς πηλικότητας αὐτῶν ποιημέθα, ὅπως μὴ μόνον ἐκτεθεμένα τά μεγέθη τῶν εὐθείων ἐχωμεν ἀνέπιστάτως, ἀλλὰ διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως τον ἑλεγχον ἐξ εὐχεροὺς μεταχειριζόμεθα. καθόλου

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sions; for dimensions must be determinate, and determinate dimensions are along perpendicular straight lines, and it is not possible to find more than three straight lines at right angles one to another, two of them determining a plane and the third measuring depth; therefore, if any other were added after the third dimension, it would be completely unmeasurable and undetermined.

(b) Table of Sines

(i.) Introduction

Ptolemy, Syntaxis i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

10. On the lengths of the chords in a circle

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of these [chords in a circle], dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set out correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considera-
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μέντοι χρησόμεθα ταῖς τῶν ἀριθμῶν ἐφόδοις κατὰ τὸν τῆς ἐξήκοντάδος τρόπον διὰ τὸ δύσχρηστον τῶν μοριασμῶν ἐτι τε τοῖς πολυπλασιασμοῖς καὶ μερισμοῖς ἀκολουθήσομεν τοῦ συνεγγίζοντος ἀεὶ καταστοχαζόμενοι, καὶ καθ' ὅσον ἄν τὸ παραλειπόμενον μηδενὶ ἀξιολόγῳ διαφέρῃ τοῦ πρὸς αἴσθησιν ἀκριβοῦς.

(ii.) sin 18° and sin 36°

Ibid. 32. 10-35. 16

"Εστω δὴ πρῶτον ἡμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΔΓ περί κέντρον τὸ Δ, καὶ ἀπὸ τοῦ Δ τῇ ΑΓ πρὸς ὀρθὰς γωνίας ἕχων ἡ ΔΒ, καὶ τετμῆσθω δίχα ἡ ΔΓ κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ EB, καὶ κείσθω αὐτῇ ἵση ἡ EZ, καὶ ἐπεζεύχθω ἡ ZB. λέγω, ὅτι ἡ μὲν ΖΔ δεκαγώνου ἐστίν πλευρά, ἡ δὲ ΒΖ πενταγώνου.

* By διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς συστάσεως Ptolemy meant more than a graphical method; the phrase indicates a rigorous proof by means of geometrical considerations, as will be seen when the argument proceeds; cf. the use of διὰ τῶν γραμμῶν ἑντα, p. 434. It may be inferred, therefore, that when Hipparchus proved "by means of lines" (διὰ τῶν γραμμῶν, On the Phaenomena of Eudoxus and Aratus, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations; in other words, he was familiar with the main formulae of spherical trigonometry.

* i.e., ΖΔ is equal to the side of a regular decagon, and ΒΖ to the side of a regular pentagon, inscribed in the circle ΑΒΓ.

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TIONS. In general we shall use the sexagesimal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

(ii.) $\sin 18^\circ$ and $\sin 36^\circ$

Ibid. 32. 10-35. 16

First, let $\text{AB}\Gamma$ be a semicircle on the diameter $\Delta\Delta\Gamma$ and with centre $\Delta$, and from $\Delta$ let $\Delta B$ be drawn perpendicular to $\Delta\Gamma$, and let $\Delta\Gamma$ be bisected at $E$, and let $EB$ be joined, and let $EZ$ be placed equal to it, and let $ZB$ be joined. I say that $Z\Delta$ is the side of a decagon, and $BZ$ of a pentagon.
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'Επει γὰρ εὐθεία γραμμὴ ἡ ΔΓ τέτμηται δίχα κατὰ τὸ Ε, καὶ πρόσκειται τις αὐτῆς εὐθεία ἡ ΔΖ, τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶν τῷ ἀπὸ τῆς ΕΖ τετραγώνω, τούτεστι τῷ ἀπὸ τῆς ΒΕ, ἐπεὶ ἵστη ἐστὶν ἡ ἘΒ τῇ ΖΕ. ἀλλὰ τῷ ἀπὸ τῆς ΕΒ τετραγώνῳ ἴσα ἐστὶ τα ἀπὸ τῶν ΕΔ καὶ ΔΒ τετράγωνα· τὸ ἀρὰ ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΔΕ τετραγώνου ἴσον ἐστὶν τοῖς ἀπὸ τῶν ΕΔ, ΔΒ τετραγώνωι. καὶ κοινῷ ἀφαιρεθέντος τοῦ ἀπὸ τῆς ΕΔ τετραγώνου λοιπὸν τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ ἴσον ἐστὶν τῷ ἀπὸ τῆς ΔΒ, τούτεστι τῷ ἀπὸ τῆς ΔΓ· ἡ ΖΓ ἀρὰ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Δ. ἐπεὶ οὖν ἡ τοῦ ἔξαγωνον καὶ ἡ τοῦ δεκαγώνου πλευρὰ τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφόμενων ἐπὶ τῆς αὐτῆς εὐθείας ἄκρον καὶ μέσον λόγον τέμνονται, ἡ ἐπὶ ΓΔ ἐκ τοῦ κέντρου ὀσα τὴν τοῦ ἔξαγωνον περιέχει πλευράν, ἡ ΔΖ ἀρὰ ἐστὶν ἵστῃ τῇ τοῦ δεκαγώνου πλευρᾷ. ὁμοίως δὲ, ἐπεὶ ἡ τοῦ πενταγώνου πλευρὰ δύναται τῇ τε τοῦ ἔξαγωνον καὶ την τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφόμενων, τοῦ δὲ ΒΔΖ ὀρθογώνιον τὸ ἀπὸ τῆς ΒΖ τετράγωνον ἴσον ἐστὶν τῷ τε ἀπὸ τῆς ΒΔ, ἡ τῆς ἐστίν ἔξαγωνον πλευρά, καὶ τῷ ἀπὸ τῆς ΔΖ, ἡ τῆς ἐστίν δεκαγώνου πλευρά, ἡ ΒΖ ἀρὰ ἐστὶν τῇ τοῦ πενταγώνου πλευρᾷ.

'Επεὶ οὖν, ὡς ἐφη, ὑποτιθέμεθα τὴν τοῦ κύκλου διάμετρον τριμάτων ἰκ, γίνεται διὰ τὰ προκειμένα ἡ μὲν ΔΕ ἡμίσεια ὀσα τῆς ἐκ τοῦ κέντρου

* Following the usual practice, I shall denote segments (τρίματα) of the diameter by *, sixtieth parts of a τρίμα by 416
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For since the straight line $\Delta \Gamma$ is bisected at E, and
the straight line $\Delta Z$ is added to it,

$$\Gamma Z \cdot \Delta \Delta + E \Delta^2 = EZ^2$$

$$= BE^2,$$

since EB = ZE.

But

$$E \Delta^2 + \Delta B^2 = EB^2;$$

[Eucl. i. 47]

therefore

$$\Gamma Z \cdot \Delta \Delta + E \Delta^2 = E \Delta^2 + \Delta B^2.$$

When the common term $E \Delta^2$ is taken away,

the remainder

$$\Gamma Z \cdot \Delta \Delta = \Delta B^2$$

i.e.,

$$= \Delta \Gamma^2;$$

therefore $Z \Gamma$ is divided in extreme and mean ratio
at $\Delta$ [Eucl. vi., Def. 3]. Therefore, since the side of
the hexagon and the side of the decagon inscribed in
the same circle when placed in one straight line are
cut in extreme and mean ratio [Eucl. xiii. 9], and $\Gamma \Delta$,
being a radius, is equal to the side of the hexagon
[Eucl. iv. 15, coroll.], therefore $\Delta Z$ is equal to the side
of the decagon. Similarly, since the square on the
side of the pentagon is equal to the rectangle con-
tained by the side of the hexagon and the side of
the decagon inscribed in the same circle [Eucl. xiii.
10], and in the right-angled triangle $B \Delta Z$ the square
on $BZ$ is equal [Eucl. i. 47] to the sum of the squares
on $B \Delta$, which is a side of the hexagon, and $\Delta Z$, which
is a side of the decagon, therefore $BZ$ is equal to the
side of the pentagon.

Then since, as I said, we made the diameter $a$ con-
sist of $120^p$, by what has been stated $\Delta E$, being half
the numeral with a single accent, and second-sixtieths by the
numeral with two accents. As the circular associations of
the system tend to be forgotten, and it is used as a general
system of enumeration, the same notation will be used for the
squares of parts.
τμημάτων Λ καὶ τὸ ἀπ’ αὐτῆς γ', ἡ δὲ ΒΔ ἐκ τοῦ κέντρου ὀψα τμημάτων ξ καὶ τὸ ἀπὸ αὐτῆς γ', τὸ δὲ ἀπὸ τῆς ΕΒ, τούτεστιν τὸ ἀπὸ τῆς EZ, τῶν ἐπὶ τὸ αὐτὸ φ. μήκει ἄρα ἦσται η EZ τμημάτων ξ, δὲ γ', ἦ γ', ἦ γ', καὶ λοιπῇ η ΔΖ τῶν αὐτῶν ΛΖ, δὲ γ'. ἦ ἄρα τοῦ δεκαγώνου πλευρά, ὑποτείνουσα δὲ περιφέρειαν τοιούτων λε, οἰῶν ἦστιν ὁ κύκλος τξ, τοιούτων ἦσται ΛΖ, δὲ γ', οἰῶν ἦ διάμετρος ρκ. πάλιν ἐπεὶ ἡ μὲν ΔΖ τμημάτων ἦστι ΛΖ, δὲ γ', τὸ δὲ ἀπὸ αὐτῆς γ' άφες δὲ γ', ἦστι δὲ καὶ τὸ ἀπὸ τῆς ΔΒ τῶν αὐτῶν ΛΖ, ἢ συνεθέντα ποίει τὸ ἀπὸ τῆς ΒΖ τετράγωνον δΧρε δ' ἢ γ', μήκει ἄρα ἦσται η ΒΖ τμημάτων ο λΒ γ', ἦ γ', καὶ ἦ τοῦ πενταγώνου ἄρα πλευρά, ὑποτείνουσα δὲ μοίρας οβζ, οἰῶν ἦστιν ὁ κύκλος τξ, τοιούτων ἦστιν ο λΒ γ', οἰῶν ἦ διάμετρος ρκ.

Φανερῶν δὲ αὐτόθεν, ὅτι καὶ ἡ τοῦ ἐξαγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ξ, καὶ ἢ τῇ ὀψα τῇ ἐκ τοῦ κέντρου, τμημάτων ἦστιν ξ. ὅμως δὲ, ἐπεὶ ἡ μὲν τοῦ τετραγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ξ, δυνάμει διπλασία ἦστιν τῆς ἐκ τοῦ κέντρου, ἢ δὲ τοῦ τριγώνου πλευρά, ὑποτείνουσα δὲ μοίρας ρκ, δυνάμει τῆς αὐτῆς ἦστιν τριπλασίων, τὸ δὲ ἀπὸ τῆς ἐκ τοῦ κέντρου τμημάτων ἦστιν γ', συναχθήσεται τὸ μὲν ἀπὸ τῆς τοῦ τετραγώνου πλευρᾶς γ', τὸ δὲ ἀπὸ τῆς τοῦ τριγώνου Μ ὦ. ἢστε καὶ μήκει ἡ μὲν τᾶς ξ μοίρας ὑποτείνουσα εὐθείᾳ τοιούτων ἦσται πδ' νά ἦ γ', ἦ γ', οἰῶν ἦ
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of the radius, consists of $30^\circ$ and its square of $900^\circ$, and $B\Delta$, being the radius, consists of $60^\circ$ and its square of $3600^\circ$, while $EB^3$, that is $EZ^2$, consists of $4500^\circ$; therefore $EZ$ is approximately $67^\circ 4' 55''$, and the remainder $\Delta Z$ is $37^\circ 4' 55''$. Therefore the side of the decagon, subtending an arc of $36^\circ$ (the whole circle consisting of $360^\circ$), is $37^\circ 4' 55''$ (the diameter being $120^\circ$). Again, since $\Delta Z$ is $37^\circ 4' 55''$, its square is $1375^\circ 4' 15''$, and the square on $\Delta B$ is $3600^\circ$, which added together make the square on $BZ$ $4975^\circ 4' 15''$, so that $BZ$ is approximately $70^\circ 32' 3''$. And therefore the side of the pentagon, subtending $72^\circ$ (the circle consisting of $360^\circ$), is $70^\circ 32' 3''$ (the diameter being $120^\circ$).

Hence it is clear that the side of the hexagon, subtending $60^\circ$ and being equal to the radius, is $60^\circ$. Similarly, since the square on the side of the square, subtending $90^\circ$, is double of the square on the radius, and the square on the side of the triangle, subtending $120^\circ$, is three times the square on the radius, while the square on the radius is $3600^\circ$, the square on the side of the square is $7200^\circ$ and the square on the side of the triangle is $10800^\circ$. Therefore the chord subtending $90^\circ$ is approximately $84^\circ 51' 10''$ (the diameter

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$a$ Theon's proof that $\sqrt{4500}$ is approximately $67^\circ 4' 55''$ has already been given (vol. i. pp. 56-61).

$b$ This is, of course, the square itself; the Greek phrase is not so difficult. We could translate, "the second power of the side of the square," but the notion of powers was outside the ken of the Greek mathematician.
Let $AB$ be a chord of a circle subtending an angle $a$ at the centre $O$, and let $AKA'$ be drawn perpendicular to $OB$ so as to meet $OB$ in $K$ and the circle again in $A'$. Then

$$\sin a = \frac{AK}{AO} = \frac{1}{2} \frac{AA'}{AO}.$$

And $AA'$ is the chord subtended by double of the arc $AB$, while Ptolemy expresses the lengths of chords as so many 120th parts of the diameter; therefore $\sin a$ is half the chord subtended by an angle $2a$ at the centre, which is conveniently abbreviated by $\frac{1}{2}$ (crd. $2a$), or, as we may alternatively express the relationship, $\sin AB$ is "half the chord subtended by
consisting of 120°), and the chord subtending 120° is 103° 55' 23''.

(iii.) \( \sin^2 \theta + \cos^2 \theta = 1 \)

_Ibid._ 35, 17–36, 12

The lengths of these chords have thus been obtained immediately and by themselves, and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending 36° was shown to be 37° 4' 55'' and its square 1375° 4' 15'', while the square on the diameter is 14400°, therefore the square on the chord subtending the remaining 144° in the semicircle is double of the arc AB, which is the Ptolemaic form; as Ptolemy means by this expression precisely what we mean by \( \sin AB \), I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that \( \cos \alpha \) \([= \sin(90 - \alpha)] = \frac{1}{2} \text{ crd.} (180° - 2\alpha)\), or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." \( \tan \alpha \) and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended Ptolemy proves that

- side of decagon \((= \text{crd.} 36° = 2 \sin 18°) = 37° 4' 55''\),
- side of pentagon \((= \text{crd.} 72° = 2 \sin 36°) = 70° 32' 3''\),
- side of hexagon \((= \text{crd.} 60° = 2 \sin 30°) = 60°\),
- side of square \((= \text{crd.} 90° = 2 \sin 45°) = 84° 51' 10''\),
- side of equilateral triangle \((= \text{crd.} 120° = 2 \sin 60°) = 103° 55' 23''\).

\(^b\) _i.e._, not deduced from other known chords.
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Μὴ γυρὸν, αὐτὴ δὲ μήκει τῶν αὐτῶν Ῥῶ ζ Λ ἐγγυστά, καὶ ἐπὶ τῶν ἀλλών ὀμοίως.

Ον δὲ τρόπον ἀπὸ τούτων καὶ αἱ λοιπαὶ τῶν κατὰ μέρος δοθῆσονται, δεῖξομεν ἑφεξῆς προεκθέ-μενοι λημμάτιον εὐχρήστον πάνυ πρὸς τὴν παροῦσαν πραγματείαν.

(iv.) **Ptolemy's Theorem**


"Εστω γὰρ κύκλος ἐγγεγραμμένον ἐχων τετρά-πλευρον τυχὼν τὸ ἈΒΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ καὶ ΒΔ. δευκτέον, ὅτι τὸ ύπὸ τῶν ΑΓ καὶ ΒΔ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ συναμφοτέρους τῷ τε ύπὸ τῶν ΑΒ, ΔΓ καὶ τῷ ύπὸ τῶν ΛΔ, ΒΓ.

Κείσθω γὰρ τῇ ύπὸ τῶν ΔΒΓ γωνία ἴση ἡ ύπὸ ΑΒΕ. ἕαν οὖν κοινὴν προσθῶμεν τὴν ύπὸ ΕΒΔ,

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* i.e., crd. 144°(=2 sin 72°)=114° 7' 37". If the given chord subtends an angle 2θ at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle (180 - 2θ), and the theorem asserts that

\[(\text{crd. } 2\theta)^2 + (\text{crd. } 180 - 2\theta)^2 = (\text{diameter})^2,\]

or

\[\sin^2 \theta + \cos^2 \theta = 1.\]

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13024° 55' 45" and the chord itself is approximately 114° 7' 37'', and similarly for the other chords."

We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(iv.) "Ptolemy's Theorem"

_Ibid._ 36. 13-37. 18

Let $\triangle ABCD$ be any quadrilateral inscribed in a circle, and let $AB$, $BC$, $CD$, and $DA$ be joined. It is required to prove that the rectangle contained by $AB$ and $BC$ is equal to the sum of the rectangles contained by $AB$, $CD$ and $DA$, $BC$.

For let the angle $ABE$ be placed equal to the angle $\triangle ABC$. Then if we add the angle $EBD$ to both, the
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ἔσται καὶ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΕΒΓ. ἔστω δὲ καὶ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΒΓΕ ἴση· τὸ γὰρ αὐτὸ τμῆμα ὑποτείνουσιν ἰσογώνιον ἀρα ἐστὶν τὸ ΑΒΔ τρίγωνον τῷ ΒΓΕ τριγώνῳ. ὥστε καὶ ἀνάλογον ἐστὶν, ὡς ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ· τὸ ἁρα ὑπὸ ΒΓ, ΑΔ ἵσον ἐστὶν τῷ ὑπὸ ΒΔ, ΓΕ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΔΒΓ γωνίᾳ, ἔστω δὲ καὶ ἡ ὑπὸ ΒΑΕ ἴση τῇ ὑπὸ ΒΔΓ, ἰσογώνιον ἀρα ἐστὶν τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ· ἀνάλογον ἀρα ἐστὶν, ὡς ἡ ΒΑ πρὸς ΑΕ, ἡ ΒΔ πρὸς ΔΓ· τὸ ἁρα ὑπὸ ΒΑ, ΔΓ ἵσον ἐστὶν τῷ ὑπὸ ΒΔ, ΑΕ. ἐδείχθη δὲ καὶ τὸ ὑπὸ ΒΓ, ΑΔ ἵσον τῷ ὑπὸ ΒΔ, ΓΕ· καὶ ὅλον ἁρα τὸ ὑπὸ ΑΓ, ΒΔ ἵσον ἐστὶν συναμφότερος τῷ τε ὑπὸ ΑΒ, ΔΓ καὶ τῷ ὑπὲρ ΑΔ, ΒΓ· ὀπερ ἐδει δεῖξαι.

(v.) \( \sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \)

Ibid. 37. 19-39. 3

Τούτου προεκτεθέντος ἔστω ἡμικύκλιον τὸ ΑΒΓΔ ἐπὶ διαμέτρου τῆς ΑΔ, καὶ ἀπὸ τοῦ Α δύο διήθθωσαν αἱ ΑΒ, ΑΓ, καὶ ἔστω ἑκατέρα αὐτῶν δοθεία τῷ μεγέθει, οἷον ἡ διάμετρος δοθεία ἡ πρ., καὶ ἐπεζεύχθω ἡ ΒΓ. λέγω, ὅτι καὶ αὐτὴ δεδοται.

Ἕπεξεύχθωσαν γὰρ αἱ ΒΔ, ΓΔ· δεδομέναι ἄρα εἰσὶν δηλονότι καὶ αὐταὶ διὰ τὸ λείπειν ἑκείνων εἰς τὸ ἡμικύκλιον. ἐπεὶ οὖν ἐν κύκλῳ τετράπλευρον ἐστὶν τὸ ΑΒΓΔ, τὸ ἁρὰ ὑπὸ ΑΒ, ΓΔ μετὰ τοῦ 424
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angle $\angle \Delta \beta \lambda = \angle \text{the angle } \beta \beta \theta$. But the angle $\Delta \beta \lambda = \angle \text{the angle } \beta \beta \theta [\text{Eucl. iii. 21}]$, for they subtend the same segment; therefore the triangle $\Delta \beta \lambda$ is equiangular with the triangle $\beta \beta \theta$.

\[ \therefore \beta \beta \gamma : \beta \beta \theta = \beta \beta \lambda : \beta \beta \lambda; \quad [\text{Eucl. vi. 4} \]
\[ \therefore \beta \beta \gamma . \beta \beta \lambda = \beta \beta \lambda . \beta \beta \theta. \quad [\text{Eucl. vi. 6} \]

Again, since the angle $\beta \beta \theta \lambda$ is equal to the angle $\beta \beta \gamma \lambda$, while the angle $\beta \beta \lambda \lambda$ is equal to the angle $\beta \beta \gamma \lambda$ $[\text{Eucl. iii. 21}]$, therefore the triangle $\beta \beta \theta \lambda$ is equiangular with the triangle $\beta \beta \lambda \gamma$

\[ \therefore \beta \beta \alpha : \beta \beta \theta = \beta \beta \lambda : \beta \beta \lambda; \quad [\text{Eucl. vi. 4} \]
\[ \therefore \beta \beta \alpha . \beta \beta \lambda = \beta \beta \lambda . \beta \beta \theta. \quad [\text{Eucl. vi. 6} \]

But it was shown that

\[ \beta \beta \gamma . \beta \beta \lambda = \beta \beta \lambda . \beta \beta \theta; \]

and $\therefore \beta \beta \alpha . \beta \beta \lambda = \beta \beta \alpha \lambda + \beta \beta \lambda . \beta \beta \theta$ $[\text{Eucl. ii. 1} \]

which was to be proved.

(v.) $\sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi$

Ibid. 37. 19-39. 3

This having first been proved, let $\beta \beta \gamma \alpha \lambda$ be a semicircle having $\beta \beta \alpha \lambda$ for its diameter, and from $\beta \beta$ let the two [chords] $\beta \beta \alpha \lambda$, $\beta \beta \gamma \lambda$ be drawn, and let each of them be given in length, in terms of the $120^\circ$ in the diameter, and let $\beta \beta \gamma$ be joined. I say that this also is given.

For let $\beta \beta \alpha$, $\beta \beta \lambda$ be joined; then clearly these also are given because they are the chords subtending the remainder of the semicircle. Then since $\beta \beta \gamma \alpha \lambda$ is a quadrilateral in a circle,
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υπὸ τῶν ΑΔ, ΒΓ ἴσον εἶστιν τῷ υπὸ ΑΓ, ΒΔ. καὶ εἶστιν τὸ τε υπὸ τῶν ΑΓ, ΒΔ δοθέν καὶ τὸ

υπὸ ΑΒ, ΓΔ· καὶ λοιπὸν ἄρα τὸ υπὸ ΑΔ, ΒΓ δοθέν εἶστιν. καὶ εἶστιν ἡ ΑΔ διάμετρος· δοθεῖσα ἄρα εἶστιν καὶ ἡ ΒΓ εὐθεία.

Καὶ φανερὸν ἡμῖν γέγονεν, ὅτι, εὰν δοθῶσιν δύο περιφέρειαι καὶ αἱ υπ' αὐτὰς εὐθείαι, δοθεῖσα εἶσται καὶ ἡ τὴν ὑπεροχὴν τῶν δύο περιφερειῶν ὑποτείνουσα εὐθεία. δῆλον δὲ, ὅτι διὰ τούτου τοῦ θεωρήματος ἄλλας τε συν ὅλιγας εὐθείας ἐγγράφωμεν ἀπὸ τῶν ἐν ταῖς καθ' αὐτὰς δεδομένων ὑπεροχῶν καὶ ὅτι καὶ τὴν υπὸ τὰς δώδεκα μοῖρας, ἐπειδὴ περ ἔχομεν τὴν τε υπὸ τὰς ξ καὶ τὴν υπὸ τὰς οβ. 426
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\[ AB \cdot \Gamma \Delta + A \Delta \cdot \Gamma \Gamma = A \Gamma \cdot B \Delta. \]

"Ptolemy's theorem"

And \( A \Gamma \cdot B \Delta \) is given, and also \( AB \cdot \Gamma \Delta \); therefore the remaining term \( A \Delta \cdot \Gamma \Gamma \) is also given. And \( A \Delta \) is the diameter; therefore the straight line \( \Gamma \Gamma \) is given.\(^a\)

And it has become clear to us that, if two arcs are given and the chords subtending them, the chord subtending the difference of the arcs will also be given. It is obvious that, by this theorem we can inscribe\(^b\) many other chords subtending the difference between given chords, and in particular we may obtain the chord subtending 12°, since we have that subtending 60° and that subtending 72°.

\(^a\) If \( A \Gamma \) subtends an angle 2\( \theta \) and \( AB \) an angle 2\( \phi \) at the centre, the theorem asserts that

\[
\text{crd. } (2\theta - 2\phi) = (\text{crd. } 180^\circ - 2\phi) - (\text{crd. } 2\phi) + (\text{crd. } 180^\circ - 2\theta)
\]

i.e.,

\[
\sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi.
\]

\(^b\) Or "calculate," as we might almost translate \( \varepsilon \gamma\gamma\rho\alpha\phi\mu\varepsilon\nu:\)

\text{cf. supra, p. 414 n. a on } \varepsilon\kappa \tau\omicron\nu \gamma\rho\alpha\mu\mu\omicron\nu.
Πάλιν προκεισθω δοθείσης τινός ευθείας ἐν κύκλῳ τὴν ὑπὸ τὸ ἡμιο τῆς ὑποτεινομένης περιφερείας εὐθείαν εὑρεῖν. καὶ ἐστι ήμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΓ καὶ δοθεία εὐθεία ἡ ΓΒ, καὶ ἡ ΓΒ περιφέρεια δίχα τετμῆσθω κατὰ τὸ Δ, καὶ ἐπεξεύθεσαν αἱ ΑΒ, ΑΔ, ΒΔ, ΔΓ, καὶ ἄπο τοῦ Δ ἐπὶ τὴν ΔΓ κάθετος ἡχθω ἡ ΔΖ. λέγω, ὅτι ἡ ΖΓ ἡμίσειά ἐστι τῆς τῶν ΑΒ καὶ ΑΓ ὑπεροχῆς.

Κείσθω γὰρ τῇ ΑΒ ἵσι ἡ ΑΕ, καὶ ἐπεξεύθεσθω ἡ ΔΕ. ἐπεὶ ἵσι ἐστὶν ἡ ΑΒ τῇ ΑΕ, κοινῇ δὲ ἡ ΑΔ, δύο δὴ αἱ ΑΒ, ΑΔ δύο ταῖς ΑΕ, ΑΔ ἰσοι εἰσὶν ἐκατέρα ἐκατέρα. καὶ γνωνία ἡ ὑπὸ ΒΑΔ γνωνία τῇ ὑπὸ ΕΑΔ ἵσι ἐστίν· καὶ βάσις ἁρα ἡ ΒΔ βάσει τῇ ΔΕ ἵσι ἐστίν. ἀλλὰ ἡ ΒΔ τῇ ΔΓ ἵσι ἐστίν· καὶ ἡ ΔΓ ἁρα τῇ ΔΕ ἵσι ἐστίν. ἐπεὶ οὖν ἴσοσκελοῦς ὄντος τριγώνου τοῦ ΔΕΓ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἴκται ἡ ΔΖ, ἵσι ἐστίν ἡ ΕΖ τῇ ΖΓ. ἀλλ' ἡ ΕΓ ὄλη ἡ ὑπεροχή ἐστιν τῶν ΑΒ καὶ ΑΓ εὐθείων· ἡ ἁρα ΖΓ ἡμίσειά ἐστιν τῆς τῶν αὐτῶν ὑπεροχῆς. ὥστε, ἐπεὶ τῆς ὑπὸ τὴν ΒΓ περιφέρειαν εὐθείας ὑποκειμένης 428
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(vi.) \( \sin^2 \frac{1}{2} \theta = \frac{1}{2} (1 - \cos \theta) \)

*Ibid.* 39. 4-41. 3

Again, given any chord in a circle, let it be required to find the chord subtending half the arc subtended by the given chord. Let \( AB \Gamma \) be a semicircle upon the diameter \( A \Gamma \) and let the chord \( \Gamma B \) be given, and let the arc \( \Gamma B \) be bisected at \( \Delta \), and let \( AB, A\Delta, B\Delta, \Delta \Gamma \) be joined, and from \( \Delta \) let \( \Delta Z \) be drawn perpendicular to \( A\Gamma \). I say that \( Z\Gamma \) is half of the difference between \( AB \) and \( A\Gamma \).

For let \( AE \) be placed equal to \( AB \), and let \( \Delta E \) be joined. Since \( AB = AE \) and \( A\Delta \) is common, [in the triangles \( AB\Delta, AE\Delta \)] the two [sides] \( AB, A\Delta \) are equal to \( AE, A\Delta \) each to each; and the angle \( BA\Delta \) is equal to the angle \( EA\Delta \) [Eucl. iii. 27]; and therefore the base \( B\Delta \) is equal to the base \( \Delta E \) [Eucl. i. 4]. But \( B\Delta = \Delta \Gamma \); and therefore \( \Delta \Gamma = \Delta E \). Then since the triangle \( \Delta E \Gamma \) is isosceles and \( \Delta Z \) has been drawn from the vertex perpendicular to the base, \( EZ = Z\Gamma \) [Eucl. i. 26]. But the whole \( E\Gamma \) is the difference between the chords \( AB \) and \( A\Gamma \); therefore \( Z\Gamma \) is half of the difference. Thus, since the chord subtending the arc \( BI \Gamma \) is given, the chord \( AB \) subtending the remainder
αὐτόθιν δέδοται καὶ ἡ λείπουσα εἰς τὸ ἣμικύκλιον ἡ ἈΒ, δοθήσεται καὶ ἡ ΖΓ ἡμίσεια οὖσα τῆς τῶν ἈΓ καὶ ἈΒ ύπεροχὴς· ἀλλ' ἐπεὶ ἐν ὀρθογώνιοις τῷ ἌΓΔ καθέτου ἀξθείσης τῆς ΔΖ ἱσογώνιον γίνεται τὸ ἌΔΓ ὀρθογώνιον τῷ ΔΓΖ, καὶ ἔστω, ὡς ἡ ἈΓ πρὸς ΓΔ, ἡ ΓΔ πρὸς ΓΖ, τὸ ἀρα ύπο τῶν ἈΓ, ΓΖ περιεχόμενον ὀρθογώνιον ἱσον ἐστὶν τῷ ἀπὸ τῆς ΓΔ τετράγωνῳ. δοθὲν δὲ τὸ ύπὸ τῶν ἈΓ, ΓΖ. δοθὲν ἂρα ἐστὶν καὶ τὸ ἀπὸ τῆς ΓΔ τετράγωνου, ὡστε καὶ μήκει ἡ ΓΔ εὐθεία δοθήσεται τὴν ἡμίσειαν ύποτεινουσα τῆς ΒΓ περιφερείας.

Καὶ διὰ τούτου δὴ πάλιν τοῦ θεωρήματος ἄλλαι τε ληφθῆσονται πλεῖστα κατὰ τὰς ἡμισείας τῶν προεκτεθειμένων, καὶ ἰδὲ καὶ ἀπὸ τῆς τὰς ἰβ μοῖρας ύποτεινούσης εὐθείας ἢ τε ύπὸ τὰς ἦ καὶ ἡ ύπὸ τὰς ἦ καὶ ἡ ύπὸ τὸν μίαν ἡμισι καὶ ἡ ύπὸ τὸν ἡμισιν τέταρτον τῆς μᾶς μοίρας. εὑρίσκομεν δὲ ἐκ τῶν ἐπιλογισμῶν τὴν μὲν ύπὸ τὴν μίαν ἡμισιν μοῖραν τοιούτων ἃ λέονται ἐν ἐγχιστά, οἷον ἔστω ἡ διάμετρος ῬΚ, τὴν δὲ ύπὸ τὸ Ζ΄ δ’ τῶν αὐτῶν Ω μίλησιν.

(vii.) \( \cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \)

Ibid. 41. 4-43. 5

Πάλιν ἐστω κύκλος ὁ ἌΒΓΔ περὶ διάμετρον μὲν τὴν ἈΔ, κέντρον δὲ τὸ Ζ, καὶ ἀπὸ τοῦ Α ἀπελθῶσαν δύο περιφέρειας δοθέσαν κατὰ τὸ ἐξῆς αἱ ἍΒ, ΒΓ, καὶ ἐπεξεύθωσαν αἱ ἍΒ, ΒΓ ύπ᾿ αὐτὰς εὐθείας καὶ αὐτὰς δεδομέναι. λέγω ὅτι, εἰς ἐπεξεύθωσαν τὴν ἌΓ, δοθήσεται καὶ αὐτῇ.

* If ΒΓ subtends an angle 2θ at the centre the proposition asserts that
of the semicircle is immediately given, and $Z\Gamma$ will also be given, being half of the difference between $A\Gamma$ and $AB$. But since the perpendicular $\Delta Z$ has been drawn in the right-angled triangle $A\Gamma\Delta$, the right-angled triangle $A\Delta\Gamma$ is equiangular with $\Delta Z$ [Eucl. vi. 8], and

$$A\Gamma : \Gamma \Delta = \Gamma \Delta : \Gamma Z,$$

and therefore

$$A\Gamma \cdot \Gamma Z = \Gamma \Delta^2.$$

But $A\Gamma \cdot \Gamma Z$ is given; therefore $\Gamma \Delta^2$ is also given. Therefore the chord $\Gamma \Delta$, subtending half of the arc $B\Gamma$, is also given.\(^a\)

And again by this theorem many other chords can be obtained as the halves of known chords, and in particular from the chord subtending $12^\circ$ can be obtained the chord subtending $6^\circ$ and that subtending $3^\circ$ and that subtending $1\frac{1}{2}^\circ$ and that subtending $\frac{1}{4}^\circ + \frac{1}{4}^\circ (= \frac{3}{4}^\circ)$. We shall find, when we come to make the calculation, that the chord subtending $1\frac{1}{2}^\circ$ is approximately $1^\circ 34' 15''$ (the diameter being $120^\circ$) and that subtending $\frac{3}{4}^\circ$ is $0^\circ 47' 8''$.\(^b\)

(vii.) $\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$

_Ibid._ 41. 4–43. 5

Again, let $AB\Gamma\Delta$ be a circle about the diameter $AD$ and with centre $Z$, and from $A$ let there be cut off in succession two given arcs $AB$, $B\Gamma$, and let there be joined $AB$, $B\Gamma$, which, being the chords subtending them, are also given. I say that, if we join $A\Gamma$, it also will be given.

$$(\text{crd. } \theta)^2 = \frac{1}{4}(\text{crd. } 180) \cdot ((\text{crd. } 180^\circ) - \text{crd. } 180^\circ - 2\theta)$$

_i.e.,_ \(\sin \frac{1}{2} \theta = \frac{1}{4}(1 - \cos \theta)\).

\(^b\) The symbol in the Greek for $\Theta$ should be noted; _v._ vol. i. p. 47 n. a.

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Διήχθω γὰρ διὰ τοῦ Β διάμετρος τοῦ κύκλου ἡ BZE, καὶ ἐπεξεύχθωσαν αἱ ΒΔ, ΔΓ, ΓΕ, ΔΕ· δὴ λογὶς αὐτὸθεν, ὅτι διὰ μὲν τὴν ΒΓ δοθῆσαι καὶ ἡ ΓΕ, διὰ δὲ τὴν ΑΒ δοθῆσαι ἡ τε ΒΔ καὶ ἡ ΔΕ· καὶ διὰ τὰ αὐτὰ τοῖς ἐμπροσθεν, ἐπεὶ ἐν κύκλῳ τετράπλευρον ἐστιν τὸ ΒΓΔΕ, καὶ διηγούμενα εἰσὶν αἱ ΒΔ, ΓΕ, τὸ ὑπὸ τῶν διηγούμενον περι-
εχόμενον ὄρθογώνιον ἴσον ἐστὶν συναμφότερος τοῖς ὑπὸ τῶν ἀπεναντίων· ὡστε, ἐπεὶ δεδομένου τοῦ ὑπὸ τῶν ΒΔ, ΓΕ δέδοται καὶ τὸ ὑπὸ τῶν ΒΓ, ΔΕ, δέδοται ἃρα καὶ τὸ ὑπὸ ΒΕ, ΓΔ. δέδοται δὲ καὶ ἡ ΒΕ διάμετρος, καὶ λοιπῇ ἡ ΓΔ ἐσται δεδομένη, καὶ διὰ τοῦτο καὶ ἡ λείπουσα εἰς τὸ ἡμικύκλιον ἡ ΓΑ· ὡστε, ἐὰν δοθῶσιν δύο περι-
ϕέρειαι καὶ αἱ ὑπ’ αὐτάς εὐθείαι, δοθῆσεται καὶ ἡ συναμφότερας τὰς περιφερείας κατὰ σύνθεσιν ὑποτείνουσα εὐθεία διὰ τοῦτο τοῦ θεωρήματος.

* * *  
* If AB subtends an angle $2\theta$ and BG an angle $2\phi$ at the centre, the theorem asserts that  
\[
\frac{\text{crd. } 180°}{\text{crd. } 180° - 2\theta - 2\phi} = \frac{\text{crd. } 180°}{\text{crd. } 180° - 2\theta} \cdot \frac{\text{crd. } 180° - 2\phi}{\text{crd. } 2\theta} \cdot \frac{\text{crd. } 2\phi}{\text{crd. } 2\phi},
\]
\[
\text{i.e., } \cos \left(\theta + \phi\right) = \cos \theta \cos \phi - \sin \theta \sin \phi.
\]

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For through $B$ let $BZE$, the diameter of the circle, be drawn, and let $B\Delta, \Delta\Gamma, \Gamma E, \Delta E$ be joined; it is then immediately obvious that, by reason of $B\Gamma$ being given, $\Gamma E$ is also given, and by reason of $AB$ being given, both $B\Delta$ and $\Delta E$ are given. And by the same reasoning as before, since $B\Gamma\Delta E$ is a quadrilateral in a circle, and $B\Delta, \Gamma E$ are the diagonals, the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides. And so, since $B\Delta, \Gamma E$ is given, while $B\Gamma, \Delta E$ is also given, therefore $BE, \Gamma \Delta$ is given. But the diameter $BE$ is given, and [therefore] the remaining term $\Gamma \Delta$ will be given, and therefore the chord $\Gamma A$ subtending the remainder of the semicircle; accordingly, if two arcs be given, and the chords subtending them, by this theorem the chord subtending the sum of the arcs will also be given.
Φανερὸν δὲ, ὅτι συντιθέντες ἀεὶ μετὰ τῶν προεκτεθειμένων πασῶν τὴν ὑπὸ ᾧ ζέ μοῖραν καὶ τὰς συναπτομένας ἐπιλογιζόμενου πάσας ἀπλῶς ἐγγράφομεν, ὅσα διὰ γνώμεναι τρίτων μέρος ἐξουσιών, καὶ μόναι ἔτι περιεισθήσονται αἱ μεταξὺ τῶν ἀνὰ ᾧ ζέ μοῖραν διαστημάτων δύο καθ' ἕκαστον ἔσομεναι, ἐπειδὴ περ ἥμμοιρίμου ποιοῦμεθα τῆν ἐγγραφήν. ὡστε, ἐὰν τὴν ὑπὸ τὸ ἥμμοιρίμου εὐθείαν εὑρωμεν, αὐτὴ κατὰ τε τὴν σύνθεσιν καὶ τὴν ὑπεροχήν τὴν πρὸς τὰς τὰ διαστήματα περιεχοῦσα καὶ διεδομένας εὐθείας καὶ τὰς λοιπὰς τὰς μεταξὺ πάσας ἡμῖν συναναπληρώσει. ἑπεὶ δὲ δοθείσης τῶν εὐθείας ὡς τῆς ὑπὸ τὴν ᾧ ζέ μοῖραν ἢ τὸ τρίτων τῆς αὐτῆς περιφερείας ὑποτείνουσα διὰ τῶν γραμμῶν οὐ δίδοται πῶς· εἰ δὲ γε δυνατὸν ἦν, εἴχομεν ἂν αὐτὸθεν καὶ τὴν ὑπὸ τὸ ἥμμοιρίμου πρότερον μεθοδεύσομεν τὴν ὑπὸ τὴν ᾧ μοῖραν ἀπὸ τε τῆς ὑπὸ τὴν ᾧ ζέ μοῖραν καὶ τῆς ὑπὸ ζέ δ' ὑποτεθέμενοι ἅμματοι, ο, καὶ μὴ πρὸ τοῦ καθόλου δύνηται τὰς πηλικότητας ὀρίζειν, ἐπὶ γε τῶν οὕτως ἐλαχίστων τὸ πρὸς τὰς ὀρισμένας ἀπαράλλακτον δύναττ' ἂν συντηρεῖν.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

Δέγω γάρ, ὅτι ἐὰν ἐν κύκλῳ διαχώσων ἀνίσοι δύο εὐθείαι, ἡ μεῖζον πρὸς τὴν ἐλάσσωνα ἐλάσσωνα λόγον ἔχει ἦπερ ἑπὶ τῆς μεῖζονος εὐθείας περιφέρεια πρὸς τὴν ἑπὶ τῆς ἐλάσσωνος.

Εστώ γάρ κύκλος ὁ ΑΒΓΔ, καὶ διήκθωσον ἐν αὐτῷ δύο εὐθείαι ἀνίσοι ἐλάσσων μὲν ἡ ΑΒ,
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It is clear that, by continually putting next to all known chords a chord subtending $1\frac{1}{4}^\circ$ and calculating the chords joining them, we may compute in a simple manner all chords subtending multiples of $1\frac{1}{2}^\circ$, and there will still be left only those within the $1\frac{1}{2}^\circ$ intervals—two in each case, since we are making the diagram in half degrees. Therefore, if we find the chord subtending $\frac{1}{2}^\circ$, this will enable us to complete, by the method of addition and subtraction with respect to the chords bounding the intervals, both the given chords and all the remaining, intervening chords. But when any chord subtending, say, $1\frac{1}{2}^\circ$, is given, the chord subtending the third part of the same arc is not given by the [above] calculations—if it were, we should obtain immediately the chord subtending $\frac{1}{4}^\circ$; therefore we shall first give a method for finding the chord subtending $1^\circ$ from the chord subtending $1\frac{1}{4}^\circ$ and that subtending $\frac{3}{4}^\circ$, assuming a little lemma which, even though it cannot be used for calculating lengths in general, in the case of such small chords will enable us to make an approximation indistinguishable from the correct figure.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

For I say that, if two unequal chords be drawn in a circle, the greater will bear to the less a less ratio than that which the arc on the greater chord bears to the arc on the lesser.

For let $AB\Delta$ be a circle, and in it let there be drawn two unequal chords, of which $AB$ is the lesser
μείζων δὲ ἡ ΒΓ. λέγω, ὅτι ἡ ΓΒ εὐθεία πρὸς τὴν ΒΑ εὐθείαν ἐλάσσονα λόγον ἔχει ἡπερ ἡ ΒΓ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τετμήσθω γὰρ ἡ ύπὸ ΑΒΓ γωνία δίχα υπὸ τῆς ΒΔ, καὶ ἐπεζεύχωσαν ἡ τε ΑΕΓ καὶ ἡ ΑΔ καὶ ἡ ΓΔ. καὶ ἐπεί ἡ ύπὸ ΑΒΓ γωνία δίχα τέτμηται υπὸ τῆς ΒΕΔ εὐθείας, ἵση μὲν ἐστὶν ἡ ΓΔ εὐθεία τῇ ΑΔ, μείζων δὲ ἡ ΓΕ τῆς ΕΑ. ἡχθὼ δὴ ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ. ἐπεὶ τοῖνυν μείζων ἐστὶν ἡ μὲν ΑΔ τῆς ΕΔ, ἡ δὲ ΕΔ τῆς ΔΖ, ὁ ἀρὰ κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεί, ὑπερπεσεῖται δὲ τὴν ΔΖ. γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσαθω ἡ ΔΖΘ. καὶ ἐπεὶ ὁ μὲν ΔΕΘ τομεὺς μείζων ἐστὶν τοῦ ΔΕΖ τριγώνου, τὸ δὲ ΔΕΑ τρίγωνον μείζουν τοῦ ΔΕΗ τομεὺς, τὸ ἀρὰ ΔΕΖ

* Lit. "let ΔΖΘ be produced."
and \( B \Gamma \) the greater. I say that

\[
\Gamma \beta : BA < \text{arc } B \Gamma : \text{arc } BA.
\]

For let the angle \( AB \Gamma \) be bisected by \( \beta \Delta \), and let

\[ A \Delta \Gamma \text{ and } A \Delta \text{ and } \Gamma \Delta \text{ be joined. Then since the angle } AB \Gamma \text{ is bisected by the chord } B \Delta \Delta, \text{ the chord } \Gamma \Delta = A \Delta [\text{Eucl. iii. 26, 29}], \text{ while } \Gamma \epsilon > EA [\text{Eucl. vi. 3}]. \]

Now let \( \Delta Z \) be drawn from \( \Delta \) perpendicular to \( A \epsilon \Gamma \). Then since \( A \Delta > E \Delta \), and \( E \Delta > \Delta Z \), the circle described with centre \( \Delta \) and radius \( \Delta E \) will cut \( A \Delta \), and will fall beyond \( \Delta Z \). Let \([\text{the arc}]\) \( H \epsilon \Theta \) be described, and let \( \Delta Z \) be produced to \( \Theta \). Then since

\[
\text{sector } \Delta \epsilon \Theta > \text{triangle } \Delta \epsilon Z,
\]

and

\[
\text{triangle } \Delta \epsilon A > \text{sector } \Delta \epsilon H,
\]

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τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον ἐλάσσονα λόγον ἔχει ἥπερ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ. ἀλλ' ὡς μὲν τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον, οὖτως ἡ ΕΖ εὐθεία πρὸς τὴν ΕΑ, ὡς δὲ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ τομέα, οὖτως ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἡ ἀρα ΖΕ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. καὶ συνθέντι ἀρα ἡ ΖΑ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΖΔΑ γωνία πρὸς τὴν ὑπὸ ΑΔΕ. καὶ τῶν ἡγομένων τὰ διπλάσια, ἡ ΓΑ εὐθεία πρὸς τὴν ΑΕ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΓΔΑ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. καὶ διελὸντι ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΓΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἀλλ' ὡς μὲν ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ, οὖτως ἡ ΓΒ εὐθεία πρὸς τὴν ΒΑ, ὡς δὲ ἡ ὑπὸ ΓΔΒ γωνία πρὸς τὴν ὑπὸ ΒΔΑ, οὖτως ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ. ἡ ΓΒ ἀρα εὐθεία πρὸς τὴν ΒΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τούτου δὴ ἦν ὑποκειμένου ἐστὼ κύκλος ὁ ΑΒΓ, καὶ διήχθωσαν ἐν αὐτῷ δύο εὐθείαι ἡ τε ΑΒ καὶ ἡ ΑΓ, ὑποκείσθω δὲ πρώτον ἡ μετὰ ΑΒ ὑποτείνουσα μᾶς μοίρας ¼ δ', ἡ δὲ ΑΓ μοῖραν ā. ἐπεὶ ἡ ΑΓ εὐθεία πρὸς τὴν ΒΑ εὐθείαν ἐλάσσονα λόγον ἔχει ἥπερ ἡ ΑΓ περιφέρεια πρὸς τὴν ΑΒ, ἡ δὲ ΑΓ περιφέρεια ἐπίτροπος ἐστὶν τῆς ΑΒ, ἡ ΓΑ ἀρα εὐθεία τῆς ΒΑ ἐλάσσων ἐστὶν ἡ ἐπίτροπος. ἀλλὰ ἡ ΑΒ εὐθεία ἐδείχθη τοιούτων ὁ μὲν ἤ, οἷον ἐστὶν ἡ διάμετρος ἑκ. ἡ ἀρα ΓΑ

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triangle ΔEZ : triangle ΔEA < sector ΔEθ : sector ΔEH.

But triangle ΔEZ : triangle ΔEA = EZ : EA,

and sector ΔEθ : sector ΔEH = angle ZΔE : angle EΔA.

\[ \therefore \quad ZE : EA < angle ZΔE : angle EΔA. \]

\[ \therefore \quad \text{componendo,} \quad ZA : EA < angle ZΔA : angle AΔE; \]

and, by doubling the antecedents,

\[ \Gamma A : AE < angle \Gamma ΔA : angle EΔA; \]

and \textit{dirimendo},

\[ \Gamma E : EA < angle \Gamma ΔE : angle EΔA. \]

But

\[ \Gamma E : EA = \Gamma B : BA, \quad [\text{Eucl. vi. 3} \]

and

angle ΓΔB : angle BΔA = arc ΓB : arc BA;

\[ [\text{Eucl. vi. 33} \]

\[ \therefore \quad \Gamma B : BA < arc \Gamma B : arc BA. \]

On this basis, then, let AΒΓ be a circle, and in it let there be drawn the two chords AΒ and AΓ, and let it first be supposed that AΒ subtends an angle of \( \frac{\pi}{3} \) and AΓ an angle of \( 1^\circ \). Then since

\[ AΓ : BA < arc AΓ : arc AΒ, \]

while

\[ \text{arc } AΓ = \frac{1}{3} \cdot \text{arc } AΒ, \]

\[ \therefore \quad \Gamma A : BA < \frac{1}{3}. \]

But the chord AΒ was shown to be \( 0^o 47' 8'' \) (the diameter being \( 120^o \)); therefore the chord ΓA

\[ * \text{If the chords } ΓB, \text{ BA subtend angles } 2θ, 2φ \text{ at the centre, this is equivalent to the formula,} \]

\[ \frac{\sin θ}{\sin φ} < \frac{θ}{φ}. \]

where \( θ < φ < \frac{1}{2}π. \)
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eὐθεῖα ἑλάσσων ἐστὶν τῶν αὐτῶν ἃ ἃ ὑ ὑπό ταῦτα γὰρ ἐπὶ τριτά ἐστὶν ἐγγύς ταῦν ὁ μὲν ἡ.

Πάλιν ἐπὶ τῆς αὐτῆς καταγραφῆς ἡ μὲν ἌΒ εὐθεῖα ὑποκείσθω ὑποτέινουσα μοῖραν ἃ, ἡ δὲ ἌΓ μοῖραν ἃ ἐ. κατὰ τὰ αὐτὰ δὴ, ἐπεὶ ἡ ἌΓ περιφέρεια τῆς ἌΒ ἐστίν ἡ ἡμιολία, ἡ ἍἈ ἀρα εὐθεῖα τῆς ἌΒ ἑλάσσων ἐστίν ἡ ἡμιόλιος. ἀλλὰ τὴν ἌΓ ἀπεδείξαμεν τοιούτων οὕσαν ἃ Ἡδ ἴπ, οἰων ἐστίν ἡ διάμετρος ἰπ. ἡ ἀρα ἌΒ εὐθεῖα μείζων ἐστίν τῶν αὐτῶν ἃ ἃ ὑ τοιτῶν γὰρ ἡμι-

όλια ἐστίν τὰ προκείμενα ἃ Ἡδ ἴπ. ὡστε, ἐπεὶ τῶν αὐτῶν ἐδείχθη καὶ μείζων καὶ ἑλάσσων ἡ τὴν μίαν μοίραν ὑποτείνουσα εὐθεῖα, καὶ ταῦτην δηλοῦν-

ὅτι ἐξομεν τοιούτων ἃ ἃ ὑ ἑγγύτα, οἰων ἐστίν ἡ διάμετρος ἰπ, καὶ διὰ τὰ προδεδειγμένα καὶ τὴν ὑπὸ τὸ ἡμιορίουν, ἡτίς εὐρίσκεται τῶν αὐτῶν
<1^\circ 2' 50''$; for this is approximately four-thirds of $0^\circ 47' 8''$.

Again, with the same diagram, let the chord $AB$ be supposed to subtend an angle of $1^\circ$, and $A\Gamma$ an angle of $1\frac{1}{2}^\circ$. By the same reasoning, since $\text{arc } A\Gamma = \frac{2}{3} \cdot \text{arc } AB$,

\[ \therefore \Gamma A : BA < \frac{2}{3}. \]

But we have proved $A\Gamma$ to be $1^\circ 34' 15''$ (the diameter being $120^\circ$); therefore the chord $AB > 1^\circ 2' 50''$; for $1^\circ 34' 15''$ is one-and-a-half times this number. Therefore, since the chord subtending an angle of $1^\circ$ has been shown to be both greater and less than [approximately] the same [length], manifestly we shall find it to have approximately this identical value $1^\circ 2' 50''$ (the diameter being $120^\circ$), and by what has been proved before we shall obtain the chord subtending $\frac{1}{2}^\circ$, which is found to be approximately
Ο Σα κε έγραστα. και συναναπληρωθήσεται τὰ λοιπά, ὡς ἔφαμεν, διαστήματα ἐκ μὲν τῆς πρὸς τὴν μίαν ἡμιοῦ μοῖραν λόγου ἐνεκεν ὡς ἐπὶ τοῦ πρῶτου διαστήματος συνθέσεως τοῦ ἡμιμοίρου δεικνυμένης τῆς ὑπὸ τὰς β μοῖρας, ἐκ δὲ τῆς ὑπεροχῆς τῆς πρὸς τὰς γ μοῖρας καὶ τῆς ὑπὸ τὰς β Λ διδομένης. ὡσαύτως δὲ καὶ ἐπὶ τῶν λοιπῶν.

(ix.) The Table

Ibid. 46. 21–63. 46

Ἡ μὲν οὖν πραγματεία τῶν ἐν τῷ κύκλῳ εὐθειῶν οὕτως ἄν οἴμαι βρώστα μεταχειρισθείη. ἦν δὲ, ὡς ἐφη, ἐφ' ἕκαστης τῶν χρειῶν ἐξ ἐτοιμου τὰς πηλικότητας ἐχομεν τῶν εὐθειῶν ἐκκειμένας, κανόνα ὑποτάξομεν ἀνὰ στίχους μὲ διὰ τοῦ συμμετροῦ, ὅτα μὲν πρῶτα μέρη περιέξει τὰς πηλικότητας τῶν περιφερειῶν καθ' ἡμιμοίρων παρηγε-μένας, τὰ δὲ δεύτερα τὰς τῶν παρακειμένων ταῖς περιφερείαις εὐθειῶν πηλικότητας ὡς τῆς διαμέτρου τῶν ρξ τμημάτων ὑποκειμένης, τὰ δὲ τρίτα τὸ λ' μέρος τῆς καθ' ἐκκοστοῦ ἡμιμοίρων τῶν εὐθειῶν παραγήσεως, ἦν ἐχοντες καὶ τὴν τοῦ ἐνὸς ἐξη-κοστοῦ μέσην ἐπιβολὴν ἀδιαφοροῦσαν πρὸς αὐθη-σιν τῆς ἀκριβοῦς καὶ τῶν μεταξὺ τοῦ ἡμίους μερῶν ἐξ ἐτοίμου τὰς ἐπιβαλλόντας πηλικότητας ἐπιλογίζεσθαι δυνώμεθα. εὐκατανόητον δ', ὅτι διὰ τῶν αὐτῶν καὶ προκειμένων θεωρημάτων, κἂν ἐν δισταγμῷ γενώμεθα γραφικῆς ἁμαρτίας περὶ τινὰ τῶν ἐν τῷ κανονίω παρακειμένων εὐ-θειῶν, ῥαδίαν ποιησόμεθα τὴν τε ἐξέτασιν καὶ τὴν
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31° 25' The remaining intervals may be completed, as we said, by means of the chord subtending $1\frac{1}{2}^\circ$—in the case of the first interval, for example, by adding $\frac{1}{2}^\circ$ we obtain the chord subtending $2^\circ$, and from the difference between this and $3^\circ$ we obtain the chord subtending $2\frac{1}{2}^\circ$, and so on for the remainder.

(iix.) The Table

Ibid. 46. 21-63. 46

The theory of the chords in the circle may thus, I think, be very easily grasped. In order that, as I said, we may have the lengths of all the chords in common use immediately available, we shall draw up tables arranged in forty-five symmetrical rows.

The first section will contain the magnitudes of the arcs increasing by half degrees, the second will contain the lengths of the chords subtending the arcs measured in parts of which the diameter contains 120, and the third will give the thirtieth part of the increase in the chords for each half degree, in order that for every sixtieth part of a degree we may have a mean approximation differing imperceptibly from the true figure and so be able readily to calculate the lengths corresponding to the fractions between the half degrees. It should be well noted that, by these same theorems now before us, if we should suspect an error in the computation of any of the chords in the table, we can easily make a test and

As there are 360 half degrees in the table, the statement appears to mean that the table occupied eight pages each of 45 rows; so Manitius, Des Claudius Ptolemaus Handbuch der Astronomie, 1st Bd., p. 35 n. a.

Such an error might be accumulated by using the approximations for $1^\circ$ and $\frac{1}{2}^\circ$; but, in fact, the sines in the table are generally correct to five places of decimals.

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Επανέρθωσιν ἦτοι ἀπὸ τῆς ὑπὸ τὴν δυσλασίων τῆς ἐπιζητουμένης ἢ τῆς πρὸς ἄλλας τινὰς τῶν δεδομένων ὑπεροχῆς ἢ τῆς τής λειτουργῶν εἰς τὸ ἡμικύκλιον περιφέρειαν ὑποτεινούσης εὐθείας. καὶ ἐστὶν ἢ τοῦ κανονίου καταγραφή τοιαύτη.

ια'. Κανόνιον τῶν ἐν κύκλῳ εὐθειῶν

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ξ | ξ | ο | ο | ο | ο | νδ | κα

| ποοο | μθ | νε | λη | λθ | λβ | ο | ο | ο | β | ξ | μζ | ιδ | νζ | μα |
| ποοζ | μθ | νζ | η | νβ | η | ο | ο | ο | ο | ο | ο | ο | ο | ο |
| ποζ | μθ | νη | νη | νε | η | ο | ο | ο | ο | ο | ο | ο | ο | ο |
| ποθ | μθ | νθ | νθ | νθ | μδ | ο | ο | ο | ο | ο | ο | ο | ο | ο |
| ποθ L' | μθ | νθ | νθ | νθ | νθ | ο | ο | ο | ο | ο | ο | ο | ο | ο |
| πθ | μθ | νθ | νθ | νθ | ο | ο | ο | ο | ο | ο | ο | ο | ο | ο |
apply a correction, either from the chord subtending double of the arc which is under investigation, or from the difference with respect to any others of the given magnitudes, or from the chord subtending the remainder of the semicircular arc. And this is the diagram of the table:

### 11. Table of the Chords in a Circle

<table>
<thead>
<tr>
<th>Ares</th>
<th>Chords</th>
<th>Sixtieths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1°</td>
<td>31'</td>
<td>25°</td>
</tr>
<tr>
<td>2°</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>3°</td>
<td>8</td>
<td>28</td>
</tr>
<tr>
<td>3½°</td>
<td>39</td>
<td>52</td>
</tr>
<tr>
<td>4°</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>4½°</td>
<td>42</td>
<td>40</td>
</tr>
<tr>
<td>60°</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>176°</td>
<td>119</td>
<td>55</td>
</tr>
<tr>
<td>176½°</td>
<td>119</td>
<td>56</td>
</tr>
<tr>
<td>177°</td>
<td>119</td>
<td>57</td>
</tr>
<tr>
<td>177½°</td>
<td>119</td>
<td>58</td>
</tr>
<tr>
<td>178°</td>
<td>119</td>
<td>58</td>
</tr>
<tr>
<td>178½°</td>
<td>119</td>
<td>59</td>
</tr>
<tr>
<td>179°</td>
<td>119</td>
<td>59</td>
</tr>
<tr>
<td>179½°</td>
<td>119</td>
<td>59</td>
</tr>
<tr>
<td>180°</td>
<td>120</td>
<td>0</td>
</tr>
</tbody>
</table>

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(c) Menelaus's Theorem

(i.) Lemmas

Ibid. 68. 14–74. 8.

ιγ'. Προλαμβανόμενα εἰς τὰς σφαιρικὰς
dεῖξεις

'Ακολούθου δ' οὖν ἀποδείξαι καὶ τὰς κατὰ
μέρος γνωμένας πηλικότητας τῶν ἀπολαμβανο-
μένων περιφερειῶν μεταξὺ τοῦ τε ἰσημερινοῦ καὶ
tοῦ διὰ μέσων τῶν Ζωδίων κύκλων τῶν γραφο-
μένων μεγίστων κύκλων διὰ τῶν τοῦ ἰσημερινοῦ
πόλων προεκθησόμεθα λημμάτια βραχέα καὶ εὐ-
χρηστα, δι' ὃν τὰς πλεῖστας σχεδὸν δεῖξεῖς τῶν
σφαιρικῶς θεωρουμένων, ὡς ἐνι μάλιστα, ἀπλοῦ-
στερον καὶ μεθοδικώτερον ποιησόμεθα.

Εἰς δύο δὴ εὐθεῖας τὰς ΑΒ καὶ ΑΓ διαχθεῖσαι
dύο εὐθείαι ἦ τε ΒΕ καὶ ἦ ΓΔ τεμνέτωσαν ἀλλήλας
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13. Preliminary matter for the spherical proofs

The next subject for investigation being to show the lengths of the arcs, intercepted between the celestial equator and the zodiac circle, of great circles drawn through the poles of the equator, we shall set out some brief and serviceable little lemmas, by means of which we shall be able to prove more simply and more systematically most of the questions investigated spherically.

Let two straight lines BE and ΓΔ be drawn so as to meet the straight lines AB and AF and to cut one
κατὰ τὸ Ζ σημεῖον. λέγω, ὅτι ὁ τῆς ΓΑ πρὸς ΑΕ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ.

"Ἡχθω γὰρ διὰ τοῦ Ε τῇ ΓΔ παράλληλος ἡ ΕΗ. ἐπεὶ παράλληλοι εἰσιν αἱ ΓΔ καὶ ΕΗ, ὁ τῆς ΓΑ πρὸς ΕΑ λόγος ὁ αὐτὸς ἐστιν τῷ τῆς ΓΔ πρὸς ΕΗ. ἔξωθεν δὲ ἡ ΖΔ· ὁ ἄρα τῆς ΓΔ πρὸς ΕΗ λόγος συγκείμενος ἐσται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ. ὦστε καὶ ὁ τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ. ἐστιν δὲ καὶ ὁ τῆς ΔΖ πρὸς ΗΕ λόγος ὁ αὐτὸς τῷ τῆς ΖΒ πρὸς ΒΕ διὰ τὸ παράλληλον πάλιν εἶναι τὰς ΕΗ καὶ ΖΔ· ὁ ἄρα τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ· ὅπερ προεκεῖτο δεῖξαι.

Κατὰ τὰ αὐτὰ δὲ δειχθῆσεται, ὅτι καὶ κατὰ διαίρεσιν ὁ τῆς ΓΕ πρὸς ΕΑ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ, διὰ τοῦ Α τῇ ΕΒ παράλληλον ἀξθείσης καὶ

* Lit. “the ratio of ΓΑ to ΑΕ is compounded of the ratio of ΓΔ to ΔΖ and ΖΒ to BE.”

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another at the point $Z$. I say that

$$\Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE).$$

For through $E$ let $EH$ be drawn parallel to $\Gamma \Delta$. Since $\Gamma \Delta$ and $EH$ are parallel,

$$\Gamma A : EA = \Gamma \Delta : EH.$$  \hspace{1cm} \text{[Eucl. vi. 4]}

But $Z\Delta$ is an external [straight line];

$$\therefore \quad \Gamma \Delta : EH = (\Gamma \Delta : \Delta Z)(\Delta Z : HE);$$

$$\therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(\Delta Z : HE).$$

But $\Delta Z : HE = ZB : BE,$  \hspace{1cm} \text{[Eucl. vi. 4]}

by reason of the fact that $EH$ and $Z\Delta$ are parallels;

$$\therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE); \quad (1)$$

which was set to be proved.

With the same premises, it will be shown by transformation of ratios that

$$\Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA),$$

a parallel to $EB$ being drawn through $A$ and $\Gamma \Delta H$. 

\[ \]
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προσεκβληθεῖσας ἐπ’ αὐτὴν τῆς ΓΔΗ. ἐπεὶ γὰρ πάλιν παράλληλός ἔστων ἡ ΑΗ τῇ EZ, ἔστω, ὡς ἡ ΓΕ πρὸς ΕΑ, ἡ ΓΖ πρὸς ΖΗ. ἀλλὰ τῆς ΖΔ ἔξωθεν λαμβανομένης ὁ τῆς ΓΖ πρὸς ΖΗ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΖΔ καὶ τοῦ τῆς ΔΖ πρὸς ΖΗ. ἔστω δὲ ὁ τῆς ΔΖ πρὸς ΖΗ λόγος ὁ αὐτὸς τῷ τῆς ΔΒ πρὸς ΒΑ διὰ τὸ εἰς παραλλήλους τὰς ΑΗ καὶ ΖΒ διήκει τὰς ΒΑ καὶ ΖΗ. οὗ ἀρα τῆς ΓΖ πρὸς ΖΗ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ. ἀλλὰ τῷ τῆς ΓΖ πρὸς ΖΗ λόγῳ ὁ αὐτὸς ἔστων ὁ τῆς ΓΕ πρὸς ΕΑ καὶ ὁ τῆς ΓΕ ἀρα πρὸς ΕΑ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ. ὡπερ ἐδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Δ, καὶ ἕλθθω ἐπὶ τὰς περιφερείας αὐτοῦ τυχόντα 450
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being produced to it. For, again, since AH is parallel to EZ,

\[ \Gamma E : EA = \Gamma Z : ZH. \]  

[Eucl. vi. 2]

But, an external straight line ZΔ having been taken,

\[ \Gamma Z : ZH = (\Gamma Z : Z\Delta)(\Delta Z : ZH); \]

and

\[ \Delta Z : ZH = \Delta B : BA, \]

by reason of BA and ZH being drawn to meet the parallels AH and ZB;

\[ \therefore \Gamma Z : ZH = (\Gamma Z : \Delta Z)(\Delta B : BA). \]

But \( \Gamma Z : ZH = \Gamma E : EA; \)  

[supra]

and \[ \therefore \Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA); \]

(2)

which was to be proved.

Again, let ABΓ be a circle with centre Δ, and let

there be taken on its circumference any three points
τρία σημεία τὰ Α, Β, Γ, ὥστε ἐκατέραν τῶν ΑΒ, ΒΓ περιφερείων ἔλασσον εἶναι ἡμικυκλίου· καὶ ἐπὶ τῶν ἔξης δὲ λαμβανομένων περιφερειῶν τὸ ὁμοιὸν ὑπακουέσθω· καὶ ἐπεξεύχθωσαν αἱ ΑΓ καὶ ΔΕΒ. λέγω, ὅτι ἔστιν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, οὕτως ἡ ΑΕ εὐθεία πρὸς τὴν ΕΓ εὐθείαν.

"Ἡχθωσαν γὰρ κάθετοι ἀπὸ τῶν Α καὶ Γ σημείων ἐπὶ τὴν ΔΒ ἢ τε ΑΖ καὶ ἡ ΓΗ. ἐπεὶ παράλληλος ἔστιν ἡ ΑΖ τῇ ΓΗ, καὶ διήκτει εἰς αὐτὰς εὐθεία ἡ ΑΕΓ, ἔστιν, ὡς ἡ ΑΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΑΕ πρὸς ΕΓ. ἀλλ' ὁ αὐτὸς ἐστιν λόγος ὁ τῆς ΑΖ πρὸς ΓΗ καὶ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ἡμίσεια γὰρ ἐκατέρα ἐκατέρας· καὶ ὁ τῆς ΑΕ ἀρα πρὸς ΕΓ λόγος ὁ αὐτὸς ἐστιν τῷ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ὀπερ ἔδει δεῖξαι.

Παρακολουθεὶ δ' αὐτόθεν, ὅτι, κἂν δοθῶσιν ἡ τε ΑΓ ὅλη περιφέρεια καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, δοθῆσεται καὶ ἐκατέρα τῶν ΑΒ καὶ ΒΓ περιφερεῖων. ἐκτεθείσης γὰρ τῆς αὐτῆς καταγραφῆς ἐπεξεύχθω ἡ ΑΔ, καὶ ἡχθω ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἢ ΔΖ. ὅτι μὲν οὖν τῆς ΑΓ περι-
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A, B, Γ, in such a manner that each of the arcs AB, BΓ is less than a semicircle; and upon the arcs taken in succession let there be a similar relationship; and let AΓ be joined and ΔEB. I say that

the chord subtended by double of the arc AB:
the chord subtended by double of the arc BΓ

\[ \text{[i.e., } \sin AB : \sin BΓ = AE : EG, \text{]} \]

For let perpendiculars AZ and ΓH be drawn from the points A and Γ to ΔB. Since AZ is parallel to ΓH, and the straight line AEG has been drawn to meet them,

\[ AZ : GH = AE : EG. \quad [\text{Eucl. vi. 4}] \]

But \( AZ : GH = \) the chord subtended by double of the arc AB:
the chord subtended by double of the arc BΓ,

for each term is half of the corresponding term;
and therefore

\[ AE : EG = \text{the chord subtended by double of the arc AB; } \]

the chord subtended by double of the arc BΓ.

\[ \text{[=sin AB : sin BΓ],} \]

which was to be proved.

It follows immediately that, if the whole arc AΓ be given, and the ratio of the chord subtended by double of the arc AB to the chord subtended by double of the arc BΓ [i.e. \( \sin AB : \sin BΓ \)], each of the arcs AB and BΓ will also be given. For let the same diagram be set out, and let AΔ be joined, and from Δ let ΔZ be drawn perpendicular to AEG. If the arc

* v. supra, p. 420 n. a.
φερείας δοθείσης ἢ τε ύπο ΑΔΖ γωνία τῆν ἡμι-
σειαν αὐτῆς ὑποτείνουσα δεδομένη ἔσται καὶ
ολον τὸ ΑΔΖ τρίγωνον, δήλου ἐπει δὲ τῆς ΑΓ

εὐθείας ὅλης δεδομένης ὑπόκειται καὶ ὁ τῆς ΑΕ
πρὸς ΕΓ λόγος ὁ αὐτὸς ὕψον τῶ τῆς ύπο τῆν διπλήν
τῆς ΑΒ πρὸς τὴν ύπο τῆν διπλήν τῆς ΒΓ, ἢ τε
ΑΕ ἔσται δοθείσα καὶ λοιπῇ ἢ ΖΕ. καὶ διὰ τοῦτο
καὶ τῆς ΔΖ δεδομένης δοθησθεῖται καὶ ἢ τε ύπο
ΕΔΖ γωνία τοῦ ΕΔΖ ὀρθογώνιον καὶ ὅλη ἢ ύπο
ΑΔΒ· ὥστε καὶ ἢ τε ΑΒ περιφερεία δοθῆσθαι
καὶ λοιπῆ ἢ ΒΓ· ὅπερ ἔδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ περὶ κέντρου τὸ Δ,
καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω τρία
σημεῖα τὰ Α, Β, Γ, ὥστε ἐκατέραν τῶν ΑΒ, ΑΓ
περιφερειῶν ἔλασσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ
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$\triangle A\Gamma$ is given, it is then clear that the angle $A\Delta Z$, subtending half the same arc, will also be given and therefore the whole triangle $A\Delta Z$; and since the whole chord $A\Gamma$ is given, and by hypothesis

$AE : E\Gamma = $ the chord subtended by double of the arc $AB$:

the chord subtended by double of the arc $B\Gamma$,

[i.e. $= \sin AB : \sin B\Gamma$],

therefore $AE$ will be given [Eucl. Dat. 7], and the remainder $ZE$. And for this reason, $\Delta Z$ also being given, the angle $E\Delta Z$ will be given in the right-angled triangle $E\Delta Z$, and [therefore] the whole angle $A\Delta B$; therefore the arc $AB$ will be given and also the remainder $B\Gamma$; which was to be proved.

Again, let $A\Gamma$ be a circle about centre $\Delta$, and let

three points $A$, $B$, $\Gamma$ be taken on its circumference so that each of the arcs $AB$, $A\Gamma$ is less than a semicircle;
τῶν ἕξις δὲ λαμβανομένων περιφερειῶν τὸ ὅμοιον ὑπακούσθω: καὶ ἐπιζευχθεῖσα ἢ τῇ ΔΔ καὶ ἢ ΓΒ ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τῷ Ε σημείον. λέγω, ὅτι ἑστών, ὡς ἢ ὑπὸ τὴν διπλήν τῆς ΓΑ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλήν τῆς AB, οὕτως ἢ ΓΕ εὐθείᾳ πρὸς τῇ BE.

'Ομοίως γὰρ τῷ πρωτέρῳ λημματίῳ, ἐὰν ἀπὸ τῶν B καὶ Γ ἀγάγωμεν καθέτους ἐπὶ τὴν ΔΔ τῆς τε ΒΖ καὶ τῆς ΓΗ, ἔσται διὰ τὸ παραλλήλους αὐτὰς εἶναι, ὡς ἢ ΓΗ πρὸς τὴν ΒΖ, οὕτως ἢ ΓΕ πρὸς τὴν ΕΒ. ὡστε καί, ὡς ἢ ὑπὸ τὴν διπλήν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλήν τῆς AB, οὕτως ἢ ΓΕ πρὸς τὴν ΕΒ. ὀπερ ἐδεί δεῖξαι.

Καὶ ἑνταῦθα δὲ αὐτόθεν παρακολουθεῖ, διότι, κἂν ἢ ΓΒ περιφέρεια μόνη δοθῇ, καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλήν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλήν τῆς AB δοθῇ, καὶ ἢ AB περιφέρεια δοθῇ· σεται. πάλιν γὰρ ἐπὶ τῆς ὀμοίας καταγράφης ἐπιζευχθείσης τῆς ΔΒ καὶ καθέτου ἀκτείσης ἐπὶ

τὴν ΒΓ τῆς ΔΖ ἢ μὲν ὑπὸ ΒΔΖ γονία τῆς ἡμι-

σείαν ὑποτείνουσα τῆς ΒΓ περιφερείας ἔσται

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and upon the arcs taken in succession let there be a similar relationship; and let $\Delta A$ be joined and let $\Gamma B$ be produced so as to meet it at the point $E$. I say that

the chord subtended by double of the arc $\Gamma A$:
the chord subtended by double of the arc $AB$

$[i.e., \sin \Gamma A : \sin AB] = \Gamma E : BE.$

For, as in the previous lemma, if from $B$ and $\Gamma$ we draw $BZ$ and $\Gamma H$ perpendicular to $\Delta A$, then, by reason of the fact that they are parallel,

$\Gamma H : BZ = \Gamma E : EB.$  [Eucl. vi. 4

$\therefore$ the chord subtended by double of the arc $\Gamma A$:
the chord subtended by double of the arc $AB$

$[i.e., \sin \Gamma A : \sin AB] = \Gamma E : EB$; . . . (4)

which was to be proved.

And thence it immediately follows why, if the arc $\Gamma B$ alone be given, and the ratio of the chord subtended by double of the arc $\Gamma A$ to the chord subtended by double of the arc $AB$ $[i.e., \sin \Gamma A : \sin AB]$, the arc $AB$ will also be given. For again, in a similar diagram let $\Delta B$ be joined and let $\Delta Z$ be drawn perpendicular to $B\Gamma$; then the angle $B\Delta Z$ subtended by half the arc $B\Gamma$ will be given; and therefore the
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dedomenh: kai olon ara to BΔZ ordoqonion. epei de kai o te tis GE pros tin EB logos dedotai kai eti he GB evdeia, dothsetai kai he te EB kai eti oly he EBZ. wste kai, epei he ΔZ dedotai, dothsetai kai he te upo EΔZ gwnia tou autou ordoqonion kai loipti he upo EΔB. wste kai he AB periferhei eystai dedomenh.

(ii.) The Theorem
Ibid. 74. 9–76. 9

Touton proleophventon gegrathwson epi sfaireistikis epifaneias megiston kyklon periferhein, wste eis duo tas AB kai AG duo grafeias tas BE kai GD teimewn alllias katta to Z smmeion: estw de ekasth auton elasson hmykukklion: to de autw kai epi pason twn katagraphwn upakouesothw.

Legeom de, osti o tis upo twn diplhn tis GE periferheias pros twn upo twn diplhn tis EA logos suniptai ek te tou tis upo twn diplhn tis ΓΖ pros twn upo twn diplhn tis ΖΔ kai tou tis upo twn diplhn tis DB pros twn upo twn diplhn tis BA.

Evlhphon gar to kentron tis sfairas kai estw to H, kai xighthsan apo tou H epi tas B, Z, E toimias ton kyklon he te HB kai he HZ kai he HE, kai epizeuxheisa he AD ekbeblhsos kai sumpitpetai tis HB ekblhisei kai auti katta to Θ smmeion, omoios de epizeuxheisa ai ΔΓ kai AG temveteis tais HZ kai HE katta to K kai Δ

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whole of the right-angled triangle $B\Delta Z$. But since the ratio $\Gamma E:EB$ is given and also the chord $\Gamma B$, therefore $EB$ will also be given and, further, the whole [straight line] $EBZ$; therefore, since $\Delta Z$ is given, the angle $E\Delta Z$ in the same right-angled triangle will be given, and the remainder $E\Delta B$. Therefore the arc $\Delta B$ will be given.

**(ii.) The Theorem**

*Ibid.* 74, 9–76, 9

These things having first been grasped, let there be described on the surface of a sphere arcs of great circles such that the two arcs $BE$ and $\Gamma \Delta$ will meet the two arcs $AB$ and $A\Gamma$ and will cut one another at the point $Z$; let each of them be less than a semicircle; and let this hold for all the diagrams.

Now I say that the ratio of the chord subtended by double of the arc $\Gamma E$ to the chord subtended by double of the arc $EA$ is compounded of (a) the ratio of the chord subtended by double of the arc $\Gamma Z$ to the chord subtended by double of the arc $Z\Delta$, and (b) the ratio of the chord subtended by double of the arc $\Delta B$ to the chord subtended by double of the arc $BA$,

\[
\left[ i.e., \frac{\sin \Gamma E}{\sin EA} = \frac{\sin \Gamma Z}{\sin Z\Delta} \cdot \frac{\sin \Delta B}{\sin BA} \right].
\]

For let the centre of the sphere be taken, and let it be $H$, and from $H$ let $HB$ and $HZ$ and $HE$ be drawn to $B, Z, E$, the points of intersection of the circles, and let $A\Delta$ be joined and produced, and let it meet $HB$ produced at the point $\Theta$, and similarly let $\Delta \Gamma$ and $A\Gamma$ be joined and cut $HZ$ and $HE$ at $K$ and the point
θημείον· ἐπὶ μᾶς δὴ γίνεται εὐθείας τὰ Θ, Κ, Λ
θημεία διὰ τὸ ἐν δυσὶν ἀμα εἶναι ἐπιτέοις τῶν ὑπὸ τοῦ ΛΔ τριγώνου καὶ τῶν ὑπὸ τοῦ ΒΖΕ κύκλου, ἦτε

ἐπίζευξθεῖσα ποιεῖ εἰς δύο εὐθείας τὰς ΘΑ καὶ
ΓΔ διηγμένας τὰς ΘΛ καὶ ΓΔ τεμνούσας ἀλλήλας
cατὰ τὸ Κ σημεῖον· ὦ ἄρα τῆς ΓΔ πρὸς ΔΑ λόγος
ανήπται ἐκ τῆς τῆς ΓΚ πρὸς ΚΔ καὶ τοῦ τῆς
ΔΘ πρὸς ΘΑ. ἀλλ' ὦς μέν ἡ ΓΔ πρὸς ΔΑ,
οὖτως ἡ ὑπὸ τὴν διπλὴν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν
dιπλὴν τῆς ΕΑ περιφερείας, ὡς ἓ δὲ ἡ ΓΚ πρὸς
ΚΔ, οὖτως ἡ ὑπὸ τὴν διπλὴν τῆς ΓΖ περιφερείας
πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΖΔ, ὡς ἓ δὲ ἡ ΘΔ
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\( \Lambda \); then the points \( \Theta, K, \Lambda \) will lie on one straight line because they lie simultaneously in two planes, that of the triangle \( \Delta \Gamma \Delta \) and that of the circle \( \text{BZE} \), and therefore we have straight lines \( \Theta \Lambda \) and \( \Gamma \Delta \) meeting the two straight lines \( \Theta \Lambda \) and \( \Gamma \Lambda \) and cutting one another at the point \( K \); therefore

\[
\Gamma \Lambda : \Delta \Lambda = (\Gamma K : K \Delta)(\Delta \Theta : \Theta \Lambda). \quad \text{[by (2)]}
\]

But \( \Gamma \Lambda : \Delta \Lambda = \) the chord subtended by double of the arc \( \Gamma \Theta \): 

the chord subtended by double of the arc \( \Theta \Lambda \)

[i.e., \( \sin \Gamma \Theta : \sin \Theta \Lambda \)],

while \( \Gamma K : K \Delta = \) the chord subtended by double of the arc \( \Gamma Z \): 

the chord subtended by double of the arc \( \Theta Z \) \quad \text{[by (3)]}

[i.e., \( \sin \Gamma Z : \sin \Theta Z \)],

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πρὸς ΘΑ, οὕτως ἦ ὑπὸ τὴν διπλὴν τῆς ΔΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΒΑ· καὶ ὁ λόγος ἄρα ὁ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΕΑ συνήπται ἐκ τε τοῦ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ τὴν διπλὴν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΒΑ.

Κατὰ τὰ αὐτὰ δὴ καὶ ὥσπερ ἐπὶ τῆς ἐπιπέδου καταγραφῆς τῶν εὐθειῶν δείκνυται, ὅτι καὶ ὁ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΕΑ λόγος συνήπται ἐκ τε τοῦ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΔ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΔΖ καὶ τοῦ τῆς ὑπὸ τὴν διπλὴν τῆς ΖΒ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΒΕ· ἀπερ προέκειτο δείξαι.

* From the Arabic version, it is known that "Menelaus's Theorem" was the first proposition in Book iii. of his Sphaerica, and several interesting deductions follow.
and $\Theta \Delta : \Theta A = \text{the chord subtended by double of the arc } \Delta B :$

the chord subtended by double of
the arc $BA$ \hspace{1cm} [\text{by (4)}$

[i.e., \sin \Delta B : \sin BA],

and therefore the ratio of the chord subtended by double of the arc $\Gamma E$ to the chord subtended by double of the arc $EA$ is compounded of $(a)$ the ratio of the chord subtended by double of the arc $\Gamma Z$ to the chord subtended by double of the arc $Z \Delta$, and $(b)$ the ratio of the chord subtended by double of the arc $\Delta B$ to the chord subtended by double of the arc $BA$,

$$\begin{bmatrix}
\text{i.e.,} & \sin \Gamma E & \sin \Gamma Z & \sin \Delta B \\
\sin EA & \sin Z \Delta & \sin BA
\end{bmatrix}.$$

Now with the same premises, and as in the case of the straight lines in the plane diagram [by (1)], it is shown that the ratio of the chord subtended by double of the arc $\Gamma A$ to the chord subtended by double of the arc $EA$ is compounded of $(a)$ the ratio of the chord subtended by double of the arc $\Gamma \Delta$ to the chord subtended by double of the arc $\Delta Z$, and $(b)$ the ratio of the chord subtended by double of the arc $ZB$ to the chord subtended by double of the chord $BE$,

$$\begin{bmatrix}
\text{i.e.,} & \sin \Gamma A & \sin \Gamma \Delta & \sin ZB \\
\sin EA & \sin \Delta Z & \sin BE
\end{bmatrix};$$

which was set to be proved.a
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XXII. MENSURATION: HERON OF ALEXANDRIA

(a) DEFINITIONS

Heron, Deff., ed. Heiberg (Heron iv.) 14. 1-24

Καὶ τὰ μὲν πρὸ τῆς γεωμετρικῆς στοιχείωσεως τεχνολογούμενα ύπογράφουσι σοι καὶ υποτυποῦμενος, ὡς ἔχει μᾶλιστα συντόμως, Διονύσιος λαμπρότατε, τὴν τε ἄρχην καὶ τὴν ὀλίγην σύνταξιν ποιήσομαι κατὰ τὴν τοῦ Εὐκλείδου τοῦ Στοιχείων τῆς ἐν γεωμετρία θεωρίας διδασκαλίαν ὁμοιὸς ὁ ὅπως οὐ μόνον τὰς ἐκείνου πραγματείας

* The problem of Heron's date is one of the most disputed questions in the history of Greek mathematics. The only details certainly known are that he lived after Apollonius, whom he quotes, and before Pappus, who cites him, say between 150 B.C. and A.D. 250. Many scraps of evidence have been thrown into the dispute, including the passage here first cited; for it is argued that the title λαμπρότατος corresponds to the Latin clarissimus, which was not in common use in the third century A.D. Both Heiberg (Heron, vol. v. p. ix) and Heath (H.G.M. ii. 306) place him, however, in the third century A.D., only a little earlier than Pappus.

The chief works of Heron are now definitively published in five volumes of the Teubner series. Perhaps the best known are his Pneumática and the Automata, in which he shows how to use the force of compressed air, water or steam; they are of great interest in the history of physics, and have led some to describe Heron as "the father of the turbine," but
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(a) Definitions

Heron, Definitions, ed. Heiberg (Heron lv.) 14. 1-24

In setting out for you as briefly as possible, O most excellent Dionysius, a sketch of the technical terms premised in the elements of geometry, I shall take as my starting point, and shall base my whole arrangement upon, the teaching of Euclid, the writer of the elements of theoretical geometry; for in this way I think I shall give you a good general understanding, as they have no mathematical interest they cannot be noticed here. Heron also wrote a Belopoeica on the construction of engines of war, and a Mechanics, which has survived in Arabic and in a few fragments of the Greek.

In geometry, Heron’s elaborate collection of Definitions has survived, but his Commentary on Euclid’s Elements is known only from extracts preserved by Proclus and an-Nairizi, the Arabic commentator. In mensuration there are extant the Metrica, Geometrica, Stereometrica, Geodaesia, Mensurae and Liber Geoponicon. The Metrica, discovered in a Constantinople ms. in 1896 by R. Schöne and edited by his son H. Schöne, seems to have preserved its original form more closely than the others, and will be relied on here in preference to them. Heron’s Dioptra, describing an instrument of the nature of a theodolite and its application to surveying, is also extant and will be cited here.

For a full list of Heron’s many works, v. Heath, H.G.M. ii. 308-310.
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εὕσυνοπτοὺς ἔσεσθαι σοι, ἀλλὰ καὶ πλείστας ἄλλας τῶν εἰς γεωμετρίαν ἀνηκόντων. ἄρξομαι τούτων ἀπὸ σημείου.

α'. Σημείων ἐστιν, οὗ μέρος οὐθέν ἡ πέρας ἀδιάστατον ἢ πέρας γραμμῆς, πέφυκε δὲ διανοία μόνη ληπτόν εἶναι ὡσανεί ἀμερές τε καὶ ἀμέγειθες τυχάνων. τοιοῦτον οὖν αὐτὸ φασιν εἶναι ὁδὸν ἐν χρόνῳ τὸ ἑνεστὸς καὶ οὗν μονάδα θέσιν ἔχουσιν. ἔστι μὲν οὖν τῇ οὐσίᾳ ταυτών τῇ μονάδι· ἀδιαύρετα γὰρ ἀμφοὶ καὶ ἀσώματα καὶ ἀμέριστα· τῇ δὲ ἐπι-

φανεία καὶ τῇ σχέσει διαφέρει· ἢ μὲν γὰρ μονάς ἀρχή ἀριθμοῦ, τὸ δὲ σημείον τῆς γεωμετρομένης οὐσίας ἀρχή, ἀρχὴ δὲ κατὰ ἐκθεσιν, οὐχ ὡς μέρος ὃν τῆς γραμμῆς, ὃς τοῦ ἀριθμοῦ μέρος ἡ μονάς, προεπιστόμενον δὲ αὐτῆς· καθήκοντος γὰρ ἡ μᾶλλον νοηθέντος ἐν ρύσει νοεῖται γραμμῆ, καὶ οὕτω σημείουν ἀρχὴν ἐστὶ γραμμῆς, ἐπιφάνεια δὲ στερεοῦ σωμάτος.

Ibid. 60. 22-62. 9

ζ'. Σπείρα γίνεται, ὅταν κύκλος ἐπὶ κύκλου τὸ κέντρον ἔχων ὅρθος ὁπλὸν πρὸς τὸ τοῦ κύκλου ἐπίπεδον περιενέχεις εἰς τὸ αὐτὸ πάλιν ἀποκατα-

σταθῇ· τὸ δὲ αὐτὸ τοῦτο καὶ κρίκος καλεῖται. διεχθῆς μὲν οὖν ἐστὶ σπείρα ἡ ἔχουσα διάλειμμα, συνεχῆς δὲ ἡ καθ' ἐν σημείοι συμπίπτουσα, ἐπι-

λάττουσα δὲ, καθ' ἤν ὁ περιφερόμενος κύκλος

                                                      1 ἦτοι Friedlein, ὅτι codd..

* The first definition is that of Euclid i. Def. 1, the third in effect that of Plato, who defined a point as ἀρχὴ γραμμῆς (Aristot. Metaph. 992 a 20); the second is reminiscent of Nicomachus, Arith. Introd. ii. 7. 1, v. vol. i. pp. 86-89.

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not only of Euclid’s works, but of many others pertaining to geometry. I shall begin, then, with the point.

1. A point is that which has no parts, or an extremity without extension, or the extremity of a line, and, being both without parts and without magnitude, it can be grasped by the understanding only. It is said to have the same character as the moment in time or the unit having position. It is the same as the unit in its fundamental nature, for they are both indivisible and incorporeal and without parts, but in relation to surface and position they differ; for the unit is the beginning of number, while the point is the beginning of geometrical being—but a beginning by way of setting out only, not as a part of a line, in the way that the unit is a part of number—and is prior to geometrical being in conception; for when a point moves, or rather is conceived in motion, a line is conceived, and in this way a point is the beginning of a line and a surface is the beginning of a solid body.

Ibid. 60. 22–62. 9

97. A spire is generated when a circle revolves and returns to its original position in such a manner that its centre traces a circle, the original circle remaining at right angles to the plane of this circle; this same curve is also called a ring. A spire is open when there is a gap, continuous when it touches at one point, and self-crossing when the revolving circle cuts itself.

αὐτὸς αὐτὸν τέμνει. γίνονται δὲ καὶ τούτων τομαὶ γραμμαῖ τινες ἰδιάζονσαι. οἱ δὲ τετράγωνοι κρίκοι ἐκπρίσματα εἰσὶ κυλίνδρων. γίνονται δὲ καὶ ἄλλα τινὰ ποικίλα πρίσματα ἐκ τε σφαιρῶν καὶ ἐκ μικτῶν ἐπιφανειῶν.

(b) Measurement of Areas and Volumes

(i.) Area of a Triangle Given the Sides.

Heron, Metr. i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

"Εστι δὲ καθολικὴ μέθοδος ὡστε τριών πλευρῶν δοθείσων οἴου ἡπτοτοῦ τριγώνου τὸ ἐμβαδὸν εὑρεῖν χωρίς καθέτου· οἷον ἐστωσαν αἱ τοῦ τριγώνου πλευραὶ μονάδων ₾, ₷, ⍺. σύνθες τὰ ₾ καὶ τὰ ₷ καὶ τὰ ⍺ γίγνεται κδ. τούτων λαβὲ τὸ ἕμμωσι· γίγνεται ἰβ. ἀφελε τὰς ₾ μονάδας· λοιπὰ ⍺. πάλιν ἀφελε ἀπὸ τῶν ἰβ τὰς ₷ λοιπὰ Δ. καὶ ἐτὶ τὰς ⍺ λοιπὰ ⍺. ποίησον τὰ ἰβ ἐπὶ τὰ ἰγ. γίγνονται ₾. ταῦτα ἐπὶ τὸν δ. γίγνονται σμ. ταῦτα ἐπὶ τὸν ⍺ γίγνεται ψκ. τούτων λαβὲ πλευρὰν καὶ ἔσται τὸ ἐμβαδὸν τοῦ τριγώνου. ἐπεὶ οὖν αἱ ψκ ῥητὶν τὴν πλευρὰν οὐκ ἐχουσί, ληψόμεθα μετὰ διαφόρου ἔλαχιστον τὴν πλευρὰν οὕτως· ἐπεὶ ὁ συνεγγίζων τῷ ψκ τετράγωνος ἐστὶν ὁ ψκθ καὶ πλευρὰν ἔχει τὸν κς, μέρισον τὰς ψκ εἰς τὸν κς· γίγνεται κς καὶ τρίτα δύο· πρόσθες τὰς κς· γίγνεται νῦν τρίτα δύο. τούτων τὸ ἕμμωσι· γίγνεται κς σγ. ἔσται ἀρα τοῦ ψκ ἡ πλευρά ἐγγίστα τὰ κς σγ. τὰ γὰρ κς σγ ἐφ' ἐαυτὰ γίγνεται ψκ λς· ὡστε τὸ διαφορὸν μονάδος ἐστὶ μόριον λς· ἕαν δὲ βουλώμεθα.
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Certain special curves are generated by sections of these spires. But the square rings are prismatic sections of cylinders; various other kinds of prismatic sections are formed from spheres and mixed surfaces.

(b) MEASUREMENT OF AREAS AND VOLUMES

(i.) Area of a Triangle Given the Sides

Heron, Metrica i. 8, ed. H. Schöne (Heron iii.) 18. 13–24. 21

There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9. Add together 7, 8 and 9; the result is 24. Take half of this, which gives 12. Take away 7; the remainder is 5. Again, from 12 take away 8; the remainder is 4. And again 9; the remainder is 3. Multiply 12 by 5; the result is 60. Multiply this by 4; the result is 240. Multiply this by 3; the result is 720. Take the square root of this and it will be the area of the triangle. Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square nearest to 720 is 729, having a root 27, divide 27 into 720; the result is \(26 \frac{2}{3}\); add 27; the result is \(53 \frac{3}{2}\). Take half of this; the result is \(26 \frac{1}{2} + \frac{1}{3}(=26 \frac{5}{6})\). Therefore the square root of 720 will be very nearly \(26 \frac{5}{6}\). For \(26 \frac{5}{6}\) multiplied by itself gives \(720 \frac{1}{3}\); so that the difference is \(\frac{1}{3}\). If we wish to make the difference less than \(\frac{1}{3}\),

* The passage should be read in conjunction with those from Proclus cited supra, pp. 360-365; note the slight difference in terminology—self-crossing for interlaced.
en elaasouni moriw tou le' twi diaforan glinveshvi, 
anti tou yekh tazoemen ta vun euredenta yek kai le', 
ka tiavta poiniasantes eurisimenev pollwi elatonta 
twvle' 1 le' twi diaforan gnominigh.

'H de gnometrik' tovto apoidezis estin yde: 
trigouin dotheiswn twv pleurwn eureiv to embadon. 
dnavaton mev ouv estin agagonwta[5] 2 
aian kathetou 
ka porissamenon autis to meladho eureiv 
twv 
trigouin to embadon, deon de estw choris 
ou 
kathetou to embadon porisastrongi.

*Estw to dothenv trigonon to ABG kai estw 
ekaistw twv AB, BG, GA dotheisa: eureiv to emba-

1 tou add. Heiberg.

* If a non-square number \( \Lambda \) is equal to \( a^2 \pm b \), Heron's method gives as a first approximation to \( \sqrt{\Lambda} \),

\[
a_1 = \frac{1}{2} \left( a + \frac{\Lambda}{a} \right),
\]

and as a second approximation,

\[
a_2 = \frac{1}{2} \left( a_1 + \frac{\Lambda}{a_1} \right).
\]

An equivalent formula is used by Rhabdas (v. vol. i. p. 30 n. b) and by a fourteenth century Calabrian monk Barlaam, who wrote in Greek and who indicated that the process could be continued indefinitely. Several modern writers have used the formula to account for Archimedes' approximations to \( \sqrt{3} \) (v. vol. i. p. 322 n. a).

* Heron had previously shown how to do this.
instead of 729 we shall take the number now found, 720$\frac{1}{3}$, and by the same method we shall find an approximation differing by much less than $\frac{1}{3}$.\(^a\)

The geometrical proof of this is as follows: In a triangle whose sides are given to find the area. Now it is possible to find the area of the triangle by drawing one perpendicular and calculating its magnitude,\(^b\) but let it be required to calculate the area without the perpendicular.

Let $\triangle AB\Gamma$ be the given triangle, and let each of $\triangle AB, \triangle B\Gamma, \triangle \Gamma A$ be given; to find the area. Let the
δών. ἐγγεγράφθω εἰς τὸ τριγώνων κύκλος οἱ ΔΕΖ, οὗ κέντρον ἦστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΒΗ, ΓΗ, ΔΗ, ΕΗ, ΖΗ. τὸ μὲν ἁρὰ υπὸ ΒΓ, ΕΗ διπλάσιον ἦστι τοῦ ΒΗΓ τριγώνου, τὸ δὲ υπὸ ΓΑ, ΖΗ τοῦ ΑΓΗ τριγώνου, <τὸ δὲ υπὸ ΑΒ, ΔΗ τοῦ ΑΒΓ τριγώνου> 1 τὸ ἁρὰ υπὸ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου καὶ τῆς ΕΗ, τούτεστι τῆς ἐκ τοῦ κέντρου τοῦ ΔΕΖ κύκλου, διπλάσιόν ἦστι τοῦ ΑΒΓ τριγώνου. ἐκβεβλήσθω η ΓΒ, καὶ τῇ ΑΔ ἰση κείσθω η ΒΘ· η ἁρὰ ΓΒΘ ἡμίσεια ἦστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου διὰ τὸ ἴσην εἶναι τὴν μὲν ΑΔ τῇ ΑΖ, τὴν δὲ ΔΒ τῇ ΒΕ, τὴν δὲ ΖΓ τῇ ΓΕ. τὸ ἁρὰ υπὸ τῶν ΓΘ, ΕΗ ἰσον ἦστι τῶν ΑΒΓ τριγώνων. ἀλλὰ τὸ υπὸ τῶν ΓΘ, ΕΗ πλευρά ἦστω τοῦ ἁπὸ τῆς ΓΘ ἐπὶ τὸ ἁπὸ τῆς ΕΗ· ἔσται ἁρὰ τοῦ ΑΒΓ τριγώνου τὸ ἐμβαδον ἐφ’ ἐαυτῷ γενόμενον ἰσον τῷ ἁπὸ τῆς ΘΓ ἐπὶ τὸ ἁπὸ τῆς ΕΗ. ἡχθω τῇ μὲν ΓΗ πρὸς ὀρθας η ΗΛ, τῇ δὲ ΓΒ η ΒΑ, καὶ ἐπεζεύχθω η ΓΑ. ἐπεὶ οὐν ὀρθή ἦστω ἐκατέρα τῶν ὑπὸ ΓΗΔ, ΓΒΔ, ἐν κύκλῳ ἁρὰ ἦστι τὸ ΓΗΒΔ τετράπλευρον· αἱ ἁρὰ υπὸ ΓΗΒ, ΓΛΒ δυσὶν ὀρθαὶς ἐσιν ισαι. εἰσὶν δὲ καὶ αἱ ὑπὸ ΓΗΒ, ΑΗΔ δυσὶν ὀρθαὶς ισαι διὰ τὸ δίγα τετμῆθαι τὰς πρὸς τῷ Η γωνίας ταῖς ΑΗ, ΒΗ, ΓΗ καὶ ισας εἶναι τὰς υπὸ τῶν ΓΗΒ, ΑΗΔ ταῖς υπὸ τῶν ΑΗΓ, ΔΗΒ καὶ τὰς πᾶσας τετραγώνων ὀρθαὶς ισας εἶναι· ἵση ἁρὰ ἦστιν ἤ υπὸ ΑΗΔ τῇ υπὸ ΓΛΒ. ἐστὶ δὲ καὶ ὀρθή ἤ υπὸ ΑΔΗ ὀρθή τῇ υπὸ ΓΒΔ ἰση· ὁμοιον ἁρὰ ἦστι τὸ ΑΗΔ τριγώνων τῷ ΓΒΔ τριγώνων. ως ἁρὰ ἤ ΒΓ πρὸς

1 τὸ δὲ ... τριγώνου: these words, along with several
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circle $\triangle E\!Z\!H$ be inscribed in the triangle with centre $H$ [Eucl. iv. 4], and let $AH, BH, GH, DH, EH, ZH$ be joined. Then

$BG \cdot EH = 2 \cdot \triangle B\!H\!G \!$, [Eucl. i. 41

$GA \cdot ZH = 2 \cdot \triangle A\!H\!G \!$, [ibid.

$AB \cdot DH = 2 \cdot \triangle A\!B\!H \!$. [ibid.

Therefore the rectangle contained by the perimeter of the triangle $\triangle ABG$ and $EH$, that is the radius of the circle $\triangle E\!Z\!H$, is double of the triangle $\triangle ABG$. Let $GB$ be produced and let $B\Theta$ be placed equal to $A\Delta$; then $B\Theta\Theta$ is half of the perimeter of the triangle $\triangle ABG$ because $A\Delta = AZ, AB = BE, Z\Gamma = \Gamma E$ [by Eucl. iii. 17]. Therefore

$G\Theta \cdot EH = \triangle ABG \!$, [ibid.

But

$G\Theta \cdot EH = \sqrt{G\Theta^2 \cdot EH^2}$

therefore

$(\triangle ABG)^2 = G\Theta^2 \cdot EH^2$.

Let $HA$ be drawn perpendicular to $GH$ and $BA$ perpendicular to $GB$, and let $\Gamma\Delta$ be joined. Then since each of the angles $GHA, GBA$ is right, a circle can be described about the quadrilateral $GHB\Delta$ [by Eucl. iii. 31]; therefore the angles $GHB, \Gamma\Delta B$ are together equal to two right angles [Eucl. iii. 22]. But the angles $GHB, A\Delta H$ are together equal to two right angles because the angles at $H$ are bisected by $AH, BH, GH$ and the angles $GHB, A\Delta H$ together with $A\!H\!G, \Delta HB$ are equal to four right angles; therefore the angle $A\Delta H$ is equal to the angle $\Gamma\Delta B$. But the right angle $A\Delta H$ is equal to the right angle $\Gamma B\Delta$; therefore the triangle $\triangle A\Delta H$ is similar to the triangle $\Gamma B\Delta$.

other obvious corrections not specified in this edition, were rightly added to the text by a fifteenth-century scribe.

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ΒΑ, ἣ ΑΔ πρὸς ΔΗ, τοιτέστιν ἣ ΒΘ πρὸς ΕΗ, καὶ ἐναλλάξ, ὡς ἢ ΓΒ πρὸς ΒΘ, ἢ ΒΑ πρὸς ΕΗ, τοιτέστιν ἢ ΒΚ πρὸς ΚΕ διὰ τὸ παράλληλον εἶναι τὴν ΒΑ τῇ ΕΗ, καὶ συνθέντι, ὡς ἢ ΓΘ πρὸς ΒΘ, οὕτως ἢ ΒΕ πρὸς ΕΚ· ὥστε καὶ ὡς τὸ ἀπὸ τῆς ΓΘ πρὸς τὸ ὑπὸ τῶν ΓΘ, ΘΒ, οὕτως τὸ ὑπὸ ΒΕΓ πρὸς τὸ ὑπὸ ΓΕΚ, τοιτέστι πρὸς τὸ ἀπὸ ΕΗ· ἐν ὀρθογώνῳ γὰρ ἀπὸ τῆς ὀρθῆς ἐπὶ τὴν βάσιν κάθετος ἦκται ἢ ΕΗ· ὥστε τὸ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ, οὐ πλευρὰ ἦν τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου, ἵσσον ἦσται τὸ ὑπὸ ΓΘΒ ἐπὶ τὸ ὑπὸ ΓΕΒ. καὶ ἦστι δοθεῖσα ἐκάστῃ τῶν ΓΘ, ΘΒ, ΒΕ, ΓΕ· ἢ μὲν γὰρ ΓΘ ἡμίσεια ἦστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου, ἢ δὲ ΒΘ ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΓΒ, ἢ δὲ BE ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΒΓ, ἢ δὲ ΕΓ ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΒ, ἐπειδὴ ἦστι ἢ μὲν ΕΓ τῇ ΕΖ, ἢ δὲ ΒΘ τῇ ΑΖ, ἐπεί καὶ τῇ ΑΔ ἦστιν ἢσ. δοθὲν ἄρα καὶ τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου.

(ii.) Volume of a Spire

Ibid. ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

"Εστω γὰρ τις ἐν ἐπιπέδῳ εὐθείᾳ ἡ AB καὶ δύο τυχόντα ἐπ' αὐτῆς σημεία. εἴληφθω δ' ΒΓΔΕ (κύκλος) ὁ ὀρθὸς ὃν πρὸς τὸ ύποκείμενον ἐπίπεδον, ἐν ὄ ἦστιν ἡ AB εὐθεία, καὶ μένοντος τοῦ Α

1 κύκλος add. H. Schöne.
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Therefore \[ \frac{BG}{BA} = \frac{\Delta\Delta}{\DeltaH} = \frac{B\Theta}{EH}, \]
and \textit{permutando}, \[ \frac{GB}{B\Theta} = \frac{BA}{EA} = \frac{B\Theta}{EK}, \]
because \(BA\) is parallel to \(EH\),
and \textit{componendo} \[ \frac{G\Theta}{B\Theta} = \frac{BE}{EK}; \]
therefore \[ \frac{G\Theta^2}{G\Theta \cdot OB} = \frac{BE \cdot EG}{GE \cdot EK}, \]
i.e. \[ = \frac{BE \cdot EG}{EH^2}, \]
for in a right-angled triangle \(EH\) has been drawn from the right angle perpendicular to the base; therefore \(G\Theta^2 \cdot EH^2\), whose square root is the area of the triangle \(ABG\), is equal to \((G\Theta \cdot OB)(GE \cdot EB)\).
And each of \(G\Theta, OB, BE, GE\) is given; for \(G\Theta\) is half of the perimeter of the triangle \(ABG\), while \(B\Theta\) is the excess of half the perimeter over \(GB\), \(BE\) is the excess of half the perimeter over \(AI\), and \(EG\) is the excess of half the perimeter over \(AB\), inasmuch as \(EG = GZ\), \(B\Theta = \Delta\Delta = AZ\). Therefore the area of the triangle \(ABG\) is given.\\

(ii.) \textit{Volume of a Spire}

\textit{Ibid.} ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

Let \(AB\) be any straight line in a plane and \(A, B\) any two points taken on it. Let the circle \(B \Gamma \Delta \Theta\) be taken perpendicular to the plane of the horizontal, in which lies the straight line \(AB\), and, while the point

* If the sides of the triangle are \(a, b, c\), and \(s = \frac{1}{2}(a + b + c)\), Heron’s formula may be stated in the familiar terms,
\[
\text{area of triangle} = \sqrt{s(s - a)(s - b)(s - c)}.
\]
Heron also proves the formula in his \textit{Dioptra} 30, but it is now known from Arabian sources to have been discovered by Archimedes.
σημείου περιφερέσθω κατά τὸ ἑπὶπέδου ἢ ΑΒ, ἀχρὶ οὗ εἰς τὸ αὐτὸ ἀποκαταστᾷ συμπεριφερο-
μένου καὶ τοῦ ΒΓΔΕ κύκλου ὀρθοῦ διαμένοντος πρὸς τὸ ὑποκείμενον ἑπὶπέδου. ἀπογεννήσει ἀρα-
τινὰ ἐπιφανείαν ἢ ΒΓΔΕ περιφέρεια, ἢν δὴ-
σπειρικὴς καλοῦσιν· κἂν μὴ ἢ δὲ ὅλος ὁ κύκλος,
ἀλλὰ τμῆμα αὐτοῦ, πάλιν ἀπογεννήσει τὸ τοῦ κύκλου τμῆμα σπειρικῆς ἐπιφανείας τμῆμα, καθ-
ἀπερ εἰσὶ καὶ αἱ ταῖς κύσεως ὑποκείμεναι σπειραῖ-
τριών γὰρ οὐσῶν ἐπιφανείων ἐν τῷ καλουμένῳ
ἀναγραφεῖ, ὃν δὴ τινες καὶ ἐμβολέος καλοῦσιν, δύο
μὲν κοίλων τῶν ἄκρων, μιᾶς δὲ μέσης καὶ κυρτῆς,
ἀμα περιφερόμεναι αἱ τρεῖς ἀπογεννώσι τὸ ἐἴδος
τῆς τοῖς κύσεις ὑποκειμένης σπειράς.

Δέον οὖν ἐστὶ τὴν ἀπογεννηθεῖσαν σπειράν ὑπὸ
τοῦ ΒΓΔΕ κύκλου μετρῆσαι. δεδοσθο ἢ μὲν
ΑΒ μονάδων κ, ἢ δὲ ΒΓ διάμετρος μονάδων ἸΒ.

εἴληφθω τὸ κέντρον τοῦ κύκλου τὸ Ζ, καὶ ἀπὸ τῶν
Α, Ζ τῶν ὑποκειμένων ἑπὶπέδω ἐν ὀρθᾶ ἡχθωσαν
αἱ ΔΖΕ, ΗΑΘ. καὶ διὰ τῶν Δ, Ε τῇ ΑΒ παράλ-
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A remains stationary, let AB revolve in the plane until it concludes its motion at the place where it started, the circle $B\Gamma\Delta E$ remaining throughout perpendicular to the plane of the horizontal. Then the circumference $B\Gamma\Delta E$ will generate a certain surface, which is called spiric; and if the whole circle do not revolve, but only a segment of it, the segment of the circle will again generate a segment of a spiric surface, such as are the spirae on which columns rest; for as there are three surfaces in the so-called anagrapheus, which some call also emboleus, two concave (the extremes) and one (the middle) convex, when the three are moved round simultaneously they generate the form of the spira on which columns rest.

Let it then be required to measure the spire generated by the circle $B\Gamma\Delta E$. Let AB be given as 20, and the diameter $B\Gamma$ as 12. Let Z be the centre of the circle, and through $A, Z$ let $HA\Theta, \Delta ZE$ be drawn perpendicular to the plane of the horizontal. And through $\Delta, E$ let $\Delta H, \Theta E$ be drawn parallel to

* The ἀναγραφεῖς or ἐμβολεύς is the pattern or templet for applying to an architectural feature, in this case an Attic-Ionic column-base. The Attic-Ionic base consists essentially of two convex mouldings, separated by a concave one. In practice, there are always narrow vertical ribbons between the convex mouldings and the concave one, but Heron ignores them. In the templet, there are naturally two concave surfaces separated by a convex, and the kind of figure Heron had in mind appears to be that here illustrated. I am indebted to Mr. D. S. Robertson, Regius Professor of Greek in the University of Cambridge, for help in elucidating this passage.

Lit. "from."
ληλού ἦχθωσαν αἱ ΔΗ, ΕΘ. δεδεικταὶ δὲ Διώνυ
σισδόρων εν τῷ Περὶ τῆς σπείρας ἐπιγραφομένῳ,
ὅτι δὲ πάνω ἔχει ὁ ΒΓΔΕ κύκλος πρὸς τὸ ήμισν
τοῦ ΔΕΘ Παραλληλόγραμμον, τούτων ἔχει καὶ
ἡ γεννηθείσα σπείρα ὑπὸ τοῦ ΒΓΔΕ κύκλου πρὸς
tὸν κύλινδρον, οὐ δὲ ἔχει δὲ καὶ η ἩΘ, ἡ δὲ ἐκ
tοῦ κέντρου τῆς βάσεως ἢ ΕΘ. ἐπεὶ οὖν ἡ ΒΓ
μονάδων ἦς ἔστιν, ἡ ἀρα ΖΓ ἐσται μονάδων 5.
ἔστι δὲ καὶ ἡ ΑΓ μονάδων ἦς ἔσται ἀρα ἡ ΑΖ
μονάδων ἦς, τούτων ἢ ΕΘ, ἦς ἔστιν ἐκ τοῦ
κέντρου τῆς βάσεως τοῦ εἰρημένου κύλινδρον.  
δοθεῖς ἀρα ἔστιν ὁ κύκλος· ἀλλὰ καὶ ὁ ἐξωθοθεῖς
ἔστιν γὰρ μονάδων ἦς, ἐπεὶ καὶ ἡ ΔΕ. ὡστε
dοθεῖς καὶ ὁ εἰρημένος κύλινδρος· καὶ ἔστι τὸ ΔΘ
παραλληλόγραμμον (δοθεῖν). ὡστε καὶ τὸ ήμισν
αὐτοῦ. ἀλλὰ καὶ ὁ ΒΓΔΕ κύκλος· δοθεῖσα γὰρ
ἡ ΓΒ διάμετρος. λόγος ἀρα τοῦ ΒΓΔΕ κύκλου
πρὸς τὸ ΔΘ παραλληλόγραμμον δοθεῖς· ὡστε καὶ
tῆς σπείρας πρὸς τὸν κύλινδρον λόγος ἔστι δοθεῖς.
καὶ ἔστι δοθεῖς ὁ κύλινδρος· δοθεῖν ἀρα καὶ τὸ
στερεόν τῆς σπείρας.

Συντεθήσεται δὴ ἀκολούθως τῇ ἀναλύει οὕτως.
άφελε ἀπὸ τῶν ἦς τὰ ἦς· λοιπὰ ἦς. καὶ πρόσθες τὰ
κ. γίγνεται ἦς· καὶ μέτρησον κύλινδρον, οὐ ἡ μὲν
dιάμετρος τῆς βάσεως ἐστὶ μονάδων ἦς, τὸ δὲ
ὑψος ἦς· καὶ γίγνεται τὸ στερεόν αὐτοῦ ἕτερον
c. καὶ μέτρησον κύκλον, οὐ διάμετρος ἐστὶ μονάδων
ἵπτερον τὸ ἐμβαδόν αὐτοῦ, καθὼς ἐμάθομεν,
ρῆ 
καὶ λαβὲ τῶν ἦς τὸ ήμισν· γίγνεται ἦς.
 Epidemic τὸ ήμισν τῶν ἦς· γίγνεται ἔτερον καὶ πολλα-

1 δοθεὶν add. H. Schöne.
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AB. Now it is proved by Dionysodorus * in the book which he wrote On the Spire that the circle $\text{BG}\Delta \text{E}$ bears to half of the parallelogram $\Delta \text{EHO}$ the same ratio as the spire generated by the circle $\text{BG}\Delta \text{E}$ bears to the cylinder having $\text{H} \Theta$ for its axis and $\text{E} \Theta$ for the radius of its base. Now, since $\text{BG}$ is 12, $\text{ZG}$ will be 6. But $\text{AG}$ is 8; therefore $\text{AZ}$ will be 14, and likewise $\text{E} \Theta$, which is the radius of the base of the aforesaid cylinder. Therefore the circle is given; but the axis is also given; for it is 12, since this is the length of $\Delta \text{E}$. Therefore the aforesaid cylinder is also given; and the parallelogram $\Delta \Theta$ is given, so that its half is also given. But the circle $\text{BG}\Delta \text{E}$ is also given; for the diameter $\Gamma \text{B}$ is given. Therefore the ratio of the circle $\text{BG}\Delta \text{E}$ to the parallelogram is given; and so the ratio of the spire to the cylinder is given. And the cylinder is given; therefore the volume of the spire is also given.

Following the analysis, the synthesis may thus be done. Take 12 from 20; the remainder is 8. And add 20; the result is 28. Let the measure be taken of the cylinder having for the diameter of its base 28 and for height 12; the resulting volume is 7392. Now let the area be found of a circle having a diameter 12; the resulting area, as we learnt, is $113\frac{1}{3}$. Take the half of 28; the result is 14. Multiply it by the half of 12; the result is 84. Now multiply

* For Dionysodorus v. supra, p. 162 n. α and p. 364 n. α.
If $\Delta \text{E} = \text{H} \Theta = 2r$ and $\text{E} \Theta = a$, then the volume of the spire bears to the volume of the cylinder the ratio $2\pi a \cdot \pi r^2 : 2r \cdot \pi a^2$ or $\pi r : a$, which, as Dionysodorus points out, is identical with the ratio of the circle to half the parallelogram, that is, $\pi r^2 : 2a$ or $\pi r : a$. 

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πλασιάσας τὰ ᾶτης ἐπὶ τὰ μὴν ἐπὶ καὶ τὰ γενόμενα
παράβαλε παρὰ τὸν ποθ. γίγνεται ἑνέχυσθαι τοῦ
ούτον ἔσται τὸ στερεὸν τῆς σπείρας.

(iii.) Division of a Circle

Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 18–174. 2

Τὸν δοθέντα κύκλον διελεῖν εἰς τρία ἵσα δυσὶν
eυθείαις. τὸ μὲν οὖν πρόβλημα ὅτι οὐ ῥητὸν ἔστι,
δῆλον, τῆς εὐχρηστίας δὲ ἐνεκεν διελούμεν αὐτὸν
ὡς ἐγγιστα σύτω. ἔστω ὁ δοθεῖς κύκλος, οὗ
κέντρον τὸ Λ, καὶ ἐνηρμόσθω εἰς αὐτὸν τρίγωνον
ἰσόπλευρον, οὗ πλευρὰ ή ΒΓ, καὶ παράλληλος αὐτῆ
ἡχω ή ΔΑΕ καὶ ἐπεξεύχθωσαν αἰ ΒΔ, ΔΓ. λέγω,
ὅτι τὸ ΔΒΓ τριήμα τρίτον ἐγγιστά ἐστι μέρος τοῦ
ὁλου κύκλου. ἐπεξεύχθωσαν γὰρ αἰ ΒΑ, ΑΓ. ὁ
ἀρα ΑΒΓΖΒ τομεῖς τρίτον ἐστὶ μέρος τοῦ ὅλου
κύκλου. καὶ ἔστω ἰσόν τὸ ΑΒΓ τρίγωνον τῷ
ΒΓΔ τριγώνῳ· τὸ ἀρα ΒΔΓΖ σχήμα τρίτον
μέρος ἔστι τοῦ ὅλου κύκλου, οὐ δὴ μείζον ἐστὶν
αὐτῶν τὸ ΔΒΓ τριήμα ἀνεπαυσθῆτον ὅντος ὡς
πρὸς τὸν ὅλον κύκλου. ὁμοίως δὲ καὶ ἔτεραν
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7392 by 113½ and divide the product by 84; the result is 9956½. This will be the volume of the spire.

(iii.) Division of a Circle

_In ibid. iii. 18, ed. H. Schöne (Heron iii.)_ 172. 13–174. 9

To divide a given circle into three equal parts by two straight lines. It is clear that this problem is not rational, and for practical convenience we shall make the division as closely as possible in this way. Let the given circle have A for its centre, and let there be inserted in it an equilateral triangle with side BΓ, and let ΔAE be drawn parallel to it, and let BΔ, ΔΓ

be joined. I say that the segment ΔΒΓ is approximately a third part of the whole circle. For let BA, ΔΓ be joined. Then the sector ABΓZB is a third part of the whole circle. And the triangle ABΓ is equal to the triangle BΓΔ [Eucl. i. 37]; therefore the figure BΔΓZ is a third part of the whole circle, and the excess of the segment ΔΒΓ over it is negligible in comparison with the whole circle. Similarly, if we
πλευράν ἵσοπλεύρου τριγώνου ἐγγράματες ἅφε-
λούμεν ἄτερον τρίτον μέρος· ὡστε καὶ τὸ κατα-
λειπόμενον τρίτον μέρος ἦσται [μέρος]¹ τοῦ ὀλου
κύκλου.

(iv.) Measurement of an Irregular Area

Heron, Diopt. 23, ed. H. Schöne (Heron iii.)
260, 18–264. 15

Τὸ δοθὲν χωρίον μετρήσαι διὰ διόπτρας. ἦστω
τὸ δοθὲν χωρίον περιεχόμενον ὑπὸ γραμμῆς

ἀτάκτου τῆς ΑΒΓΔΕΖΗΘ. ἐπεὶ οὖν ἐμάθομεν
diὰ τῆς κατασκευασθείσης διόπτρας διάγενιν πάση
tῆ δοθείσῃ εὐθείᾳ (ἐτέραν)² πρὸς ὀρθάς, ἔλαβον
τι σημεῖον ἐπὶ τῆς περιεχούσης τὸ χωρίον γραμμῆς

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inscribe another side of the equilateral triangle, we may take away another third part; and therefore the remainder will also be a third part of the whole circle.a

(iv.) Measurement of an Irregular Area

Heron, *Dioptre* b 23, ed. H. Schöne (Heron iii.)
260. 18–264. 15

To measure a given area by means of the dioptra. Let the given area be bounded by the irregular line ΑΒΓΔΕΖΗΘ. Since we learnt to draw, by setting the dioptra, a straight line perpendicular to any other straight line, I took any point B on the line en-

a Euclid, in his book *On Divisions of Figures* which has partly survived in Arabic, solved a similar problem—to draw in a given circle two parallel chords cutting off a certain fraction of the circle; Euclid actually takes the fraction as one-third. The general character of the third book of Heron’s *Metrics* is very similar to Euclid’s treatise.

It is in the course of this book (iii. 20) that Heron extracts the cube root of 100 by a method already noted (vol. i. pp. 60-63).

b The dioptra was an instrument fulfilling the same purposes as the modern theodolite. An elaborate description of the instrument prefaces Heron’s treatise on the subject, and it was obviously a fine piece of craftsmanship, much superior to the “parallactic” instrument with which Ptolemy had to work—another piece of evidence against an early date for Heron.

1 μέpos om. H. Schöne.
2 τρέπαν add. H. Schöne.
τό Β, καὶ ἡγαγον εὐθείαν τυχόσαν διὰ τῆς
dιόπτρας τῆς ΒΗ, καὶ ταυτή πρὸς ὀρθάς τήν ΒΓ,
(καὶ ταυτή) ἔτεραν πρὸς ὀρθάς τήν ΓΖ, καὶ
ἀμοίως τῇ ΓΖ πρὸς ὀρθάς τήν ΖΘ. καὶ ἐλαβον
ἐπὶ τῶν ἀκθεισῶν εὐθείων συνεχῆ σημεία, ἐπὶ μὲν
τῆς ΒΗ τὰ Κ, Λ, Μ, Ν, Ξ, Ο. ἐπὶ δὲ τῆς ΒΓ τὰ
Π, Ρ. ἐπὶ δὲ τῆς ΓΖ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω,
ἐπὶ δὲ τῆς ΖΘ τὰς σημεία, καὶ ἀπὸ τῶν ληφθέντων
σημείων ταῖς εὐθείαις, ἐφ’ ὧν ἐστὶ τὰ σημεία,
πρὸς ὀρθάς ἡγαγον τὰς Κγ, ΛΑ, Μ,Α, Ν,Β,
Ξ,Γ, Ο,Δ, Π,Ε, Ρ,Ζ, Σ,Ζ, Τ,Η, Υ,Θ, Φ,Δ,
ΧΜ, Ψ,Μ, ΩΕ, ΞΜ, ΩΜ ὁτις ὠστε τὰς ἐπὶ
tὰ πέρατα τῶν ἀκθεισῶν πρὸς ὀρθάς ἐπὶ ζευγνυμένας]
ἀπολαμβάνειν γραμμὰς ἀπὸ τῆς περιεχούσης τὸ
χωρίον γραμμῆς σύνεχος εὐθείας· καὶ τούτων
gενηθέντων ἔσται δυνατὸν τὸ χωρίον μετρεῖν. τὸ
μὲν γὰρ ΒΓΖΜ παραλληλογράμμων ὀρθογώνιον
ἔστω· ἐπειτὰ τὰς πλευρὰς ἀλλοι ἔστω· καὶ σχοινίω
βεβασισμένως, τοιτέστων μῆτ’ ἐκτείνοντα μῆτε
συστελλέσθαι δυνατές, μετρήσαντες ἔξομεν τὸ
ἐμβαδὸν τοῦ παραλληλογράμμων. τὰ δ’ ἐκτὸς
tοῦτον τρίγωνα ὀρθογώνια καὶ τραπέζια ὀμοίως
μετρήσομεν, ἔχοντες τὰς πλευρὰς αὐτῶν· ἔσται
gὰρ τρίγωνα μὲν ὀρθογώνια τὰ ΒΚγ, ΒΠ,Ε,
ΓΡ,Ζ, ΓΣ,Ζ, ΖΩΕ, ΖΞΜ, ΘΗΜ· τὰ δὲ λοιπὰ
tραπέζια ὀρθογώνια. τὰ μὲν οὖν τρίγωνα με-
tρέεται τῶν περὶ τῆς ὀρθῆς γωνίαν πολλαπλασια-
ζομένων ἐπ’ ἄλληλα· καὶ τοῦ γενομένου τὸ ἡμιν.
tὰ δὲ τραπέζια· συναρμοτέρων τῶν παραλλήλων
τὸ ἡμιν ἐπὶ τῆν ἐπ’ αὐτὰς κάθετον ὀўσαν, οὕδον
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closing the area, and by means of the dioptra drew any straight line BH, and drew BG perpendicular to it, and drew another straight line GZ perpendicular to this last, and similarly drew ZΘ perpendicular to GZ. And on the straight lines so drawn I took a series of points—on BH taking K, Λ, Μ, N, Ξ, Ω, on BG taking Π, P, on GZ taking Σ, Τ, Υ, Φ, Χ, Ψ, Ω, and on ZΘ taking σ, ζ. And from the points so taken on the straight lines designated by the letters, I drew the perpendiculars Κγ, ΛΑ, ΜΑ, ΝΒ, ΞΓ, ΩΔ, ΠΕ, Ρζ, ΣΖ, ΤΗ, ΥΩ, ΦΔ, ΧΜ, ΨΜ, ΩΕ, σΜ, ζΜ in such a manner that the extremities of the perpendiculars cut off from the line enclosing the area approximately straight lines. When this is done it will be possible to measure the area. For the parallelogram BGZM is right-angled; so that if we measure the sides by a chain or measuring-rod, which has been carefully tested so that it can neither expand nor contract, we shall obtain the area of the parallelogram. We may similarly measure the right-angled triangles and trapezia outside this by taking their sides; for BKγ, BΠΕ, ΓΡζ, ΓΣΖ, ΖΩΕ, ΖζΜ, ΘΗΜ are right-angled triangles, and the remaining figures are right-angled trapezia. The triangles are measured by multiplying together the sides about the right angle and taking half the product. As for the trapezia—take half of the sum of the two parallel sides and multiply it by the perpendicular upon

1 καὶ ταύτη ομ. H. Schöne.
2 τὰς εἰς om. H. Schöne.
3 ἐπιζευγνωμόνας om. H. Schöne.
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tων ΚΞ, ΑΛ τὸ ήμισο ἐπὶ τὴν ΚΛ. καὶ τῶν λαυτῶν δε ὀμοίως. ἔσται ἄρα μεμετρημένον ὅλον τὸ χωρίον διὰ τε τοῦ μέσου παραλληλογράμμου καὶ τῶν ἐκτὸς αὐτῶν τριγώνων καὶ τραπεζίων.

ἐάν δὲ τὴν ποτὲ μεταξὺ αὐτῶν τῶν ἀχθεισῶν πρὸς ὀρθὰς ταῖς τοῦ παραλληλογράμμου πλευραῖς καμπύλη γραμμῆ πῆ συνεγγίζουσα εὐθεία (οἷον μεταξὺ τῶν Ξ,Γ, Ο,Δ γραμμῆ ἡ Γ,Δ), ἀλλὰ περιφερεῖ, μετρήσομεν οὕτως: ἀγαγόντες τὴν τῇ

Ο,Δ πρὸς ὀρθὰς τὴν ,ΔΜ, καὶ ἐπ' αὐτῆς λαβόντες σημεία συνεχῆ τὰ Μ, Μ, καὶ ἀπ' αὐτῶν πρὸς ὀρθὰς ἀγαγόντες τῇ ΜΔ τὰς ΜΜ, ΜΜ, ὥστε τὰς μεταξὺ τῶν ἀχθεισῶν σύνεγγυς εὐθείας εἶναι, πάλιν μετρήσομεν τὸ τε ΜΞΟ,Δ παραλληλόγραμ-

μον καὶ τὸ ΜΜ,Δ τρίγωνον, καὶ τὸ ΓΜΜΜ τραπέζιον, καὶ ἐτὶ τὸ ἐτερὸν τραπέζιον, καὶ ἔξομεν τὸ περιεχόμενον χωρίον ὑπὸ τε τῆς ,ΓΜΜ,Δ γραμμῆς καὶ τῶν ,ΓΣ, <ΞΟ,> Ο,Δ εὐθείῶν μεμετρημένον.

(c) Mechanics

Heron, Diopt. 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

Τῆ δοθεισῆ δυνάμει τὸ δοθὲν βάρος κινῆσαι

1 τῇ add. H. Schöne.

2 ΞΟ add. H. Schöne.

* Heron’s Mechanics in three books has survived in Arabic, but has obviously undergone changes in form. It begins with the problem of arranging toothed wheels so as to move
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them, as, for example, half of $\kappa \gamma$, $\Lambda \Delta$ by $K \Delta$; and similarly for the remainder. Then the whole area will have been measured by means of the parallelogram in the middle and the triangles and trapezia outside it. If perchance the curved line between the perpendiculars drawn to the sides of the parallelogram should not approximate to a straight line (as, for example, the curve $\Gamma \Delta$ between $\Xi \Gamma$, $O \Delta$), but to an arc, we may measure it thus: Draw $\Delta M$ perpendicular to $O \Delta$, and on it take a series of points $M, M$, and from them draw $M M, M M$ perpendicular to $\Delta$, so that the portions between the straight lines so drawn approximate to straight lines, and again we can measure the parallelogram $M \Xi O \Delta$ and the triangle $M M, \Delta$, and the trapezium $\Gamma M M M$, and also the other trapezium, and so we shall obtain the area bounded by the line $\Gamma M M, \Delta$ and the straight lines $\Gamma \Xi$, $\Xi O$, $O \Delta$.

(c) Mechanics

Heron, Dioptra 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

With a given force to move a given weight by th

a given weight by a given force. This account is the same as that given in the passage here reproduced from the Dioptra, and it is obviously the same as the account found by Pappus (viii. 19, ed. Hultsch 1060. 1–1068. 23) in a work of Heron's (now lost) entitled Baroulokos ("weight-lifter")—though Pappus himself took the ratio of force to weight as 4 : 160 and the ratio of successive diameters as 2 : 1. It is suggested by Heath (H.G.M. ii, 346-347) that the chapter from the
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dià τυμπάνων ὁδοντωτῶν παραδέσεως. κατε-
σκευάσθω πῆγμα καθάπερ γλωσσόκομον ἐἰς τοὺς
μακροὺς καὶ παραλλήλους τοῖχους διακείσθωσαν
ἀξόνες παράλληλοι ἑαυτοῖς, ἐν διαστήματι κείμενοι

ὡστε τὰ συμφυὴ αὐτοῖς ὀδοντωτὰ τύμπανα παρα-
κεῖσθαι καὶ συμπεπλέχθαι ἀλλήλοις, καθὰ μέλλομεν
δηλοῦν. ἔστω τὸ εἰρημένον γλωσσόκομον τὸ
ΑΒΓΔ, ἐν ὧν ἀξών ἔστω διακείμενος, ὡς εἰρηται,
καὶ δινάμενος εὐλύτως στρέφεσθαι, ὁ ΕΖ. τούτῳ
δὲ συμφυῆς ἔστω τύμπανον ὡδοντωμένον τὸ ἩΘ
ἐξον τὴν διάμετρον, εἰ τύχοι, πενταπλασίων ἡ
tῆς τοῦ ΕΖ ἄξονος διαμέτρου. καὶ Ἰνα ἐπὶ παραδείγ-
ματος τὴν κατασκευὴν ποιησώμεθα, ἔστω τὸ μὲν
ἀγομένον βάρος ταλάντων χιλίων, ἡ δὲ κινοῦσα
dύναμις ἔστω ταλάντων ἐ, τούτῃ τὸ ἐκ κινῶν
ἀνθρώπος ἡ παιδάριον, ὡστε δύνασθαι καθ' ἑαυτόν
ἀνε ἡμχανής ἐλκεῖν τάλαντα ἐ. οὐκοῦν ἐὰν τὰ
ἐκ τοῦ φορτίου ἐκδιδεμένα ὑπὰ διὰ τῶν ἡπῆς

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juxtaposition of toothed wheels. Let a framework be prepared like a chest; and in the long, parallel walls let there lie axles parallel one to another, resting at such intervals that the toothed wheels fitting on to them will be adjacent and will engage one with the other, as we shall explain. Let ΑΒΓΔ be the aforesaid chest, and let EZ be an axle lying in it, as stated above, and able to revolve freely. Fitting on to this axle let there be a toothed wheel ΗΘ whose diameter, say, is five times the diameter of the axle EZ. In order that the construction may serve as an illustration, let the weight to be raised be 1000 talents, and let the moving force be 5 talents, that is, let the man or slave who moves it be able by himself, without mechanical aid, to lift 5 talents. Then if the rope holding the load passes through some aperture in

barouλՁς was substituted for the original opening of the Mechanics, which had become lost.

Other problems dealt with in the Mechanics are the paradox of motion known as Aristotle's wheel, the parallelogram of velocities, motion on an inclined plane, centres of gravity, the five mechanical powers, and the construction of engines. Edited with a German translation by L. Nix and W. Schmidt, it is published as vol. ii. in the Teubner Heron.

a Perhaps "rollers."

1 τῆς add. Vincentius.
οὖσης) ἐν τῷ ΑΒ τοῖχω ἐπειδήθη περὶ τὸν ΕΖ άξονα (.....) κατειλούμενα τὰ ἕκ τοῦ φορτίου ὀπλα κινήσει τὸ βάρος. ἦν δὲ κινηθῇ τὸ ΗΘ τύμπανον, (δεὶ διναὶ) μετὰ ύπάρχειν πλέον ταλάντων διακοσίων, διὰ τὸ τὴν διάμετρον τοῦ τυμπάνου τῆς διαμέτρου τοῦ άξονος, ὡς ὑπεθέμεθα, πενταπλῆς (ἐναι). ταῦτα γὰρ ἀπεδείχθη ἐν ταῖς τῶν ἐ δυνάμεων ἀποδείξεων. ἀλλ' (.....) ἔχομεν τί τὴν δύναμιν ταλάντων διακοσίων, ἀλλὰ πέντε. γεγονεῖτο οὖν ἔτερος άξων (παράλληλος) διακειμένος τῷ ΕΖ, ὃ ΚΛ, ἕχων συμφυνεῖς τύμπανον ὀδοντωμένον τὸ ΜΝ. ὀδοντώδες δὲ καὶ τὸ ΗΘ τύμπανον, ὡστε ἑναρμόζειν ταῖς ὀδοντώσει τοῦ ΜΝ τυμπάνου. τῷ δὲ αὐτῷ άξον τῷ ΚΛ συμφυνεῖς τύμπανον τὸ ΞΟ, ἔχων ὁμοίως τὴν διάμετρον πενταπλασίων τῆς τοῦ ΜΝ τυμπάνου διαμέτρου. διὰ δὴ τοῦτο δεῆσε τὸν βουλόμενον κινεῖν διὰ τοῦ ΞΟ τυμπάνου τὸ βάρος ἔχειν δύναμιν ταλάντων μ., ἐπειδήηπερ τῶν ὅ ταλάντων τὸ πέμπτον ἐστὶ τάλαντα μ., πάλιν οὖν παρακείσθω (τῷ ΞΟ τυμπάνῳ ὀδοντωμένῳ) τύμπανον ὀδοντωθέν ἔτερον (τῷ ΠΡ, καὶ ἔστω τῷ) τύμπανῳ ὀδοντωμένῳ τῷ ΠΡ συμφυνεῖ ἔτερον τύμπανον τὸ ΣΤ ἔχων ὁμοίως πενταπλῆς τὴν διάμετρον τῆς ΠΡ τυμπάνου διαμέτρου. ἦ δὲ ἄναλογος ἐσται δύναμις τοῦ ΣΤ τυμπάνου ἢ ἔχουσα τὸ βάρος ταλάντων ἦ·

1 ὁπῆς αὐς ἢς add. Hultsch et H. Schöne.
2 After άξονα there is a lacuna of five letters.
3 τὰ ἕκ τοῦ φορτίου ὀπλα κινήσει τὸ βάρος H. Schöne, τὰ ἕκ τοῦ φορτίου ἐπιλακών εν τισι τὸ βάρος cod.
4 δεὶ δυνάμει—" septem litteris madore absumptis, supplevi dubitantem," H. Schöne.
5 εἰναι add. H. Schöne.
the wall AB and is coiled round the axle EZ, the rope holding the load will move the weight as it winds up. In order that the wheel HO may be moved, a force of more than 200 talents is necessary, owing to the diameter of the wheel being, as postulated, five times the diameter of the axle; for this was shown in the proofs of the five mechanical powers. We have [not, however . . .] a force of 200 talents, but only of 5. Therefore let there be another axle KA, lying parallel to EZ, and having the toothed wheel MN fitting on to it. Now let the teeth of the wheel HO be such as to engage with the teeth of the wheel MN. On the same axle KA let there be fitted the wheel ΞO, whose diameter is likewise five times the diameter of the wheel MN. Now, in consequence, anyone wishing to move the weight by means of the wheel ΞO will need a force of 40 talents, since the fifth part of 200 talents is 40 talents. Again, then, let another toothed wheel IP lie alongside the toothed wheel ΞO, and let there be fitted to the toothed wheel IP another toothed wheel ST whose diameter is likewise five times the diameter of the wheel IP; then the force needing to be applied to the wheel ST will be 8 talents; but the force actually available

8 The wheel and axle, the lever, the pulley, the wedge and the screw, which are dealt with in Book ii. of Heron's Mechanics.

8 After ἀλλ' is a special sign and a lacuna of 22 letters.
7 παράλληλος add. H. Schöne.
8 τῷ ΞΟ τυμπάνῳ ἀδυντωμένος add. H. Schöne.
9 τῷ ΠΠ, καὶ ἐστὶν τῷ add. H. Schöne.
10 τύμπανον τῷ ΣΤ, so I read in place of the συμφωνεῖ in Schöne's text.
11 ἀνάλογος ἐσται δύναμι—so H. Schöne completes the lacuna.

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αλλ' ἡ ὑπάρχουσα ἡμῶν δύναμις δέδοται ταλάντων ἐ. ὁμοίως ἐτερον παρακείσθω τύμπανον ὕδοντωμένον τὸ ΓΦ τῷ ΣΤ ὕδοντωμένον τοῦ τυμπάνου τῷ ἄξονι συμφιές ἐστὶ τύμπανον τὸ ΧΨ ὕδοντωμένον, οὕτω δὲ διάμετρος πρὸς τὴν τοῦ ΓΦ τυμπάνου λόγον ἐχέτω, ὃν τὰ ὀκτὼ τάλαντα πρὸς τὰ τῆς δοθείσης δυνάμεως τάλαντα ἐ.

Καὶ τούτων παρασκευασθέντων, εάν ἐπινοήσωμεν τὸ ΑΒΓΔ ἡ γλωσσόκομον1 μετέωρον κείμενον, καὶ ἐκ μὲν τοῦ ΕΖ ἄξονος τὸ βάρος ἐξαίσιμεν, ἐκ δὲ τοῦ ΧΨ τυμπάνου τὴν ἐλκουσαν δύναμιν, οὐδοπότερον αὐτῶν κατενεχθήσεται, εὐλύτως στρεφομένων τῶν ἄξωνών, καὶ τῆς τῶν τυμπάνων παρασκευασμένης καλῶς ἀρμοζούσης, ἀλλ' ὅσπερ ξυγοῦ τινος ἑορροπήσει ἡ δύναμις τῷ βάρει. εάν δὲ ἐν αὐτῶν προσθάμωμεν ὅλων ἐτερον βάρος, καταρρέψωμεν καὶ ἐνεχθήσεται έφ' ὃ προσετέθη βάρος, ὡστε εάν ἐν τῶν ε ταλάντων δυνάμει (. . . . . .) εἰ τόχοι μναιαίοι προσετέθη βάρος, κατακρατήσηται καὶ ἐπισπάσηται τὸ βάρος. αὕτι δὲ τῆς προσβεσεως τούτω παρακείσθω κοχλίας ἔχων τὴν ἐλκα ἀρμοστήν τοὺς ὁδοὺς τοῦ τυμπάνου, στρεφόμενοι εὐλύτως περὶ τόρμους ἐνότας ἐν τρήμασι στρογγυλοῖς, ὅπως ἡ μὲν ἐτερον ὑπερεχέτω εἰς τὸ ἐκτὸς μέρος τοῦ γλωσσόκομον κατὰ τὸν ΓΔ <τοίχων τὸν παρακείμενον> τῷ κοχλία ἡ ἄρα ὑπεροχὴ τετραγωνισθεῖσα λαβέτω χειρολάβην τὴν ζζ', δι' ἡς ἐπιλαμβανόμενος τις καὶ ἐπιστρέφων ἐπιστρέφει τὸν κοχλίαν καὶ τὸ ΧΨ τύμπανον, ὡστε καὶ τὸ ΓΦ συμφιές αὐτώ. διὰ δὲ τούτο καὶ τὸ παρακείμενον τὸ ΣΤ ἐπιστραφήσεται, καὶ τὸ συμφιές αὐτῷ τὸ ΠΡ, καὶ τὸ τούτῳ παρακείμενον τὸ ΣΩ, 494
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to us is 5 talents. Let there be placed another toothed wheel $\Phi T$ engaging with the toothed wheel $\Sigma T$; and fitting on to the axle of the wheel $\Phi T$ let there be a toothed wheel $\Phi \Psi$, whose diameter bears to the diameter of the wheel $\Phi T$ the same ratio as 8 talents bears to the given force 5 talents.

When this construction is done, if we imagine the chest $A\beta\Gamma \Delta$ as lying above the ground, with the weight hanging from the axle $E\Sigma$ and the force raising it applied to the wheel $\Phi \Psi$, neither of them will descend, provided the axles revolve freely and the juxtaposition of the wheels is accurate, but as in a beam the force will balance the weight. But if to one of them we add another small weight, the one to which the weight was added will tend to sink down and will descend, so that if, say, a mina is added to one of the 5 talents in the force it will overcome and draw the weight. But instead of this addition to the force, let there be a screw having a spiral which engages the teeth of the wheel, and let it revolve freely about pins in round holes, of which one projects beyond the chest through the wall $\Gamma \Delta$ adjacent to the screw; and then let the projecting piece be made square and be given a handle $\zeta r$. Anyone who takes this handle and turns, will turn the screw and the wheel $\Phi \Psi$, and therefore the wheel $\Phi T$ joined to it. Similarly the adjacent wheel $\Sigma T$ will revolve, and $\Pi P$ joined to it, and then the adjacent wheel $\Xi O$, and then $MN$ fitting

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1 γλυσσόκομον add. H. Schöne.
2 After δυνάμει is a lacuna of seven letters.
3 In Schöne’s text $\delta \varepsilon$ is printed after τούτω.
4 τοίχον τὸν παρακελμένον add. H. Schöne.
Greek Mathematics

καὶ τὸ τούτῳ συμφιές τὸ ΜΝ, καὶ τὸ τούτῳ παρακείμενον τὸ ὙΘ, ὡστε καὶ ὁ τούτῳ συμφιές ἄξων ὁ EZ, περὶ ὄν ἐπειλούμενα τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος. ὡτι γὰρ κινήσει, πρόδηλον ἐκ τοῦ προστεθῆναι ἐτέρα δυνάμει τῇ τῆς χειρολάβης, ἦτις περιγράφει κύκλον τῆς τοῦ κοχλίου περιμέτρου μεῖζονα. ἀπεδείχθη γὰρ ὡτι οἱ μεῖζονες κύκλοι τῶν ἑλαστόνων κατακρατοῦσιν, ὅταν περὶ τὸ αὐτὸ κέντρον κυλἰον ὑποτασσονται.

(d) Optics: Equality of Angles of Incidence and Reflection


Ἀπεδείξε γὰρ ὁ μηχανικὸς Ὡρων ἐν τοῖς αὐτοῖς Κατοπτρικοῖς, ὡτι αἱ πρὸς ἱσας γωνίας κλάμεναι εὐθεῖαι ἐλάχισται εἰσὶ πασῶν2 τῶν ἀπὸ τῆς αὐτῆς καὶ ὁμοιομερῶς γραμμῆς πρὸς τὰ αὐτὰ κλωμένων [πρὸς ἄνισους γωνίας].3 τοῦτο δὲ ἀποδείξεις φῆσιν ὡτι ἄνεκαλο ἢ φύσις μάτην περιάγειν τὴν ἡμετέραν ὑφιν, πρὸς ἱσας αὐτῆς ἀνακλάσει γωνίας.

Olympiod. In Meteor. iii. 2 (Aristot. 371 b 18), ed. Stüve 212. 5-213. 21

Ἐπείδη γὰρ τοῦτο ὠμολογημένου ἐστὶ παρὰ πᾶσιν, ὡτι οὐδὲν μάτην ἑργάζεται ἡ φύσις οὐδὲ ματαιοποιεῖ, ἐὰν ἡ δύσωμεν πρὸς ἱσας γωνίας γίνεσθαι τὴν ἀνάκλασιν, πρὸς ἄνισους ματαιοποιεῖ

1 τῆν add. H. Schöne.
2 πασῶν G. Schmidt, τῶν μέσων codd.
3 πρὸς ἄνισους γωνίας om. R. Schöne.
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on to this last, and then the adjacent wheel ΗΘ, and so finally the axle EZ fitting on to it; and the rope, winding round the axle, will move the weight. That it will move the weight is obvious because there has been added to the one force that moving the handle which describes a circle greater than that of the screw; for it has been proved that greater circles prevail over lesser when they revolve about the same centre.

(d) OPTICS: EQUALITY OF ANGLES OF INCIDENCE AND REFLECTION

Damianus, On the Hypotheses in Optics 14, ed. R. Schöne 20. 12-18

For the mechanician Heron showed in his Catoptrica that of all [mutually] inclined straight lines drawn from the same homogenous straight line [surface] to the same [points], those are the least which are so inclined as to make equal angles. In his proof he says that if Nature did not wish to lead our sight in vain, she would incline it so as to make equal angles.

Olympiodorus, Commentary on Aristotle’s Meteora iii. 2 (371 b 18), ed. Stüve 212. 5-213. 21

For this would be agreed by all, that Nature does nothing in vain nor labours in vain; but if we do not grant that the angles of incidence and reflection are equal, Nature would be labouring in vain by following

* Damianus, or Heliodorus, of Larissa (date unknown) is the author of a small work on optics, which seems to be an abridgement of a large work based on Euclid’s treatise. The full title given in some mss.—Δαμιανος φιλοσοφος του Ἡλιοδόρου Λαρισαίου Περὶ ὀπτικῶν ὑποθέσεων βιβλία β leaves uncertain which was his real name.
ἡ φύσις, καὶ ἀντὶ τού διὰ βραχείας περιόδου φθάσαι τὸ ὄρωμενον τὴν ὅψιν, διὰ μακρὰς περιόδου τούτο φανήσεται καταλαμβάνουσα. εὐρεθήσονται γὰρ αἱ τὰς ἀνίσους γωνίας περιέχουσαι εὐθείαι, αἰτίνες ἀπὸ τῆς ὁβεσὶς [περιέχουσαι] φέρονται πρὸς τὸ κάτοπτρον κάκειθεν πρὸς τὸ ὄρωμενον, μείζονες οὐσίως τῶν τὰς ἰσας γωνίας περιεχούσων εὐθείων. καὶ ὅτι τούτῳ ἀληθεὶς, δῆλον ἑντεῦθεν.

Ὑποκείσθω γὰρ τὸ κάτοπτρον εὐθεία τις ἡ ΑΒ, καὶ ἔστω τὸ μὲν ὄρον Γ, τὸ δὲ ὄρωμενον τὸ Δ, τὸ δὲ Ε σημεῖον τοῦ κατόπτρου, ἐν δὲ προσπὶπτοσα ἡ ὅψις ἀνακλάται πρὸς τὸ ὄρωμενον, ἔστω,

καὶ ἐπεζεύχθω ἡ ΓΕ, ΕΔ. λέγω ὅτι ἡ ύπὸ ΑΕΓ γωνία ἵση ἐστὶ τῇ ύπὸ ΔΕΒ.
MENSURATION: HERON OF ALEXANDRIA unequal angles, and instead of the eye apprehending the visible object by the shortest route it would do so by a longer. For straight lines so drawn from the eye to the mirror and thence to the visible object as to make unequal angles will be found to be greater than straight lines so drawn as to make equal angles. That this is true, is here made clear.

For let the straight line $AB$ be supposed to be the mirror, and let $\Gamma$ be the observer, $\Delta$ the visible object, and let $E$ be a point on the mirror, falling on which the sight is bent towards the visible object, and let $\Gamma E, E\Delta$ be joined. I say that the angle $\Delta EI'$ is equal to the angle $\Delta EB$.\textsuperscript{a}

\textsuperscript{a} Different figures are given in different mss., with corresponding small variants in the text. With G. Schmidt, I have reproduced the figure in the Aldine edition.

\textsuperscript{1} καταλαμβάνοντα om. Ideler.
\textsuperscript{2} περιέχοντα om. R. Schöne, περιέχοναι Ideler, Stüve.
\textsuperscript{3} φέρονται R. Schöne, φέρομένας codd.
Εἰ γὰρ μὴ ἔστων ἴση, ἔστω ἔτερον σημεῖον τοῦ κατόπτρου, ἐν οἷ ἐπροσπίπτουσα ἡ ὀβὺς πρὸς ἀνίσους γωνίας ἀνακλᾶται, τὸ Ζ, καὶ ἐπεξεύρηκα ἡ ΓΖ, ΖΔ. δὴ ὅτι ἡ ὑπὸ ΓΖΑ γωνία μεῖζον ἔστι τῆς ὑπὸ ΔΖΕ γωνίας. λέγω ὅτι αἱ ΓΖ, ΖΔ εὐθείας, αὐτὶ τὰς ἀνίσους γωνίας περιέχουσιν ὑποκειμένης τῆς ΑΒ εὐθείας, μεῖζονες ἐσι τῶν ΓΕ, ΕΔ εὐθείων, αὐτὶ τὰς ἰσὰς γωνίας περιέχουσι μετὰ τῆς ΑΒ. ἦχθω γὰρ κάθετος ἀπὸ τοῦ Δ ἐπὶ τὴν ΑΒ κατὰ τὸ Ἡ σημεῖον καὶ ἐκβεβλήθη τὸ εὐθείας ὡς ἐπὶ τὸ Θ. φανερὸν δὴ ὅτι αἱ πρὸς τῷ Ἡ γωνίαι ἱσαί εἰσί· ὀρθαὶ γὰρ εἰσί. καὶ ἔστω ἡ ΔΗ τῇ ἜΘ ἴση, καὶ ἐπεξεύρηκα ἡ ΘΖ καὶ ἡ ΘΕ. αὕτη μὲν ἡ κατασκευή. ἔπει ὅν τὴν ἴση ἔστὶν ἡ ΔΗ τῇ ἘΘ, ἀλλὰ καὶ ἡ ὑπὸ ΔΗΕ γωνία τῇ ὑπὸ ΘΗΕ γωνίας ἴση ἔστι, κοινὴ δὲ πλευρά τῶν δύο τριγώνων ἡ ΗΕ, [καὶ βάσις ἡ ΘΕ βάσει τῇ ΕΔ ἴση ἔστι, καὶ] τὸ ΗΕΕ τριγώνον τῷ ΔΗΕ τριγώνῳ ἴσον ἔστι, καὶ τοὺς λοιποὺς τῶν δύο τριγώνων εἰσὶν ἵσαί, υφ' ὅς αἱ ἰσαί πλευραί ὑποτείνουσιν. ἴση ἄρα ἡ ΘΕ τῇ ΕΔ. πάλιν ἐπείδη τῇ ἘΘ ἴση ἔστιν ἡ ΔΗ καὶ γωνία ἡ ὑπὸ ΔΗΖ γωνία τῇ ὑπὸ ΘΗΖ ἴση ἔστι, κοινὴ δὲ ἡ ΗΖ τῶν δύο τριγώνων τῶν ΔΗΖ καὶ ΘΗΖ, [καὶ βάσις ἄρα ἡ ΘΖ βάσει τῇ ΖΔ ἴση ἔστι, καὶ] τὸ ΖΗΔ τριγώνον τῷ ΘΗΖ τριγώνῳ ἴσον ἔστιν. ἴση ἄρα ἐστὶν ἡ ΘΖ τῇ ΖΔ. καὶ ἐπεὶ ἴση ἔστιν ἡ ΘΕ τῇ ΕΔ, κοινὴ προσκείσθω ἡ ΕΓ. δύο ἄρα αἱ ΓΕ, ΕΔ δυοὶ ταῖς ΓΕ, ΕΘ ἱσαί εἰσίν. ὁλὴ ἄρα ἡ ΓΘ δυοὶ ταῖς ΓΕ, ΕΔ ἴση ἔστι. καὶ ἐπεὶ πάντως τριγώνον αἱ δύο πλευραί
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For if it be not equal, let there be another point Z, on the mirror, falling on which the sight makes unequal angles, and let ΓΖ, ΖΔ be joined. It is clear that the angle ΓΖΑ is greater than the angle ΔΖΕ. I say that the sum of the straight lines ΓΖ, ΖΔ which make unequal angles with the base line AB, is greater than the sum of the straight lines ΓΕ, ΕΔ, which make equal angles with AB. For let a perpendicular be drawn from Δ to AB at the point H and let it be produced in a straight line to Θ. Then it is obvious that the angles at H are equal; for they are right angles. And let ΔΗ = ΗΘ, and let ΘΖ and ΘΕ be joined. This is the construction. Then since ΔΗ = ΗΘ, and the angle ΔΗΕ is equal to the angle ΘΗΕ, while HE is a common side of the two triangles, the triangle HΘΕ is equal to the triangle ΔΗΕ, and the remaining angles, subtended by the equal sides are severally equal one to the other [Eucl. i. 4]. Therefore ΘΕ = ΕΔ. Again, since ΗΔ = ΗΘ and angle ΔΗΖ = angle ΘΗΖ, while HZ is common to the two triangles ΔΗΖ and ΘΗΖ, the triangle ΖΗΔ is equal to the triangle ΘΗΖ [ibid.]. Therefore ΘΖ = ΖΔ. And since ΘΕ = ΕΔ, let ΕΓ be added to both. Then the sum of the two straight lines ΓΕ, ΕΔ is equal to the sum of the two straight lines ΓΕ, ΕΘ. Therefore the whole ΓΘ is equal to the sum of the two straight lines ΓΕ, ΕΔ. And since in any triangle the sum of two sides is always greater than

1 καί...καί. These words are out of place here and superfluous.
2 καί add. Schmidt. But possibly καί...ἐποτείνουσι, being superfluous, should be omitted.
3 καί...καί. These words are out of place here and superfluous.
GREEK MATHEMATICS

τῆς λουσῆς μεῖζονες εἰσὶ πάντη μεταλαμβανόμεναι, τριγώνου ἀρα τοῦ ΘΣΓ αἱ δύο πλευραί αἱ ΘΖ, ΖΓ μᾶς τῆς ΓΘ μεῖζονες εἰσὶν. ἀλλ' ἦ ΘΖ ἦση ἐστὶ ταῖς ΓΕ, ΕΔ· αἱ ΘΖ, ΖΓ ἀρα μεῖζονες εἰσὶ τῶν ΓΕ, ΕΔ. ἀλλ' ἦ ΘΖ τῇ ΖΔ ἦστιν ἦση· αἱ ΖΖ, ΖΔ ἀρα τῶν ΓΕ, ΕΔ μεῖζονες εἰσί· καὶ εἰσὶν αἱ ΓΖ, ΖΔ αἰ τὰς ἀνίσους γωνίας περιέχουσαι· αἱ ἀρα τὰς ἀνίσους γωνίας περιέχουσαι μεῖζονες εἰσὶ τῶν τὰς ἴσας γωνίας περιέχουσών· ὅπερ ἐδεί δείξαι.

(c) Quadratic Equations

Heron, Geom. 21. 9-10, ed. Heiberg (Heron iv.) 380. 15-31

Δοθέντων συναμφοτέρων τῶν ἀριθμῶν ἣγουν τῆς διαμέτρου, τῆς περιμέτρου καὶ τοῦ ἐμβαδοῦ τοῦ κύκλου ἐν ἀριθμῷ ἐνὶ διαστέλλω καὶ εὑρεῖν ἕκαστον ἀριθμὸν. ποίει οὕτως· ἐστώ ὁ δοθεῖς ἀριθμὸς μονάδες αἱβ. ταῦτα ἀεὶ ἐπὶ τὰ ρνδ. γίνονται μυριάδες γ καὶ βχθ. τούτων προστίθει καθολικῶς ὁμια· γίνονται μυριάδες τρεῖς καὶ γνπθ. ὅν πλευρὰ τετράγωνος γίνεται ῥπγ· ἀπὸ τούτων κούφισον κθ· λοιπὰ ρνδ· ὅν μέρος ια· γίνεται ῦδ· τοσοῦτον ἦ διάμετρος τοῦ κύκλου. ἐὰν δὲ θέλησι καὶ τὴν περιφέρειαν εὑρεῖν, ὑφειλοῦ τὰ κθ ἀπὸ τῶν ῥπγ· λοιπὰ ρνδ· ταῦτα ποίησον δίς· γίνονται τῇ· τούτων λαβὲ μέρος ζ· γίνονται μδ· τοσοῦτον ἦ.

* The proof here given appears to have been taken by Olympiodorus from Heron's Catoptrica, and it is substantially identical with the proof in De Speculis 4. This work was formerly attributed to Ptolemy, but the discovery of Ptolemy's Optics in Arabic has encouraged the belief, now 502
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the remaining side, in whatever way these may be taken [Eucl. i. 20], therefore in the triangle $\theta Z\Gamma'$ the sum of the two sides $\theta Z, Z\Gamma'$ is greater than the one side $\Gamma\theta$. But

$$\Gamma\theta = \Gamma E + E\Delta;$$

$$\therefore \theta Z + Z\Gamma' > \Gamma E + E\Delta.$$  

But  

$$\theta Z = Z\Delta;$$

$$\therefore Z\Gamma + Z\Delta > \Gamma E + E\Delta.$$  

And $\Gamma Z, Z\Delta$ make unequal angles; therefore the sum of straight lines making unequal angles is greater than the sum of straight lines making equal angles; which was to be proved.\(^a\)

(c) Quadratic Equations

Heron, Geometrica 21. 9-10, ed. Helberg
(Heron iv.) 380. 15-31

Given the sum of the diameter, perimeter and area of a circle, to find each of them separately. It is done thus: Let the given sum be 212. Multiply this by 154; the result is 32648. To this add 841, making 33489, whose square root is 183. From this take away 29, leaving 154, whose eleventh part is 14; this will be the diameter of the circle. If you wish to find the circumference, take 29 from 183, leaving 154; double this, making 308, and take the seventh part, which is 44; this will be the perimeter. To

usually held, that it is a translation of Heron's Catoptrica. The translation, made by William of Moerbeke in 1269, can be shown by internal evidence to have been made from the Greek original and not from an Arabic translation. It is published in the Teubner edition of Heron's works, vol. ii. part i.

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περίμετρος: τὸ δὲ ἐμβαθὸν εὑρεῖν. ποίει οὗτος: τὰ ἑν τῆς διαμέτρου ἐπὶ τὰ μὲν τῆς περιμέτρου: γίνονται χιλ. τούτων λαβὲ μέρος τέταρτον: γίνονται ρυθ. τοσοῦτον τὸ ἐμβαθὸν τοῦ κύκλου. ὁμοῦ τῶν τριῶν ἄριθμῶν μονάδες σιβ.

(f) Indeterminate Analysis

Heron, Geom. 24. 1, ed. Heiberg
(Heron iv.) 414. 28-415. 10

Εὑρεῖν δύο χωρία τετράγωνα, ὡς τὸ τοῦ πρῶτον ἐμβαθὸν τοῦ τοῦ δευτέρου ἐμβαθὸν ἐσται τριπλάσιον. ποιῶ οὗτος: τὰ γά κύβισον: γίνονται

* If d is the diameter of the circle, then the given relation is that

\[ d + \frac{32}{7} d + \frac{11}{14} d^2 = 212, \]

i.e.

\[ \frac{11}{14} d^2 + \frac{20}{7} d = 212. \]

To solve this quadratic equation, we should divide by 11 so as to make the first term a square: Heron makes the first term a square by multiplying by the lowest requisite factor, in this case 154, obtaining the equation

\[ 11^2 d^2 + 2 \cdot 29 \cdot 11d = 154 \cdot 212. \]

By adding 841 he completes the square on the left-hand side

\[ (11d + 29)^2 = 154 \cdot 212 + 841 \]

\[ = 32648 + 841 \]

\[ = 33489. \]

\[ \therefore \quad 11d + 29 = 183. \]

\[ \therefore \quad 11d = 154. \]

and \[ d = 14. \]

The same equation is again solved in Geom. 24. 46 and a similar one in Geom. 24. 47. Another quadratic equation is solved in Geom. 24. 3 and the result of yet another is given in Metr. iii. 4.

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find the area. It is done thus: Multiply the diameter, 14, by the perimeter, 44, making 616; take the fourth part of this, which is 154; this will be the area of the circle. The sum of the three numbers is 212.a

(f) Indeterminate Analysis b

Heron, Geometrica 24. 1, ed. Heiberg
(Heron iv.) 414. 28–415. 10

To find two rectangles such that the area of the first is three times the area of the second.c I proceed thus:

a The Constantinople ms. in which Heron’s Metrica was found in 1896 contains also a number of interesting problems in indeterminate analysis; and two were already extant in Heron’s Geéponicus. The problems, thirteen in all, are now published by Heiberg in Heron iv. 414. 28–426. 29.

c It appears also to be a condition that the perimeter of the second should be three times the perimeter of the first. If we substitute any factor u for 3 the general problem becomes: To solve the equations

\[ u + v = n(x + y) \quad \ldots \quad (1) \]
\[ xy = n \cdot uv \quad \ldots \quad (2) \]

The solution given is equivalent to

\[ x = 2u^2 - 1, \quad y = 2u^3 \]
\[ u = n(4n^3 - 2), \quad v = n. \]

Zeuthen (Bibliotheca mathematica, viii. (1907–1908), pp. 118-134) solves the problem thus: Let us start with the hypothesis that \( v = n \). It follows from (1) that \( u \) is a multiple of \( n \), say \( nz \). We have then

\[ x + y = 1 + z, \]

while by (2)

\[ xy = n^2 z, \]

whence

\[ xy = n^3(x + y) - n^3 \]

or

\[ (x - n^3)(y - n^3) = n^3(n^3 - 1). \]

An obvious solution of this equation is

\[ x - n^3 = n^3 - 1, \quad y - n^3 = n^3, \]

which gives \( z = 4n^3 - 2 \), whence \( u = n(4n^3 - 2) \). The other values follow.

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κατα δισε γίνονται νο. νῦν ἄρον μονάδα α. λοιπόν γίνονται νῦ. ἔστω οὖν η μὲν μία πλευρὰ ποδῶν νῦ, ἡ δὲ ἑτέρα πλευρὰ ποδῶν νῦ. καὶ τὸν ἄλλον χωρίον οὕτως. θέσω ὁμοί τὰ νῦ καὶ τὰ νῦ. γίνοντα πόδες ρζ. ταῦτα ποίει ἐπὶ τὰ γ... λοιπὸν γίνονται πόδες τη. ἔστω οὖν η τοῦ προτέρου πλευρὰ ποδῶν τη. ἡ δὲ ἑτέρα πλευρὰ ποδῶν γ' τὰ δὲ ἑμβαδὰ τοῦ ἐνὸς γίνεται ποδῶν ξηδ καὶ τοῦ ἄλλου ποδῶν βωξβ.

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15-424. 5

Τριγώνου ὀρθογώνιου τὸ ἑμβαδὸν μετὰ τῆς περιμέτρου ποδῶν οπί. ἀποδιαστείλαι τὰς πλευρὰς καὶ εὑρεῖν τὸ ἑμβαδὸν. ποιῶ οὕτως. αἰ τῇ τοὺς ἀπαρτίζοντας ἀριθμοὺς ἀπαρτίζει δὲ τὸν οπί ὁ δἰς τὸν βί. ὁ δ' τὸν δ', ὁ ε' τὸν β', ὁ ζ' τὸν μ', ὁ η' τὸν λε. ὁ ι' τὸν κη. ὁ ιδ' τὸν κ. ἔσκεψάμην, ὅτι ὁ η' καὶ λε πούσαν στὸ δοβὴν ἐπίταγμα. τῶν οπί τοῦ η'. γίνονται πόδες λε. διὰ παντὸς λάμβανε δυνάκα τῶν η'. λοιπῶν μένουσιν ζ'. πόδες. τὰ οὖν λε καὶ τὰ θ' ὁμοί γίνονται πόδες μα. ταῦτα ποίει ἐφ' ἐαυτά. γίνονται πόδες αχπ. τὰ λε ἐπὶ τὰ ζ'. γίνονται πόδες ατ' ταῦτα ποίει ἐφ' ἐπὶ τὰ η'. γίνονται πόδες αχπ. ταῦτα ἄρον ἀπὸ τῶν αχπ. λοιπῶν μένει α' ὅν πλευρὰ τετραγωνικὴ γίνεται α'. ἄρτι βες τὰ μα καὶ ἄρον μονάδα α'. λοιπὸν μ' ὃν ζ' γίνεται κ' τούτῳ ἐστὶν ἡ κάθετος, ποδῶν κ. καὶ βες πάλιν τὰ μα καὶ πρόοδες α' γίνονται πόδες μβ'. ὅτι ζ' γίνεται πόδες κα. ἐστὶν ἡ βάσις ποδῶν κα. καὶ βες τὰ λε καὶ ἄρον τὰ ζ'. λοιπὸν μένουσι

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Take the cube of 3, making 27; double this, making 54. Now take away 1, leaving 53. Then let one side be 53 feet and the other 54 feet. As for the other rectangle, [I proceed] thus: Add together 53 and 54, making 107 feet; multiply this by 3, [making 321; take away 3], leaving 318. Then let one side be 318 feet and the other 3 feet. The area of the one will be 954 feet and of the other 2862 feet. 

*Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15-424. 5

In a right-angled triangle the sum of the area and the perimeter is 280 feet; to separate the sides and find the area. I proceed thus: Always look for the factors; now 280 can be factorized into 2.140, 4.70, 5.56, 7.40, 8.35, 10.28, 14.20. By inspection, we find 8 and 35 fulfil the requirements. For take one-eighth of 280, getting 35 feet. Take 2 from 8, leaving 6 feet. Then 35 and 6 together make 41 feet. Multiply this by itself, making 1681 feet. Now multiply 35 by 6, getting 210 feet. Multiply this by 8, getting 1680 feet. Take this away from the 1681, leaving 1, whose square root is 1. Now take the 41 and subtract 1, leaving 40, of which the half is 20; this is the perpendicular, 20 feet. And again take 41 and add 1, getting 42 feet, of which the half is 21; and let this be the base, 21 feet. And take 35 and subtract 6, leaving 29 feet. Now multiply

*The term "feet," ποδές, is used by Heron indiscriminately of lineal feet, square feet and the sum of numbers of lineal and square feet.

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πόδες κθ. ἀρτι θέσ τὴν κάθετον ἐπὶ τὴν βάσιν; ὅν ζ' γίνεται πόδες τοι· καὶ αἱ τρεῖς πλευραὶ περιμετροῦμεναι ἔχουσι πόδας τοῖ· ὁμοί σύνθες μετὰ τοῦ ἑμβαδοῦ· γίνονται πόδες τοῖ τοῦ ἑμβαδοῦ.

* Heath (H.G.M. ii. 446-447) shows how this solution can be generalized. Let a, b be the sides of the triangle containing the right angle, c the hypotenuse, S the area of the triangle, r the radius of the inscribed circle; and let

\[ s = \frac{1}{2}(a + b + c). \]

Then

\[ S = rs = \frac{1}{2}ab, r + s = a + b, c = s - r. \]

Solving the first two equations, we have

\[ \frac{a}{b} = \frac{1}{2}[r + s \pm \sqrt{(r + s)^2 - 8rs}], \]

and this formula is actually used in the problem. The
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the perpendicular and the base together, [getting 420], of which the half is 210 feet; and the three sides comprising the perimeter amount to 70 feet; add them to the area, getting 280 feet.a

method is to take the sum of the area and the perimeter \( S + 2s \), separated into its two obvious factors \( s(r+2) \), to put \( s(r+2) = A \) (the given number), and then to separate \( A \) into suitable factors to which \( s \) and \( r+2 \) may be equated. They must obviously be such that \( sr \), the area, is divisible by 6.

In the given problem \( A = 280 \), and the suitable factors are \( r+2 = 8 \), \( s = 35 \), because \( r \) is then equal to 6 and \( rs \) is a multiple of 6. Then

\[
a = \frac{1}{4}(6 + 35 - \sqrt{(6 + 35)^2 - 8 \cdot 6 \cdot 35}) = \frac{1}{4}(41 - 1) = 20,
\]

\[
b = \frac{1}{3}(41 + 1) = 21,
\]

\[
c = 35 - 6 = 29.
\]

This problem is followed by three more of the same type.
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XXIII. ALGEBRA: DIOPHANTUS

(a) General

Anthol. Palat. xiv. 126, The Greek Anthology, ed. Paton (L.C.L.) v. 92-93

Οὐτός τοι Διόφαντον ἦχει τάφος· ἃ μέγα θαύμα· καὶ τάφος ἐκ τέχνης μέτρα βίου λέγει. ἐκτην κουρίζειν βιότου θεος ὁπασε μοῖρην· ὁδεκάτην δ’ ἐπιθεῖς, μῆλα πόρεν χνοάειν· τῇ δ’ ἄφ’ ἑφ’ ἐβδομάτῃ τὸ γαμήλιον ἦματο φέγγος, ἐκ δὲ γάμων πέμπτῳ παῖδ’ ἐπένευσεν ἔτει. αἰαῖ, τηλύγετον δειλὸν τέκος, ἦμησυ πατρὸς τοῦδε καὶ ἡ κρυερὸς μέτρον ἐλῶν βιότου. πένθος δ’ αὕτ’ πισύρεσσι παρηγορέων ἐνιαυτοῖς τῇδε πόσου σοφιή τέρμ’ ἐπέρησε βίου.

* There are in the Anthology 46 epigrams which are algebraical problems. Most of them (xiv. 116-146) were collected by Metrodorus, a grammarian who lived about A.D. 500, but their origin is obviously much earlier and many belong to a type described by Plato and the scholiast to the Charmides (v. vol. i. pp. 16, 20).

Problems in indeterminate analysis solved before the time of Diophantus include the Pythagorean and Platonic methods of finding numbers representing the sides of right-angled triangles (v. vol. i. pp. 90-95), the methods (also Pythagorean) of finding "side- and diameter-numbers" (vol. i. pp. 132-139), Archimedes' Cattle Problem (v. supra, pp. 202-205) and Heron's problems (v. supra, pp. 504-509).
(a) General

*Palatine Anthology* a xiv. 126, *The Greek Anthology, ed.* Paton (L.C.L.) v. 92-93

This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.\(^b\)


\(^b\) If \(x\) was his age at death, then

\[
\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{5}x + 4 = x,
\]

whence

\[
x = 84.
\]
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Theon Alex. in Ptol. Math. Syn. Comm. i. 10, ed. Rome, Studi e Testi, lxxii. (1936); 433. 4-6

Καθ’ ἄ καὶ Διοφαντός φησι: “τῆς γὰρ μονάδος ἀμεταθέτου οὐσίας καὶ ἐστώσης πάντοτε, τὸ πολλαπλασιαζόμενον ἔδος ἐπ’ αὐτὴν αὐτὸ τὸ ἔδος ἐσται.”

Dioph. De polyg. num. [5]. Dioph. ed. Tannery i. 470. 27-472. 4

Καὶ ἀπεδείχθη τὸ παρὰ Ψικλεῖ ἐν ὄρῳ λεγόμενον, ὅτι, “εὰν ὄσιν ἀριθμοὶ ἀπὸ μονάδος ἐν ἒσῃ ὑπεροχῇ ὅποιοιον, μονάδος μενοῦσης τῆς ὑπεροχῆς, ὁ σύμπας ἐστιν (<τρίγωνος, δωδέκαν δὲ>, τετράγωνος, τριάδος δὲ, πεντάγωνος: λέγεται δὲ τὸ πλῆθος τῶν γωνιῶν κατὰ τὸν δωδέκα μεῖζον τῆς ὑπεροχῆς, πλευραὶ δὲ αὐτῶν τὸ πλῆθος τῶν ἐκτεθέντων σὺν τῇ μονάδι.”


Περὶ δὲ τῆς Ἀλγυπτικῆς μεθόδου ταύτης Διοφαντος μὲν διέλαβεν ἀκριβέστερον, ὁ δὲ λογισσαῖος Ἄνατολος τὰ συνεκτικῶτα μέρη τῆς κατ’

1 τρίγωνος, δωδέκαν δὲ add. Bachet.

* Cf. Dioph. ed. Tannery i. 8. 13-15. The word ἔδος, as will be seen in due course, is regularly used by Diophantus for a term of an equation.

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Theon of Alexandria, Commentary on Ptolemy's Syntaxis i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 453. 4-6

As Diophantus says: "The unit being without dimensions and everywhere the same, a term that is multiplied by it will remain the same term."  

Diophantus, On Polygonal Numbers [5], Dioph. ed. Tannery i. 470. 27–472. 4

There has also been proved what was stated by Hypsicles in a definition, namely, that "if there be as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square number; when 3, a pentagonal number [ ; and so on]. The number of angles is called after the number which exceeds the common difference by 2, and the sides after the number of terms including 1."  

Michael Psellus, A Letter, Dioph. ed. Tannery ii. 38. 22–39. 1

Diophantus dealt more accurately with this Egyptian method, but the most learned Anatolius collected the most essential parts of the theory as stated by

\[ \text{i.e., the } n\text{th } a\text{-gonal number (1 being the first) is } \frac{1}{2}n^2+(n-1)(a-2); \text{ v. vol. i. p. 98 n. a.} \]

\[ \text{Michael Psellus, "first of philosophers" in a barren age, flourished in the latter part of the eleventh century A.D. There has survived a book purporting to be by Psellus on arithmetic, music, geometry and astronomy, but it is clearly not all his own work. In the geometrical section it is observed that the most favoured method of finding the area of a circle is to take the mean between the inscribed and circumscribed squares, which would give } \pi = \sqrt{8} = 2.8284271. \]
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ёκεινον ἐπιστήμης ἀπολεξάμενον ἑτέρως¹ Διοφάντων συνοπτικώτατα προσεφώνησε.


Νῦν δ’ ἔπι τὰς προτάσεις χωρίσωμεν ὅδον, πλείοτην ἔχοντες τὴν ἐπ’ αὐτοῖς τοῖς εἰδεσι συν-
ηθρουσμένην ὠλην. πλείστων δ’ ὄντων τῷ ἀριθμῷ καὶ μεγίστων τῷ ὄγκῳ, καὶ διὰ τοῦτο βραδέως 
βεβαιούμενων ὑπὸ τῶν παραλαμβανόντων αὐτὰ 
καὶ ὄντων ἐν αὐτοῖς δυσμηνομενων, ἐδοκίμασα 
τὰ ἐν αὐτοῖς ἐπιδεχόμενα διαίρετα, καὶ μάλιστα τὰ 
ἐν ἀρχῇ ἔχοντα στοιχεῖόν ἄπο ἀπλουστέρων ἐπὶ 
σκολιώτερα διελεῖν ὡς προσήκεν. οὗτος γὰρ 
eὐδευτὰ γενήσεται τοῖς ἀρχομένοις, καὶ ἡ ἀγωγὴ 
aὐτῶν μημονευθήσεται, τῆς πραγματείας αὐτῶν 
ἐν τρισκαίδεκα βιβλίοις γεγενημένης.

Ibid. v. 3, Dioph. ed. Tannery i. 316. 6

"Ἐξομεν ἐν τοῖς Πορίσμασιν.

¹ ἑτέρως Tannery, ἑτέρῳ codd.

The two passages cited before this one allow us to infer that Diophantus must have lived between Hypsicles and Theon, say 150 B.C. to A.D. 350. Before Tannery edited Michael Psellus's letter, there was no further evidence, but it is reasonable to infer from this letter that Diophantus was a contemporary of Anatolius, bishop of Laodicea about A.D. 280 (v. vol. i. pp. 2-3). For references by Plato and a scholiast to the Egyptian methods of reckoning, v. vol. i. pp. 16, 20.

² Of these thirteen books in the Arithmetica, only six
him in a different way and in the most concise form, and dedicated his work to Diophantus.a


Now let us tread the path to the propositions themselves, which contain a great mass of material compressed into the several species. As they are both numerous and very complex to express, they are only slowly grasped by those into whose hands they are put, and include things hard to remember; for this reason I have tried to divide them up according to their subject-matter, and especially to place, as is fitting, the elementary propositions at the beginning in order that passage may be made from the simpler to the more complex. For thus the way will be made easy for beginners and what they learn will be fixed in their memory; the treatise is divided into thirteen books.b

_Ibid._ v. 3, Dioph. ed. Tannery i. 316. 6

We have it in the Porisms.c

have survived. Tannery suggests that the commentary on it written by Hypatia, daughter of Theon of Alexandria, extended only to these first six books, and that consequently little notice was taken of the remaining seven. There would be a parallel in Eutocius's commentaries on Apollonius's _Conics_. Nesselmann argues that the lost books came in the middle, but Tannery (Dioph. ii. xix-xxi) gives strong reasons for thinking it is the last and most difficult books which have been lost.

Whether this collection of propositions in the Theory of Numbers, several times referred to in the _Arithmetica_, formed a separate treatise from, or was included in, that work is disputed; Hultsch and Heath take the former view, in my opinion judiciously, but Tannery takes the latter.
Τὴν εὑρέσιν τῶν ἐν τοῖς ἀριθμοῖς προβλημάτων, τιμώτατε μοι Διονύσιε, γυνώσκων σε σπουδαίωσ ἔχοντα μαθεῖν, [ὁργανώσαι τὴν μέθοδον] ἐπειράθην, ἀρξάμενος ἀφ’ ὅν συνέστηκε τὰ πράγματα θεμελίων, ὑποστήσαι τὴν ἐν τοῖς ἀριθμοῖς φύσιν τε καὶ δύναμιν.

"Ἅσως μὲν οὖν δοκεῖ τὸ πράγμα δυσχερέστερον, ἐπειδὴ μῆπως γυνώρμον ἔστω, δυσέλπιστοι γὰρ εἰς κατόρθωσιν εἰσέναι τῶν ἀρχωμένων ψυχαί, ὅμως δὲ εὐκατάληπτον σοι γενήσεται, διὰ τὴν σὴν προθυμίαν καὶ τὴν ἐμὴν ἀπόδειξιν ταχεῖα γὰρ εἰς μάθησιν ἐπιθυμία προσλαβοῦσα διδαχῆν.

'Αλλὰ καὶ πρὸς τούτῳ γυνώσκοντι σοι πάντας τοὺς ἀριθμοὺς συγκεκριμένους ἐκ μονάδων πλῆθος των, φανερὸν καθέστηκεν εἰς ἀπειρον ἔχειν τὴν ὑπαρξιν. τυγχανόντων δὴ οὖν ἐν τούτοις ὅτι μὲν τετραγώνων, οὐ εἰσὶν εἰς ἀριθμοὺς τῶν ἐφ’ ἐαυτὸν πολυπλασιασθέντων. οὕτως δὲ ὁ ἀριθμὸς καλεῖται πλευρὰ τοῦ τετραγώνου.

ὅτι δὲ κύβων, οὐ εἰσὶν ἐκ τετραγώνων ἐπὶ τὰς αὐτῶν πλευρὰς πολυπλασιασθέντων,

ὅτι δὲ δυναμικῶν, οὐ εἰσὶν ἐκ τετραγώνων ἐφ’ ἐαυτοὺς πολυπλασιασθέντων,

ὅτι δὲ δυναμικὸς, οὐ εἰσίν ἐκ τετραγώνων ἐπὶ
(b) Notation a

Ibid. i., Preface, Dioph. ed. Tannery i. 2. 3-6. 21

Knowing that you are anxious, my most esteemed Dionysius, to learn how to solve problems in numbers, I have tried, beginning from the foundations on which the subject is built, to set forth the nature and power in numbers.

Perhaps the subject will appear to you rather difficult, as it is not yet common knowledge, and the minds of beginners are apt to be discouraged by mistakes; but it will be easy for you to grasp, with your enthusiasm and my teaching; for keenness backed by teaching is a swift road to knowledge.

As you know, in addition to these things, that all numbers are made up of some multitude of units, it is clear that their formation has no limit. Among them are—

squares, which are formed when any number is multiplied by itself; the number itself is called the side of the square b;

cubes, which are formed when squares are multiplied by their sides,

square-squares, which are formed when squares are multiplied by themselves;

square-cubes, which are formed when squares are

---

a This subject is admirably treated, with two original contributions, by Heath, *Diophantus of Alexandria*, 2nd ed., pp. 34-53. Diophantus's method of representing large numbers and fractions has already been discussed (vol. i. pp. 44-45). Among other abbreviations used by Diophantus are $\sq$, declined throughout its cases, for τετράγωνος; and $\s$ (apparently $\sigma$ in the archetype) for the sign $=$, connecting two sides of an equation.

b Or "square root."
τοὺς ἀπὸ τῆς αὐτῆς αὐτοῖς πλευρᾶς κύβους πολυπλασιασθέντων,

ὅν δὲ κυβοκύβων, οἱ εἰσὶν ἐκ κύβων ἐφ' ἕαυτοὺς πολυπλασιασθέντων,

ἐκ τε τῆς τούτων ἦτοι συνθέσεως ἡ ὑπεροχής ἡ πολυπλασιασμὸς ἡ λόγου τοῦ πρὸς ἀλλήλους ἡ καὶ ἐκάστων πρὸς τὰς ἰδίας πλευρὰς συμβαίνει πλέκεσθαι πλείστα προβλήματα ἀριθμητικά· λύεται δὲ βαδίζοντος σου τὴν ύποδειγμησομένην ὅδον.

Ἐδοκιμάσθη οὖν ἐκαστὸς τούτων τῶν ἀριθμῶν συντομωτέραν ἐπωνυμίαν κτησάμενος στοιχεῖον τῆς ἀριθμητικῆς θεωρίας εἶναι· καλεῖται οὖν ὁ μὲν τετράγωνος δύναμις καὶ ἔστιν αὐτῆς σημείον τὸ Δ ἐπίσημον ἔχον Υ, Δ' δύναμις·

ὁ δὲ κύβος καὶ ἔστιν αὐτοῦ σημείου Κ ἐπίσημον ἔχον Υ, Κ' κύβος·

ὁ δὲ ἐκ τετραγώνου ἐφ' ἕαυτὸν πολυπλασιασθέντος δυναμοδύναμις καὶ ἔστιν αὐτοῦ σημείου δέλτα δύο ἐπίσημον ἔχοντα Υ, Δ''Δ δυναμοδύναμις·

ὁ δὲ ἐκ τετραγώνου ἐπὶ τῶν ἀπὸ τῆς αὐτῆς αὐτῶ τῆς πλευρᾶς κύβου πολυπλασιασθέντος δυναμόκυβος καὶ ἔστιν αὐτοῦ σημείου τὰ ΔΚ ἐπίσημον ἔχοντα Υ, ΔΚ' δυναμόκυβος·

ὁ δὲ ἐκ κύβου ἕαυτὸν πολυπλασιασμοῦ κυβοκύβος καὶ ἔστιν αὐτοῦ σημείου δύο κάππα ἐπίσημον ἔχοντα Υ, Κ'Κ κυβοκύβος.
multiplied by the cubes formed from the same side;

\textit{cube-cubes}, which are formed when cubes are multiplied by themselves;

and it is from the addition, subtraction, or multiplication of these numbers or from the ratio which they bear one to another or to their own sides that most arithmetical problems are formed; you will be able to solve them if you follow the method shown below.

Now each of these numbers, which have been given abbreviated names, is recognized as an element in arithmetical science; the \textit{square} [of the unknown quantity]\(^*\) is called \textit{dynamis} and its sign is $\Delta$ with the index $\Upsilon$, that is $\Delta^\Upsilon$;

the cube is called \textit{cubus} and has for its sign $\kappa$ with the index $\Upsilon$, that is $\kappa^\Upsilon$;

the square multiplied by itself is called \textit{dynamodynamis} and its sign is two deltas with the index $\Upsilon$, that is $\Delta^\Upsilon\Delta$;

the square multiplied by the cube formed from the same root is called \textit{dynamocubus} and its sign is $\Delta\kappa$ with the index $\Upsilon$, that is $\Delta\kappa^\Upsilon$;

the cube multiplied by itself is called \textit{cubocubus} and its sign is two kappas with the index $\Upsilon$, $\kappa^\Upsilon\kappa$.

\(^*\) It is not here stated in so many words, but becomes obvious as the argument proceeds that $\delta\nu\alpha\mu\varsigma$ and its abbreviation are restricted to the square of the \textit{unknown} quantity; the square of a determinate number is $\tau\epsilon\rho\alpha\gamma\omega\alpha\varsigma$. There is only one term, $\kappa\beta\sigma\varsigma$, for the cube both of a determinate and of the unknown quantity. The higher terms, when written in full as $\delta\nu\alpha\mu\delta\nu\alpha\mu\varsigma$, $\delta\nu\alpha\mu\kappa\beta\sigma\varsigma$ and $\kappa\beta\sigma\kappa\beta\sigma\varsigma$, are used respectively for the fourth, fifth and sixth powers both of determinate quantities and of the unknown, but their abbreviations, and that for $\kappa\beta\sigma\varsigma$, are used to denote powers of the unknown only.
'Ο δὲ μηδὲν τούτων τῶν ἰδιωμάτων κτησάμενος, ἔχων δὲ ἐν ἑαυτῷ πλῆθος μονάδων ἀόριστον, ἀριθμός καλεῖται καὶ ἐστὶν αὐτοῦ σημεῖον τὸ 5.

"Εστὶ δὲ καὶ ἑτερον σημεῖον τὸ ἀμετάθετον τῶν ὁμοιόμενων, ἡ μονᾶς, καὶ ἐστὶν αὐτῆς σημείον τὸ Μ ἐπίσημον ἔχου τὸ Ο, Ἔ.

"Ωσπερ δὲ τῶν ἀριθμῶν τὰ ὁμοίωμα μόρια παρομοίως καλεῖται τοῖς ἀριθμοῖς, τοῦ μὲν τρία τὸ τρίτον, τοῦ δὲ τέσσαρα τὸ τέταρτον, οὕτως καὶ τῶν νῦν ἐπονομαζόμενων ἀριθμῶν τὰ ὁμοίωμα μόρια κληθήσεται παρομοίως τοῖς ἀριθμοῖς.

τοῦ μὲν ἀριθμοῦ τὸ ἀριθμοστῶν,
τῆς δὲ δυνάμεως τὸ δυναμοστῶν,
τοῦ δὲ κύβου τὸ κυβοστῶν,
τῆς δὲ δυναμοδυνάμεως τὸ δυναμοδυναμοστῶν,
τοῦ δὲ δυναμοκύβου τὸ δυναμοκυβοστῶν,
τοῦ δὲ κυβοκύβου τὸ κυβοκυβοστῶν.

ἐξει δὲ ἑκαστὸν αὐτῶν ἐπὶ τὸ τοῦ ὁμονύμου ἀριθμοῦ σημείον γραμμῆν Χ διαστέλλουσαν τὸ είδος.

* I am entirely convinced by Heath's argument, based on the Bodleian ms. of Diophantus and general considerations, that this symbol is really the first two letters of ἀριθμός; this suggestion brings the symbol into line with Diophantus's abbreviations for δύναμις, κύβος, and so on. It may be declined throughout its cases, e.g., ἅριθμος for the genitive plural, infra p. 552, line 5.

Diophantus has only one symbol for an unknown quantity, but his problems often lead to subsidiary equations involving other unknowns. He shows great ingenuity in isolating these subsidiary unknowns. In the translation I shall use.
The number which has none of these characteristics, but merely has in it an undetermined multitude of units, is called *arithmos*, and its sign is $\varepsilon \ [x]^a$.

There is also another sign denoting the invariable element in determinate numbers, the unit, and its sign is $\text{M}$ with the index 0, that is $\text{M}_0$.

As in the case of numbers the corresponding fractions are called after the numbers, a *third* being called after 3 and a *fourth* after 4, so the functions named above will have reciprocals called after them:

\[
\begin{align*}
\text{arithmos } [x] & & \text{arithmoston } \left[ \frac{1}{x} \right], \\
\text{dynamis } [x^2] & & \text{dynamoston } \left[ \frac{1}{x^2} \right], \\
\text{cubus } [x^3] & & \text{cuboston } \left[ \frac{1}{x^3} \right], \\
\text{dynamodynamis } [x^4] & & \text{dynamodynamoston } \left[ \frac{1}{x^4} \right], \\
\text{dynamocubus } [x^5] & & \text{dynamocuboston } \left[ \frac{1}{x^5} \right], \\
\text{cubocubus } [x^6] & & \text{cubocuboston } \left[ \frac{1}{x^6} \right].
\end{align*}
\]

And each of these will have the same sign as the corresponding process, but with the mark $\chi$ to distinguish its nature.\(^b\)

different letters for the different unknowns as they occur, for example, $x, z, m$.

Diophantus does not admit negative or zero values of the unknown, but positive fractional values are admitted.

\(^b\) So the symbol is printed by Tannery, but there are many variants in the mss.
GREEK MATHEMATICS

Ibid. i., Praef., Dioph. ed. Tannery i. 12. 19-21

Λεύψις ἐπὶ λεύψιν πολλαπλασιασθείσα ποιεῖ ὑπαρξίν, λεύψις δὲ ἐπὶ ὑπαρξίν ποιεῖ λεύψιν, καὶ τῆς λεύψεως σημείου Ψ ἐλλιπέσε κάτω νεῦον, Λ.

(c) Determinate Equations

(i.) Pure Determinate Equations

Ibid. i., Praef., Dioph. ed. Tannery i. 14. 11-20

Μετὰ δὲ ταῦτα ἔαν ἀπὸ προβλήματος των γένηται εἴδη τινὰ ἢ σα εἴδεις τοῖς αὐτοῖς, μὴ ὀμοπληθῇ δὲ, ἀπὸ ἐκατέρω τῶν μερῶν δεῖσει ἀφαιρεῖν τὰ ὀμοια ἀπὸ τῶν ὀμοίων, ἕως ἂν ἐν εἴδος ἐνὶ εἴδει ἵπτεν γένηται. ἔαν δὲ πως ἐν ὀποτέρῳ ἐνυπάρχῃ ἢ ἐν ἀμφότεροις ἐν ἔλλειψεί τινα εἴδη, δεῖσει προσθεῖναι τὰ λείποντα εἴδη ἐν ἀμφότεροις τοῖς μέρεσιν, ἕως ἂν ἐκατέρω τῶν μερῶν τὰ εἴδη ἐνυπάρχοντα γένηται, καὶ πάλιν ἀφελεῖν τὰ ὀμοια ἀπὸ τῶν ὀμοίων, ἕως ἂν ἐκατέρω τῶν μερῶν ἐν εἴδος καταλειφθῇ.

* Lit. "a deficiency multiplied by a deficiency makes a forthcoming."
* The sign has nothing to do with Ψ, but I see no reason why Diophantus should not have described it by means of Ψ.
ALGEBRA: DIOPHANTUS

Ibid. i., Preface, Dioph. ed. Tannery i. 12. 19-21

A minus multiplied by a minus makes a plus, a minus multiplied by a plus makes a minus, and the sign of a minus is a truncated Ψ turned upside down, that is Λ.

(c) DETERMINATE EQUATIONS

(i.) Pure *Determinate Equations

Ibid. i., Preface, Dioph. ed. Tannery i. 14. 11-20

Next, if there result from a problem an equation in which certain terms are equal to terms of the same species, but with different coefficients, it will be necessary to subtract like from like on both sides until one term is found equal to one term. If perchance there be on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides become positive, and again to subtract like from like until on each side one term only is left.

and cannot agree with Heath (H.G.M. ii. 459) that "the description is evidently interpolated." But Heath seems right in his conjecture, first made in 1885, that the sign Λ is a compendium for the root of the verb λαίθαι, and is, in fact, a Λ with an Ι placed in the middle. When the sign is resolved in the manuscripts into a word, the dative λείθαι is generally used, but there is no conclusive proof that Diophantus himself used this non-classical form.

* A pure equation is one containing only one power of the unknown, whatever its degree; a mixed equation contains more than one power of the unknown.

* In modern notation, Diophantus manipulates the equation until it is of the form Λx^n = B; as he recognizes only one value of x satisfying this equation, it is then considered solved.
Εὑρεῖν τρεῖς ἁριθμοὺς ὡς ἡ ὑπεροχὴ τοῦ μεῖζονος καὶ τοῦ μέσου πρὸς τὴν ὑπεροχήν τοῦ μέσου καὶ τοῦ ἐλάσσονος λόγον ἔχῃ δεδομένον, ἐτι δὲ καὶ σὺν δύο λαμβανόμενοι, ποιώσαι τετράγωνον.

'Επιτετάχθω δὴ τὴν ὑπεροχήν τοῦ μεῖζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἑλαχίστου εἶναι γν.

'Επει δὲ συναμφότερος ὁ μέσος καὶ ὁ ἐλάσσων ποιεῖ \( m \), ποιεῖτω \( Mδ \). ὁ ἀρα μέσος μεῖζων ἐστὶ δυνάς. ἐστω \( Ξ \) \( Α \) \( Μ \) \( Β \). ὁ ἀρα ἑλάχιστος ἐστι \( Μ \) \( Β \) \( Λ \) \( Ξ \).

Καὶ ἐπειδὴ ἡ ὑπεροχή τοῦ μεῖζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἑλαχίστου \( \gamma \). \( \epsilon \) στὶ, καὶ ἡ ὑπεροχή τοῦ μέσου καὶ τοῦ ἑλαχίστου \( \Xi \) \( Β \), ἡ ἀρα ὑπεροχή τοῦ μεῖζονος καὶ τοῦ μέσου ἐσται \( \Xi \) \( Ξ \), καὶ ὁ μεῖζων ἀρα ἐσται \( \Xi \) \( Ξ \) \( Μ \) \( Β \).

Δοιπότον ἐστὶ δύο ἐπιτάγματα, τὸ τε συναμφότερον \( τὸν μεῖζονα καὶ τὸν ἑλάχιστον ποιεῖ \( \Box \), καὶ τὸ τὸν μεῖζονα \). καὶ τὸν μέσον ποιεῖ \( \Box \), καὶ γίνεται μοι διπλὴ ἡ ἰσότης:

\[ \Xi Η \] \( Μ \) \( Δ \) \( ίσος. \Box \), καὶ \( \Xi \) \( Ξ \) \( Μ \) \( Δ \) \( ίσος. \Box \).

καὶ διὰ τὸ τὰς \( M \) εἶναι τετράγωνικάς, εὐχερῆς ἐστιν ἡ ἰσώσις.

1 ἐστὶ add. Bachet.
2 τὸν μεῖζονα... τὸν μεῖζονα add. Tannery.
(ii.) Quadratic Equations

Ibid. iv. 39, Dioph. ed. Tannery i. 298. 7-306. 8

To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.

Let it be laid down that the difference of the greatest and the middle has to the difference of the middle and the least the ratio 3 : 1.

Since the sum of the middle term and the least makes a square, let it be 4. Then the middle term > 2. Let it be $x + 2$. Then the least term = 2 - $x$.

And since the difference of the greatest and the middle has to the difference of the middle and the least the ratio 3 : 1, and the difference of the middle and the least is 2$x$, therefore the difference of the greatest and the middle is 6$x$, and therefore the greatest will be $7x + 2$.

There remain two conditions, that the sum of the greatest and the least make a square and the sum of the greatest and the middle make a square. And I am left with the double equation

\[ 8x + 4 = \text{a square}, \]
\[ 6x + 4 = \text{a square}. \]

And as the units are squares, the equation is convenient to solve.

a The quadratic equation takes up only a small part of this problem, but the whole problem will give an excellent illustration of Diophantus’s methods, and especially of his ingenuity in passing from one unknown to another. The geometrical solution of quadratic equations by the application of areas is treated in vol. i. pp. 192-215, and Heron’s algebraical formula for solving quadratics, supra, pp. 502-505.

b For double equations, v. infra p. 543 n. b.
Πλάσω αριθμούς δύο ἓνα ὀ ὑπ’ αὐτῶν ὡς β, καθὼς ἴσον διπλῆν ἴσότητα· ἐστι οὖν ὡς γε γιὰ τὸν β καὶ δ. καὶ γίνεται ὡς ἴσον γιὰ τὸν θέλω οὖν τὸν ζ εὑρεθήναι ἐλάσσονα β, ὥστε καὶ ζ ὡς δ. ἐλάσσονις ἐσονται δ. εἰς γάρ ἡ δύνα ἐπὶ ζ γένεται καὶ προσ-λάβῃ δ, ποτε δ. ἕνεκ.

Ἐπεὶ οὖν ζητῶ ζ βδ ἴσος. καὶ δ. καὶ ζ δ. ὡς. ἄλλα καὶ ὃ ἀπὸ τῆς δύνας, τοντέστι δ. ὡς. ἕστι, γεγόνασι τρεῖς ὡς, βδ, καὶ ζ δ. καὶ δ. καὶ ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γ. μέρος ἐστίν. ἀπήκται οὖν μοι εἰς τὸ εὑρεῖν τρεῖς τετραγώνους, ὡς ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γ. μέρος ὡς, ἓτε δὲ ὁ μὲν ἐλαχίστος ὡς δ, ὁ δὲ μέσος ἐλάσσων δ. εἰς.

1 τρεῖς add. Bachet.

* If we put

\[8x + 4 = (p + q)^2,\]
\[6x + 4 = (p - q)^2,\]

on subtracting,

\[2x = 4pq,\]

Substituting \(p = \frac{1}{2}x,\) \(2q = 4\) (i.e., \(p = \frac{1}{2}x,\) \(q = 2\)) in the first equation we get

\[8x + 4 = (\frac{1}{2}x + 2)^2,\]

or

\[112x = x^2,\]

whence

\[x = 112.\]
ALGEBRA: DIOPHANTUS

I form two numbers whose product is $2x$, according to what we know about a double equation; let them be $\frac{1}{2}x$ and 4; and therefore $x = 112$. But, returning to the conditions, I cannot subtract $x$, that is 112, from 2; I desire, then, that $x$ be found <2, so that $6x + 4 < 16$. For $2 \cdot 6 + 4 = 16$.

Then since I seek to make $8x + 4 = a$ square, and $6x + 4 = a$ square, while $2 \cdot 2 = 4$ is a square, there are three squares, $8x + 4$, $6x + 4$, and 4, and the difference of the greatest and the middle is one-third of the difference of the middle and least. My problem therefore resolves itself into finding three squares such that the difference of the greatest and the middle is one-third of the difference of the middle and least, and further such that the least = 4 and the middle <16.

This method of solving such equations is explicitly given by Diophantus in ii. 11, Dioph. ed. Tannery i. 96. 8-14: ἦσται ἄρα ὁ μὲν Ἐ Ἰ ἌΜ ὑ, ὁ δὲ Ἰ ἌΜ γ, Ἰ. ὅ. καὶ τούτῳ τὸ εἴδος καλεῖται διπλοῦστος ἢ ἐσθεναὶ τὸ τρόπον τούτουν. ἤδη ἤν τὴν ὑπεροχήν, ἐπεί δύο ἀριθμοὺς ἣν τὸ ὑπ’ αὐτῶν ποίη τὴν ὑπεροχήν εἶναι ὑπ’ ἌΜ δ καὶ Ἰ. ὅ. δ’ τούτων ἦτοι τῆς ὑπεροχῆς τὸ Μ’ ἐφ’ ἐαυτό ἢν ἤτοι τῷ ἐλάσσονι, ἦ τῆς συνθεσεως τὸ Μ’ ἐφ’ ἐαυτῷ ἢν ὑπ’ τῷ μείζονι—"The equations will then be $x + 2 = a$ square, $x + 3 = a$ square; and this species is called a double equation.

It is solved in this manner: observe the difference, and seek two [suitable] numbers whose product is equal to the difference; they are 4 and 1. Then, either the square of half the difference of these numbers is equated to the lesser, or the square of half the sum to the greater."

The ratio of the differences in this subordinate problem has, of course, nothing to do with the ratio of the differences in the main problem; the fact that they are reciprocals may lead the casual reader to suspect an error.
Τετάχθων ὁ μὲν ἐλάχιστος Ὄ. ἦ ὃς τοῦ μέσου τιν. αὐτὸς ἀρα ἔσται ὃ. Γ. αὐτὸς ἄρα ἔσται ὧ.

Ἐπεὶ οὖν ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γ' μέρος ἐστίν, καὶ ἔστιν ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου Γ. αὐτοῦ, ἵστε ἡ ὑπεροχὴ τοῦ μεγίστου καὶ τοῦ μέσου ἔσται Γ. γ' ἐπὶ γ'. καὶ ἔστιν ὃ μέσος Γ. αὐτοῦ Ὀ. ἢ ἀρα μεγίστος ἔσται Γ. αὐτοῦ ἐπὶ γ'. ἔστιν Γ. ἄρα Γ. ἐπὶ μ. ἦ. ἐστιν Γ. ἀνα 

Ἐπὶ δὲ τότε τοῦ μέσου τετράγωνον ἐλάσσονα εἶναι Ὄ. καὶ τὴν π. δηλαδὴ ἐλάσσονος Ὅ. ἦ. δὲ πλευρά τοῦ μέσου ἐστὶν τις. αὐτοῦ ἐλάσσονος εἰς Ὅ. καὶ κοινῶν ἀφαιρεθεισῶν τῶν ἄ. ὃ ἔσται ἐλάσσονος Ὅ. 

Γέγονεν οὖν μοι Γ. ἔσται Ὅ. ἐστι. ποιήσαι ὃ. θάλασσα ὃ. γινόμενοι τοῖς τοῦ ἄπο. ἱποτώς. καὶ γίνεται ὃ ἐκ τοῦ ἀριθμοῦ ἐκ. γενομένου καὶ προσλαβόντως τοῦ ἡ. συν. εἰς ὃ. ἵστε τῆς ἀριθμοῦ τῆς ἐπὶ, καὶ μερισθέντος εἰς τὴν ἑπεροχήν ἤ ὑπερέχει ὃ ἀπὸ τοῦ ἀριθμοῦ τῶν ἐν τῇ ἑπεροχήν ἢ. ἀπήκται οὖν μοι εἰς τὸ εὐθεῖα ἔπεις τοῦ ἀριθμοῦ, ὃς ἐκ. γενόμενος καὶ προσλαβόν τοῦ ἐλάσσονος ἀριθμοῦ καὶ μερισθέντος εἰς τὴν ἑπεροχήν ἢ ὑπερέχει ὃ ἀπὸ τοῦ αὐτοῦ τριάδος, ποιεῖ τὴν παραβολήν ἐλάσσονος Ὅ.
ALGEBRA: DIOPHANTUS

Let the least be taken as 4, and the side of the middle as \( z + 2 \); then the square is \( z^2 + 4z + 4 \).

Then since the difference of the greatest and the middle is one-third of the difference of the middle and the least, and the difference of the middle and the least is \( z^2 + 4z \), so that the difference of the greatest and the least is \( \frac{1}{3}z^2 + 1\frac{1}{3}z \), while the middle term is \( z^2 + 4z + 4 \), therefore the greatest term = \( 1\frac{1}{3}z^2 + 5\frac{1}{3}z + 4 \) = a square. Multiply throughout by 9:

\[
12z^2 + 48z + 36 = \text{a square} ;
\]

and take the fourth part:

\[
3z^2 + 12z + 9 = \text{a square}.
\]

Further, I desire that the middle square < 16, whence clearly its side < 4. But the side of the middle square is \( z + 2 \), and so \( z + 2 < 4 \). Take away 2 from each side, and \( z < 2 \).

My equation is now

\[
3z^2 + 12z + 9 = \text{a square}.
\]

\[
= (mz - 3)^2, \text{ say.}^a
\]

Then

\[
z = \frac{6m + 12}{m^2 - 3}.
\]

and the equation to which my problem is now resolved is

\[
\frac{6m + 12}{m^2 - 3} < 2,
\]

i.e.,

\[
\frac{2}{1} < 1
\]

^a As a literal translation of the Greek at this point would be intolerably prolix, I have made free use of modern notation.
"Εστω ο ζητούμενος $\delta a$. ούτως $\nu\nu\nu$ γενόμενος και προσλαβομεν $\bar{M}ib$, ποιει $\bar{\xi} Mib$. δε δε ἀπ' αὐτοῦ, $\Lambda \bar{M} \gamma$, ποιει $\Delta \alpha \Lambda \bar{M} \gamma$, θελω οὖν $\bar{\xi} Mib$ μεριζεσθαι εἰς $\Delta \alpha \Lambda \bar{M} \gamma$ και ποιεν τὴν παραβολὴν ἐλάσσονος $\bar{M}b$. ἀλλα καὶ ώ $ib$ μεριζόμενος εἰς $\bar{M}a$, ποιει τὴν παραβολὴν $\alpha$: ἵστη $\bar{\xi} Mib$ πρὸς $\Delta \alpha \Lambda \bar{M} \gamma$ ἐλάσσονα λόγον ἑξουσιών ἤπερ $\beta$ πρὸς $\delta$.

Καὶ χωρίων χωρίῳ ἄνυσιν εἰς ἀρα ὑπὸ $\bar{\xi} \bar{M}ib$ καὶ $\bar{M}a$ ἐλάσσων ἔστιν τοῦ ὑπὸ δυᾶδος καὶ $\Delta \alpha \Lambda \bar{M} \gamma$, τοῦτον $\bar{\xi} \bar{M}ib$ ἐλάσσωνες εἰς ὄν $\Delta \beta \Lambda \bar{M} \xi$. καὶ κοιναὶ προσκείσθωσαν αἱ $\bar{\xi} \bar{M}ib$. $\bar{\xi} \bar{M}i\eta$ ἐλάσσωνες $\Delta \beta$.

"Οταν δὲ ποιατὴν ἰσωσιν ἰσώσωμεν, ποιοῦμεν τῶν $\bar{\xi}$ τὸ $\Delta$ ἐφ' εαυτῷ, γίνεται $\beta$, καὶ τὰς $\Delta \beta$ ἐπὶ τὰς $\bar{M}i\eta$, γίνονται $\bar{\xi}$ προσθέτος τοῖς $\beta$, γίνονται $\bar{\xi}$, ὅπως πλὴν οὐκ ἔλαττον ἐστὶ $\bar{M}i\xi$ προσθέτος τὸ ἡμίσεσμα τῶν $\bar{\xi}$, (γίνεται οὐκ ἔλαττον $\bar{M}i$ καὶ μέρισον εἰς τὰς $\Delta \gamma$ γίνεται οὐκ ἔλαττον $\bar{M}i\xi$.

Γέγονεν οὖν μοι $\Delta \gamma \bar{\xi}ib \bar{M} \theta$ ἰσο. $\bar{\xi} \gamma$ τῷ ἀπὸ $\bar{M} \gamma \Lambda \xi \bar{\xi}$, καὶ γίνεται $\delta \bar{M}ib$ μὲ τοῦτον καὶ $\bar{\xi} \mu$. $\bar{\xi}\bar{M}ib$ γίνεται... τὰς $\Delta$ add. Tannery.

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*This is not strictly true. But since $\sqrt{45}$ lies between 6 and 7, no smaller integral value than 7 will satisfy the conditions of the problem.

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ALGEBRA: DIOPHANTUS

The inequality will be preserved when the term are cross-multiplied,

\[ (6m + 12) \cdot 1 < 2 \cdot (m^2 - 3); \]
\[ i.e., \quad 6m + 12 < 2m^2 - 6. \]

By adding 6 to both sides,

\[ 6m + 18 < 2m^2. \]

When we solve such an equation, we multiply half the coefficient of \(x\) [or \(m\)] into itself—getting 9; then multiply the coefficient of \(x^2\) into the units—2 \(\cdot\) 18 = 36; add this last number to the 9—getting 45; take the square root—which is \(\sqrt{45}\); add half the coefficient of \(x\)—making a number \(\frac{9}{2}\); and divide the result by the coefficient of \(x^2\)—getting a number \(\frac{9}{5}\).

My equation is therefore

\[ 3x^2 + 12x + 9 = \text{a square on side } (3 - 5x), \]

and

\[ x = \frac{42}{22} = \frac{21}{11}. \]

I have made the side of the middle square to be

\[ \frac{9}{5}. \]

This shows that Diophantus had a perfectly general formula for solving the equation

\[ ax^2 + bx + c, \]

namely

\[ x = \frac{-b + \sqrt{b^2 + 4ac}}{2a}. \]

From vi. 6 it becomes clear that he had a similar general formula for solving

\[ ax^4 + bx = c, \]

and from v. 10 and vi. 22 it may be inferred that he had a general solution for

\[ ax^2 + c = bx. \]
GREEK MATHEMATICS

ἐσται ἦ τοῦ □ου πα. Μα. αὐτὸς δὲ ὁ □ος
Μ ῥακ.

,αμμθ.

"Ερχομαι οὖν ἐπὶ τὸ ἔξ ἀρχῆς καὶ τάσσω
Μ ῥκα, ἀμμθ. ὡντα □ος, Ἵθ. τοῖς ζ-μ.δ. καὶ πάντα
eis ῥκα: καὶ γίνεται ὁ ζ ψικς
,ατξε', καὶ ἐστιν ἐλάσσων
dυάδος.

'Επὶ τὰς ὑποστάσεις τοῦ προβλήματος τοῦ ἔξ
ἀρχῆς: ὑπέστημεν δὴ τὸν μὲν μέσον ἢ ἄμβ, τὸν
dὲ ἐλάχιστον ἄμβ Λζ ἂ, τὸν dὲ μέγιστον ζ-μ.β.
ἐσται ὁ μὲν μέγιστος α..αξ, ὁ dὲ βος
,βωις, ὁ
dὲ ἐλάχιστος ὁ γος πξ. καὶ εἶπε τὸ μόριον, ἐστὶ τὸ
ψικς, οὖκ ἐστιν □ος, ςος dὲ ἐστιν αὐτοῦ, εἀν
λάβωμεν ρκα, ὁ ἐστι □ος, πάντων οὖν τὸ ςος,
καὶ ὁμοίως ἐσται ὁ μὲν aος ρκαων
,αωδζ', ὁ dὲ
βος υξθζ', ὁ dὲ γος ιδζ'.

Καὶ ἐὰν ἐν ὀλοκλήρως θέλης ἵνα μὴ τὸ ζ' ἐπι-
τρέχῃ, εἰς δ' ἐμβαλε. καὶ ἐσται ὁ aος υπδ
,ζτλη', ὁ dὲ
βος
,αωνη', ὁ dὲ γος υπδ. καὶ ἡ ἀπόδειξις φανερά.

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$x + 2$; therefore the side will be $\frac{43}{11}$ and the square itself $\frac{1849}{121}$.

I return now to the original problem and make $\frac{1849}{121}$, which is a square, $= 6x + 4$. Multiplying by 121 throughout, I get $x = \frac{1365}{726}$, which is $<2$.

In the conditions of the original problem we made the middle term $= x + 2$, the least $= 2 - x$, and the greatest $7x + 2$.

Therefore

the greatest $= \frac{11007}{726}$,

the middle $= \frac{2817}{726}$,

the least $= \frac{87}{726}$.

Since the denominator, 726, is not a square, but its sixth part is, if we take 121, which is a square, and divide throughout by 6, then similarly the numbers are

$1834\frac{1}{2}$, $469\frac{1}{2}$, $14\frac{1}{2}$,

$121$, $121$, $121$.

And if you prefer to use integers only, avoiding the $\frac{1}{2}$, multiply throughout by 4. Then the numbers will be

$7338$, $1878$, $58$,

$484'$, $484'$, $484$.

And the proof is obvious.
(iii.) Simultaneous Equations Leading to a Quadratic

*Ibid.* i. 28, Dioph. ed. Tannery i. 62. 20-64. 10

Εὑρεῖν δύο ἀριθμοὺς ὅπως καὶ ἡ σύνθεσις αὐτῶν καὶ ἡ σύνθεσις τῶν ἀπ’ αὐτῶν τετραγώνων ποὺ Ἰς δοθέντας ἀριθμοὺς.

Δεῖ δὴ τοὺς δίς ἀπ’ αὐτῶν τετραγώνων τοῦ ἀπὸ συναμφοτέρου αὐτῶν τετραγώνου ὑπερέχειν τετρα- γώνων. ἔστι δὲ καὶ τοῦτο πλασματικόν.

'Ἐπιτετάχθω δὴ τὴν μὲν σύνθεσιν αὐτῶν ποιεῖν Ἐκ, τὴν δὲ σύνθεσιν τῶν ἀπ’ αὐτῶν τετραγώνων ποιεῖν Ἐψη.

Τετάχθω δὴ ἡ ὑπεροχὴ αὐτῶν ᾳβ. καὶ ἐστω ὁ μείζων ἀα καὶ Ἐκ, τῶν ἡμίσεων πάλιν τοῦ συνθέματος, ὁ δὲ ἐλάσσων ἘκἈα. καὶ μένει πάλιν τὸ μὲν σύνθεμα αὐτῶν Ἐκ, ἡ δὲ ὑπεροχή ἐβ.

Ἀοιτὸν ἔστι καὶ τὸ σύνθεμα τῶν ἀπ’ αὐτῶν τετραγώνων ποιεῖν Ἐψη. ἀλλὰ τὸ σύνθεμα τῶν ἀπ’ αὐτῶν τετραγώνων ποιεῖ Ἐβ’Ἐκ. ταῦτα ἱσα Ἐψη, καὶ γίνεται ὁ ὡ Ἐκβ.

Ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν μείζων Ἐκβ, ὁ δὲ ἐλάσσων Ἐψη. καὶ ποιοῦσι τὰ τῆς προτάσεως.

*a* In general terms, Diophantus's problem is to solve the simultaneous equations

\[ \xi + \eta = 2a \]
\[ \xi^2 + \eta^2 = \Lambda. \]

He says, in effect, let

\[ \xi - \eta = 2x; \]

then

\[ \xi = a + x, \eta = a - x, \]
(iii.) *Simultaneous Equations Leading to a Quadratic*

*Ibid.* i. 28, Dioph. ed. Tannery i. 62. 20–64. 10

To find two numbers such that their sum and the sum of their squares are given numbers.\(^a\)

It is a necessary condition that double the sum of their squares exceed the square of their sum by a square. This is of the nature of a formula.\(^b\)

Let it be required to make their sum 20 and the sum of their squares 208.

Let their difference be \(2x\), and let the greater \(= x + 10\) (again adding half the sum) and the lesser \(= 10 - x\).

Then again their sum is 20 and their difference \(2x\).

It remains to make the sum of their squares 208. But the sum of their squares is \(2x^2 + 200\).

Therefore \[2x^2 + 200 = 208,\]

and \[x = 2.\]

To return to the hypotheses—the greater = 12 and the lesser = 8. And these satisfy the conditions of the problem.

\[\text{and } (a + x)^2 + (a - x)^2 = \Lambda,\]

\[\text{i.e., } 2(a^2 + x^2) = \Lambda.\]

A procedure equivalent to the solution of the pair of simultaneous equations \(\xi + \eta = 2a, \xi \eta = \Lambda\), is given in i. 27, and a procedure equivalent to the solution of \(\xi - \eta = 2a, \xi \eta = \Lambda\), in i. 30.

\(^a\) In other words, \(2(\xi^2 + \eta^2) - (\xi + \eta)^2 = \text{a square; it is, in fact, } (\xi - \eta)^3.\) I have followed Heath in translating ἐστι δὲ καὶ τοῦτο πλασματικὸν as "this is of the nature of a formula." Tannery evades the difficulty by translating "est et hoc formativum," but Bachet came nearer the mark with his "effictum allunde." The meaning of πλασματικὸν should be "easy to form a mould," i.e. the formula is easy to discover.
Εὑρεῖν τρίγωνον ὀρθογώνιον ὅπως ὁ ἐν τῷ ἐμβαδῷ αὐτοῦ, προσαλβὼν τὸν ἐν τῇ ὑποτευνοῦσῃ, ποιῆ τετράγωνον, ὁ δὲ ἐν τῇ περιμέτρῳ αὐτοῦ ἣ κύβος.

Τετάχθω ὁ ἐν τῷ ἐμβαδῷ αὐτοῦ ζά, ὁ δὲ ἐν τῇ ὑποτευνοῦσῃ αὐτοῦ Μ τινῶν τετραγωνικῶν Λζά, ἐστιν Μζά Λζά.

Ἀλλ' ἐπεὶ ὑπεθέμεθα τὸν ἐν τῷ ἐμβαδῷ αὐτοῦ εἶναι ζά, ὁ ἄρα ὑπὸ τῶν περὶ τὴν ὀρθὴν αὐτοῦ γίνεται ζΒ. ἄλλα ΒΒ' περιέχονται ὑπὸ ζά καὶ ΜΒ' ἐάν οὖν τάξωμεν μῖαν τῶν ὀρθῶν ΜΒ', ἐστιν ἡ ἐτέρα ζά.

Καὶ γίνεται ἡ περιμέτρου Μη καὶ οὐκ ἔστι κύβος· ὁ δὲ η γέγονεν ἐκ τινός ζών καὶ ΜΒ' δεῖσαι ἃ ἐν ἐφείσει ζϊν ταῦτα, ὅσον προσαλβὼν ΜΒ', ποιεῖ κύβον, ὡστε κύβον ζ' ὑπερέχειν ΜΒ'.

Τετάχθω ὁν ἡ μὲν τοῦ ζ' ζάΜά, ἡ δὲ τοῦ κύβου ζάΛΜά. γίνεται ὁ μὲν ζών, Δ'άζβΜά, ὁ δὲ κύβος, Κ'άζγΛΔ'γΜά. θέλω οὖν τὸν κύβον τοῦ ζών ὑπερέχειν δυᾶδι· ὁ ἄρα ζών μετὰ δυάδος, τούτων Δ'άζβΜγ, ἔστιν ισος Κ'άζγΛΔ'γΜά, δὲν ὁ ζ' εὐρίσκεται Μδ.

"Εσται οὖν ἡ μὲν τοῦ ζών Μέ, ἡ δὲ τοῦ
(iv.) Cubic Equation


To find a right-angled triangle such that its area, added to one of the perpendiculare, makes a square, while its perimeter is a cube.

Let its area = \(x\), and let its hypotenuse be some square number minus \(x\), say \(16 - x\).

But since we supposed the area = \(x\), therefore the product of the sides about the right angle = \(2x\). But \(2x\) can be factorized into \(x\) and 2; if, then, we make one of the sides about the right angle = 2, the other = \(x\).

The perimeter then becomes 18, which is not a cube; but 18 is made up of a square \([16] + 2\). It shall be required, therefore, to find a square number which, when 2 is added, shall make a cube, so that the cube shall exceed the square by 2.

Let the side of the square = \(m + 1\) and that of the cube \(m - 1\). Then the square = \(m^2 + 2m + 1\) and the cube = \(m^3 + 3m - 3m^2 - 1\). Now I want the cube to exceed the square by 2. Therefore, by adding 2 to the square,

\[m^2 + 2m + 3 = m^3 + 3m - 3m^2 - 1,\]

whence \(m = 4\).

Therefore the side of the square = 5 and that of

*This is the only example of a cubic equation solved by Diophantus. For Archimedes’ geometrical solution of a cubic equation, *v. supra*, pp. 126-163.*
GREEK MATHEMATICS

κύβου Μ." αὐτοὶ ἀρα ὁ μὲν □οτε Μ ἐκ, ὁ δὲ κύβος Μ κε.

Μεθυφισταμαι οὖν το ὀρθογώνιον, καὶ τάξια αὐτοῦ τὸ ἐμβαθόν ζα, τάσσω τὴν ὑποτεινουσαν Μ κε Λ εζα. μένει δὲ καὶ ἡ βάσις Μ β, ἡ δὲ κάθετος εζα.

Λοιπὸν ἐστιν τὸν ἀπὸ τῆς ὑποτεινούσης ἴσον ἐλναι τοῖς ἀπὸ τῶν περὶ τὴν ὀρθὴν γίνεται δὲ Δε ιό Μ χκε Λ ζζν- ἐσται ἴση Δε ιό Μ δ. ὦθεν ὁ ζ Μ ὑν

'Επὶ τὰς ὑποστάσεις καὶ μένει.

(d) INDETERMINATE EQUATIONS

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

Ibid. ii. 20, Dioph. ed. Tannery I. 114. 11-22

Εὑρεῖν δύο ἀριθμοὺς ὅπως ὁ ἀπὸ τοῦ ἑκατέρου αὐτῶν τετράγωνων, προσλαβὴν τὸν λοιπὸν, ποιῆ τετράγωνων.

Τετάχθω ὁ αοτε ζα, ὁ δὲ βοτε Μ αζβ, ἐνα ὁ ἀπὸ τοῦ αοτε □οτε, προσλαβὴν τὸν βοτε, ποιῆ □οτε. λοιπὸν ἐστι καὶ τὸν ἀπὸ τοῦ βοτε □οτε, προσλαβὸντα τὸν αοτε, ποιεῖν □οτε. ἀλλ’ ὁ ἀπὸ τοῦ βοτε □οτε, προσλαβὸν τὸν αοτε, ποιεῖ Δε δζε Μδα. ταῦτα ἴσα □οτε.

* Diophantus makes no mention of indeterminate equations of the first degree, presumably because he admits 540
the cube = 3; and hence the square is 25 and the cube 27.

I now transform the right-angled [triangle], and, assuming its area to be $x$, I make the hypotenuse $= 25 - x$; the base remains = 2 and the perpendicular = $x$.

The condition is still left that the square on the hypotenuse is equal to the sum of the squares on the sides about the right angle;

\[ x^2 + 625 - 50x = x^2 + 4, \]

whence

\[ x = \frac{621}{50}. \]

This satisfies the conditions.

(d) Indeterminate Equations

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

Ibid. ii. 20, Dioph. ed. Tannery i. 114. 11-22

To find two numbers such that the square of either, added to the other, shall make a square.

Let the first be $x$, and the second $2x + 1$, in order that the square on the first, added to the second, may make a square. There remains to be satisfied the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is $4x^2 + 5x + 1$; and therefore this must be a square.

Rational fractional solutions, and the whole point of solving an indeterminate equation of the first degree is to get a solution in integers.
GREEK MATHEMATICS

Πλάσσω τὸν □ος ἀπὸ ζβΛΜβ. αὐτὸς ἄρα ἐσται Δ' δΜδΛζη. καὶ γίνεται ὦ ζ γ'.

"Εσται ὦ μὲν αος 'γ', ὦ δὲ βος 'γ', καὶ ποιοῦσι τὸ πρόβλημα.

(β) Double Equations

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

Δοθέντα ἀριθμὸν διελείν εἰς τρεῖς ἀριθμοὺς ὡς ὁ ὑπὸ τοῦ πρῶτου καὶ τοῦ δεύτερου, εάν τε προσλάβη τὸν τρίτον, εάν τε λεῖψῃ, ποιή τετράγωνον.

"Ἐστω ὦ δοθεῖς ὦ ζ.'

Τετάχθω ὦ γος ζ α', καὶ ὦ βος Μ ἐλασσόνων τοῦ ζ'. ἔστω Μβ. ὁ ἄρα αος ἐσται ΜδΛζά'. καὶ λοιπά ἐστι δύο ἐπιτάγματα, τὸν ὑπὸ αου καὶ βους, εάν τε προσλάβη τὸν γος, εάν τε λεῖψῃ, ποιεῖν □ος. καὶ γίνεται διπλή χ ἴσοτης: ΜηΛζά ἴσ. □γ' καὶ ΜηΛζγ' ἴσ. □γ' καὶ οὔ ῥητὸν

* The problem, in its most general terms, is to solve the equation

\[ Ax^2 + Bx + C = y^2. \]

Diophantus does not give a general solution, but takes a number of special cases. In this case \( A \) is a square number \( (=a^2, \text{say}) \), and in the equation

\[ a^2x^2 + Bx + C = y^2 \]

he apparently puts \( y^2 = (ax - m)^2 \),

where \( m \) is some integer,

whence \( x = \frac{m^2 - C}{2am + B} \).
ALGEBRA: DIOPHANTUS

I form the square from $2x - 2$; it will be $4x^2 + 4 - 8x$; and $x = \frac{3}{13}$.

The first number will be $\frac{3}{13}$, the second $\frac{19}{13}$, and they satisfy the conditions of the problem.²

(β) Double Equations ²

*Ibid.* iv. 32, Dioph. ed. Tannery 268. 18–272. 15

To divide a given number into three parts such that the product of the first and second ± the third shall make a square.

Let the given number be 6.

Let the third part be $x$, and the second part any number <6, say 2; then the first part $= 4 - x$; and the two remaining conditions are that the product of the first and second ± the third = a square. There results the double equation

$$8 - x = \text{a square},$$

$$8 - 3x = \text{a square}.$$  

And this does not give a rational result since the ratio

² Diophantus’s term for a double equation is διπλοϊνώτης, διπλή ισότης or διπλή ισοων. It always means with him that two different functions of the unknown have to be made simultaneously equal to two squares. The general equations are therefore

$$A_1x^2 + B_1x + C_1 = u_1^2,$$

$$A_2x^2 + B_2x + C_2 = u_2^2.$$  

Diophantus solves several examples in which the terms in $x^2$ are missing, and also several forms of the general equation.
εστι δια το μή είναι τοὺς ζ πρὸς ἀλλήλους λόγου ἑξοντας ὅν □οι ἀριθμός πρὸς □ος ἀριθμόν.

Ἀλλὰ ὁ ἦ οἱ μονάδι ἐλάσσων τοῦ β, οὶ δὲ ζ γ ὁμοίως μεῖζων Ἄτο τοῦ β. ἀπήκταν οὖν μοι εἰς τὸ εὐρεῖν ἀριθμόν τινα, ὡς τὸν β, ἢν ὁ Ἄτο αὐτοῦ μείζων, πρὸς τὸν Ἄτο (αὐτοῦ ἐλάσσων, λόγου ἑξον ὅν □ος ἀριθμός πρὸς)¹ □ος ἀριθμόν.

"Εστώ ἢ ζητούμενος ζ ἃ, καὶ ὁ Ἄτο αὐτοῦ μεῖζων ἐσται ζ Ἄτο ἃ, ὁ δὲ Ἄτο αὐτοῦ ἐλάσσων ζ Ἄτο Λ Ἄτο ἃ. θέλομεν οὖν αὐτοῦ πρὸς ἀλλήλους λόγου ἑχει ὅν □ος ἀριθμός πρὸς □ος ἀριθμόν. ἐστὶν ὅν ζ πρὸς ἃ. ὥστε ζ Ἄτο Λ Ἄτο ἃ ἐπὶ Λ ἃ γίνονται ζ δ Λ Ἄτο δ. καὶ ζ Ἄτο ἃ ἐπὶ τὴν Ἄτο (γίνονται ζ Ἄτο ἃ).² καὶ εἰσὶν οὖν οἱ ἐκκείμενοι ἀριθμοὶ λόγου ἑχοντες πρὸς ἀλλήλους ὅν ἑχει □ος ἀριθμός πρὸς □ος ἀριθμόν. νῦν ζ δ Λ Ἄτο δ ἵστ. ζ Ἄτο ἃ, καὶ γίνεται ὁ Ἄτο ζ ἅ.

Τάσιον οὖν τὸν βων ζ ἅ. ὁ γὰρ ἄρα ἅ ἐστὶν ζ. ὁ ἀρα ζ ἐστὶν ζ ἅ. ἀρα ἅ.

Λοιπὸν δὲ οἵ εἰναι τὸ ἐπίταγμα, ἐστὶν τὸν ὑπὸ αων καὶ βων, προσλαβόντα τὸν γων, ποιεῖν □ος, καὶ λειψάντα τὸν γων, ποιεῖν □ος. ἀλλ' ὁ ὑπὸ αων καὶ βων, προσλαβών τὸν γων, ποιεῖ Ζ ἅ θ Λ ἃ γον ἵστ. □ος. Λ ἃ. τοῦ γων, ποιεῖ Ζ ἅ. θ Λ ἃ γον ἵστ. □ος. καὶ 544
of the coefficients of \( x \) is not the ratio of a square to
a square.

But the coefficient 1 of \( x \) is 2\(-1\) and the co-
efficient 3 of \( x \) likewise is 2\(+1\); therefore my pro-
blem resolves itself into finding a number to take the
place of 2 such that (the number \(+1\)) bears to (the
number \(-1\)) the same ratio as a square to a square.

Let the number sought be \( y \); then (the num-
ber \(+\)) \( = y + 1 \), and (the number \(-\)) \( = y - 1 \). We
require these to have the ratio of a square to a square,
say 4:1. Now \((y - 1) \cdot 4 = 4y - 4\) and \((y + 1) \cdot 1 = y + 1\).
And these are the numbers having the ratio of a
square to a square. Now I put

\[
4y - 4 = y + 1,
\]

giving

\[
y = \frac{5}{3}.
\]

Therefore I make the second part \( \frac{5}{3} \), for the
third = \( x \); and therefore the first = \( \frac{13}{3} - x \).

There remains the condition, that the product of
the first and second \( \pm \) the third = a square. But the
product of the first and second \( + \) the third =

\[
\frac{65}{9} - \frac{2}{3} x = \text{a square},
\]

and the product of the first and second \( - \) the third =

\[
\frac{65}{9} - 2\frac{2}{3} x = \text{a square}.
\]
GREEK MATHEMATICS

πάντα ἐπὶ τὸν θ., καὶ γίνονται Μξε ΛΣζ ισ. □ν., καὶ Μξε ΛΣζ κὸδ ισ. □ν. καὶ ξεισῶ, τοὺς Σ τῆς μεῖζονος ἰσότητος ποιήσας δκα, καὶ ἔστι
Μσξ ΛΣζ κὸδ ισ. □ν καὶ Μξε ΛΣζ κὸδ ισ. □ν.

Νῦν τούτων λαμβάνω τὴν ὑπεροχὴν καὶ ἔστι
Μρξε· καὶ ἐκτίθεμαι δύο ἀριθμοὺς ὅν τὸ ὑπὸ ἔστι
Μρξε, καὶ εἰσὶ ἕν καὶ ἕν· καὶ τῆς τούτων ὑπεροχῆς
tὸ ζ' ἑφ' ἕαυτο ἰσον ἔστι τῷ ἐλάσσον □ν, καὶ
gάνεται ὁ ζ γνν η.

'Επὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν αος ἐ, ὁ δὲ
βος ἐ, ὁ δὲ γος η. καὶ ἡ ἀπόδειξις φανερά.

* These are a pair of equations of the form

\[ am^2x + a = u^2, \]
\[ an^2x + b = v^2. \]

Multiply by \( n^2, m^2 \) respectively, getting, say

\[ am^2n^2x + an^2 = u^2, \]
\[ am^2n^2x + bm^2 = v^2. \]

\[ \therefore \]
\[ an^2 - bm^2 = u^2 - v^2. \]

Let
\[ an^2 - bm^2 = pq, \]
and put
\[ u' + v' = p, \]
\[ u' - v' = q; \]
\[ \therefore \]
\[ u'^2 = \frac{1}{4}(p + q)^2, \]
\[ v'^2 = \frac{1}{4}(p - q)^2, \]
and so
\[ am^2n^2x + an^2 = \frac{1}{4}(p + q)^2, \]
\[ am^2n^2x + bm^2 = \frac{1}{4}(p - q)^2; \]

whence, from either,

\[ x = \frac{1}{4}(p^2 + q^2) - \frac{1}{4}(an^2 + bm^2). \]

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Multiply throughout by 9, getting

\[ 65 - 6x = \text{a square} \]

and

\[ 65 - 24x = \text{a square}^a \]

Equating the coefficients of \( x \) by multiplying the first equation by 4, I get

\[ 260 - 24x = \text{a square} \]

and

\[ 65 - 24x = \text{a square} \]

Now I take their difference, which is 195, and split it into the two factors 15 and 13. Squaring the half of their difference, and equating the result to the lesser square, I get \( x = \frac{8}{3} \).

Returning to the conditions—the first part will be \( \frac{5}{3} \), the second \( \frac{5}{3} \), and the third \( \frac{8}{3} \). And the proof is obvious.

---

This is the procedure indicated by Diophantus. In his example,

\[ p = 15, \quad q = 13, \]

and

\[ \frac{1}{4}(15 - 13)^2 = 65 - 24x, \]

whence

\[ 24x = 64, \quad \text{and} \quad x = \frac{8}{3} \]
Εὐρεῖν δύο ἀριθμοὺς, ὅπως ὁ ἀπὸ τοῦ πρώτου κύβος προσλαβὼν τὸν δεύτερον ποιῇ κύβον, ὁ δὲ ἀπὸ τοῦ δευτέρου τετράγωνοι προσλαβὼν τὸν πρώτον ποιῇ τετράγωνον.

Τετάχθω ο a<sup>α</sup> ζα· ο ἄρα β<sup>α</sup> ἠσται Μ κυβικαί η ΛΚ<sup>α</sup> λ. καὶ γίνεται ο ἀπὸ τοῦ a<sup>α</sup> κύβος, προσλαβὼν τὸν β<sup>α</sup>, κύβος.

Λοιπὸν ἠστὶ καὶ τὸν ἀπὸ τοῦ β<sup>α</sup> προσλαβόντα τὸν α<sup>α</sup>, ποιεῖν Δ<sup>α</sup> ὁ ἀπὸ τοῦ β<sup>α</sup>, προσλαβών τὸν α<sup>α</sup>, ποιεῖ K<sup>α</sup> Μ ξιδώ ΛΚ<sup>α</sup> τῇ τῷ ἀπὸ π<sup>α</sup> K<sup>α</sup> Μ η, τουτέστι K<sup>α</sup> Μ ξιδώ Μ ξιδώ· καὶ κοινῶν προστιθεμένων τῶν λειπομένων καὶ ἀφαίρεσιμών τῶν ὁμοίων ἀπὸ ὁμοίων, λοιποὶ K<sup>α</sup> Μ λβ ἵσαι εις α· καὶ πάντα παρὰ 5· Δ<sup>α</sup> λβ ἵσαι Μ α.

Καὶ ἠστιν η Μ παίζει, καὶ Δ<sup>α</sup> λβ εἰ ἵσαι διαίτης ἐκ τῶν κύκλων τῶν δις K<sup>α</sup> τῇ· οἱ δὲ K<sup>α</sup> τῇ εἰσιν ὑπὸ τῶν δις Μ η

1 ταύτα ορθαί. Μ ξιδώ add. Bachet.

* As with equations of the second degree, these may be single or double. Single equations always take the form that an expression in x, of a degree not exceeding the sixth, is to be made equal to a square or cube. The general form is therefore

\[ A_0 x^6 + A_1 x^5 + \ldots + A_n = y^2 \text{ or } y^3. \]

Diophantus solves a number of special cases of different degrees.

In double equations, one expression is made equal to a
(ii.) Indeterminate Equations of Higher Degree

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

To find two numbers such that the cube of the first added to the second shall make a cube, and the square of the second added to the first shall make a square.

Let the first number be $x$. Then the second will be a cube number less $x^3$, say $8 - x^3$. And the cube of the first, added to the second, makes a cube.

There remains the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is $x^6 + x + 64 - 16x^3$. Let this be equal to $(x^3 + 8)^2$, that is to $x^6 + 16x^3 + 64$. Then, by adding or subtracting like terms,

$$32x^3 = x;$$

and, after dividing by $x$,

$$32x^2 = 1.$$

Now 1 is a square, and if $32x^2$ were a square, my equation would be soluble. But $32x^2$ is formed from $2 \cdot 16x^3$, and $16x^3$ is $(2 \cdot 8)(x^3)$, that is, it is formed cube and the other to a square, but only a few simple cases are solved by Diophantus.

The general type of the equation is

$$x^6 - Ax^3 + Bx + c^2 = y^2.$$

Put $y = x^3 + c$, then

$$x^3 = \frac{B}{A + 2c},$$

and if the right-hand expression is a square, there is a rational solution.

In the case of the equation $x^6 - 16x^3 + x + 64 = y^2$ it is not a square, and Diophantus replaces the equation by another, $x^6 - 128x^3 + x + 4096 = y^2$, in which it is a square.
καὶ τοῦ Κυ ἄ, τούτου διὰ τῶν Ἰη ὅστε αἱ Λ̈ Δ̈ ἐκ δεί τῶν ἸΗ. γέγονεν οὖν μοι εὑρεῖν κύβον δὲς δεί γενόμενος ποιεῖ □ν.

"Εστῶ οICLE γενόμενος Κυ ἄ. οὕτως δεί γενόμενος ποιεῖ Κ̈ Δ̈ ἢς. □ν. ἔστω Δ̈ ἢς· καὶ γίνεται ο̄ Ζ ἸΗ. ἐπὶ τὰς ὑποστάσεις· ἔσται ο̄ Κ̈ Μ̈ ἢς.

Τάσσω ἀρα τὸν βν Μ̈ ἢς Δ̈ ΛΚ̈ ἄ. καὶ λοιπὸν ἔστι τὸν ἀπὸ τοῦ βν □ν προσλαβόντα τὸν αν̈ ποιεῖν □ν. ἀλλὰ ο̄ ἀπὸ τοῦ βν̈ προσλαβόν τὸν αν̈ ποιεῖ Κ̈ Κ̈ Λ̈ ἸΗ. □ν. τῷ ἀπὸ π'· Κ̈ ἸΗ. ἢς· καὶ γίνεται ο̄ □ν Κ̈ Κ̈ Κ̈ ἸΗ. □ν. καὶ γίνονται λοιποὶ Κ̈ ἸΗ. Ζ. καὶ γίνεται ο̄ Ζ ἢς ἢς ἢς.

"Επὶ τὰς ὑποστάσεις· ἔσται ο̄ αν̈ ἢς ἢς, ὁ δὲ βμ̈ το̄ δεί βτμή· κατεργάζεται τετράγωνον διελέιν εἰς δύο τετράγωνοις.

(c) Theory of Numbers: Sums of Squares
Ibid. ii. 6, Dioph. ed. Tannery i. 90. 9-21.

Τὸν ἐπιτάχθεντα τετράγωνον διελέιν εἰς δύο τετράγωνοις.

* It was on this proposition that Fermat wrote a famous note: "On the other hand, it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which, however, the margin is not large enough.
from 2.8. Therefore $32x^2$ is formed from 4.8. My problem therefore becomes to find a cube which, when multiplied by 4, makes a square.

Let the number sought be $y^3$. Then $4y^3 = a$ square $= 16y^2$ say; whence $y = 4$. Returning to the conditions— the cube will be 64.

I therefore take the second number as $64 - x^3$. There remains the condition that the square on the second added to the first shall make a square. But the square on the second added to the first:

$$x^4 + 4096 + x - 128x^3 = a$$

$$= (x^3 + 64)^2, \text{ say},$$

$$= x^6 + 4096 + 128x^3.$$

On taking away the common terms,

$$256x^3 = x,$$

and

$$x = \frac{1}{16}.$$

Returning to the conditions—

first number $= \frac{1}{16}$, second number $= \frac{262143}{4096}$.

(c) **Theory of Numbers: Sums of Squares**

*Ibid.* ii. 8, Dioph. ed. Tannery i. 90. 9-21

*To divide a given square number into two squares.*

to contain." Fermat claimed, in other words, to have proved that $x^n + y^n = z^n$ cannot be solved in rational numbers if $n > 2$. Despite the efforts of many great mathematicians, a proof of this general theorem is still lacking.

Fermat's notes, which established the modern Theory of Numbers, were published in 1670 in Bachet's second edition of the works of Diophantus.
'Επιτετάχθω δὴ τὸν ἴς διελεῖν εἰς δύο τετράγωνους.

Καὶ τετάχθω ὁ αὖ Δγ ἅ, ὁ ἁρὰ ἐτερος ἑσται ὁ ἐπὶ τοῦ ἴς ΛΔγ ἅ ἀνεῖσυ ἁρὰ Ἰς ἴς ΛΔγ ἅ ἑσται ἱς χρῆος ἃ τῶν ἴς Λ ἑστάλτων ἵς ἑστὶν ἃ τῶν ἴς Λ πλευρά ἑστω ζυλόταϊ. ἀντὶς ἁρὰ ὁ αὖ ἑσται Δγ διδί οἰκία ἴς ΛΖ ἵς ἑστὶν ἡ λείψης καὶ ἀπὸ ὁμοίων ὁμοίων.
Δγ ἁρὰ ἵς ἱς χρῆος ζυλόταϊ, καὶ γίνεται ὁ ζυλόταϊ πέμπτων.

'Εσται ὁ μέν κε ἰς ἱς, ὁ δὲ κε μενδι, καὶ οἱ δύο συντεθέντες ποιοῦσιν κε ἰς ἵς, καὶ ἑστῶν ἐκάτερος τετράγωνος.

Ibid. v. 11, Dioph. ed. Tannery l. 342. 13-346. 12

Μονάδα διελεῖν εἰς τρεῖς ἁρίθμους καὶ προσθεῖναι ἐκάστων αὐτῶν πρότερον τὸν αὐτὸν δοθέντα καὶ ποιεῖν ἐκάστον τετράγωνον.

Δεὶ δὴ τὸν διδόμενον ἁρίθμον μήτε δυάδα εἴναι μήτε τινὰ τῶν ἀπὸ δύαδος ὁδεῖς παραμεινεὶς.

'Επιτετάχθω δὴ τὴν ἴς διελεῖν εἰς τρεῖς ἁρίθμους καὶ προσθεῖναι ἐκάστων ἴς ἵς καὶ ποιεῖν ἐκάστον ἴς ἱς.
ALGEBRA: DIOPHANTUS

Let it be required to divide 16 into two squares.
And let the first square = $x^2$; then the other will be $16 - x^2$; it shall be required therefore to make

$$16 - x^2 = \text{a square.}$$

I take a square of the form $a (mx - 4)^2$, $m$ being any integer and 4 the root of 16; for example, let the side be $2x - 4$, and the square itself $4x^2 + 16 - 16x$. Then

$$4x^2 + 16 - 16x = 16 - x^2.$$

Add to both sides the negative terms and take like from like. Then

$$5x^2 = 16x,$$

and

$$x = \frac{16}{5}.$$

One number will therefore be $\frac{256}{25}$, the other $\frac{144}{25}$, and their sum is $\frac{400}{25}$ or 16, and each is a square.

Ibid. v. 11, Dioph. ed. Tannery i. 342. 13-346. 12

To divide unity into three parts such that, if we add the same number to each of the parts, the results shall all be squares.

It is necessary that the given number be neither 2 nor any multiple of 8 increased by 2. $^b$

Let it be required to divide unity into three parts such that, when 3 is added to each, the results shall all be squares.

numbers not of this form which also are not the sum of three squares. Fermat showed that, if $3a + 1$ is the sum of three squares, then it cannot be of the form $4^n (24k + 7)$ or $4^n (8k + 7)$, where $k = 0$ or any integer.
Πάλιν δεί τὸν ἵ διελέην εἰς τρεῖς □οος ὡπως ἐκαστὸς αὐτῶν μεῖζων ἢ Μγ. ἔαν οὖν πάλιν τὸν ἵ διέλωμεν εἰς τρεῖς □οος, τῇ τῆς παριστήτης ἀγωγῇ, ἔσται ἐκάστος αὐτῶν μεῖζων τριάδος καὶ δυνησόμεθα, ἀφ’ ἐκάστου αὐτῶν ἀφελόντες Μγ, ἔχειν εἰς οὖς Μ διαιρεῖται.

Λαμβάνομεν ἀρτι τοῦ ἵ τὸ γο, γλ. γγχ, καὶ ζητοῦμεν τί προστιθέντες μόριον τετραγωνικὸν ταῖς Μγγχ, ποιήσομεν □οο. πάντα θει. δεί καὶ τῷ ἱ προσθέιναι τί μόριον τετραγωνικὸν καὶ ποιεῖν τὸν ὅλον □οο.

*Εστω τὸ προστιθέμενον μόριον Δγχ ἀ. καὶ πάντα ἐπὶ Δγ. γύνονται ΔγΛΜ α ἰσ. □ο. τῷ ἀπὸ πλευρᾶς εἰΜά γίνεται ὁ □ος ΔγκεῖιΜά ἰσ. ΔγΛΜ α. οὖν ὁ ἵ ΜΒ, ἡ ΔγΜδ, τὸ Δγχ Μδχ.

Εἰ οὖν ταῖς Μλ προστιθεται Μδχ, ταῖς Ἡγγχ προστεθήσεται λχ καὶ γίνεται ἱκά. δεῖ οὖν τὸν ἵ διελεῖν εἰς τρεῖς □οος ὡπως ἐκάστου □ον ἡ πλευρά πάρισος ἢ Μδ ϊ.

*Αλλὰ καὶ ὁ ἵ σύγκειται ἐκ δύο □οο, τοῦ τε Ἰ καὶ τῆς Μ. διαιροῦμεν τὴν Μ εἰς δύο □οο τά τε Ἰ καὶ τά Ἰ, ὡστε τὸν ἵ σύγκεισθαι ἐκ τριῶν □οο.*

---

* The method has been explained in v. 19, where it is proposed to divide 13 into two squares each > 6. It will be sufficiently obvious from this example. The method is also used in v. 10, 12, 13, 14.

554
ALGEBRA: DIOPHANTUS

Then it is required to divide 10 into three squares such that each of them > 3. If then we divide 10 into three squares, according to the method of approximation, each of them will be > 3 and, by taking 3 from each, we shall be able to obtain the parts into which unity is to be divided.

We take, therefore, the third part of 10, which is $3\frac{1}{3}$, and try by adding some square part to $3\frac{1}{3}$ to make a square. On multiplying throughout by 9, it is required to add to 30 some square part which will make the whole a square.

Let the added part be $\frac{1}{x^2}$; multiply throughout by $x^2$; then

$$30x^2 + 1 = \text{a square}.$$ 

Let the root be $5x + 1$; then, squaring,

$$25x^2 + 10x + 1 = 30x^2 + 1;$$

whence

$$x = 2, x^2 = 4, \frac{1}{x^2} = \frac{1}{4}.$$

If, then, to 30 there be added $\frac{1}{4}$, to $3\frac{1}{3}$ there is added $\frac{1}{36}$, and the result is $\frac{121}{36}$. It is therefore required to divide 10 into three squares such that the side of each shall approximate to $\frac{11}{6}$.

But 10 is composed of two squares, 9 and 1. We divide 1 into two squares, $\frac{9}{25}$ and $\frac{16}{25}$, so that 10 is composed of three squares, $9, \frac{9}{25}$ and $\frac{16}{25}$. It is there-
Εκ τε τοῦ θ καὶ τοῦ θε καὶ τοῦ θε ιε. δει οὖν εκάστην τῶν πλευρῶν παρασκευάσαι πάρισσον ἵνα.

'Αλλὰ καὶ αἱ πλευρὰς τῆς ἐσθῆ Μ ἱγ καὶ Μ ἅ καὶ Μ ἀκ καὶ πάντα λεῖκα καὶ γίνονται Μ ἵγ καὶ Μ ἱδ καὶ Μ ἱγ. τὰ δὲ ἐν εἰς γίνονται Μ ἱδ. δει οὖν εκάστην πλευρᾶς κατασκευάσαι γε.

Πλάσσομεν ἐνὸς πλευρᾶς Ἐ Μ ἱγ ἱδ λεῖκα, ἐτέρου δὴ ἐπὶ Μ ἱε, τοῦ δὲ ἐτέρου ξ Μ ἱε. γίνονται ὁ ἀπὸ τῶν εἰρημένων, Δ, γραμμένο Μ ἱ λεῖκα. ταύτα ἢσα Μ ἱ. ὅτες εὑρίσκεται ὁ γραμμένο.

'Επὶ τὰς ὑποστάσεις καὶ γίνονται αἱ πλευρὰς τῶν τετράγωνων δοθεῖσαι, ὅστε καὶ αὐτοὶ. τὰ λοιπὰ δηλα.


Εὑρεῖν τέσσαρας ἀριθμοὺς (τετράγωνους), οἱ συντεθέντες καὶ προσολαβόντες τὰς ίδιας πλευρᾶς συντεθείσας ποιούσι δοθέντα ἀριθμόν.
fore required to make each of the sides approximate to $\frac{11}{6}$.

But their sides are $3, \frac{4}{5}$ and $\frac{3}{5}$. Multiply throughout by 30, getting 90, 24 and 18; and $\frac{11}{6}$ [when multiplied by 30] becomes 55. It is therefore required to make each side approximate to 55.

[Now $3 > \frac{55}{30}$ by $\frac{35}{30}, \frac{4}{5} < \frac{55}{30}$ by $\frac{31}{30}$, and $\frac{3}{5} < \frac{55}{30}$ by $\frac{37}{30}$.]

If, then, we took the sides of the squares as $3 - \frac{35}{30}$, $\frac{4}{5} + \frac{31}{30}$, $\frac{3}{5} + \frac{37}{30}$, the sum of the squares would be $3 \cdot \left(\frac{11}{6}\right)^2$

or $\frac{363}{36}$, which > 10.

Therefore we take the side of the first square as $3 - 35x$; of the second as $\frac{4}{5} + 31x$, and of the third as $\frac{3}{5} + 37x$. The sum of the aforesaid squares

$$3555x^2 + 10 - 116x = 10;$$

whence

$$x = \frac{116}{3555}.$$

Returning to the conditions—as the sides of the squares are given, the squares themselves are also given. The rest is obvious.a


To find four square numbers such that their sum added to the sum of their sides shall make a given number.
"Εστω δή τὸν μία.

Έπει πάς Ἰος προσλαβών τὴν ἴδιαν παρακολουθούσαν, καὶ ἠλευθεροῦσα, οὐ ἢ παρακολουθούσαν, ὁποῖοι ἀριθμοὶ ἀρέτης πλευρά, οἱ τέσσαρες ἀριθμοὶ ἀρά, προσλαβώντες μὲν τὰς ἴδιας παρακολουθοῦσαν, προσλαβώντες δὲ καὶ ἰσοτρόπως, ποιοῦσι ὁμοιὸς Ἱμία, προσλαβώντες δὲ καὶ ἰσοτρόπως, εἰσὶν δὲ καὶ αἱ Ἰμία μετὰ ἰσοτρόπως, ὃ ἐστὶ Ἰμά, Ἰμία γ. τὰς Ἰμία ἄρα Ἰμία διαφέρειν δεῖ εἰς τέσσαρας ἰσοτρόπως, καὶ ἀπὸ τῶν πλευρῶν, ἀφελῶν ἀπὸ ἐκάστης παρακολουθοῦσαν, Ἰμία γ., ἐξω τῶν ἰσοτρόπως τὰς παρακολουθοῦσαν.

Διαφέρειται δὲ ὁ Ἰμία εἰς δύο ἰσοτρόπως, τὸν ἰσοτρόπως καὶ θ. καὶ πάλιν ἐκάτερος τούτων διαφέρειται εἰς δύο ἰσοτρόπως, εἰς καὶ κατὰ καὶ ρηματικό καὶ παρακολουθοῦσαν, ἐκ τῆς πλευρᾶς ἰσοτρόπως καὶ ἀριθμοῦ ἀπὸ ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολουθοῦσαν, ἐκ τῶν ἀριθμῶν ἐκάτερος τούτων πλευρῶν Ἰμία γ., καὶ ἐσονται αἱ παρακολο
ALGEBRA: DIOPHANTUS

Let it be 12.

Since any square added to its own side and \( \frac{1}{4} \) makes a square, whose side \textit{minus} \( \frac{1}{4} \) is the number which is the side of the original square,\(^a\) and the four numbers added to their own sides make 12, then if we add \( 4 \cdot \frac{1}{4} \) they will make four squares. But

\[
12 + 4 \cdot \frac{1}{4} \text{ (or 1)} = 13.
\]

Therefore it is required to divide 13 into four squares, and then, if I subtract \( \frac{1}{4} \) from each of their sides, I shall have the sides of the four squares.

Now 13 may be divided into two squares, 4 and 9. And again, each of these may be divided into two squares, \( \frac{64}{25} \) and \( \frac{86}{25} \), and \( \frac{144}{25} \) and \( \frac{81}{25} \). I take the side of each \( \frac{8}{5}, \frac{6}{5}, \frac{12}{5}, \frac{9}{5} \), and subtract half from each side, and the sides of the required squares will be

\[
\begin{array}{cccc}
11 & 7 & 19 & 13 \\
10 & 10 & 10 & 10
\end{array}
\]

The squares themselves are therefore respectively

\[
\begin{array}{cccc}
121 & 49 & 361 & 169 \\
100 & 100 & 100 & 100
\end{array}
\]

\(^a\) \textit{i.e.}, \( x^2 + x + \frac{1}{4} = (x + \frac{1}{2})^2 \).

\(^b\) In iv. 30 and v. 14 it is also required to divide a number into four squares. As \textit{every number is either a square or the sum of two, three or four squares} (a theorem stated by Fermat and proved by Lagrange), and a square can always be divided into two squares, it follows that any number can be divided into four squares. It is not known whether Diophantus was aware of this.
GREEK MATHEMATICS

(f) POLYGONAL NUMBERS

Dioph. De polyg. num., Praef., Dioph. ed. Tannery
1. 450. 3-19

"Εκαστος τῶν ἀπὸ τῆς τριάδος ἀριθμῶν αὐξομένων μονάδι, πολύγωνός ἐστι πρῶτος ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὅσον ἐστὶν τὸ πλῆθος τῶν ἐν αὐτῷ μονάδων: πλευρά τε αὐτοῦ ἐστὶν ὁ ἕξης τῆς μονάδος ἀριθμός, ὁ β. ἔσται δὲ ὁ μὲν γ τριγώνων, ὁ δὲ δ τετράγωνων, ὁ δὲ ἐ πεντάγωνως, καὶ τούτο ἕξης.

Τῶν δὴ τετράγωνων προδήλων ὃντων ὧτι καθεστήκασι τετράγωνοι διὰ τὸ γεγονέναι αὐτοὺς ἐξ ἀριθμῶν τινω ἐφ' ἐαυτὸν πολλαπλασιασθέντος, ἑδοκιμάσθη ἐκαστὸν τῶν πολυγώνων, πολυπλασιαζόμενον ἐπὶ τινα ἀριθμὸν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνίων αὐτοῦ, καὶ προσλαβόντα τετράγωνον τινα πάλιν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνίων αὐτῶν, φαίνεσθαι τετράγωνον: ὃ δὴ παραστήσωμεν ὑποδείξαντες πῶς ἀπὸ δοθείσης πλευρᾶς ἐπιταχθεὶς πολύγωνος εὑρίσκεται, καὶ πῶς δοθεντὶ πολυγώνῳ ἡ πλευρὰ λαμβάνεται.

1 πρῶτος Bachet, πρῶτον codd.

*A fragment of the tract On Polygonal Numbers is the only work by Diophantus to have survived with the Arithmetica. The main fact established in it is that stated in Hypsicles' definition, that the a-gonal number of side n is
(f) Polygonal Numbers

Diophantus, *On Polygonal Numbers*, Preface, Dioph. ed. Tannery i. 450. 3-19

From 3 onwards, every member of the series of natural numbers increasing by unity is the first (after unity) of a particular species of polygon, and it has as many angles as there are units in it; its side is the number next in order after the unit, that is, 2. Thus 3 will be a triangle, 4 a square, 5 a pentagon, and so on in order.\(^b\)

In the case of squares, it is clear that they are squares because they are formed by the multiplication of a number into itself. Similarly it was thought that any polygon, when multiplied by a certain number depending on the number of its angles, with the addition of a certain square also depending on the number of its angles, would also be a square. This we shall establish, showing how any assigned polygonal number may be found from a given side, and how the side may be calculated from a given polygonal number.

\[
\frac{1}{4}n\{2+(n-1)(a-2)\} \quad (v. \, supra, \, p. \, 396 \, n. \, a, \, and \, vol. \, i. \, p. \, 98 \, n. \, a). \, The \, method \, of \, proof \, contrasts \, with \, that \, of \, the \, \textit{Arithmetica} \, in \, being \, geometrical. \, For \, polygonal \, numbers, \, v. \, vol. \, i. \, pp. \, 86-99.
\]

\(^b\) The meaning is explained in vol. i. p. 86 n. a, especially in the diagram on p. 89. In the example there given, 5 is the first (after unity) of the series of pentagonal numbers 1, 5, 12, 22 \ldots \, It has 5 angles, and each side joins 2 units.
XXIV. REVIVAL OF GEOMETRY:
PAPPUS OF ALEXANDRIA
XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, s.v. Πάππος

Πάππος, Ἀλεξάνδρεύς, φιλόσοφος, γεγονὼς κατὰ τὸν πρεσβύτερον Θεοδόσιον τὸν βασιλέα, ὁτὲ καὶ Θέων ὁ φιλόσοφος ἤκμαζεν, ὁ γράφας εἰς τὸν Πτολεμαίον Κανόνα. βιβλία δὲ αὐτοῦ Χωρογραφία οἰκουμενικῆ, Εἰς τὰ δ ἡ βιβλία τῆς Πτολεμαίου

* Theodosius I reigned from A.D. 379 to 395, but Suidas may have made a mistake over the date. A marginal note opposite the entry Dioecletian in a Leyden ms. of chronological tables by Theon of Alexandria says, “In his time Pappus wrote”; Dioecletian reigned from A.D. 284 to 305. In Rome’s edition of Pappus’s commentary on Ptolemy’s *Syntaxis* (*Studii e Testi*, liv. pp. x-xiii), a cogent argument is given for believing that Pappus actually wrote his *Collection* about A.D. 320.

Suidas obviously had a most imperfect knowledge of Pappus, as he does not mention his greatest work, the *Synagoge* or *Collection*. It is a handbook to the whole of Greek geometry, and is now our sole source for much of the history of that science. The first book and half of the second are missing. The remainder of the second book gives an account of Apollonius’s method of working with large numbers (v. *supra*, pp. 352-357). The nature of the remaining books to the eighth will be indicated by the passages here cited. There is some evidence (v. *infra*, p. 607 n. a) that the work was originally in twelve books.

The edition of the *Collection* with ancillary material published in three volumes by Friedrich Hultsch (Berlin, 1876-564)
XXIV. REVIVAL OF GEOMETRY:
PAPPUS OF ALEXANDRIA

(a) General

Suidas, s.v. Pappus

Pappus, an Alexandrian, a philosopher, born in the time of the Emperor Theodosius I, when Theon the philosopher also flourished, a who commented on Ptolemy's Table. His works include a Universal Geography, a Commentary on the Four Books of

1878) was a notable event in the revival of Greek mathematical studies. The editor's only major fault is one which he shares with his generation, a tendency to condemn on slender grounds passages as interpolated.

Pappus also wrote a commentary on Euclid’s Elements; fragments on Book x. are believed to survive in Arabic (e. vol. i. p. 456 n. a). A commentary by Pappus on Euclid's Data is referred to in Marinus's commentary on that work. Pappus (e. vol. i. p. 301) himself refers to his commentary on the Analemma of Diodorus. The Arabic Fihrist says that he commented on Ptolemy's Planisphaerium.

The separate books of the Collection were divided by Pappus himself into numbered sections, generally preceded by a preface, and the editors have also divided the books into chapters. References to the Collection in the selections here given (e.g., Coll. iii. 11. 28, ed. Hultsch 68. 17–70. 8) are first to the book, then to the number or preface in Pappus's division, then to the chapter in the editors' division, and finally to the page and line of Hultsch's edition. In the selections from Book vii. Pappus's own divisions are omitted as they are too complicated, but in the collection of lemmas the numbers of the propositions in Hultsch's edition are added as these are often cited.
GREEK MATHEMATICS

Μεγάλης συντάξεως ύπόμνημα, Ποταμώς τούς ἐν Λιβύη, Ὀνειροκριτικά.

(b) Problems and Theorems

Papp. Coll. iii., Praef. 1, ed. Hultsch 30. 3-32. 3

Οἱ τὰ ἐν γεωμετρίᾳ ζητούμενα βουλόμενοι τεχνικῶτερον διακρίνειν, ὁ κράτιστος Πανδροσίων, πρόβλημα μὲν ἀξιοῦσι καλεῖν ἐφ' οὐ προβάλλεται τι ποιήσαι καὶ κατασκευάσαι, θεωρημα δὲ ἐν ᾧ τινῶν ὑποκειμένων τὸ ἐπόμενον αὐτοῖς καὶ πάντως ἐπισυμβαίνον θεωρεῖται, τῶν παλαιῶν τῶν μὲν προβλήματα πάντα, τῶν δὲ θεωρήματα εἶναι φασκώτων. οὐ μὲν οὖν τὸ θεώρημα προτείνων, συνιῶν ὄντως τρόπον, τὸ ἀκόλουθον τοῦτω ἀξιῶν ζητεῖν καὶ οὐκ ἂν ἄλλως ὑγιῶς προτείνωι, δὲ τὸ πρόβλημα προτείνων [ἂν μὲν ἄμαθης ἢ καὶ παντάπασιν ἰδιώτης], κἂν ἀδύνατον πως κατασκευασθήναι προστάξῃ, σύγγνωστὸς ἐστι καὶ ἀνυπεύθυνος. τοῦ γὰρ ζητοῦντος ἔργον καὶ τοῦτο διορίσαι, τὸ τε δυνατὸν καὶ τὸ ἀδύνατον, κἂν ἢ δυνατὸν, πότε καὶ πῶς καὶ ποσαχώς δυνατὸν. ἐὰν δὲ προσποιούμενος ἢ τὰ μαθήματα πως ἀπείρως προβάλλων, οὐκ ἐστὶν αἰτίας ἔξω. πράγμα γοῦν τινὲς τῶν τὰ μαθήματα προσποιούμενων εἰδέναι διὰ σοῦ τὰς τῶν προβλημάτων προτάσεις ἀμαθῶς ἤμιν ὄρισαν. περὶ ἄν ἔδει καὶ τῶν

1 ἂν ... ἰδιώτης om. Hultsch.

* Suidas seems to be confusing Ptolemy's Μαθηματικὴ τετράβιβλος σύνταξις (Tetrabiblos or Quadripartitum) which was in four books but on which Pappus did not comment, with the Μαθηματικὴ σύνταξις (Syntaxis or Almagest), which was the subject of a commentary by Pappus but extended to 566
REVIVAL OF GEOMETRY: PAPPUS


(b) PROBLEMS AND THEOREMS

Pappus, Collection iii., Preface 1, ed. Hultsch 30. 3-32. 3

Those who favour a more exact terminology in the subjects studied in geometry, most excellent Panderision, use the term problem to mean an inquiry in which it is proposed to do or to construct something, and the term theorem an inquiry in which the consequences and necessary implications of certain hypotheses are investigated, but among the ancients some described them all as problems, some as theorems. Therefore he who propounds a theorem, no matter how he has become aware of it, must set for investigation the conclusion inherent in the premises, and in no other way would he correctly propound the theorem; but he who propounds a problem, even though he may require us to construct something which is in some way impossible, is free from blame and criticism. For it is part of the investigator's task to determine the conditions under which a problem is possible and impossible, and, if possible, when, how and in how many ways it is possible. But when a man professing to know mathematics sets an investigation wrongly he is not free from censure. For example, some persons professing to have learnt mathematics from you lately gave me a wrong enunciation of problems. It is desirable that I should state some of the proofs of thirteen books. Pappus's commentary now survives only for Books v. and vi., which have been edited by A. Rome, Studi e Testi, liv., but it certainly covered the first six books and possibly all thirteen.

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παραπλησίων αυτοῖς ἀποδείξεις τινὰς ἡμᾶς εἶπεν εἰς ὑφέλειαν σήν τε καὶ τῶν φιλομαθούντων ἐν τῷ τρίτῳ τούτῳ τῆς Συναγωγῆς βιβλίῳ. τὸ μὲν οὖν πρῶτον τῶν προβλημάτων μέγας τις γεωμέτης εἶναι δοκῶν ἄριστεν ἀμαθῶς. τὸ γὰρ δύο δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον ἐν συνεχεὶ ἀναλογίᾳ λαβεῖν ἐφασκεν εἰδέναι δι’ ἐπιπέδου θεωρίας, ἥξιον δὲ καὶ ἡμᾶς ὁ ἀνήρ ἐπισκεψαμένος ἀποκρίνασθαι περὶ τῆς ὑπ’ αὐτοῦ γενηθείσης κατασκευῆς, ἣτις ἔχει τὸν τρόπον τούτον.

(c) The Theory of Means

Ibid. iii. 11. 28, ed. Hultsch 68. 17–70. 8

Τὸ δὲ δεύτερον τῶν προβλημάτων ἦν τόδε.

Ἐν ἡμικυκλίῳ τᾶς τρεῖς μεσότητας λαβεῖν ἄλλος τις ἐφασκεν, καὶ ἡμικύκλιον τὸ ΑΒΓ ἐκθεμένου, οὗ κέντρον τὸ Ε, καὶ τυχὸν σημείον ἐπὶ τῆς ΑΓ λαβὼν τὸ Δ, καὶ ἀπ’ αὐτοῦ πρὸς ορθάς ἄγαγῶν τῇ ΕΓ τὴν ΔΒ, καὶ ἐπιζεύξας τὴν ΕΒ, καὶ αὐτῇ κάθετον ἄγαγῶν ἀπὸ τοῦ Δ τὴν ΔΖ, τὰς τρεῖς μεσότητας ἔλεγεν ἄπλως ἐν τῷ ἡμικυκλίῳ ἐκτεθείσαι, τὴν μὲν ΕΓ μέσην ἀριθμητικὴν, τὴν δὲ ΔΒ μέσην γεωμετρικὴν, τὴν δὲ ΒΖ ἀρμονικὴν.

"Ὅτι μὲν οὖν ἡ ΒΔ μέση ἐστὶ τῶν ΑΔ, ΔΓ ἐν

* The method, as described by Pappus, but not reproduced here, does not actually solve the problem, but it does furnish a series of successive approximations to the solution, and deserves more kindly treatment than it receives from him.

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these and of matters akin to them, for the benefit both of yourself and of other lovers of this science, in this third book of the Collection. Now the first of these problems was set wrongly by a person who was thought to be a great geometer. For, given two straight lines, he claimed to know how to find by plane methods two means in continuous proportion, and he even asked that I should look into the matter and comment on his construction, which is after this manner.\(^a\)

(c) The Theory of Means

*Ibid.* iii. 11. 28, ed. Hultsch 68. 17–70. 8

The second of the problems was this:

A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle \(AB\Gamma\), with centre \(E\), and taking any point \(\Delta\) on \(A\Gamma\), and from it drawing \(\Delta B\) perpendicular to \(E\Gamma\), and joining \(EB\), and from \(\Delta\) drawing \(\Delta Z\) perpendicular to it, he claimed simply that the three means had been set out in the semicircle, \(E\Gamma\) being the arithmetic mean, \(\Delta B\) the geometric mean and \(BZ\) the harmonic mean.

That \(B\Delta\) is a mean between \(A\Delta\), \(\Delta \Gamma\) in geometrical
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τῇ γεωμετρικῇ ἀναλογίᾳ, ἢ δὲ ΕΓ τῶν ΑΔ, ΔΓ ἐν τῇ ἀριθμητικῇ μεσότητι, φανερὸν. ἔστι γὰρ ὃς μὲν ἢ ΑΔ πρὸς ΔΒ, ἢ ΔΒ πρὸς ΔΓ, ὡς δὲ ἢ ΑΔ πρὸς εαυτὴν, οὕτως ἢ τῶν ΑΔ, AE ὑπεροχή, τούτεστιν ἢ τῶν ΑΔ, ΕΓ, πρὸς τὴν τῶν ΕΓ, ΓΔ. πώς δὲ καὶ ἢ ΖΒ μέση ἐστὶν τῆς ἄρμονικῆς μεσοτητος, ἢ ποιῶν εὐθείων, οὐκ ἐπεν, μόνον δὲ ὅτι τρίτη ἀνάλογον ἐστὶν τῶν ΕΒ, ΒΔ, ἀγνοῶν ὅτι ἀπὸ τῶν ΕΒ, ΒΔ, ΖΒ ἐν τῇ γεωμετρικῇ ἀναλογίᾳ οὐσῶν πλάσσεται ἡ ἄρμονική μεσότης. δεικθῆ- σεται γὰρ υφ᾿ ἡμῶν ὑστερον ὅτι δύο αἱ ΕΒ καὶ τρεῖς αἱ ΔΒ καὶ μία ἡ ΖΒ ὡς μία συντεθεῖσαι ποιοῦσι τὴν μείζων ἀκραν τῆς ἄρμονίκης μεσοτητος, δύο δὲ αἱ ΒΔ καὶ μία ἡ ΖΒ τὴν μέσην, μία δὲ ἡ ΒΔ καὶ μία ἡ ΖΒ τὴν ἐλαχίστην.

(d) The Paradoxes of Erycinus

Ibid. iii. 24. 58, ed. Hultsch 104. 14—106. 9

Τὸ δὲ τρίτον τῶν προβλημάτων ἢ τὸ τέδε.

"Εστιν τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν

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proportion, and $ET$ between $AD$, $GT$ in arithmetical proportion, is clear. For

$$AD : AB = AB : GT,$$  [Eucl. iii. 31, vi. 8 Por.

and

$$AD : AD = (AD - AE) : (ET - GT)$$
$$= (AD - ET) : (ET - GT).$$

But how $ZB$ is a harmonic mean, or between what kind of lines, he did not say, but only that it is a third proportional to $EB$, $DB$, not knowing that from $EB$, $DB$, $BZ$, which are in geometrical proportion, the harmonic mean is formed. For it will be proved by me later that a harmonic proportion can thus be formed—

greater extreme $= 2EB + 3DB + BZ,$
mean term $= 2DB + BZ,$
lesser extreme $= DB + BZ.$

(d) THE PARADOXES OF ERYCINUS

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

The third of the problems was this:
Let $ABGT$ be a right-angled triangle having the

It is Pappus, in fact, who seems to have erred, for $BZ$ is a harmonic mean between $AD$, $GT$, as can thus be proved:

Since $BDAE$ is a right-angled triangle in which $AZ$ is perpendicular to $BE$, 

$$BZ : BDA = BDA : BE,$$
i.e.,

$$BZ : BE = BDA : ADG.$$

But

$$BE = \frac{1}{2}(AD + GT);$$

$$BZ(AD + GT) = 2AD : ADG,$$
i.e.,

$$AD(BZ - GT) = GT(AD - BZ),$$
i.e.,

$$AD : GT = (AD - BZ) : (BZ - GT),$$

and $\therefore$ $BZ$ is a harmonic mean between $AD$, $GT$.

The three means and the several extremes have thus been
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ἐχον τὴν Β γωνίαν, καὶ διήκθω τις ἡ ΑΔ, καὶ κείσθω τῇ ΑΒ ἤσον ἡ ΔΕ, καὶ δίχα τιμηθείσης τῆς ΕΑ κατὰ τὸ Ζ, καὶ ἐπιζευγχείσης τῆς ΖΓ δείξαι συναμφότερας τὰς ΔΖΓ δύο πλευράς ἐντὸς τοῦ τριγώνου μεῖζονας τῶν ἐκτὸς συναμφότερων τῶν ΒΑΓ πλευρῶν.

Καὶ ἐστι δήλον. ἐπεὶ γὰρ αἱ ΓΖΑ, τοιτέστων αἱ ΓΖΕ, τῆς ΓΑ μεῖζονές εἰστιν, ἴση δὲ ἡ ΔΕ τῇ ΑΒ, αἱ ΓΖΔ ἀρα δύο τῶν ΓΒΕ μεῖζονές εἰσιν. . . .

'Ἀλλ' ὅτι τοῦτο μὲν, ὅπως ἂν τὸς έθέλεις προτείνεις, ἀπειραχώς δείκνυται δήλον, οὐκ ἀκαίρου δὲ καθολικώτερον περὶ τῶν τοιούτων προβλημάτων διαλαβεῖν ἀπὸ τῶν φερομένων παραδόξων Ἐρυκίνου προτείνοντας οὕτως.

(e) The Regular Solids

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

Εἰς τὴν δοθείσαν σφαῖραν ἐγγράφατο τὰ πέντε πολύεδρα, προγράφηται δὲ τάδε.

'Εστω εὖ σφαῖρα κύκλος ὁ ΑΒΓ, οὗ διάμετρος ἡ ΑΓ καὶ κέντρον τὸ Δ, καὶ προκείσθω εἰς τὸν

represented by five straight lines (ΕΒ, ΒΖ, ΑΔ, ΔΓ, ΒΔ). Pappus takes six lines to solve the problem. He proceeds to define the seven other means and to form all ten means as linear functions of three terms in geometrical progression (v. vol. i. pp. 124-129).
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angle B right, and let AΔ be drawn, and let ΔE be placed equal to AB, then if EA be bisected at Z, and ZΓ be joined, to show that the sum of the two sides ΔZ, ZΓ within the triangle, is greater than the sum of the two sides BA, AΓ without the triangle.

And it is obvious. For

since

\[ \Gamma Z + ZA > \Gamma A, \]

[i.e.,]

\[ \Gamma Z + ZE > \Gamma A, \]

while

\[ \Delta E = AB, \]

\[ \therefore \quad [\Gamma Z + ZE + E\Delta = \Gamma A + A \]

[i.e.,]

\[ \Gamma Z + Z\Delta > \Gamma A + AB. \ldots \]

But it is clear that this type of proposition, according to the different ways in which one might wish to propound it, can take an infinite number of forms, and it is not out of place to discuss such problems more generally and [first] to propound this from the so-called paradoxes of Erycinus.①

(e) THE REGULAR SOLIDS ②

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

In order to inscribe the five polyhedra in a sphere, these things are premised.

Let AБΓ be a circle in a sphere, with diameter AΓ and centre A, and let it be proposed to insert in the

① Nothing further is known of Erycinus. The propositions next investigated are more elaborate than the one just solved.
② This is the fourth subject dealt with in Coll. iii. For the treatment of the subject by earlier geometers, v. vol. i. pp. 216-225, 466-479.
κύκλον ἐμβαλεῖν εὐθείαν παράλληλον μὲν τῇ ΑΓ διαμέτρῳ, ἵστην δὲ τῇ δοθείσῃ μὴ μείζονι οὐσῆ τῆς ΑΓ διαμέτρου.

Κείσθω τῇ ἡμισεία τῆς δοθείσης ἴση τῇ ΕΔ, καὶ τῇ ΑΓ διαμέτρῳ ἢχθων πρὸς ὀρθὰς τῇ ΕΒ, τῇ δὲ ΑΓ παράλληλος ἢ ΒΖ, ἦτε ίση ἐσται τῇ δοθείσῃ διπλῇ γάρ ἐστιν τῆς ΕΔ, ἐπεὶ καὶ ἴση τῇ ΕΗ, παράλληλον ἀκθείσης τῆς ΖΗ τῇ ΒΕ.

(f) Extension of Pythagoras's Theorem

_Ibid._ iv. 1. 1, ed. Hultsch 176. 9–178. 13

Ἐὰν γὰρ τρίγωνον τὸ ΑΒΓ, καὶ ἀπὸ τῶν ΑΒ, ΒΓ ἀναγραφῆ τυχόντα παράλληλόγραμμα τὰ ΑΒΔΕ, ΒΓΖΗ, καὶ αἱ ΔΕ, ΖΗ ἐκβληθῶσιν ἐπὶ τὸ Θ, καὶ ἐπιζευχὴ τῇ ΘΒ, γίνεται τὰ ΑΒΔΕ,
circle a chord parallel to the diameter \(A\Gamma\) and equal to a given straight line not greater than the diameter \(A\Gamma\).

Let \(E\Delta\) be placed equal to half of the given straight line, and let \(EB\) be drawn perpendicular to the diameter \(A\Gamma\), and let \(BZ\) be drawn parallel to \(A\Gamma\); then shall this line be equal to the given straight line. For it is double of \(E\Delta\), inasmuch as \(ZH\), when drawn, is parallel to \(BE\), and it is therefore equal to \(EH\).*

(f) Extension of Pythagoras's Theorem


If \(AB\Gamma\) be a triangle, and on \(AB\), \(B\Gamma\) there be described any parallelograms \(AB\Delta E\), \(B\Gamma ZH\), and \(\Delta E\). \(ZH\) be produced to \(\Theta\), and \(\Theta B\) be joined, then the

* This lemma gives the key to Pappus's method of inscribing the regular solids, which is to find in the case of each solid certain parallel circular sections of the sphere. In the case of the cube, for example, he finds two equal and parallel circular sections, the square on whose diameter is two-thirds of the square on the diameter of the sphere. The squares inscribed in these circles are then opposite faces of the cube. In each case the method of analysis and synthesis is followed. The treatment is quite different from Euclid's.
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ΒΓΖΗ παραλληλόγραμμα ἵσα τῷ ὑπὸ τῶν ΑΓ, ΘΒ περιεχόμενον παραλληλογράμμων ἐν γωνίᾳ ή ἐστιν ἵση συναμφοτέρω τῇ ὑπὸ ΒΑΓ, ΔΘΒ.

'Εκβεβλῆσθω γὰρ ἡ ΘΒ ἐπὶ τὸ K, καὶ διὰ τῶν A, Γ τῇ ΘΚ παράλληλοι ἤχθωσαν αἱ AA, ΓΜ, καὶ ἐπεξεύθεω ἡ ΔΜ. ἔπει παραλληλόγραμμόν ἐστιν τὸ ΑΛΘΒ, αἱ AA, ΘΒ ἵσαι τέ εἰσιν καὶ παράλληλοι. ὅμοιος καὶ αἱ MG, ΘΒ ἵσαι τέ εἰσιν καὶ παράλληλαι, ὡστε καὶ αἱ AA, MG ἰσαι τέ εἰσιν καὶ παράλληλαι. καὶ αἱ ΔΜ, ΑΓ ἄρα ἵσαι τε καὶ παράλληλαι εἰσιν: παραλληλόγραμμον ἀρα ἐστιν τὸ ΛΑΜΓ ἐν γωνίᾳ τῇ ὑπὸ ΔΑΓ, τοῦτο ἐστιν συναμφοτέρω τῇ τε ὑπὸ ΒΑΓ καὶ ὑπὸ ΔΘΒ. ἵση γὰρ ἐστιν ἡ ὑπὸ ΔΘΒ τῇ ὑπὸ ΛΑΒ. καὶ ἔπει τὸ ΔΑΒΕ παραλληλόγραμμον τῷ ΔΑΒΘ ἰσον ἐστιν (ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἐστιν 576
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parallelograms $AB\Delta E$, $B\Gamma ZH$ are together equal to the parallelogram contained by $A\Gamma$, $\Theta B$ in an angle which is equal to the sum of the angles $BA\Gamma$, $\Delta\Theta B$.

For let $\Theta B$ be produced to $K$, and through $A$, $\Gamma$ let $AA$, $\Gamma M$ be drawn parallel to $\Theta K$, and let $\Lambda M$ be joined. Since $A\Lambda\Theta B$ is a parallelogram, $AA$, $\Theta B$ are equal and parallel. Similarly $M\Gamma$, $\Theta B$ are equal and parallel, so that $A\Lambda$, $M\Gamma$ are equal and parallel. And therefore $\Lambda M$, $A\Gamma$ are equal and parallel; therefore $A\Lambda M \Gamma$ is a parallelogram in the angle $A\Lambda A\Gamma$, that is an angle equal to the sum of the angles $B A \Gamma$ and $\Delta \Theta B$; for the angle $\Delta \Theta B = \text{angle } A\Lambda A$. And since the parallelogram $A\Lambda A B E$ is equal to the parallelogram $A\Lambda A B \Theta$ (for they are upon the same base $AB$ and in the

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τῆς ΑΒ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΒ, ΔΘ), ἀλλὰ τὸ ΛΑΒΘ τῷ ΛΑΚΝ ἵσον ἔστιν (ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἔστιν τῆς ΛΑ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΔΑ, ΘΚ), καὶ τὸ ΑΔΕΒ ἀρα τῷ ΛΑΚΝ ἵσον ἔστιν. διὰ τὰ αὐτὰ καὶ τὸ ΒΗΖΓ τῷ ΝΚΓΜ ἵσον ἔστιν· τὰ ἀρα ΔΑΒΕ, ΒΗΖΓ παραλληλόγραμμα τῷ ΛΑΓΜ ἵσα ἔστιν, τούτεστιν τῷ ὑπὸ ΑΓ, ΘΒ ἐν γωνίᾳ τῇ ὑπὸ ΛΑΓ, ἦ ἔστιν ὅση συναμφοτέραις ταῖς ὑπὸ ΒΑΓ, ΒΘΔ. καὶ ἔστι τοῦτο καθολικώτερον πολλῷ τοῦ ἐν τοῖς ὀρθογώνιοις ἐπὶ τῶν τετραγώνων ἐν τοῖς Στουχεῖοις δεδειγμένου.

(g) Circles Inscribed in the ἄρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

Φέρεται ἐν τισιν ἀρχαίᾳ πρότασις τουατη· ὑποκείσθω τρία ἡμικύκλια ἐφαπτόμενα ἀλλήλων τὰ ΑΒΓ, ΑΔΕ, ΕΖΓ, καὶ εἰς τὸ μεταξὺ τῶν περιφερείων αὐτῶν χωρίων, ὁ δὴ καλούσιν ἄρβηλον, 578
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same parallels AB, ΔΘ), while ΔΔΘ = ΔΛΚΝ (for they are upon the same base ΔΑ and in the same parallels ΔΑ, ΘΚ), therefore ΔΔΕΒ = ΔΛΚΝ. By the same reasoning BHΖΓ = ΝΚΓΜ; therefore the parallelograms ΔΑΒΕ, BHΖΓ' are together equal to ΔΑΓΜ, that is, to the parallelogram contained by ΑΓ, ΘΒ in the angle ΔΑΓ', which is equal to the sum of the angles ΒΑΓ, ΒΘΔ. And this is much more general than the theorem proved in the Elements about the squares on right-angled triangles.  

(g) CIRCLES INSCRIBED IN THE ἄρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

There is found in certain [books] an ancient proposition to this effect: Let ΑΒΓ', ΔΔΕ, ΕΖΓ' be supposed to be three semicircles touching each other, and in the space between their circumferences, which

* Eucl. i. 47, v. vol. i. pp. 178-185. In the case taken by Pappus, the first two parallelograms are drawn outwards and the third, equal to their sum, is drawn inwards. If the areas of parallelograms drawn outwards be regarded as of opposite sign to the areas of those drawn inwards, the theorem may be still further generalized, for the algebraic sum of the three parallelograms is equal to zero.

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εἴς εἰς ἐφαστόμενοι τῶν τε ἡμι-
κυκλών καὶ ἀλλήλων ὅσοιδῃστούν, ὡς οἱ περὶ
κέντρα τὰ Ἡ, Θ, Κ, Λ. δεῖξαι τὴν μὲν ἀπὸ τοῦ Ἡ
κέντρου κάθετον ἐπὶ τὴν ΑΓ ὄσην τῇ διαμέτρῳ
τοῦ περὶ τὸ Ἡ κύκλου, τὴν δ’ ἀπὸ τοῦ Θ κάθετον
dιπλασίαν τῆς διαμέτρου τοῦ περὶ τὸ Θ κύκλου,
tὴν δ’ ἀπὸ τοῦ Κ κάθετον τριπλασίαν, καὶ τὰς
ἐξῆς καθέτους τῶν ὀγκείων διαμέτρων πολλα-
πλασίας κατὰ τοὺς ἐξῆς μονάδες ἀλλήλων ὑπερ-
έχονται ἄρμιμοι ἐπὶ ἀπειρον γινομένης τῆς τῶν
κύκλων ἐγγραφῆς.

(h) SPIRAL ON A SPHERE

Ibid. iv. 35. 33-56, ed. Hultsch 264. 3-268. 21

"Ωσπερ ἐν ἐπιτέθῳ νοεῖται γινομένη τις ἐξ
φερομένου σημείου κατ’ εὐθείας κύκλων περιγρα-
φούσης, καὶ ἐπὶ στερεών φερομένου σημείου κατὰ
μᾶς πλευράς τιν’ ἐπιφάνειαν περιγραφούσης,
οὔτως δὴ καὶ ἐπὶ σφαίρας ἐλίκα νοεῖν ἀκόλουθον
ἔστι γραφομένην τὸν τρόπον τοῦτον.

"Εστι τοῦ ἐν σφαίρα μέγιστος κύκλος ὁ ΚΛΜ περὶ
πόλον τὸ Θ σημεῖον, καὶ ἀπὸ τοῦ Θ μεγίστου

* Three propositions (Nos. 4, 5 and 6) about the figure
known as the ἄρβηλος from its resemblance to a leather-
worker’s knife are contained in Archimedes’ Liber Assump-
torum, which has survived in Arabic. They are included as
particular cases in Pappus’s exposition, which is unfortunately
too long for reproduction here. Professor D’Arcy W. Thomp-
son (The Classical Review, lvi. (1942), pp. 75-76) gives reasons
for thinking that the ἄρβηλος was a saddler’s knife rather
than a shoemaker’s knife, as usually translated.

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is called the "leather-worker's knife," let there be inscribed any number whatever of circles touching both
the semicircles and one another, as those about the
centres $H$, $\Theta$, $K$, $\Lambda$; to prove that the perpendicular
from the centre $H$ to $AI$ is equal to the diameter of
the circle about $H$, the perpendicular from $\Theta$ is double
of the diameter of the circle about $\Theta$, the perpen-
dicular from $K$ is triple, and the [remaining] perpen-
diculars in order are so many times the diameters
of the proper circles according to the numbers in a
series increasing by unity, the inscription of the
circles proceeding without limit.\(^a\)

(h) Spiral on a Sphere\(^b\)

*Ibid.* iv. 35. 33-56, ed. Hultsch 264. 3–268. 21

Just as in a plane a spiral is conceived to be generated
by the motion of a point along a straight line
revolving in a circle, and in solids [, such as the cylinder
or cone,]\(^c\) by the motion of a point along one straight
line describing a certain surface, so also a correspond-
ing spiral can be conceived as described on the sphere
after this manner.

Let $KAM$ be a great circle in a sphere with pole $\Theta$,
and from $\Theta$ let the quadrant of a great circle $\Theta NK$ be

\(^a\) After leaving the $\alpha$π\\βηλος, Pappus devotes the remainder
of Book iv. to solutions of the problems of doubling the cube,
squaring the circle and trisecting an angle. This part has
been frequently cited already (e. vol. i. pp. 298-309, 336-363).
His treatment of the spiral is noteworthy because his method
of proof is often markedly different from that of Archimedes;
and in the course of it he makes this interesting digression.

\(^c\) Some such addition is necessary, as Commandinus,
Chasles and Hultsch realized.
κύκλου τεταρτημορίουν γεγράφθω τὸ ΘΝΚ, καὶ ἡ μὲν ΘΝΚ περιφέρεια, περὶ τὸ Θ μένον φερομένη κατὰ τῆς ἑπιφανείας ὡς ἐπὶ τὰ Λ, Μ μέρη,

ἀποκαθιστάσθω πάλιν ἐπὶ τὸ αὐτὸ, σημείον δὲ τι φερόμενον ἐπὶ αὐτῆς ἀπὸ τοῦ Θ ἐπὶ τὸ Κ παραγινέσθω γράφει δή τινα ἐπὶ τῆς ἑπιφανείας ἔλικα, οίᾳ ἐστίν ἡ ΘΟΙΚ, καὶ ἦτις ἃν ἀπὸ τοῦ Θ γραφῇ μεγίστον κύκλου περιφέρεια, πρὸς τὴν ΚΛ περιφέρειαν λόγων ἔχει ὅν ἡ ΛΘ πρὸς τὴν ΘΘ· λέγω δή ὅτι, ἂν ἐκτεθῇ τεταρτημορίουν τοῦ μεγίστου ἐν τῇ σφαῖρᾳ κύκλου τὸ ΑΒΓ περὶ κέντρον τὸ Δ, καὶ ἑπιζευχῇ ἡ ΓΑ, γίνεται ὡς ἡ τοῦ ἡμισφαίριου ἑπιφάνεια πρὸς τὴν μεταξὺ τῆς ΘΟΙΚ ἔλικος καὶ τῆς ΚΝΘ περιφερείας ἀπολαμβανομένην ἑπιφάνειαν, οὕτως ὁ ΑΒΓΔ τομεὺς πρὸς τὸ ΑΒΓ τμῆμα.

"Ηχθῶ γὰρ ἐφαπτομένη τῆς περιφερείας ἡ ΓΖ, καὶ περὶ κέντρον τὸ Γ διὰ τοῦ Α γεγράφθω περιφέρεια ἡ ΑΕΖ· ἵσος ἀρα ὁ ΑΒΓΔ τομεὺς τῷ 582
described, and, $\Theta$ remaining stationary, let the arc $\Theta N K$ revolve about the surface in the direction $\Lambda, M$

and again return to the same place, and [in the same time] let a point on it move from $\Theta$ to $K$; then it will describe on the surface a certain spiral, such as $\Theta O K$, and if any arc of a great circle be drawn from $\Theta$ [cutting the circle $K \Lambda M$ first in $\Lambda$ and the spiral first in $O$], its circumference will bear to the arc $K \Lambda$ the same ratio as $\Lambda \Theta$ bears to $\Theta O$. I say then that if a quadrant $A B \Gamma$ of a great circle in the sphere be set out about centre $\Lambda$, and $\Gamma A$ be joined, the surface of the hemisphere will bear to the portion of the surface intercepted between the spiral $\Theta O K$ and the arc $K N \Theta$ the same ratio as the sector $A B \Gamma \Delta$ bears to the segment $A B \Gamma$.

For let $\Gamma Z$ be drawn to touch the circumference, and with centre $\Gamma$ let there be described through $A$ the arc $A E Z$; then the sector $A B \Gamma \Delta$ is equal to the

* Or, of course, the circumference of the circle $K \Lambda M$ to which it is equal.
AEZΓ (διπλασία μὲν γὰρ ἢ πρὸς τῷ Δ γωνία τῆς ὑπὸ ΑΓΖ, ήμευν δὲ τὸ ἀπὸ ΔΑ τοῦ ἀπὸ ΑΓ). οὕτω καὶ ὃς οἱ εἰρημέναι ἐπιφάνειαι πρὸς ἄλληλας, οὕτως ὁ ΑΕΖΓ τομεὺς πρὸς τὸ ΑΒΓ τμῆμα.

"Εστώ, ὁ μέρος ἡ ΚΛ περιφέρεια τῆς ὅλης τοῦ κύκλου περιφερειάς, καὶ τὸ αὐτὸ μέρος περιφέρεια ἡ ΖΕ τῆς ΖΑ, καὶ ἔπεζεύχω ἡ ΕΓ· ἔσται δὴ καὶ ἡ ΒΓ τῆς ΑΒΓ τοῦ αὐτὸ μέρος. ὁ δὲ μέρος ἡ ΚΛ τῆς ὅλης περιφερειάς, τὸ αὐτὸ καὶ ἡ ΘΟ τῆς ΘΟΔ. καὶ ἔστω ἵσος ἡ ΘΟΔ τῆς ΑΒΓ· ἵσον ἄρα καὶ ἡ ΘΟ τῆς ΒΓ. γεγράφθω περὶ πόλον τοῦ Θ διὰ τοῦ Θ περὶ περὶ τὸ Γ κέντρον ἡ ΒΗ. ἐπεὶ οὖν ὃς ἡ ΛΚΘ σφαιρικὴ ἐπιφάνεια πρὸς τὴν ΘΟΝ, ἡ ὅλη τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν τοῦ τμῆματος ἐπιφάνειαν οὕτως ἔστω τὸ ἀπὸ τῆς τὰ Θ, Λ ἐπιζεύχωνον ἐνθεῖας τετράγωνον πρὸς τὸ ἀπὸ τῆς ἐπὶ τὰ Θ, Ο, ἡ τὸ ἀπὸ τῆς ΕΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΒΓ, ἔσται ἄρα καὶ ὃς ὁ ΚΛΘ τομεῦς ἐν τῇ ἐπιφάνεια πρὸς τὴν ΘΟΝ, οὕτως ὁ ΕΖΓ τομεῦς πρὸς τὴν ΒΗΓ. ὁμοίως δείκουμεν ὅτι καὶ ὃς πάντες οἱ ἐν τῷ ἡμισφαιρίῳ τομεῖς οἱ ἱσοὶ τῷ ΚΛΘ, οἱ

* Pappus's method of proof is, in the Archimedean manner, to circumscribe about the surface to be measured a figure consisting of sectors on the sphere, and to circumscribe about the segment ABΓ a figure consisting of sectors of circles; in the same way figures can be inscribed. The divisions need, therefore, to be as numerous as possible. The conclusion can then be reached by the method of exhaustion.
REVIVAL OF GEOMETRY: PAPPUS

sector $\triangle EZ\Gamma$ (for angle $\angle A\Delta \Gamma = 2$, angle $A\Gamma Z$, and $\Delta A\triangle = \frac{1}{2} \Delta A\Gamma^2$); I say, then, that the ratio of the aforesaid surfaces one towards the other is the same as the ratio of the sector $\triangle EZ\Gamma$ to the segment $AB\Gamma$.

Let $ZE$ be the same [small] $a$ part of $ZA$ as $KA$ is of the whole circumference of the circle, and let $EI$ be joined; then the arc $BI$ will be the same part of the arc $ABI$. $b$ But $\theta O$ is the same part of $\Omega \Omega \Delta$ as $KA$ is of the whole circumference [by the property of the spiral]. And arc $\Omega \Omega \Delta$ = arc $ABI$ [ex constructione]. Therefore $\theta O = BI$. Let there be described through $O$ about the pole $\theta$ the arc $ON$, and through $B$ about centre $I$ the arc $BH$. Then since the [sector of the] spherical surface $\Delta K \theta$ bears to the [sector] $O \Omega N$ the same ratio as the whole surface of the hemisphere bears to the surface of the segment with pole $\theta$ and circular base $ON$, $c$ while the surface of the hemisphere bears to the surface of the segment the same ratio as $\theta \Omega \triangle$ to $\theta \Omega \triangle$, or $EI\triangle$ to $BI\triangle$, therefore the sector $KA\theta$ on the surface [of the sphere] bears to $O \Omega N$ the same ratio as the sector $EZ\Gamma$ [in the plane] bears to the sector $BH\Gamma$. Similarly we may show that all the sectors [on the surface of] the hemi-

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For arc $ZA$: arc $ZE$ = angle $ZFA$: angle $ZGE$. But angle $ZFA = \frac{1}{4}$, angle $A\Delta \Gamma$, and angle $ZGE = \frac{1}{4}$, angle $B\Delta \Gamma$ [Eucl. iii. 32, 32]. $: \therefore$ arc $ZA$: arc $ZE$ = arc $ABI$: arc $BI$.

Because the arc $AK$ is the same part of the circumference $KAM$ as the arc $ON$ is of its circumference.

The square on $\theta \Omega$ is double the square on the radius of the hemisphere, and therefore half the surface of the hemisphere is equal to a circle of radius $\theta \Omega$ [Archim. De sph. et cyl. i. 33]; and the surface of the segment is equal to a circle of radius $\theta O$ [ibid. i. 42]: and as circles are to one another as the squares on their radii [Eucl. xii. 2], the surface of the hemisphere bears to the surface of the segment the ratio $\Theta A^2$ : $\Theta O^2$. 

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𝑒ἰς ἡ δὴ τοῦ ἡμισφαίριον ἐπιφάνεια, πρὸς τοὺς περιγραφομένους περὶ τὴν ἑλικα τομέας ὁμοταγεῖς τῷ ὈΘΝ, οὕτως πάντες οἱ ἐν τῷ ΑΖΓ τομεῖς οἱ ὁσοὶ τῷ ΕΖΓ, τοὐτέστι οὗ ὁ ΑΖΓ τομεύς, πρὸς τοὺς περιγραφομένους περὶ τὸ ΑΒΓ τμῆμα τοὺς ὁμοταγεῖς τῷ ΓΒΗ. τῷ δ’ αὐτῷ τρόπῳ δειχθῆ-
σεται καὶ ὡς ἡ τοῦ ἡμισφαίριον ἐπιφάνεια πρὸς τοὺς ἐγγραφομένους τῇ ἑλικα τομέας, οὕτως ὁ ΑΖΓ τομεύς πρὸς τοὺς ἐγγραφομένους τῷ ΑΒΓ τμῆματος τομέας, ὡστε καὶ ὡς ἡ τοῦ ἡμισφαίριον ἐπιφάνεια πρὸς τὴν ὑπὸ τῆς ἐλικος ἀπολαμβανο-
μένην ἐπιφάνειαν, οὕτως ὁ ΑΖΓ τομεύς, τοῦτον τὸ ΑΒΓΔ τεταρτημόριον, πρὸς τὸ ΑΒΓ τμῆμα. συνάγεται δὲ διὰ τούτου ἢ μὲν ἀπὸ τῆς ἐλικος ἀπολαμβανομένη ἐπιφάνεια πρὸς τὴν ὌΝΚ περι-
φέρειαν ὀκταπλασία τοῦ ΑΒΓ τμήματος (ἐπεὶ καὶ ἡ τοῦ ἡμισφαίριον ἐπιφάνεια τοῦ ΑΒΓΔ τομέως), ἢ δὲ μεταξὺ τῆς ἐλικος καὶ τῆς βάσεως τοῦ ἡμισφαίριον ἐπιφάνεια ὀκταπλασία τοῦ ΑΓΔ τριγώνου, τοῦτον ἢ ὡς τῷ ἀπὸ τῆς διαμέτρου τῆς σφαίρας τετραγώνων.

* This would be proved by the method of exhaustion. It is proof of the great part played by this method in Greek geometry that Pappus can take its validity for granted.

* For the surface of the hemisphere is double of the circle of radius AD [Archim. De sph. et cyl. i. 33] and the sector ABGD is one-quarter of the circle of radius AD.

* For the surface between the spiral and the base of the hemisphere is equal to the surface of the hemisphere less the surface cut off from the spiral in the direction OKN.

i.e. Surface in question = surface of hemisphere =

8 segment ABΓ,

= 8 sector ABGD – 8 segment ABΓ.
sphere equal to $K\Lambda\Theta$, together making up the whole
surface of the hemisphere, bear to the sectors de-
scribed about the spiral similar to $O\Theta\Lambda$ the same ratio
as the sectors in $AZT\Gamma$ equal to $EZT\Gamma$, that is the whole
sector $AZT\Gamma$, bear to the sectors described about the
segment $AB\Gamma\Lambda$ similar to $\Gamma\Lambda\Pi$. In the same manner
it may be shown that the surface of the hemisphere
bears to the [sum of the] sectors inscribed in the
spiral the same ratio as the sector $AZT\Gamma$ bears to the
[sum of the] sectors inscribed in the segment $AB\Gamma\Lambda$,
so that the surface of the hemisphere bears to the
surface cut off by the spiral the same ratio as the
sector $AZT\Gamma$, that is the quadrant $AB\Gamma\Lambda\Delta$, bears to
the segment $AB\Gamma\Lambda$.a From this it may be deduced
that the surface cut off from the spiral in the direction
of the arc $\Theta\Lambda\Pi\Lambda$ is eight times the segment $AB\Gamma\Lambda$ (since
the surface of the hemisphere is eight times the
sector $AB\Gamma\Lambda\Delta$),b while the surface between the spiral
and the base of the hemisphere is eight times the
triangle $\Lambda\Pi\Delta$, that is, it is equal to the square on
the diameter of the sphere.c

\[=8\ \text{triangle } \Lambda\Pi\Delta\]
\[=4\Lambda\Delta^2\]
\[=(2\Lambda\Delta)^2,\]

and $2\Lambda\Delta$ is the diameter of the sphere.
Heath (\textit{H.G.M.} ii. 384-385) gives for this elegant pro-
position an analytical equivalent, which I have adapted to the
Greek lettering. If $\rho$, $\omega$ are the spherical co-ordinates of $O$
with reference to $\Theta$ as pole and the arc $\Theta\Lambda\Pi\Lambda$ as polar axis,
the equation of the spiral is $\omega=4\rho$. If $A$ is the area of the
spiral to be measured, and the radius of the sphere is taken
as unity, we have as the element of area

\[dA = d\omega(1 - \cos \rho) = 4d\rho(1 - \cos \rho).\]
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(i) ISOPERIMETRIC FIGURES

Ibid. v., Ptol. 1-3, ed. Hultsch 304. 5-308. 5

Σοφίας καὶ μαθημάτων ἐννοεῖν ἀρίστην μὲν καὶ
teleiotáthn ἀνθρώποις θεός ἐδωκεν, ὦ κράτιστε
Μεγεβίον, ἐκ μέρους δέ που καὶ τῶν ἁλόγων ζῴων
μοίραν ἀπένεμεν τισιν. ἀνθρώποις μὲν οὖν ἢ
τοὺς λογικοῖς οὔσι τὸ μετὰ λόγου καὶ ἀποδείξεως
παρέσχεν ἑκαστα ποιεῖν, τοῖς δὲ λοιποῖς ζῷοις
ἀνευ λόγου τὸ χρῆσιμον καὶ βιωφελές αὐτὸ μόνον
κατά τινα φυσικὴν πρόνοιαν ἑκάστους ἔχειν ἐδωρή-
σατο. τούτῳ δὲ μάθοι τις ἂν ἦπάρχον καὶ ἐν ἐτέρους
μὲν πλείστοις γένεσιν τῶν ζῴων, οὐχ ἦκιστα δὲ
καὶ ταῖσ μελίσσαις. ἦ τε γὰρ εὐταξὶ καὶ πρὸς
tὰς ἡγομένας τῆς ἐν αὐταῖς πολυτελείας εὐπέθεια
θαυμαστῆ τις, ἦ τε φιλοτιμία καὶ καθαρότης ἢ
περὶ τὴν τοῦ μέλιτος συναγωγῆν καὶ ἢ περὶ τὴν
φυλακῆς αὐτοῦ πρόνοια καὶ οἰκονομία πολὺ μᾶλλον
θαυμασιωτέρα. πεπιστευμέναι γὰρ, ὅς εἰκός,
παρὰ θεῶν κομίζειν τοῖς τῶν ἀνθρώπων μουσικοῖς

\[ A = \int_{\pi}^{\frac{3}{2}\pi} 4d\rho (1 - \cos \rho) \]

\[ = 2\pi - 4. \]

\[ A \text{ surface of hemisphere} = \frac{2\pi - 4}{2\pi} \]

\[ = \frac{\frac{2\pi}{4}}{\frac{1}{2}} \]

\[ = \text{segment } ABΓ \text{ sector } ABΓΔ \]

* The whole of Book v. in Pappus’s Collection is devoted
to isoperimetry. The first section follows closely the exposition
of Zenodorus as given by Theon (v. supra, pp. 386-395),
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REVIVAL OF GEOMETRY: PAPPUS

(i) ISOPERIMETRIC FIGURES

Ibid. v., Preface 1-3, cd. Hultsch 304. 5-308. 5

Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of except that Pappus includes the proposition that of all circular segments having the same circumference the semicircle is the greatest. The second section compares the volumes of solids whose surfaces are equal, and is followed by a digression, already quoted (supra, pp. 194-197) on the semi-regular solids discovered by Archimedes. After some propositions on the lines of Archimedes' De sph. et cyl., Pappus finally proves that of regular solids having equal surfaces, that is greatest which has most faces.

The introduction, here cited, on the sagacity of bees is rightly praised by Heath (H.G.M. ii. 389) as an example of the good style of the Greek mathematicians when freed from the restraints of technical language.
τῆς ἀμβροσίας ἀπόμοιράν των ταύτην οὐ μάτην ἐκχεῖν εἰς γῆν καὶ ξύλον ἡ τινα ἐτέραν ἀσχήμονα καὶ ἀτακτὸν ὑλὴν ἡξίωσαν, ἀλλ' ἐκ τῶν ἱδίωτων ἐπὶ γῆς φυομένων ἀνθέων συνάγουσα τὰ κάλλιστα κατασκευάζουσα ἐκ τούτων εἰς τὴν τοῦ μελιτος ύποδοχήν ἀγγεία τα καλούμενα κηρία πάντα μὲν ἀλλήλους ἵσα καὶ ὀμοια καὶ παρακείμενα, τῷ δὲ τχήματι ἐξάγονα.

Τούτο δ' ὅτι κατὰ τινα γεωμετρικὴν μηχανίωνται πρόνοιαν οὐτως ἂν μάθοιμεν. πάντως μὲν γὰρ ὄντος δειν τὰ σχήματα παρακείσθαι τε ἀλλήλοις καὶ κοινωνεῖν κατὰ τὰς πλευρὰς, ἵνα μὴ τοῖς μεταξὺ παραπληρώμασιν ἐμπίπτοντά τινα ἐτερα λυμήνηται αὐτῶν τὰ ἔργα· τρία δὲ σχήματα εὐθύγραμμα τὸ προκείμενον ἐπιτελεῖν ἐδύνατο, λέγω δὲ τεταγμένα τὰ ἱσόπλευρα τε καὶ ἱσογώνια, τὰ δ' ἀνόμοια ταῖς μελίσσαις οὐκ ἤρεσεν. τα μὲν οὖν ἱσόπλευρα τρίγωνα καὶ τετράγωνα καὶ τὰ ἱσόγωνα χωρὶς ἀνομοίων παραπληρωμάτων ἀλλήλους δύναται παρακείμενα τὰς πλευρὰς κοινῶς ἔχειν [ταῦτα] γὰρ δύναται συμπληρών ἐξ αὐτῶν τὸν περὶ τὸ αὐτὸ σημείον τόπον, ἐτέρῳ δὲ τεταγμένω σχήματι τούτῳ ποιεῖν ἀδύνατον. 3 ο γὰρ περὶ τὸ αὐτὸ σημείον τόπος ὑπὸ ἦ ὡς καὶ τριγώνων ἱσόπλευρων καὶ διὰ ἔγων ὁ, ὅν ἐκάστη διμοιρόν ἐστίν ὅθες, συμπληροῦται, τεσσάρων δὲ τετραγώνων καὶ ὁ ὅρθων γωνιῶν [αὐτοῦ], 3 τριῶν δὲ ἐξαγώνων καὶ ἐξαγώνου γωνιῶν τριῶν, ὅν ἐκάστῃ αὖ ἐστίν ὅθες· πεντάγωνα δὲ τὰ τρία μὲν οὐ φθάνει συμπληρώσαι τὸν περὶ τὸ αὐτὸ σημείον τόπον, ὑπερβάλλει δὲ τὰ τέσσαρα· τρεῖς μὲν γὰρ τοῦ πενταγώνου γωνίαι δ' ὅρθον ἐλάσσονες εἰσιν
ambrosia in this form, they do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical foreshadowing we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilineal figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. For equilateral triangles and squares and hexagons can lie adjacent to one another and have their sides in common without irregular interstices. For the space about the same point can be filled by six equilateral triangles and six angles, of which each is $\frac{\pi}{3}$ right angle, or by four squares and four right angles, or by three hexagons and three angles of a hexagon, of which each is $1\frac{1}{3}$ right angle. But three pentagons would not suffice to fill the space about the same point, and four would be more than sufficient; for three angles of the pentagon are less than four right angles (inasmuch

1 ταύτα . . . ἀδύνατον om. Hultsch.
2 "αὐτοῦ" spurium, nisi forte αὐτῶν dedit scriptor"—Hultsch.
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(έκαστη γὰρ γωνία μιᾶς καὶ ε’ ἐστὶν ὀρθὴ), τέσσαρες δὲ γωνίαι μείζους τῶν τεσσάρων ὀρθῶν, ἔπταγονα δὲ οὔδὲ τρία περὶ τὸ αὐτὸ σημεῖον δύναται τίθεσθαι κατὰ τὰς πλευρὰς ἄλλης παρακείμενα. τρεῖς γὰρ ἐπταγώνου γωνίαι τεσσάρων ὀρθῶν μείζονες (έκαστη γὰρ ἐστὶν μιᾶς ὀρθῆς καὶ τριῶν ἐβδομών). ἔτι δὲ μᾶλλον ἐπὶ τῶν πολυγωνοτέρων ὁ αὐτὸς ἐφαρμόσαι δυνήσεται λόγος. ὅντων ὅτι οὐν τριῶν σχημάτων τῶν ἐς αὐτῶν δυναμένων συμπληρώσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, τριγώνου τε καὶ τετραγώνου καὶ ἕξαγωνου, τὸ πολυγωνοτέρον εἴλαντο διὰ τὴν σοφίαν αἰ μέλισσαι πρὸς τὴν παρασκευὴν, ἀτε καὶ πλείον ἐκάτερον τῶν λοιπῶν αὐτὸ χωρεῖν ὑπολαβάνουσα μὲλι.

Καὶ αἱ μελισσαὶ μὲν τὸ χρήσιμον αὐταῖς ἐπίστανται μόνον τοῦτ’ ὅτι τὸ ἕξαγωνον τοῦ τετραγώνου καὶ τοῦ τριγώνου μείζον ἐστιν καὶ χωρήσαε δύναται πλεῖον μέλι τῆς ἑσις εἰς τὴν ἑκάστον κατασκευὴν ἀναληλυκειμένης ὑλῆς, ἡμεῖς δὲ πλέον τῶν μελισσῶν σοφίας μέρος ἔχειν ὑπισχυομέναι ξητήσιμαν τι καὶ περισσότερον. τῶν γὰρ ἑσιν εὕοντον περίμετρον ἰσοπλεύρων τε καὶ ἰσογωνίων ἐπιπέδων σχημάτων μείζον ἐστιν ἀεὶ τὸ πολυγωνοτέρον, μέγιστος δ’ ἐν πάσιν ὁ κύκλος, ὅταν ἑσιν αὐτοῖς περίμετρον ἑξῆς.

(j) APPARENT FORM OF A CIRCLE

Ibid. vi. 48. 90-91, ed. Hultsch 580. 12-27

"Ἔστω κύκλος ὁ ἈΒΓ, οὖ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἐστω τῷ τοῦ κύκλου ἐπι- 592"
as each angle is $1\frac{1}{2}$ right angle), and four angles are greater than four right angles. Nor can three heptagons be placed about the same point so as to have their sides adjacent to each other; for three angles of a heptagon are greater than four right angles (inasmuch as each is $1\frac{3}{7}$ right angle). And the same argument can be applied even more to polygons with a greater number of angles. There being, then, three figures capable by themselves of filling up the space around the same point, the triangle, the square and the hexagon, the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.

\textit{(f) Apparent Form of a Circle}

\textit{Ibid. vi.} 48. 90-91, ed. Hultsch 580. 12-27

Let \text{ABEI} be a circle with centre \text{E}, and from \text{E} let \text{EZ} be drawn perpendicular to the plane of the circle;

\text{a Most of Book vi. is astronomical, covering the treatises in the Little Astronomy (v. supra, p. 408 n. b). The proposition here cited comes from a section on Euclid's Optics.}
πέδω ἡ EZ· λέγω, ὅτι ἐὰν ἐπὶ τῆς EZ τὸ ὄμμα
tεθῇ ἵσαι αἱ διάμετροι φαίνονται τοῦ κύκλου.

Τούτο δὲ δήλον· ἀπασαί γὰρ αἱ ἀπὸ τοῦ Z πρὸς
tὴν τοῦ κύκλου περιφέρειαν προσπίπτουσαι εὐθεῖαι
ίσαι εἰσίν ἄλληλαις καὶ ἴσαι γωνίαις περιέχουσιν.
Μὴ ἔστω δὲ ἡ EZ πρὸς ὅρθας τῶν τοῦ κύκλου
ἐπιτέθη, ἵσῃ δὲ ἔστω τῇ ἐκ τοῦ κέντρου τοῦ
κύκλου· λέγω, ὅτι τοῦ ὄμματος ὄντος πρὸς τῷ Z
σημεῖο καὶ οὕτως αἱ διάμετροι ἵσαι ὀρθώνται.
"Χθὼσαν γὰρ δύο διάμετροι αἱ AG, BD, καὶ
ἐπεξεύθωσαν αἱ ZA, ZB, ZΓ, ZΔ. ἐπεὶ αἱ
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I say that, if the eye be placed on EZ, the diameters of the circle appear equal.

This is obvious; for all the straight lines falling from Z on the circumference of the circle are equal one to another and contain equal angles.

Now let EZ be not perpendicular to the plane of the circle, but equal to the radius of the circle; I say that, if the eye be at the point Z, in this case also the diameters appear equal.

For let two diameters AΓ, BΔ be drawn, and let ZA, ZB, ZΓ, ZΔ be joined. Since the three straight

* As they will do if they subtend an equal angle at the eye.

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καλούμενος ἀναλυόμενος, Ἐρμόδωρε τέκνον, κατὰ σύλληψιν ἰδία τίς ἐστὶν ὑλή παρεσκευασμένη μετὰ τὴν τῶν κοιών στοιχείων ποίησαν τοῖς βουλομένοις ἀναλαμβάνειν ἐν γραμμαῖς δύναμιν εὐρετικής τῶν προτεινόμενων αὐτοῖς προβλημάτων, καὶ εἰς τούτο μόνον χρήσιμη καθεστώσα. γέγραφται δὲ ὑπὸ τριῶν ἀνδρῶν, Εὐκλείδου τε τοῦ Στοιχειωτοῦ καὶ Ἀπολλωνίου τοῦ Περγαῖου καὶ Ἀρισταίου τοῦ πρεσβυτέρου, κατὰ ἀνάλυσιν καὶ σύνθεσιν ἔχουσα τὴν ἐφοδιον.

Ἀνάλυσις τούτων ἐστὶν ὅδος ἀπὸ τοῦ ζητούμενον ὡς ὀμολογούμενον διὰ τῶν ἔξης ἀκολούθων ἐπὶ τῇ ὀμολογούμενον συνθέσει: ἐν μὲν γὰρ τῇ ἀναλύσει τὸ ζητούμενον ὡς γεγονός ὑποθέμενοι τὸ εὖ ὑπὸ τούτο συμβαίνει σκοπούμεθα καὶ πάλιν ἐκεῖνῳ τὸ προηγούμενον, ἐως ἂν οὕτως ἀναποδίζοντες καταντήσωμεν εἰς τὶ τῶν ἡδον γνωριζομένων ἡ τάξιν ἀρχῆς ἐχόντων· καὶ τὴν τοιαύτην ἐφόδιον ἀνάλυσιν καλούμενην, οἶνον ἀνάπαλιν λύσιν.

Ἐν δὲ τῇ συνθέσει ἐξ ὑποστροφῆς τὸ ἐν τῇ ἀναλύσει καταλήφθην ὑστατὸν ὑποστηθάμενοι γεγονός ἡδη, καὶ ἔπομενα τὰ ἐκεῖ [ἐνταῦθα] ἀντιθετὴν ὀμ. Ηultzsch.
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lines EA, EΓ, EZ are equal, therefore the angle AZΓ is right. And by the same reasoning the angle BZΔ is right; therefore the diameters ΑΓ, BΔ appear equal. Similarly we may show that all are equal.

(k) The "Treasury of Analysis"

Ibid. vii., Preface 1-3, ed. Hultsch 634. 3-636. 30

The so-called Treasury of Analysis, my dear Hermodorus, is, in short, a special body of doctrine furnished for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves, and for this purpose only is it useful. It is the work of three men, Euclid the writer of the Elements, Apollonius of Perga and Aristaeus the elder, and proceeds by the method of analysis and synthesis.

Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.

But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural

* Or, perhaps, "to give a complete theoretical solution of problems set to them"; v. supra, p. 414 n. a.
ηγούμενα κατὰ φύσιν τάξαντες καὶ ἄλληλοις ἐπισυνθέντες, εἰς τέλος ἀφικνούμεθα τῆς τοῦ ζητούμενον κατασκευῆς· καὶ τούτο καλοῦμεν σύνθεσιν.

Διήνοον δ' ἐστὶν ἀναλύσεως γένος τὸ μὲν ζητητικῶν τάληθοι, δ' καλεῖται θεωρητικόν, τὸ δ' ἔρωμα τοῦ προθετέντος [λέγειν], 1 δ' καλεῖται προβληματικόν. ἐπὶ μὲν οὖν τοῦ θεωρητικοῦ γένους τὸ ζητούμενον ὡς ὑποθέμενοι καὶ ὡς ἀληθὲς, εἰσὶ διὰ τῶν ἔξις ἀκολούθων ὡς ἀληθῶν καὶ ὡς ἔστιν καθ' ὑπόθεσιν προελθόντες ἐπὶ τι τὸ όμολογούμενον, ἐὰν μὲν ἀληθὲς ἢ ἐκεῖνο τὸ όμολογούμενον, ἀληθὲς ἐστι καὶ τὸ ζητούμενον, καὶ ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐὰν δὲ ψευδή όμολογούμενον ἐντύχομεν, ψευδὸς ἐστι καὶ τὸ ζητούμενον. ἐπὶ δὲ τοῦ προβληματικοῦ γένους τὸ προταθὲν ὡς γνωσθὲν ὑποθέμενοι, εἰσὶ διὰ τῶν ἔξις ἀκολούθων ὡς ἀληθῶν προελθόντες ἐπὶ τι τὸ όμολογούμενον, ἐὰν μὲν τὸ όμολογούμενον δυνατόν ἢ καὶ ποριστόν, δ' καλοῦσιν οἱ ἀπὸ τῶν μαθημάτων δοθέν, δυνατὸν ἐστι καὶ τὸ προταθὲν, καὶ πάλιν ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐὰν δὲ ἀδυνατῶ όμολογούμενον ἐντύχομεν, ἀδύνατον ἐστι καὶ τὸ πρόβλημα.

Τοσοῦτα μὲν οὖν περὶ ἀναλύσεως καὶ συνθέσεως.

Τῶν δὲ προειρημένων τοῦ ἀναλυμένου βιβλίων ἡ τάξεις ἐστὶν τουαύτη. Εὐκλείδου Δεδομένων βιβλίων ἂ, Ἀπολλωνίου Λόγου ἀποτομῆς β, Χωρίου ἀποτομῆς β, Διωρισμένης τομῆς δύο, Ἐπαφῶν δύο, Εὐκλείδου Ποιημάτων τριά, Ἀπολλωνίου Νεώσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο,
order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.

Now analysis is of two kinds, one, whose object is to seek the truth, being called theoretical, and the other, whose object is to find something set for finding, being called problematical. In the theoretical kind we suppose the subject of the inquiry to exist and to be true, and then we pass through its consequences in order, as though they also were true and established by our hypothesis, to something which is admitted; then, if that which is admitted be true, that which is sought will also be true; and the proof will be the reverse of the analysis, but if we come upon something admitted to be false, that which is sought will also be false. In the problematical kind we suppose that which is set as already known, and then we pass through its consequences in order, as though they were true, up to something admitted; then, if what is admitted be possible and can be done, that is, if it be what the mathematicians call given, what was originally set will also be possible, and the proof will again be the reverse of the analysis, but if we come upon something admitted to be impossible, the problem will also be impossible.

So much for analysis and synthesis.

This is the order of the books in the aforesaid Treasury of Analysis. Euclid's Data, one book, Apollonius's Cutting-off of a Ratio, two books, Cutting-off of an Area, two books, Determinate Section, two books, Contacts, two books, Euclid's Porisms, three books, Apollonius's Vergings, two books, his Plane Loci, two books, Conics, eight books, Aristaeus's
Κωνικῶν ἦ, Ἄρισταίου Τόπων στερεῶν πέντε, Εὐκλείδου Τόπων τῶν πρὸς ἐπιφανεία δύο, Ἐρατοσθένους Περὶ μεσοτήτων δύο. γίνεται βιβλία ἕν, ὅν τὰς περιοχὰς μέχρι τῶν Ἀπολλωνίου Κωνικῶν ἐξεθέμην σοι πρὸς ἐπίσκεψιν, καὶ τὸ πλῆθος τῶν τόπων καὶ τῶν διορισμῶν καὶ τῶν πτωσεων καθ' ἑκαστὸν βιβλίον, ἀλλὰ καὶ τὰ λήμματα τὰ ξητούμενα, καὶ οὐδεμίαν ἐν τῇ πραγ- ματείᾳ τῶν βιβλίων καταλέλοιπα ζήτησιν, ὡς ἑνόμιζον.

(I) LOCUS WITH RESPECT TO FIVE OR SIX LINES

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

'Εὰν ἀπὸ τινὸς σημείου ἐπὶ θέσει δεδομένας εὐθείας πέντε καταχθῶσιν εὐθείαι ἐν δεδομέναις γωνίαις, καὶ λόγος ἢ δεδομένος τοῦ ὑπὸ τριῶν κατηγομένων περιεχομένου στερεοῦ παραλληλεπίπεδου ὀρθογώνιου πρὸς τὸ ὑπὸ τῶν λοιπῶν δύο κατηγομένων καὶ δοθέσθη τινὸς περιεχόμενον παραλληλεπίπεδον ὀρθογώνιον, ἀψῆται τὸ σημεῖον θέσει δεδομένης γραμμῆς. εὰν τε ἐπὶ ζ, καὶ λόγος ἢ δοθεῖ τοῦ ὑπὸ τῶν τριῶν περιεχομένου προερημένου στερεοῦ πρὸς τὸ ὑπὸ τῶν λοιπῶν τριῶν, πάλιν τὸ σημεῖον ἀψῆται θέσει δεδομένης. εὰν δὲ ἐπὶ πλείονας τῶν ἢ, οὐκέτι μὲν ἐχούσι λέγειν, "ἐὰν λόγος ἢ δοθεῖ τοῦ ὑπὸ τῶν δ ἔν περιεχομένου τινὸς πρὸς τὸ ὑπὸ τῶν λοιπῶν," ἐπεὶ οὐκ ἔστι τι

* These propositions follow a passage on the locus with respect to three or four lines which has already been quoted (v. vol. i. pp. 486-489). The passages come from Pappus's 600
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Solid Loci, five books, Euclid's Surface Loci, two books, Eratosthenes' On Means, two books. In all there are thirty-three books, whose contents as far as Apollonius's Conics I have set out for your examination, including not only the number of the propositions, the conditions of possibility and the cases dealt with in each book, but also the lemmas which are required; indeed, I believe that I have not omitted any inquiry arising in the study of these books.

(l) LOCUS WITH RESPECT TO FIVE OR SIX LINES

Ibid. vii. 38-40, ed. Hultsch 680, 2-30

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say "if the ratio be given between some figure contained by four of them to some figure contained by the remainder," since no figure can be contained in more account of the Conics of Apollonius, who had worked out the locus with respect to three or four lines. It was by reflection on this passage that Descartes evolved the system of co-ordinates described in his Géométrie.
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περιεχόμενον ὑπὸ πλειόνων ἢ τριῶν διαστάσεων. συγκεκριμένα δὲ ἐαυτοῖς οἱ βραχὺ πρὸ ἡμῶν ἔρμηνεύων τὰ τοιαῦτα, μηδὲ ἐν μηδαμῶς διάλεξα τοὺς σημαίνωντες, τὸ ὑπὸ τῶν ἐπὶ τὸ ὑπὸ τῶν παρῆν δὲ διὰ τῶν συνημμένων λόγων τούτα καὶ λέγειν καὶ δεικνύναι καθόλου καὶ ἐπὶ τῶν προειρημένων προτάσεων καὶ ἐπὶ τούτων τὸν τρόπον τούτον· εὰν ἀπὸ τῶν σημείων ἐπὶ θέσει δεδομένας εὐθείας καταχωσίων εὐθείαι ἐν δεδομέναις γωνίαις, καὶ δεδομένοις ὃ λόγος ὁ συνημμένος εἰς οὗ ἔχει μία κατηγομένη πρὸς μίαν καὶ ἄλλη πρὸς ἄλλην, καὶ ἄλλη πρὸς ἄλλην, καὶ ἡ λοιπὴ πρὸς λοιπὴν, ἐὰν ὡς ἐς ἐὰν δὲ ἢ, καὶ ἡ λοιπὴ πρὸς λοιπὴν, τὸ σημεῖον ἀφεται θέσει δεδομένης γραμμῆς· καὶ ὀρμοίως ὅσα ἃν ὡς περισσαὶ ἢ ἄρτια τὸ πλῆθος τούτων, ὡς ἐφην, ἐπομένων τῷ ἐπὶ τέσσαρας τόπων οὔτε ἐν συνθε折磨ισιν, ἀμφετῇ γραμμῇ ντεῖναι.

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than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions,\textsuperscript{a} but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner: If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line— if there be seven, or, if there be eight, that which the fourth bears to the fourth—the point will lie on a curve given in position; and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.\textsuperscript{b}

\textsuperscript{a} As Heron in his formula for the area of a triangle, given the sides (supra, pp. 476-477).

\textsuperscript{b} The general proposition can thus be stated: If $p_1$, $p_2$, $p_3$ \ldots $p_n$ be the lengths of straight lines drawn to meet $n$ given straight lines at given angles (where $n$ is odd), and $a$ be a given straight line, then if

$$\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \ldots \frac{p_n}{a} = \lambda,$$

where $\lambda$ is a constant, the point will lie on a curve given in position. This will also be true if $n$ is even and

$$\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \ldots \frac{p_{n-1}}{p_n} = \lambda.$$
(m) Anticipation of Guldin's Theorem

Ibid. vii. 41-42, ed. Hultsch 680. 30-682. 20

Ταύτ' οί βλέποντες ἰκιστα ἐπαιρονται, καθάπερ οί πάλαι καί τών τά κρείττονα γραφάντων ἔκαστον ἐγὼ δὲ καὶ πρὸς ἀρχαῖς ἐτι τῶν μαθημάτων καί τῆς ὑπὸ φύσεως προκειμένης ζητημάτων ὕλης κινουμένους ὅρων ἀπαντας, αἰδούμενος ἐγὼ καί δείξας γε πολλῷ κρείσσονα καί πολλὴν προφερόμενα ὠφελείαν . . . ἵνα δὲ μὴ κεναὶς χεροὶ τοῦτο φθεγξάμενος ὥδε χωρισθῶ τοῦ λόγου, ταῦτα δῶσω ταῖς ἀναγνώσιν. ο μὲν τῶν τελείων ἀμφουστικῶν λόγος συνήπται ἐκ τε τῶν ἀμφουσμάτων καί τῶν ἐπὶ τῶν ἄξονας ὁμοίως κατηγμένων εὐθείων ἀπὸ τῶν ἐν αὐτοῖς κεντροβαρικῶν σημείων, ὁ δὲ τῶν ἄτελῶν ἐκ τε τῶν ἀμφουσμάτων καί τῶν περιφερειῶν, ὅσα ἐποίησεν τὰ ἐν τούτοις κεντροβαρικὰ σημεῖα, ὁ δὲ τούτων τῶν περιφερειῶν λόγος συνήπται δὴλον ὡς ἐκ τε τῶν κατηγμένων καί ὅν περιέχουσιν αἱ τούτων ἀκραὶ, εἰ καὶ εἰεν πρὸς τοὺς ἄξονας ἀμφουστικῶν, γωνίων. περι-

a Paul Guldin (1577-1643), or Guldinus, is generally credited with the discovery of the celebrated theorem here enunciated by Pappus. It may be stated: If any plane figure revolve about an external axis in its plane, the volume of the solid figure so generated is equal to the product of the area of the figure and the distance travelled by the centre of gravity of the figure. There is a corresponding theorem for the area.

b The whole passage is ascribed to an interpolator by Hultsch, but without justice; and, as Heath observes (H.G.M. ii. 403), it is difficult to think of any Greek mathematician after Pappus's time who could have discovered such an advanced proposition.

Though the meaning is clear enough, an exact translation 604
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(m) ANTICIPATION OF GULDIN'S THEOREM

*ibid.* vii. 41-42, ed. Hultsch 680. 30-682. 20

The men who study these matters are not of the same quality as the ancients and the best writers. Seeing that all geometers are occupied with the first principles of mathematics and the natural origin of the subject matter of investigation, and being ashamed to pursue such topics myself, I have proved propositions of much greater importance and utility ... and in order not to make such a statement with empty hands, before leaving the argument I will give these enunciations to my readers. Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (a) of the ratio [of the areas] of the figures, and (b) of the ratio of the straight lines similarly drawn to the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in a ratio compounded (a) of the ratio [of the areas] of the figures, and (b) of the ratio of the arcs described by the centres of gravity of the respective figures, the ratio of the arcs being itself compounded (1) of the ratio of the straight lines similarly drawn [from the respective centres of gravity to the axes of rotation] and (2) of the ratio of the angles contained about the axes of revolution by the extremities of these straight lines.

These propositions, which are practi-

is impossible; I have drawn on the translations made by Halley (v. Papp. Coll., ed. Hultsch 683 n. 2) and Heath (H.G.M. ii. 402-403). The obscurity of the language is presumably the only reason why Hultsch brackets the passage, as he says: "exciderunt autem in eodem loco pauciora plurave genuina Pappi verba."

* i.e., drawn to meet at the same angles.

* The extremities are the centres of gravity.

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échousi dé autai ai protáseis, schedon oussai mia, pléistata òsa kai pantoia theowrímatata graamíon te kai épifaneiówn kai stereów, pánth' amá kai mia deîzai kai tà mu'pw dedeugmeína kai tà ëdhe òsw kai tà ën tà ðwð doidékátow tònve tòn stoicheiów.

(n) Lemmas to the Treatises

(i.) To the "Determinate Section" of Apollonius

Ibid. vii. 115, ed. Hultsch, Prop. 61, 756. 28–760. 4

Τριών δοθειόν πε̂θειόν τῶν ΑΒ, ΒΓ, ΓΔ, ἕαν γένηται ως τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ,

οὕτως τὸ ἀπὸ ΒΕ πρὸς τὸ ἀπὸ ΕΓ, μοναχὸς λόγος καὶ ἐλάχιστός ἐστιν ὁ τοῦ ὑπὸ ΑΕΔ πρὸς τὸ ὑπὸ

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ally one, include a large number of theorems of all sorts about curves, surfaces and solids, all of which are proved simultaneously by one demonstration, and include propositions never before proved as well as those already proved, such as those in the twelfth book of these elements.a

(n) Lemmas to the Treatises b

(i.) To the "Determinate Section" of Apollonius

Ibid. vii. 115, ed. Hultsch, Prop. 61, 756. 28-760. 4

Given three straight lines AB, BΓ, ΓΔ, e if AB . BΔ : AΓ . ΓΔ = BE2 : EI2, then the ratio AE . EΔ : BE . EI

a If the passage be genuine, which there seems little reason to doubt, this is evidence that Pappus's work ran to twelve books at least.

b The greater part of Book vii. is devoted to lemmas required for the books in the Treasury of Analysis as far as Apollonius's Conies, with the exception of Euclid's Data and with the addition of two isolated lemmas to Euclid's Surface-Loci. The lemmas are numerous and often highly interesting from the mathematical point of view. The two here cited are given only as samples of this important collection: the first lemma to the Surface-Loci, one of the two passages in Greek referring to the focus-directrix property of a conic, has already been given (vol. i. pp. 492-503).

c It is left to be understood that they are in one straight line ΔΔ.
ΓΕΩΡΓΙΩΝ ΠΕΡΙ ΤΗΝ ΑΔ ΚΥΚΛΟΣ, ΚΑΙ ἩΧΘΩΣΑΝ ὌΡΘΑΙ ΑΙ ΒΖ, ΓΗ. ἘΠΕΙ ΟΥΝ ΕΎΣΤΙΝ ΩΣ ΤΟ ὩΠΟ ΑΒΔ ΠΡΟΣ ΤΟ ὩΠΟ ΑΓΔ, ΤΟΥΣΤΕΣΤΙΝ ὩΣ ΤΟ ὩΠΟ ΒΖ ΠΡΟΣ ΤΟ ὩΠΟ ΓΗ, ΟΥΤΩΣ Η ΒΕ ΠΡΟΣ ΤΗΝ ΓΗ, ΟΥΤΩΣ Ἡ ΒΕ ΠΡΟΣ ΤΗΝ ΕΓ· ΕΥΘΕΙΑ ἌΡΑ ΕΎΣΤΙΝ Ἡ ΔΙΑ ΤΩΝ Ζ, Ε, Η. ΕΎΣΤΙΝ Ἡ ΖΕΗ, ΚΑΙ ΕΚΒΕΒΛΗΘΩΝ Ἡ ΜΕΝ ΗΓ ΕΠΙ ΤΟ Θ, ΕΠΙΖΕΥΧΘΕΙΣΑ ΔΕ Ἡ ΖΘ ΕΚΒΕΒΛΗΘΩΝ ΕΠΙ ΤΟ Κ, ΚΑΙ ΕΠ' ΑΥΤΗΝ ΚΑΘΕΤΟΣ ἩΧΘΩΝ Ἡ ΔΚ. ΚΑΙ ΔΙΑ ΔΗ ΤΟΝ ΠΡΟΥΓΕΡΑΜΜΕΝΟΝ ΛΗΜΜΑ ΓΙΝΕΤΑΙ ΤΟ ΜΕΝ ὩΠΟ ΑΓ, ΒΔ ΊΣΟΝ ΤΟ ὩΠΟ ΖΚ, ΤΟ ὩΠΟ ΑΒ, ΓΔ ΤΟ ΑΠΟ ΘΚ. ΛΟΙΤΗ ἌΡΑ Ἡ ΖΘ ΕΎΣΤΙΝ Ἡ ΥΠΕΡΟΧΗ Ἡ ΥΠΕΡΕΧΕΙ Η ΔΥΝΑΜΕΙΝ ΤΟ ὩΠΟ ΑΓ, ΒΔ ΤΗΣ ΔΥΝΑΜΕΙΝ ΤΟ ὩΠΟ ΑΒ, ΓΔ. ἩΧΘΩΝ ΟΥΝ ΔΙΑ ΤΟΥ ΚΕΝΤΡΟΥ Η ΖΔ, ΚΑΙ ΕΠΕΞΕΥΧΘΩΝ Ἡ ΘΔ. ΕΠΕΙ ΟΥΝ ΌΡΘΗ Ἡ ὩΠΟ ΖΘΛ ὌΡΘΗ ΤΗ ὩΠΟ ΕΓΗ ἘΎΣΤΙΝ ΊΣΗ, ΕΎΣΤΙΝ ΔΕ ΚΑΙ Ἡ ΠΡΟΣ ΤΩ Λ ΤΗ ΠΡΟΣ ΤΩ Η ΓΩΝΙΑ ΊΣΗ, ΙΣΟΓΟΝΙΑ ἌΡΑ ΤΑ ΤΡΙΓΩΝΑ. ΕΎΣΤΙΝ ἌΡΑ ὩΣ Ἡ ΛΖ ΠΡΟΣ ΤΗΝ ΘΖ, ΤΟΥΣΤΕΣΤΙΝ ὩΣ Ἡ ΑΔ ΠΡΟΣ ΤΗΝ ΖΘ, ΟΥΤΩΣ Ἡ ΕΗ ΠΡΟΣ ΤΗΝ ΕΓ· ΚΑΙ ὩΣ ἌΡΑ ΤΟ ὩΠΟ ΑΔ ΠΡΟΣ ΤΟ ὩΠΟ ΖΘ, ΟΥΤΩΣ ΤΟ ὩΠΟ ΕΗ ΠΡΟΣ ΤΟ ὩΠΟ ΕΓ, ΚΑΙ ΤΟ ὩΠΟ ΗΕ, ΕΖ, ΤΟΥΣΤΕΣΤΙΝ ΤΟ ὩΠΟ ΑΕ, ΕΔ, ΠΡΟΣ ΤΟ ὩΠΟ ΒΕ, ΕΓ. ΚΑΙ ΕΎΣΤΙΝ Ο ΜΕΝ ΤΟΥ ὩΠΟ ΑΕ, ΕΔ ΠΡΟΣ ΤΟ ὩΠΟ ΒΕ,

* For, because BZ: ΓΗ = BE: EG, the triangles ZEB, NEΓ are similar, and angle ZEB = angle ΗΕΓ; . . . Θ is in the same straight line with B, E [Eucl. I. 13, Conv.].

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is singular and a minimum; and I say that this ratio is equal to \( \Delta \Delta^2 : (\sqrt{\Delta \Gamma \cdot B\Delta} - \sqrt{AB \cdot \Gamma \Delta})^2 \).

Let a circle be described about \( \Delta \Delta \), and let \( BZ, \Gamma H \) be drawn perpendicular [to \( \Delta \Delta \)]. Then since

\[
\frac{AB \cdot B\Delta}{\Delta \Delta} : \frac{A\Gamma \cdot \Gamma \Delta}{\Gamma \Delta} = \frac{BE^2}{\Gamma \Gamma}, \quad \text{[ex hyp. i.e.,]}
\]

\[
\frac{BZ^2}{\Gamma \Gamma} = \frac{BE^2}{\Gamma \Gamma}
\]

[Eucl. x. 33, Lemma]

\[
\therefore \quad \frac{BZ}{\Gamma \Gamma} = \frac{BE}{\Gamma \Gamma}
\]

Therefore \( Z, E, H \) lie on a straight line.² Let it be \( ZEH \), and let \( H\Gamma \) be produced to \( \Theta \), and let \( Z\Theta \) be joined and produced to \( K \), and let \( \Delta K \) be drawn perpendicular to it. Then by the lemma just proved [Lemma 19]

\[
\frac{A\Gamma \cdot B\Delta}{\Delta \Delta} = \frac{ZK^2}{\Delta \Delta},
\]

\[
\frac{AB \cdot \Gamma \Delta}{\Delta \Delta} = \frac{\Theta K^2}{\Delta \Delta};
\]

[on taking the roots and] subtracting,

\[
\frac{ZK - \Theta K}{\Delta \Delta} = \frac{Z\Theta = \sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot \Gamma \Delta}}{\Delta \Delta}.
\]

Let \( Z\Delta \) be drawn through the centre, and let \( \Theta \Delta \) be joined. Then since the right angle \( Z\Theta \Delta = \) the right angle \( \Gamma \Gamma H \), and the angle at \( \Delta = \) the angle at \( H \), therefore the triangles \([Z\Theta \Delta, \Gamma \Gamma H]\) are equiangular;

\[
\therefore \quad \frac{\Delta Z}{\Theta \Gamma} = \frac{EH}{\Gamma \Gamma},
\]

i.e.,

\[
\frac{\Delta \Delta}{Z\Theta} = \frac{EH}{\Gamma \Gamma};
\]

\[
\therefore \quad \frac{\Delta \Delta^2}{Z\Theta^2} = \frac{EH^2}{\Gamma \Gamma^2}
\]

\[
= \frac{HE \cdot EZ}{BE \cdot \Gamma \Gamma} = \frac{AE \cdot E\Delta}{BE \cdot \Gamma \Gamma}.
\]

[Eucl. iii. 35]

And [therefore] the ratio \( \frac{AE \cdot E\Delta}{BE \cdot \Gamma \Gamma} \) is

² Because, on account of the similarity of the triangles \( H\Gamma E, ZBE \), we have \( HE : \Gamma \Gamma = EZ : EB \).
ΕΓ μοναχός καὶ ἐλάσσων λόγος, ἡ δὲ ΖΘ ἡ ὑπεροχὴ ἡ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ τῶν ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ [τουτέστων τὸ ἀπὸ τῆς ΖΚ τοῦ ἀπὸ τῆς ΘΚ], ὥστε ὁ μοναχὸς καὶ ἐλάσσων λόγος ὁ αὐτὸς ἔστιν τῷ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ἡ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ, ὅπερ :-

(ii.) To the "Porisms" of Euclid


Καταγραφή ἡ ΑΒΓΔΕΖΗΘΚΔΑ, ἔστω δὲ ὡς τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΒ, ΓΖ, οὕτως τὸ ὑπὸ ΑΖ, ΔΕ πρὸς τὸ ὑπὸ ΑΔ, EZ· ὅτι εὐθεία ἔστω ἡ διὰ τῶν Θ, Η, Ζ σημείων.
REVIVAL OF GEOMETRY: PAPPUS

singular and a minimum, while [as proved above,]

\[ z_\theta = \sqrt{AG \cdot BD} - \sqrt{AB \cdot TD}, \] so that the same

singular and minimum ratio =

\[ \frac{AD^2}{(\sqrt{AG \cdot BD} - \sqrt{AB \cdot TD})^2}. \quad \text{Q.E.D.} \]

(ii.) To the "Porisms" of Euclid


Let \( ABGDZH\theta K \Delta \) be a figure, and let \( AZ \cdot BI' \):

\[ AB \cdot FZ = AZ \cdot DE : AD \cdot EZ ; \] [I say] that the line

through the points \( \Theta, H, Z \) is a straight line.

* Notice the sign: ~ used in the Greek for εν δεύτερα. In all Pappus proves this property for three different positions of the points, and it supports the view (c. supra, p. 341 n. a) that Apollonius's work formed a complete treatise on involution.

1 v. vol. i. pp. 478-485.

* Following Breton de Champ and Hultsch I reproduce the second of the eight figures in the mss., which vary according to the disposition of the points.

---

1 *tou<λετω> ... τῆς ΘΩ om. Hultsch.
'Επει' έστιν ως τὸ ὑπὸ AZ, BG πρὸς τὸ ὑπὸ AB, ΓΖ, οὖτως τὸ ὑπὸ AZ, ΔΕ πρὸς τὸ ὑπὸ AD, EZ ἐναλλάξ ἐστιν ως τὸ ὑπὸ AZ, BG πρὸς τὸ ὑπὸ AZ, ΔΕ, τούτεστιν ως ἡ BG πρὸς τὴν ΔΕ, οὖτως τὸ ὑπὸ AB, ΓΖ πρὸς τὸ ὑπὸ AD, EZ. ἀλλ' ὁ μὲν τῆς BG πρὸς τὴν ΔΕ συνήπται λόγος, εὰν διὰ τοῦ K τῇ AZ παράλληλος ἀχθῇ ἢ KM, ἐκ τοῦ τῆς BG πρὸς KN καὶ τῆς KN πρὸς KM καὶ ἐτι τοῦ τῆς KM πρὸς ΔΕ, ὁ δὲ τοῦ ὑπὸ AB, ΓΖ πρὸς τὸ ὑπὸ AD, EZ συνήπται ἐκ τοῦ τῆς BA πρὸς AD καὶ τοῦ τῆς ΓΖ πρὸς τὴν ZE. κοινὸς ἐκκεκροῦσθω ὁ τῆς BA πρὸς AD ὁ αὐτὸς ὁ ὅθων τῷ τῆς NK πρὸς KM· λοιπὸν ἀρα ὁ τῆς ΓΖ πρὸς τὴν ΖΕ συνήπται ἐκ τοῦ τῆς BG πρὸς τὴν KN, τούτεστιν τοῦ τῆς ΘΓ πρὸς τὴν ΚΘ, καὶ τοῦ τῆς KM πρὸς τὴν ΔΕ, τούτεστιν τοῦ τῆς KH πρὸς τὴν ΗΕ· εὐθεία ἀρα ἡ διὰ τῶν Θ, Η, Ζ.

'Εάν γὰρ διὰ τοῦ E τῇ ΘΓ παράλληλον ἀγάμω τὴν EZ, καὶ ἐπιζευγθείσα ἡ ΘΗ εἰκληθῇ ἐπὶ τὸ Ε, ὁ μὲν τῆς KH πρὸς τὴν HE λόγος ὁ αὐτὸς ἐστιν τῷ τῆς ΘΓ πρὸς τὴν EZ, ὁ δὲ συνημμένος ἐκ τε τοῦ τῆς ΘΓ πρὸς τὴν ΘΚ καὶ τοῦ τῆς ΘΚ πρὸς τὴν EZ μεταβάλλεται εἰς τὸν τῆς ΘΓ πρὸς EZ λόγου, καὶ ὁ τῆς ΓΖ πρὸς ZE λόγος ὁ αὐτὸς τῷ τῆς ΓΘ πρὸς τὴν EZ· παράλληλον οὕσης τῆς ΓΘ τῇ EZ· εὐθεία ἀρα ἐστιν ἡ διὰ τῶν Θ, Ε, Ζ (τούτο γὰρ φανερόν), ὡστε καὶ ἡ διὰ τῶν Θ, Η, Ζ εὐθεία ἐστιν.

* It is not perhaps obvious, but is easily proved, and is in fact proved by Pappus in the course of iv. 21, ed. Hultsch 212. 4-13, by drawing an auxiliary parallelogram.

* Conversely, if ΗΘΚΛ be any quadrilateral, and any
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Since \( AZ \cdot BG : AB \cdot \Gamma Z = AZ \cdot \Delta E : A\Delta \cdot EZ, \)
permutando
\( AZ \cdot BG : AZ \cdot \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ, \)
i.e.,
\( BG \cdot \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ. \)
But, if \( KM \) be drawn through \( K \) parallel to \( AZ, \)
\( BG \cdot \Delta E = (BG : KN) \cdot (KN : KM) \cdot (KM : \Delta E), \)
and
\( AB \cdot \Gamma Z : A\Delta \cdot EZ = (BA : A\Delta) \cdot (\Gamma Z : ZE). \)
Let the equal ratios \( BA : A\Delta \) and \( NK : KM \) be eliminated;
then the remaining ratio
\( \Gamma Z : ZE = (BG : KN) \cdot (KM : \Delta E), \)
i.e.,
\( \Gamma Z : ZE = (\Theta \Gamma : K\Theta) \cdot (KH : HE); \)
then shall the line through \( \Theta, H, Z \) be a straight line.
For if through \( E \) I draw \( E\Xi \) parallel to \( \Theta \Gamma, \) and if \( \Theta H \) be joined and produced to \( \Xi, \)
\( KH : HE = K\Theta : E\Xi, \)
and \( (\Theta \Gamma : \Theta K) \cdot (\Theta K : E\Xi) = \Theta \Gamma : E\Xi, \)
\( \therefore \Gamma Z : ZE = \Theta \Gamma : E\Xi; \)
and since \( \Theta \Theta \) is parallel to \( E\Xi, \) the line through \( \Theta, \)
\( \Xi, Z \) is a straight line (for this is obvious\(^a\)), and there-
fore the line through \( \Theta, H, Z \) is a straight line.\(^b\)

transversal cut pairs of opposite sides and the diagonals in
the points \( A, Z, \Delta, \Gamma, B, E, \) then \( BG \cdot \Delta E = AB \cdot \Gamma Z : A\Delta \cdot EZ. \)
This is one of the ways of expressing the proposition enunci-
ated by Desargues: The three pairs of opposite sides of a
complete quadrilateral are cut by any transversal in three
pairs of conjugate points of an involution (v. L. Cremona,
Elements of Projective Geometry, tr. by C. Leudesdorf, 1885,
pp. 106-108). A number of special cases are also proved by
Pappus.
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(o) Mechanics

_Ibid._ viii., Praef. 1-3, ed. Hultsch 1022. 3-1028. 3

'Ἡ μηχανικὴ θεωρία, τέκνον Ἑρμὸδορε, πρὸς πολλὰ καὶ μεγάλα τῶν ἐν τῷ βίῳ χρήσιμοι ύπο-άρχουσα πλείστης εἰκότως ἀποδοχὴς ἦσσωται πρὸς τῶν φιλοσόφων καὶ πάσι τοῖς ἀπὸ τῶν μαθημάτων περιπούδαστός ἔστιν, ἐπειδὴ σχεδὸν πρώτη τῆς περὶ τὴν ὕλην τῶν ἐν τῷ κόσμῳ στοιχείων φυσιολογίας ἀπτεταί. στάσεως γὰρ καὶ φορᾶς σωμάτων καὶ τῆς κατὰ τόπον κινήσεως ἐν τοῖς ὁλοῖς θεωρηματικὴ τυχανούσα τὰ μὲν κινούμενα κατὰ φύσιν αἰτιολογεῖ, τὰ δὲ ἀναγκάζουσα παρὰ φύσιν ἑξω τῶν οἰκείων τόπων εἰς ἐναντίας κινήσεως μεθίστησιν ἐπιμηχανωμένη διὰ τῶν ἐξ αὐτῆς τῆς ὕλης ὑποπάπτοντων αὐτῇ θεωρημάτων. τῆς δὲ μηχανικῆς τὸ μὲν εἶναι λογικὸν τὸ δὲ χειρουργικὸν οἱ περὶ τὸν Ἡρωνα μηχανικοὶ λέγουσιν καὶ τὸ μὲν λογικὸν συνεσταῖνε μέρος ἐκ τῆς γεωμετρίας καὶ ἁρμηνευτικῆς καὶ ἀστρονομίας καὶ τῶν φυσικῶν λόγων, τὸ δὲ χειρουργικὸν ἐκ τῆς καλκευτικῆς καὶ οἰκοδομικῆς καὶ τεκτονικῆς καὶ ξυγραφικῆς καὶ τῆς ἐν τούτῳ κατὰ χείρα ἀσκήσεως τῶν μὲν οὖν ἐν ταῖς προειρημέναις ἐπιστήμαις ἐκ παῖδος γενόμενον καὶ ταῖς προειρημέναις τέχναις ἐξιν εἰληφότα πρὸς δὲ τούτους φύσιν εὐκίνητον ἔχουσα, κράτιστον ἐπεσθαὶ μηχανικῶν ἔργων εὑρετὴν καὶ ἀρχιτέκτονα φασίν. μὴ δυνατοῦ δ' δόντος τὸν αὐτὸν μαθημάτων

* After the historical preface here quoted, much of Book viii. is devoted to arrangements of toothed wheels, already encountered in the section on Heron (_supra_, pp. 488-497). A 614
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(o) Mechanics a

Ibid. viii., Preface 1-3, ed. Hultsch 1022, 3-1028, 3

The science of mechanics, my dear Hermodorus, has many important uses in practical life, and is held by philosophers to be worthy of the highest esteem, and is zealously studied by mathematicians, because it takes almost first place in dealing with the nature of the material elements of the universe. For it deals generally with the stability and movement of bodies [about their centres of gravity], b and their motions in space, inquiring not only into the causes of those that move in virtue of their nature, but forcibly transferring [others] from their own places in a motion contrary to their nature; and it contrives to do this by using theorems appropriate to the subject matter. The mechanicians of Heron's school c say that mechanics can be divided into a theoretical and a manual part; the theoretical part is composed of geometry, arithmetic, astronomy and physics, the manual of work in metals, architecture, carpentering and painting and anything involving skill with the hands. The man who had been trained from his youth in the aforesaid sciences as well as practised in the aforesaid arts, and in addition has a versatile mind, would be, they say, the best architect and inventor of mechanical devices. But as it is impossible for the same person to familiarize himself with such

number of interesting theoretical problems are solved in the course of the book, including the construction of a conic through five points (viii. 13-17, ed. Hultsch 1072, 30-1084, 2).

b It is made clear by Pappus later (vii., Praef. 5, ed. Hultsch 1030, 1-17) that φορά has this meaning.

c With Pappus, this is practically equivalent to Heron himself: cf. vol. i. p. 184 n. b.
τε τοσούτων περιγενέσθαι καὶ μαθεῖν ἁμα τάς προειρημένας τέχνας παραγγέλλουσι τῷ τά μηχανικά ἔργα μεταχειρίζεσθαι βουλομένω χρῆσθαι ταῖς οἰκείαις τέχναις ὑποχειρίοις εν ταῖς παρ’ ἐκαστα χρείαις.

Μάλιστα δὲ πάντων ἀναγκαίοταται τέχναι τυγχάνουσι πρὸς τήν τοῦ βίου χρείαιν [μηχανικὴ προηγουμένη τῆς ἀρχιτεκτονῆς] 1 ἢ τα τών μαγγαναρίων, μηχανικών καὶ αὐτῶν κατὰ τοὺς ἀρχαίους λεγομένων (μεγάλα γὰρ οὖν θυρίδα διὰ μηχανῶν παρὰ φύσιν εἰς ύφος ἀνάγουσιν ἐλάττονι δύναμει κινοῦντες), καὶ ἢ τῶν ὀργανοποίων τῶν πρὸς τὸν πόλεμον ἀναγκαίων, καλομένων δὲ καὶ αὐτῶν μηχανικών (Βέλη γὰρ καὶ λιθίνα καὶ σιδήρα καὶ τὰ παραπλήσια τούτωσι εξαποστελλεί εἰς μακρὸν οὐδὲ μήκος τοῖς ὑπ’ αὐτῶν γινομένων ὀργανοῖς καταπαλτικοῖς), πρὸς δὲ ταύτας ἢ τῶν ἐδώς πάλιν καλομένων μηχανοποίων (ἐκ βάθους γὰρ πολλοῦ ὕδωρ εὐκολώτερον ἀνάγεται διὰ τῶν ἀντληματικῶν ὀργάνων ἐν αὐτοῖς κατασκευάζοντω). καλοῦσι δὲ μηχανικοὺς οἱ παλαιοὶ καὶ τοὺς θαυμασιούργους, ὃν οἱ μὲν διὰ πνευμάτων φιλοτεχνοῦσιν, ὡς Ἡρων Πνευματικὸς, οἱ δὲ διὰ νευρίων καὶ σπάρτων ἐμφύχων κίνησεις δοκοῦσι μιμεῖσθαι, ὡς Ἡρων Αὐτομάτος καὶ Ζυγίος, ἄλλοι δὲ διὰ τῶν ἐφ’ ὕδατος ὄχουμένων, ὡς Ἀρχιμήδης Ὀχουμένοις, ἢ τῶν δὲ ὕδατος ὀρολογίων, ὡς Ἡρων Τρεῖος, ᾗ δὴ καὶ τῇ γνωμονικῇ

1 μηχανικὴ ... ἀρχιτεκτονῆς om. Hultsch.

* μάγγανον is properly the block of a pulley, as in Heron's
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mathematical studies and at the same time to learn the above-mentioned arts, they instruct a person wishing to undertake practical tasks in mechanics to use the resources given to him by actual experience in his special art.

Of all the [mechanical] arts the most necessary for the purposes of practical life are: (1) that of the makers of mechanical powers, they themselves being called mechanicians by the ancients—for they lift great weights by mechanical means to a height contrary to nature, moving them by a lesser force; (2) that of the makers of engines of war, they also being called mechanicians—for they hurl to a great distance weapons made of stone and iron and suchlike objects, by means of the instruments, known as catapults, constructed by them; (3) in addition, that of the men who are properly called makers of engines—for by means of instruments for drawing water which they construct water is more easily raised from a great depth; (4) the ancients also describe as mechanicians the wonder-workers, of whom some work by means of pneumatics, as Heron in his Pneumatica, some by using strings and ropes, thinking to imitate the movements of living things, as Heron in his Automata and Balancings, some by means of floating bodies, as Archimedes in his book On Floating Bodies, or by using water to tell the time, as Heron in his Hydria, which appears to have affinities with the

Beloposacea, ed. Schneider 84. 12, Greek Papyri in the British Museum iii. (ed. Kenyon and Bell) 1164 n. 8.

* v. supra, p. 466 n. a.

* v. supra, pp. 242-237.

* This work is mentioned in the Pneumatica, under the title Περὶ ὄραν ὀροσκοπεῖων, as having been in four books. Fragments are preserved in Proclus (Hypotyposis 4) and in Pappus's commentary on Book v. of Ptolemy's Syntaxis.
GREEK MATHEMATICS

θεωρία κοινωνοῦτα φαίνεται. μηχανικῶς δὲ καλοῦσιν καὶ τοὺς τὰς σφαιροποιίας [ποιεῖν] ἐπισταμένους, ὧν ἐκιών τοῦ οὐρανοῦ κατασκευᾶται δὲ ὀμαλῆς καὶ ἐγκυκλιών κινήσεως ύδατος.

Πάντων δὲ τούτων τὴν αἰτίαν καὶ τὸν λόγον ἐπεγνωκέναι φασίν τινες τὸν Συρακόσιον Ἀρχιμήδην μόνον γὰρ οὗτος ἐν τῷ καθ' ἡμᾶς βίῳ ποικίλῃ πάντα κέχρηται τῇ φύσει καὶ τῇ ἐπινοίᾳ, καθὼς καὶ Γέμινος ὁ μαθηματικὸς ἐν τῷ Περὶ τῆς τῶν μαθημάτων τάξεως φησιν. Κάρπος δὲ ποιοὶ φησιν ὁ Ἀντιοχεὺς Ἀρχιμήδη τὸν Συρακόσιον ἐν μόνον βιβλίον συντεταχθαί νομικῶς τὸ κατὰ τὴν σφαιροποίαν, τῶν δὲ ἄλλων οὐδὲν ἡξιωκέναι συντάξαι. καίτοι παρὰ τοῖς πολλοῖς ἐπὶ μηχανικῆς δοξασθεὶς καὶ μεγαλοφυὴς τῆς γενομένης ὁ θαυμαστὸς ἐκεῖνος, ὅπου διαμειναι παρὰ πάσιν ἀνθρώπωσ ὑπερβαλλόντως ὑμνοῦμενος, τῶν τε προηγουμένων γεωμετρικῆς καὶ ἀριθμητικῆς ἐξομένων θεωρίας τὰ βραχύτατα δοκοῦνται εἶναι σπουδαῖος συνέγραφεν ὃς φαίνεται τὰς εἰρημένας ἐπιστήμας οὕτως ἀγαπήσας ὡς μηδὲν ἔξωθεν ὑπομένειν αὐταῖς ἐπεισάγει. αὐτός δὲ Κάρπος καὶ ἄλλοι τινὲς συνεχήσαντο γεωμετρία καὶ εἰς τέχνας τινὰς εὐλόγως γεωμετρία γὰρ οὐδὲν βλάπτεται, συμματοποιῶν πεφυκώμενο πολλὰς τέχνας, διὰ τοῦ συνεῖθαν αὐταῖς [μὴν ὅπερ οὖσα τεχνῶν οὐ βλάπτεται διὰ τοῦ προωτίζειν ὀργανικῆς καὶ ἀρχιτεκτονικῆς οὐδὲ γὰρ διὰ τὸ συνεῖπα γεωμετρία καὶ γνωμονικῆ καὶ μηχανικῆ καὶ σχημα- γραφία βλάπτεται τι], τούναντιον δὲ προάγουσα

¹ ποιεῖν om. Hultsch. ² μήτηρ . . . τι om. Hultsch.
science of sun-dials; (5) they also describe as mechan-
icians the makers of spheres, who know how to make
models of the heavens, using the uniform circular
motion of water.

Archimedes of Syracuse is acknowledged by some
to have understood the cause and reason of all these
arts; for he alone applied his versatile mind and
inventive genius to all the purposes of ordinary life, as
Geminus the mathematician says in his book On the
Classification of Mathematics. a Carpus of Antioch b
says somewhere that Archimedes of Syracuse wrote
only one book on mechanics, that on the construction
of spheres, c not regarding any other matters of this
sort as worth describing. Yet that remarkable man
is universally honoured and held in esteem, so that his
praises are still loudly sung by all men, but he himself
on purpose took care to write as briefly as seemed
possible on the most advanced parts of geometry and
subjects connected with arithmetic; and he obviously
had so much affection for these sciences that he
allowed nothing extraneous to mingle with them.
Carpus himself and certain others also applied geo-
metry to some arts, and with reason; for geometry is
in no way injured, but is capable of giving content to
many arts by being associated with them, and, so far
from being injured, it is obviously, while itself

a For Geminus and this work, v. supra, p. 370 n. c.
b Carpus has already been encountered (vol. i. p. 334) as
the discoverer (according to Iamblichus) of a curve arising
from a double motion which can be used for squaring the
circle. He is several times mentioned by Proclus, but his
date is uncertain.
c This work is not otherwise known.
With the great figure of Pappus, these selections illustrating the history of Greek mathematics may appropriately come to an end. Mathematical works continued to be written in Greek almost to the dawn of the Renaissance, and
REVIVAL OF GEOMETRY: PAPPUS

advancing those arts, appropriately honoured and adorned by them.\textsuperscript{a}

they serve to illustrate the continuity of Greek influence in the intellectual life of Europe. But, after Pappus, these works mainly take the form of comment on the classical treatises. Some, such as those of Proclus, Theon of Alexandria, and Eutocius of Ascalon have often been cited already, and others have been mentioned in the notes.
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INDEX OF GREEK TERMS

The purpose of this index is to give one or more typical examples of the use of Greek mathematical terms occurring in these volumes. Non-mathematical words, and the non-mathematical uses of words, are ignored, except occasionally where they show derivation. Greek mathematical terminology may be further studied in the Index Graecitatis at the end of the third volume of Hultsch’s edition of Pappus and in Heath’s notes and essays in his editions of Euclid, Archimedes and Apollonius. References to vol. i. are by page alone, to vol. ii. by volume and page. A few common abbreviations are used. Words should be sought under their principal part, but a few cross-references are given for the less obvious.

"Aγει, to draw; εἰδείαν γραμμήν αὐγείν, to draw a straight line, 442 (Eucl.); έπεις έπιμαίονοι אקחיווע, if tangents be drawn, ii. 64 (Archim.); παράλληλος חףוה ח AK, let AK be drawn parallel, ii. 312 (Apollon.) ἄγεωμέτρητος, or, ignorant of or unversed in geometry, 386 (Tzetzes) ἄδιαιρος, or, undivided, indivisible, 366 (Aristot.) ἄδηνατος, or, impossible, 394 (Plat.), ii. 566 (Papp.) οπερ εἰσὶν α., often without εἰσίν, which is impossible, a favourite conclusion to reasoning based on false premises, ii. 192 (Archim.); οἱ διὰ τοῦ δ. περαιόντες, those who argue per impossibile, 110 (Aristot.) ἄθροισμα, στος, τό, collection; ἀ. φιλοσοφότατος, a collection most skilfully framed, 480 (Papp.) Αἰγυπτιακός, η, ὁ, Egyptian; οἱ Αἰ. καλούμεναι μέθοδοι ἐν πολλαπλασιασμοῖς, 16 (Schol. in Plat. Charm.) αἴρειν, to take away, subtract, ii. 506 (Heron) αἴτειν, to postulate, 442 (Eucl.), ii. 206 (Archim.) αἴτημα, στος, τό, postulate, 420 (Aristot.), 440 (Eucl.), ii. 366 (Procl.) ἀκίνητος, or, that cannot be moved, immobile, fixed, 394 (Aristot.), ii. 246 (Archim.) ἀκολουθεῖν, to follow, ii. 414 (Ptol.)
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ἀκόλουθος, ὁ, following, consequent, corresponding, ii. 580 (Papp.) as subst., ἀκόλουθος, τὸ, consequence, ii. 566 (Papp.)
ἀκόλουθος, adv., consistently, consequentially, in turn, 458 (Eucl.), ii. 384 (Procl.)
ἀκονσηματικός, ὁ, ὁ, eager to hear; as subst., ἀ., ὁ, hearer, exoteric member of Pythagorean school, 3 n. d (Iambi.)
ἀκριβής, ἐς. exact, accurate, precise, ii. 414 (Ptol.)
ἀκρός, ὁ, ὁ, at the farthest end, extreme, ii. 270 (Cleom.) of extreme terms in a proportion, 122 (Nicom.) ἀ. καὶ μέδος λόγος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.)
ἄλλος, alternatively, 356 (Papp.)
ἀλογος, ὁ, irrational, 420 (Aristot.), 452 (Eucl.), 456 (Eucl.) ἄτι ἀlogen, by irrational means, 388 (Plut.)
ἀμβλυώνος, ὁ, obtuse-angled; ἀ. τρίγωνος, 440 (Eucl.) ἄ. κώνος, ii. 278 (Eutoc.)
ἀμφί cola, ἢ, ἢ, obtuse; ἢ. γωνία, often without γωνία, obtuse angle, 438 (Eucl.)
ධμεταβλετός, ὁ, unaltered, immutable; μονάδος ἡ. ὁμοίως, ii. 514 (Dioph.)
ἀμήχανος, ὁ, impracticable, 298 (Eutoc.)
ἀμφούμα, ὁ, ὁ, revolving figure, ii. 604 (Papp.)
ἀμφοστικός, ὁ, ὁ, described by revolution; ἀμφοστικόν, τὸ, figure generated by revolution, ii. 604 (Papp.)
ἀναγράφειν, to describe, construct, 180 (Eucl.), ii. 68 (Archim.)
ἀνακλάν, to bend back, incline, reflect (of light), ii. 496 (Damian.)
ἀναλημμα, ὁ, τὸ, a representation of the sphere of the heavens on a plane, analemma; title of work by Diodorus, 300 (Papp.)
ἀναλογία, ἡ, proportion, 446 (Eucl.): κύριος ἃ καὶ πρότη, proportion par excellence and primary, i.e., the geometric proportion, 125 n. a; συνεχής ἃ, continued proportion, 262 (Eutoc.)
ἀναλογον, adv., proportionally, but nearly always used adjectively, 70 (Eucl.), 446 (Eucl.)
ἀναλύων, to solve by analysis, ii. 160 (Archim.); ὃ ἀναλύμενος τόπος, the Treasury of Analysis, often without τόπος, e.g., ὃ καλομένος ἀναλόγειμονος, ii. 596 (Papp.)
ἀναλύων, ἢ, ἢ, solution of a problem by analytical methods, analysis, ii. 596 (Papp.)
ἀναλυτικός, ὁ, ὁ, analytical, 158 (Procl.)
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ἀνάθαλ, adv., in a reverse direction; transformation of a ratio known as inverting, 448 (Eucl.)

ἀναποδακτικὸς, adv., independently of proof, ii. 370 (Procl.)

ἀναστρέφειν, to convert a ratio according to the rule of Eucl. v. Def. 16; ἀναστρέφαντι, lit. to one who has converted, converting, 466 (Eucl.)

ἀναστροφή, ἔμετρον; conversion of a ratio according to the rule of Eucl. v. Def. 16, 448 (Eucl.)

ἀνεναιόητος, ov, unperceived, imperceptible; hence, negligible, ii. 482 (Heron)

ἀνίμος, ov, unequal, 444 (Eucl.), ii. 50 (Archim.)

ἀνατόμαι, to set up, erect, ii. 78 (Archim.)

ἀνταλλοψία, ἔμετρον; reciprocity, 76 (Theol. Arith.)

ἀντικείμενα, to be opposite, 114 (Nicom.); τοιαί ἀντικείμενα, opposite branches of a hyperbola, ii. 322 (Apollon.)

ἀντιπάλοχος, to be reciprocally proportional, 114 (Nicom.); ἀντιπροσώποτος, adv., reciprocally, ii. 208 (Archim.)

ἀντιστροφή, ἔμετρον; conversion, converse, ii. 140 (Archim. ap. Eutoc.)

ἀξίομα, ἀριθμός, τό, axiōm, postulate, ii. 42 (Archim.)

ἀξόν, ovos, δ, axis; of a cone, ii. 286 (Apollon.); of any plane curve, ii. 288 (Apollon.); of a conic section, 282 (Eutoc.); συνογείς δ., conjugate axes, ii. 288 (Apollon.)

ἀόριστος, ov, without boundaries, undefined, πλῆθος μονάδων ἄ., ii. 522 (Dioph.)

ἀπαγογή, ἔμετρον; reduction of one problem or theorem to another, 252 (Procl.)

ἀπαρίθμειν, to make even; of ἀπαριθμητικὲς ἀριθμούς, factors, ii. 506 (Heron)

ἀπειράγησις, in an infinite number of ways, ii. 572 (Papp.)

ἀπειρος, ov, infinite; as subst., ἀπειρον, τό, the infinite, 424 (Aristot.); ἀπειρον, to infinity, indefinitely, 440 (Eucl.); ἐνί ἄ., ii. 580 (Papp.)

ἀπενεκτίον, adv. used adjectivally, opposite; ἀβάλαμα, 444 (Eucl.)

ἀπέκειν, to be distant, 470 (Eucl.), ii. 6 (Aristarch.)

ἀπλανῆς, ἐς, motionless, fixed, ii. 2 (Archim.)

ἀπλανής, ἐς, without breadth, 436 (Eucl.)

ἀπλοῦς, ἔμετρον, contr. ἀπλοῦς, ἔμετρον, simple; ἄ. γραμμή, ii. 360 (Procl.)
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<td>ἀπλωσις, εως, ἡ, simplification, explanation; A. ἐπιφάνειας σφαιρᾶς, Explanation of the Surface of a Sphere, title of work by Ptolemy, ii. 408 (Suidas)</td>
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<td>ἀπό, from; τὸ ἀπὸ τῆς διαμέτρου τετράγωνον, the square on the diameter, 332 (Archim.); τὸ ἀπὸ ΓΗ (sc. τετράγωνον), the square on ΓΗ, 268 (Eutoc.)</td>
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<td>ἀποδεικτικός, ἡ, ὁ, affording proof, demonstrative, 420 (Aristot.), 158 (Procl.)</td>
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<td>ἀποδεικτικῶς, adv., theoretically, 260 (Eutoc.)</td>
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<td>ἀποδείξες, εως, ἡ, proof, demonstration, ii. 42 (Archim.), ii. 566 (Papp.)</td>
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<td>ἀποκαθιστάω, to re-establish, restore; pass., to return to an original position, ii. 182 (Archim.)</td>
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<td>ἀπολαμβάνω, to cut off; ἡ ἀπολαμβανόμενη περιφέρεια, 440 (Eucl.)</td>
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<td>ἀπορία, ἡ, difficulty, perplexity, 256 (Theon Smyr.)</td>
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<td>ἀπόστιμα, atos, τὸ, distance, interval, ii. 6 (Aristarch.)</td>
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<td>ἀποτομή, ἡ, cutting off, section; Δόγων ἀποτομῆ, Ἡχορίου ἀποτομῆ, works by Apollonius, ii. 598 (Papp.)</td>
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<td>with negative sign, apotome, 456 (Eucl.)</td>
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<td>ἀποτευκτενος, ὁ, to fasten to; mid., ἀποτευκταν, to be in contact, meet, 438 (Eucl.), ii. 106 (Archim.)</td>
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<td>ἀρα, therefore, used for the steps in a proof, 180 (Eucl.)</td>
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<td>ἄρθρος, ὁ, semicircular knife used by leather-workers, a geometrical figure used by Archimedes and Pappus, ii. 578 (Papp.)</td>
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<td>ἀριθμεῖν, to number, reckon, enumerate, ii. 198 (Archim.), 90 (Luc.)</td>
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<td>ἀριθμητικός, ἡ, ὁ, of or for reckoning or numbers; ἡ ἀριθμητικὴ (sc. τέχνη), arithmetic, 6 (Plat.), 420 (Aristot.); ἡ ἀριθμητικὴ μέσον (sc. εὐθείαν), arithmetic mean, ii. 568 (Papp.): ἡ μεσῶτης, 110 (Iamb.)</td>
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<td>ἀριθμητός, ἡ, ὁ, that can be counted, numbered, 16 (Plat.)</td>
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<td>ἄριθμος, ὁ, number, 6 (Plat.), 66 (Eucl.); πρῶτος ὁ, prime number, 68 (Eucl.); πρῶτοι, δεύτεροι, τρίτοι, τέταρτοι, πέμπτοι ὁ, numbers of the first, second, third, fourth, fifth order, ii. 198-199 (Archim.); μελῖνης ὁ, problem about a number of sheep, 16 (Schol. in Plat. Charm.); φιάλης ὁ, problem about a number of bowls (ibid.)</td>
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ἀριθμοστὸν, τὸ, fraction whose denominator is unknown [?], ii. 522 (Dioph.)
ἀρμοζευ, to fit together, ii. 494 (Heron)
ἀρμονία, ἡ, musical scale, octave, music, harmony, 404 (Plat.); used to denote a square and a rectangle, 398 (Plat.)
ἀρμονικός, ᾗ, ὁ, skilled in music, musical; ἡ ἄρμονική (sc. ἑπιστήμη), mathematical theory of music, harmonic; ἡ ἄρμονικὴ μέση, harmonic mean, 112 (Iamb.)
ἀρτιάς, adv., an even number of times; ἀ. ἀρτιος ἀρτιὸς, even-times even number, 66 (Eucl.)
ἀρτιόπλευρος, ὁ, having an even number of sides; πολύγωνον ἀ., ii. 88 (Archim.)
ἀρτιος, ὁ, ὁ, complete, perfect; ἀ. ἀρτιὸς, even number, 66 (Eucl.)
ἀρχή, ἡ, beginning or principle of a proof or science, 418 (Aristot.); beginning of the motion of a point describing a curve; ἀρ. τῆς έλικος, origin of the spiral, ii. 182
ἀρχικός, ὁ, ὁ, principal, fundamental; ἀ. σύμπτωμα, principal property of a curve, ii. 282 (Archim.)
ἀρχικάτατος, ὁ, sovereign, fundamental; ἀ. βίβλος, 90 (Nicom.)
ἀρχιτεκτονικός, ὁ, ὁ, of or for an architect; ἡ ἀρχιτεκτονική (sc. τέχνη), architecture, ii. 616 (Papp.)
ἀστρολογία, ἡ, astronomy, 388 (Aristox.)
ἀστρολόγος, ὁ, astronomer, 378 (Suidas)
ἀστρονομία, ἡ, astronomy, 14 (Plat.)
ἀσύμμετρος, ὁ, incommensurable, irrational, 110 (Aristot.), 452 (Eucl.), ii. 214 (Archim.)
ἀσύμπτωτος, ὁ, not falling in, non-secant, asymptotic, ii. 374 (Procl.); ὁ (sc. γραμμή), ᾗ, asymptote, ii. 292 (Apollon.)
ἀσύμβατος, ὁ, incomposite; ἀ. γραμμή, ii. 360 (Procl.)
ἀτακτος, ὁ, unordered; Περὶ ἀτ. ἀλγων, title of work by Apollonius, ii 350 (Procl.)
ἀτελής, ἡ, incomplete; ἀ. ἄμφωστικα, figures generated by an incomplete revolution, ii. 604 (Papp.)
ἀτομος, ὁ, indivisible; ἄτομοι γραμμαί, 424 (Aristot.)
ἀτομος, ὁ, out of place, absurd; ὁπερ ἄτομον, which is absurd, a favourite conclusion to a piece of reasoning based on a false premise, e.g. ii. 114 (Archim.)
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αὔξανει, to increase, to multiply; τρίς αὔξθεις, 398 (Plat.)

αὔξη, ἡ, increase, dimension, 10 (Plat.)

αὔξησις, εώς, ἡ, increase, multiplication, 398 (Plat.)

αὐτόματα, ἡ, on, self-acting; Αὐτόματα, τά, title of work by Heron, ii. 616 (Papp.)

ἀφαιρεῖν, to cut off, take away, subtract, 444 (Eucl.)

ἀφή, ἡ, point of concourse of straight lines; point of contact of circles or of a straight line and a circle, ii. 64 (Archim.)

'Αχιλλεύς, ἐως, ὁ, Achilles, the first of Zeno's four arguments on motion, 368 (Aristot.)

Βάρος, ὑς, Ion. ἔως, τό, weight, esp. in a lever, ii. 206 (Archim.), or system of pulleys, ii. 490 (Heron); τό κέντρον τοῦ βάρους, centre of gravity, ii. 208 (Archim.)

βαρωνίς (sc. μηχανή), ἡ, lifting-screw invented by Archimedes, title of work by Heron, ii. 489 n. a

βάσις, ἔως, ἡ, base; of a geometrical figure; of a triangle, 318 (Archim.); of a cube, 222 (Plat.); of a cylinder, ii. 42 (Archim.); of a cone, ii. 304 (Apollon.); of a segment of a sphere, ii. 40 (Archim.)

Γεωμετρία, ἡ, land dividing, mensuration, geodesy, 18 (Anatolius)

γεωμετρεῖν, to measure, to practise geometry; ἄκη γ. τῶν θεῶν, 386 (Plat.);

γεωμετρουμένη ἐπιφάνεια, geometric surface, 292 (Eutoc.), γεωμετρουμένη ἀπόδειξις, geometric proof, ii. 228 (Archim.)

γεωμέτρης, ὁ, ὁ, land measurer, geometry, 258 (Eutoc.)

γεωμετρία, ἡ, land measurement, geometry, 256 (Theon Smyr.), 144 (Procl.)

γεωμετρικός, ὁ, ὁ, pertaining to geometry, geometrical, ii. 590 (Papp.), 298 (Eutoc.)

γεωμετρικός, adv., geometrical, ii. 222 (Archim.)

γίνεθαι, to be brought about; γεγονέτω, let it be done, a formula used to open a piece of analysis; of curves, to be generated, ii. 468 (Heron); to be brought about by multiplication, i.e., the result (of the multiplication) is, ii. 480 (Heron); τὸ γεγομένον, τὰ γεγομένα, the product, ii. 482 (Heron)

γεωμετρικός, ὁ, ὁ, of or concerning sundials, ii. 616 (Papp.)

γεωμετρός, ὁ, ὁ, carpenter's
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square; pointer of a sundial, ii. 268 (Cleom.); geometrical figure known as gnomon, number added to a figured number to get the next number, 98 (Iamb.)

γραμμή, ἴ, line, curve, 436 (Eucl.); εἰθὲνα γ., (often without γ.), straight line, 438 (Eucl.); ἐκ τῶν γραμμῶν, rigorous proof by geometrical arguments, ii. 412 (Ptol.)

γραμμικός, ἴ, ὁ, linear, 348 (Papp.)

γράφειν, to describe, 442 (Eucl.), ii. 582 (Papp.), 298 (Eutoc.); to prove, 380 (Plat.), 260 (Eutoc.)

γραφή, ἴ, description, account, 260 (Eutoc.); writing, treatise, 260 (Eutoc.)

γωνία, ἴ, angle; ἐπίπεδος γ., plane angle (presumably including angles formed by curves), 438 (Eucl.); εἴθενα γραμμος γ., rectilinear angle (formed by straight lines), 438 (Eucl.); ὀβόδα, ἀμβλεία, ἀξεία γ., right, obtuse, acute angle, 440 (Eucl.)

Δείκνυαι, to prove; δεδακται γάρ τοῦτο, for this has been proved, ii. 220 (Archim.); δακτέον ὅτι, it is required to prove that, ii. 168 (Archim.)

δείν, to be necessary, to be required; δείν ἐστιν, let it be required; δείν ἐστιν, quod erat demonstrandum, which was to be proved, the customary ending to a theorem, 184 (Eucl.); δείν: ὅτι = δείν ἐστιν, ii. 610 (Papp.)

δεκάγωνον, τὸ, a regular plane figure with ten angles, decagon, ii. 196 (Archim.)

δῆλος, ἴ, ὁ, also ὁς, ὁ, manifest, clear, obvious; ὅπι μὲν ὁν ἀπο αὐτὰ συμπέραντα, δῆλον, ii. 192 (Archim.)

διάγειν, to draw through, 190 (Eucl.), 290 (Eutoc.)

διαγραμμα, ἄτον, τὸ, figure, diagram, 428 (Aristot.)

διαιρεῖν, to divide, cut, ii. 286 (Apollon.); διηρμήνεσθαι, or, divided: δ. ἀναλογία, discrete proportion, 262 (Eutoc.); διελέγειν, lit. to one having divided, dividendo (or, less correctly, dividendo), indicating the transformation of the ratio a : b into a - b : b according to Eucl. v. 15, ii. 130 (Archim.)

διαιρέως, ἐστι, ἴ, division, separation, 368 (Aristot.); δ. λόγου, transformation of a ratio dividendo, 448 (Eucl.)

διαμένειν, to remain, to remain stationary, 258 (Eutoc.)

διάστροφος, ὁ, diagonal, diametrical: as subst., δ. (sc. γραμμή), ἴ, diagonal; of a parallelogram, ii. 218
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(Archim.); diameter of a circle, 438 (Eucl.); of a sphere, 466 (Eucl.); principal axis of a conic section in Archim., ii. 148 (Archim.); diameter of any plane curve in Apollon., ii. 286 (Apollon.); πλαγία δ., transverse diameter, ii. 286 (Apollon.); συνυσίδ δ., conjugate diameters, ii. 288 (Apollon.)

dιάστασις, εως, η, dimension, 412 (Simpl.)
dιαστάλλειν, to separate, ii. 502 (Heron)
dιάστημα, ατος, το, interval; radius of a circle, ii. 192 (Archim.), 442 (Eucl.); interval or distance of a conchoid, 300 (Papp.); in a proportion, the ratio between terms, το των μειζόνων άρων δ., 112 (Archytas ap. Porph.);
dimension, 88 (Nicom.)
dιαφορά, η, difference, 114 (Nicom.)
dιόδων, to give; nom. part., δοθείκας, είος, εν, given, ii. 598 (Papp.); Δεδομένα, τα, Data, title of work by Euclid, ii. 588 (Papp.); θέσι τι και μεγεθε δεδομις, to be given in position and magnitude, 478 (Eucl.)
dισλόντι, ι. διαφέρων
dιεχθής, ες, discontinuous; σπείρα δ., open spire, ii. 364 (Procl.)
dιορίζειν, to determine, ii. 566 (Papp.); Διωρισμένη
tομή, Determinate Section, title of work by Apollonius, ii. 598 (Papp.)
dιορισμός, δ., statement of the limits of possibility of a solution of a problem, dìorismos, 150 (Procl.)
dιπλασιάζειν, to double, 258 (Eutoc.)
dιπλασιασμός, δ., doubling, duplication; κόβον δ., 258 (Eutoc.)
dιπλάσιος, α., ου, double, 302 (Papp.); δ. λόγος, duplicate ratio, 446 (Eucl.)
dιπλάσιον, ου, later form for διπλάσιο, double, 326 (Archim.)
dιπλός, η, ου, contr. διπλούς, η, ουν, twofold, double, 326 (Archim.); δ. ισότης, double equation, ii. 528 (Dioph.)
dίχα, adv., in two (equal) parts, 66 (Eucl.); δ. τικετ, to bisect, 440 (Eucl.)
dιχοτομία, η, dividing in two; point of bisection, ii. 216 (Archim.); Dichotomy, first of Zeno’s arguments on motion, 368 (Aristot.)
dιχοτόμος, ου, cut in two, halved, ii. 4 (Aristarch.)
dινάμες, εως, η, power, force, ii. 488 (Heron), ii. 616 (Papp.); άι πέντε δ., the five mechanical powers (wheel and axle, lever, pulley, wedge, screw), ii. 492 (Heron); power in
the algebraic sense, esp. square; δυνάμει, in power, i.e., squared, 322 (Archim.); δυνάμει σώματος, commensurable in square, 450 (Eucl.); δυνάμει αὐξήσεως δυνατεύωμαι, 398 (Plat.)

δυνατός, ἕ, ὅν, possible, ii. 566 (Papp.)

δοκιμοευκάνταδρον, τό, solid with ninety-two faces, ii. 196 (Archim.)

δοκιμεξύκανταδρον, τό, solid with sixty-two faces, ii. 196 (Archim.)

δοκιμωτριακάνταδρον, τό, solid with thirty-two faces, ii. 196 (Archim.)

δωδεκάδρον, ὁ, with twelve faces; as subst., δωδεκάδρον, τό, body with twelve faces, dodecahedron, 472 (Eucl.), 216 (Aët.)

Ἐβδομηκοστόμονος, ὁ, seventy-first; τό ἐτ., seventy-first part, 320 (Archim.)

ἐγγράφειν, to inscribe, 470 (Eucl.), ii. 46 (Archim.)

ἐγκύκλιος, ὁ, also a, ὁ, circular, ii. 618 (Papp.)

εἴδος, ἄν. Ιων. ὁς, τό, shape or form of a figured number, 94 (Aristot.); figure giving the property of a conic section, viz., the rectangle contained by the diameter and the parameter, ii. 317 n. a, 358 (Papp.), 282 (Eutoc.); term in an equation, ii. 524 (Dioph.); species—of number, ii. 522 (Dioph.), of angles 390 (Plat.)
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eikooadpos, ov, having twenty faces; eikooadpos, to, body with twenty faces, icosahedron, 216 (Aet.)
eikooaplpos, ov, twenty-fold, ii. 6 (Aristarch.)
ekatontas, ados, the number one hundred, ii. 198 (Archim.)

ekballoes, to produce (a straight line), 442 (Eucl.), ii. 8 (Aristarch.), 352 (Papp.)
ekkakeikooadpos, to, solid with twenty-six faces, ii. 196 (Archim.)

ekkwothai, used as pass. of ekthwenv, to be set out, be taken, ii. 95 (Archim.), 298 (Papp.)

ekpooues, to take away, eliminate, ii. 612 (Papp.)

ekptasma, atos, to, that which is spread out, unfolded; 'Ekptasma, title of work by Democritus dealing with projection of armillary sphere on a plane, 239 n. a

ekptasia, atos, to, section sawn out of a cylinder, prismatic section, ii. 470 (Heron)

ekthwenv, to set out, ii. 568 (Papp.)

ektos, adv., without, outside: as prep., to, to, kuxlo, 314 (Alex. Aphr.): adv. used adjectively, the external straight line, 314 (Simpl.): the external

gle of the triangle, ii. 310 (Apollon.)

elasow, ov, smaller, less, 320 (Archim.); proi meloun estin, ii. 112 (Archim.); orbeh, less than a right angle, 438 (Eucl.); the, (sc. eideia), minor in Euclid's classification of straight lines, 458 (Eucl.)

ellagostos, of, smallest, least, ii. 44 (Archim.)

ellis, elkos, spiral, helix, ii. 182 (Archim.): spiral on a sphere, ii. 580 (Papp.)

elleuma, atos, to, defect, deficiency, 206 (Eucl.)

elleineu, to fall short, be deficient, 394 (Plat.), 188 (Procl.)

ellein, eos, falling short, deficiency, 186 (Procl.):

the conic section ellipse, so called because the square on the ordinate is equal to a rectangle whose height is equal to the abscissa applied to the parameter as base but falling short (elleineu), ii. 316 (Apollon.), 188 (Procl.)

emados, to, area, ii. 470 (Heron)

emabaleu, to throw in, insert, ii. 574 (Papp.): multiply, ii. 534 (Dioph.)

em pain, to fall on, to meet, to cut, 442 (Eucl.), ii. 58 (Archim.)

empleceu, to plait or weave in; opeira emplegeomai.
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interlaced spire, ii. 364 (Procl.)

ἐναλλάξ, adv., often used adjectively, transformation of a ratio according to the rule of Eucl. v. Def. 12, permutando, 448 (Eucl.), ii. 144 (Archim.)

ἐναντίος, a, ov, opposite; κατ' ἐ., ii. 216 (Archim.)

ἐναρμόζειν, to fit in, to insert, 284 (Eutoc.)

ἐναρασις, ἐως, ἡ, inscription, 396 (Plat.)

ἐνεκλήσ, ἐσ, perfect, complete; τρίγωνον ἐ., 90 (Procl.)

ἐντός, adv. used adjectively, within, inside, interior; at ἐ. γωνία, 442 (Eucl.)

ἐνυπάρχειν, to exist in; ἔδη ἐνυπάρχοντα, τά, positive terms, ii. 524 (Dioph.)

ἐξαγωνικός, ἡ, ὅν, hexagonal; ε. ἀρετώμος, 96 (Nicom.)

ἐξάγωνος, ὁ, as subst. ἐξάγωνον, τό, hexagon, 470 (Eucl.)

ἐξακοστός, ὁ, ὁν, sixtieth; in astron., πρώτον ἐξακοστόν, τό, first sixtieth, minute, δεύτερον ἐ., second sixtieth, second, 50 (Theon Alex.)

ἐξής, adv., in order, successively, ii. 566 (Papp.)

ἐνατή, ἡ, touching, tangency, contact, 314 (Simpl.);

Ἐνατη, On Tangencies, title of a book by Apollonius, ii. 336 (Papp.)

ἐπεθανα, to be or come after, follow; τό ἐπόμενον, con-

sequence, ii. 566 (Papp.);

τά ἐπόμενα, rearward elements, ii. 184 (Apollon.);

in theory of proportion, τά ἐπόμενα, following terms, consequents, 448 (Eucl.)

ἐπί, prep. with acc., upon, on to, on, εἰδεῖα ἐπ’ εἰδείαν σταθεῖα, 438 (Eucl.)

ἐπιέναγωναι, to join up, ii. 608 (Papp.);

ai ἐπιενεχθεῖσα εἰδεία, connecting lines, 272 (Eutoc.)

ἐπιλογιζομαι, to reckon, calculate, 60 (Theon Alex.)

ἐπιλογισμός, ὁ, reckoning, calculation, ii. 412 (Ptol.)

ἐπισεδός, ὁ, plane; ε. ἐπιφάνεια, 438 (Eucl.);

ἐκ σφών, 388 (Eucl.);

ἐκ σχήμα, 438 (Eucl.);

ἐκ ἀριθμός, 70 (Eucl.);

ἐπισεδός, adv., plane-wise, 88 (Nicom.)

ἐπιστάμενς, ἐσ, flat, broad; ὁμαροποιεῖσ ἐ., ii. 164 (Archim.)

ἐπιγραμματικός, ὁ, ὁν, srtiction: condition, ii. 50 (Archim.), ii. 526 (Dioph.);

ποιεῖν τό ἐ., to satisfy the condition; subdivision of a problem, ii. 340 (Papp.)

ἐπιτρυπτος, or, containing an integer and one-third, in the ratio 4 : 3, ii. 222 (Archim.)

ἐπιφάνεια, ἡ, surface, 438 (Eucl.); κοινῆ ἡ, conical surface (double cone), ii. 286 (Apollon.)
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ἐπιφανέω, to touch, ii. 190 (Archim.); ἡ ἐπιφανεία (sc. εἴδεια), tangent, ii. 64 (Archim.)

ἐπερομήχης, c., with unequal sides, oblong, 440 (Eucl.)

ἐπίπλογμαρμοσ, or, rectilinear; εὐ. γωνία, 438 (Eucl.); εὐ. σχήμα, 440 (Eucl.); as subst., εὐπλογμαρμον, τό, rectilinear figure, 318 (Archim.)

ἐὐδος, εῖα, ι, straight; εὐ. γραμμή, straight line, 438 (Eucl.); εἴδεια (sc. γραμμή), η, straight line, ii. 44 (Archim.); chord of a circle, ii. 412 (Ptol.); distance (first, second, etc.) in a spiral, ii. 182 (Archim.); κατ' εὐδος, along a straight line, ii. 580 (Papp.)

ἐπαραγώγητος, or, readily admissible, easily obvious; εὐ. λήμματα, ii. 230 (Archim.)

ἐφρευς, εῶς, η, discovery, solution, ii. 518 (Dioph.), 260 (Eutoc.)

ἐφήμα, ατος, τό, discovery, ii. 380 (Schol. in Eucl.)

ἐφροκεν, to find, discover, solve, ii. 526 (Dioph.), 340 (Papp.), 262 (Eutoc.); ὅπερ ἐδεί εὐφρευς, which was to be found, 282 (Eutoc.)

ἐφερης, εἶα, easy to solve, ii. 526 (Dioph.)

ἐφαπτοσθαί, to touch, ii. 224 (Archim.); ἐφαπτομένη, ἡ (sc. εἴδεια), tangent, 322 (Archim.)

ἐφαρμογή, ἡ, coincidence of geometrical elements, 340 (Papp.)

ἐφαρμοζων, to fit exactly, coincide with, 444 (Eucl.), ii. 208 (Archim.), 298 (Papp.); pass. ἐφαρμοζων, to be applied to, ii. 208 (Archim.)

ἐφεξῆς, adv., in order, one after the other, successively, 312 (Them.); used adjectively, as at εὐ. γωνία, the adjacent angles, 483 (Eucl.)

ἐφιστάναι, to set up, erect; perf., ἐφιστηκέναι, intr., stand, and perf. part. act., ἐφιστηκός, ὦς, ὡς, standing, 438 (Eucl.)

ἐφοδος, η, method, ii. 596 (Papp.); title of work by Archimedes

ἐχειν, to have; λόγον ἡ, to have a proportion or ratio, ii. 14 (Aristarch.); γέων ἡ, to be generated (of a curve), 348 (Papp.)

ἐῶς, as far as, to, ii. 290 (Apollon.)

Ζητεῖν, to seek, investigate, ii. 222 (Archim.); ζητουμένον, τό, the thing sought, 158 (Procl.), ii. 596 (Papp.)

ζητήσας, εῶς, η, inquiry, investigation, 152 (Procl.)

ζώγιον, τό = ζωγόν, τό, ii. 234 (Archim.)
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ζυγόν, τό, beam of a balance, balance, ii. 234 (Archim.)
ζωίδιον, τό, dim. of ζωή, lit. small figure painted or carved; hence sign of the Zodiac; οἱ τῶν Ζ. κύκλοι, Zodiac circle, ii. 394 (Hypsicles)

Πηγέσθαι, to lead; ηγούμενα, τά, leading terms in a proportion, 448 (Eucl.)
ημικύκλος, or, semicircular; as subst., ημικύκλον, τό, semicircle, 440 (Eucl.), ii. 568 (Papp.)
ημικυλαδρος, ὁ, half-cylinder, 260 (Eutoc.): dim. ἡμικυλαδρόν, τό, 286 (Eutoc.)
ημόλιος, ἄ, ov, containing one and a half, half as much or as large again, one-and-a-half times, ii. 42 (Archim.)
ημίους, ες, v, half, ii. 10 (Aristarch.); as subst., ημιος, τό, 320 (Archim.)

Θέας, εος, ἡ, setting, position, 268 (Eutoc.); θέασι δεδόθαι, to be given in position, 478 (Eucl.)
θεωρεῖν, to look into, investigate, ii. 222 (Archim.)
θεώρημα, τό, theorem, 228 (Archim.), ii. 566 (Papp.), 150 (Procl.), ii. 366 (Procl.)
θεωρητικός, ὁ, ὁ, able to perceive, contemplative, speculative, theoretical; applied to species of analysis, ii. 598 (Papp.)
θεωρία, ἡ, inquiry, theoretical investigation, theory, ii. 222 (Archim.), ii. 568 (Papp.)
θυρεός, ὁ, shield, 490 (Eucl.);
ὁ (sc. γραμμή) τοῦ θ., ellipse, ii. 360 (Procl.)

Ἰσίκες, adv., the same number of times, as many times; τά ἑ τολλαπλάσια, equimultiples, 446 (Eucl.)
ἰσοβάρης, ἐς, equal in weight, ii. 250 (Archim.)
ἰσογκος, ου, equal in bulk, equal in volume, ii. 250 (Archim.)
ἰσογώνιος, ου, equiangular, ii. 608 (Papp.)
ἰσομέγκης, ἐς, equal in length, 398 (Plat.)
ἰσοπερίμετρος, ου, of equal perimeter, ii. 386 (Theon Alex.)
ἰσόπλευρος, ου, having all its sides equal, equilateral; i. τρίγωνον, 440 (Eucl.), i. τετράγωνον, 440 (Eucl.), i. πολύγωνον, ii. 54 (Archim.)
ἰσοπληθής, ἐς, equal in number, 454 (Eucl.)
ἰσορροπεῖν, to be equally balanced, be in equilibrium, balance, ii. 206 (Archim.)
Ἰσορροπεῖν, τά, title of work on equilibrium by Archimedes, ii. 226 (Archim.)
ἰσόρροπος, ου, in equilibrium, ii. 226 (Archim.)
ἰσος, η, ου, equal, 268 (Eutoc.); ἐκ ἵσου, evenly,
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488 (Eucl.); δι' ἴσου, ex aequali, transformation of a ratio according to the rule of Eucl. v. Def. 17, 448 (Eucl.)

ἰσοσκελῆς, ἢς, with equal legs, having two sides equal, isosceles: ἰ. τρίγωνον, 440 (Eucl.); ἰ. κώνος, ii. 58 (Archim.)

ἰσοτὰγήως, uniformly, ii. 182 (Archim.)

ἰσότης, ἡ, equality, equation, ii. 526 (Dioph.)

ἰσόταγη, to set up: εἴδεια ἐπ' εἴδειαν σταθείσσα, 438 (Eucl.)

ἰσοβοῶς, ἤς, ἢ, making equal, equation, ii. 526 (Dioph.)

Κάθετος, ὁ, let down, perpendicular; ἴ. κ. (σκ. γραμμή), perpendicular, 438 (Eucl.), ii. 580 (Papp.)

καθολικός, ὁ, ἐν, general: κ. μέθοδος, ii. 470 (Heron)

καθολικός, generally; καθολικότερον, more generally, ii. 572 (Papp.)

καθόλου, adv., on the whole, in general; τὰ κ. καλούμενα θεωρήματα, 152 (Procl.)

καμπύλος, ὁ, ov, curved; κ. γραμμαί, ii. 42 (Archim.); 260 (Eutoc.)

κανόνον, τὸ, table, ii. 444 (Ptol.)

κανονικός, ὁ, ὁν, of or belonging to a rule; ἴ. κανονική (sc. τέχνη), the mathematical theory of music, theory of musical intervals, canonic, 18 (Anatolius);

κ. ἑθεάις, display in the form of a table, ii. 412 (Ptol.)

κανών, ὁνος, ὁ, straight rod, bar, 308 (Aristoph.), 264 (Eutoc.); rule, standard, table, ii. 408 (Suidas)

κατάγειν, to draw down or out, ii. 600 (Papp.)

καταγραφή, ἡ, construction, 188 (Procl.); drawing, figure, ii. 158 (Eutoc.), ii. 444 (Ptol.), ii. 610 (Papp.)

καταλαμβάνειν, to overtake, 368 (Aristot.)

καταλείπειν, to leave, 454 (Eucl.), ii. 218 (Archim.), ii. 524 (Dioph.); τὰ καταλαμβάνειν, the remainders, 444 (Eucl.)

καταμετρεῖν, to measure, i.e., to be contained in an integral number of times, 444 (Eucl.)

κατασκευᾶσθαι, to construct, 264 (Eutoc.), ii. 566 (Papp.)

κατασκευή, ἡ, construction, ii. 500 (Heron)

καταστροφαιμός, ὁ, placing among the stars: Καταστροφαιμός, of, title of work wrongly attributed to Eratosthenes, ii. 262 (Suidas)

κατατομή, ἡ, cutting, section; κ. κανών, title of work by Cleonides, 151 (P.)

κατάπτρικός, ὁ, ὁν, of or in a mirror; Κατάπτρικα, τά, title of work ascribed to Euclid, 156 (Procl.)
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κάτωτρον, τό, mirror, ii. 498 (Heron)
κείονται, to lie, ii. 268 (Cleom.): of points on a straight line, 438 (Eucl.): as pass. of πιθέων, to be placed or made; of an angle, 326 (Archim.): ὀροιος κ., to be similarly situated, ii. 208 (Archim.)
κεντροβαρικός, ἡ, of or pertaining to a centre of gravity; κ. σημεῖα, ii. 604 (Papp.)
κέντρον, τό, centre; of a circle, 438 (Eucl.), ii. 8 (Aristarch.), ii. 572 (Papp.): of a semicircle, 440 (Eucl.); ἣ (sc. γραμμή or εὐθεία) ἐκ τοῦ κ., radius of a circle, ii. 40 (Archim.); κ. τοῦ βάρκος, centre of gravity, ii. 208 (Archim.)
κινεῖσθαι, to move, 264 (Eutoc.)
κίνησις, εἰς ἡ, motion, 264 (Eutoc.)
κύκκος, ἢ, Att. κύκκος, ἢ, like ivy; κ. γραμμή, cissoid, 276 n. a
κλάνει, to bend, to incline, 420 (Aristot.), 358 (Papp.); κλάμεναι εὐθείαι, inclined straight lines, ii. 496 (Damian).
κλίνειν, to make to lean; pass., to incline, ii. 252 (Archim.)
κλίνος, εἰς ἡ, inclination; τῶν γραμμῶν κ., 438 (Eucl.)
κοινός, ἡ, common, 412 (Aristot.): κ. πλευρά, ii. 500 (Heron); κ. ἐννοοῖ, 444 (Eucl.); κ. τομή, ii. 290 (Apollon.): τό κοινόν, common element, 306 (Papp.)
κορυφή, ἡ, vertex: of a cone, ii. 286 (Apollon.): of a plane curve, ii. 286 (Apollon.): of a segment of a sphere, ii. 40 (Archim.)
κόκκλας, οὖ, ὁ, snail with spiral shell; hence anything twisted spirally; screw, ii. 496 (Heron); screw of Archimedes, ii. 34 (Diod. Sic.): Περὶ τοῦ κ., work by Apollonius, ii. 350 (Procl.)
κοχλοειδής, ἢ, of or pertaining to a shell fish; ἣ (sc. γραμμή), cochlroid, 334 (Simpl.): also κοχλοειδῆς, ἢ, as ἣ κ. γραμμή, 302 (Papp.): probably anterior to ἣ κοχλοειδῆς γραμμή with same meaning
κρίκος, ὁ, ring; τετράγωνοι κ., prismatic sections of cylinders, ii. 470 (Heron)
κυβίσταλεν, to make into a cube, cube, raise to the third power, ii. 504 (Heron)
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κύβικος, ἧν. ὁν. of or for a cube, cubic, 222 (Plat.)
κυβοκυβικός, ὁ, cube multiplied by a cube, sixth power of the unknown quantity \( x^6 \), ii. 522 (Dioph.)
κυβοκυβοστόν, τό, the fraction \( \frac{1}{x^2} \), ii. 522 (Dioph.)
κύβος, ὁ, cube, 258 (Eutoc.)
cubic number, ii. 518 (Dioph.);
third power of unknown, ii. 523 (Dioph.)
κυβοστόν, τό, the fraction \( \frac{1}{x^2} \), ii. 522 (Dioph.)
κυκλικός, ὁν., circular, ii. 360 (Procl.)
κύκλος, ὁ, circle, 392 (Plat.), 438 (Eucl.);
μέγιστος κ.,
great circle (of a sphere), ii. 8 (Aristarch.), ii. 42 (Archim.)
kυλινδρικός, ὁ, on., cylindrical, 286 (Eutoc.)
kύλινδρος, ὁ, cylinder, ii. 42 (Archim.)
kυρίως, adv., in a special sense;
κ. ἀνάλογη, proportion par excellence, i.e., the geometric proportion, 125 n. a
κωνικός, ὁ, on., conical, conic;
k. ἐπιφάνεια, conical surface (double cone), ii. 286 (Apollon.)
kωνοειδής, ἐς, conical; as subst. κωνοειδός, τό, conoid;
ὅψη, τῆς, right-angled conoid, i.e., paraboloid of revolution, ii. 164; ἀμβλυγάνων κ., obtuse-angled

conoid, i.e., hyperboloid of revolution, ii. 164
κώνος, ὁν., ὁ, cone, ii. 286 (Apollon.)
κωνοτομεῖν, to cut the cone, 236 (Eratos. ap. Eutoc.)

Λαμβάνειν, to take, ii. 112 (Archim.);
ἐλεύθερα τὰ κέντρα, let the centres be taken, ii. 388 (Theon Alex.);
λ. τὰς μέσας, to take the means, 294 (Eutoc.);
to receive, postulate, ii. 44 (Archim.)

λέγειν, to choose, ii. 166 (Archim.)
λείπειν, to leave, ii. 62 (Archim.);
λέιποντα εἰδή,
ta, negative terms, ii. 524 (Dioph.)

λιθος, ἔως, ὁ, negative term, minus, ii. 524 (Dioph.)

λήμμα, ατος, τό, auxiliary theorem assumed in proving the main theorem, lemma, ii. 608 (Papp.)
λημμάτιον, τό, dim. of λήμμα,

λύφα, ἔως, ὁ, taking hold, solution, 260 (Eutoc.)

λογικός, ὁ, on., endowed with reason, theoretical, ii. 614 (Papp.)

λογιστικός, ὁ, on., skilled or practised in reasoning or calculating; ὁ λογιστική (sc. τέχνη), the art of manipulating numbers, practical arithmetic, logistic, 17 (Schol. ad Plat. Charm.)
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λόγος, ὁ, ratio, 444 (Eucl.)

Δόγων ἀποτομή, Cutting-off of a Ratio, title of work by Apollonius, ii. 598 (Papp.)

λ. συνημένως, compound ratio, ii. 602 (Papp.)

λ. μοναχός, singular ratio, ii. 606 (Papp.)

ἄκρος καὶ μέσος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.)

λοιπός, ὁ, ὁ, remaining, ii. 600 (Papp.) as subst., λοιπόν, τῷ, the remainder, ii. 506 (Papp.), 270 (Eutoc.)

λοξός, ὁ, ὁ, oblique, inclined; κατὰ λ. κύκλον, ii. 4 (Plut.)

λύσις, εἰς, ἡ, solution, ii. 596 (Papp.)

Μαγγανάριος, ὁ, mechanical engineer, maker of mechanical powers, ii. 616 (Papp.)

μάγγανον, τῷ, block of a pulley, ii. 616 (Papp.)

μάθημα, τό, study, 8 (Plat.), 4 (Archytas); μαθηματα, τά, mathematics; τά δὲ καλοῦμενα ἰδιών μ. 2 (Anatolius); 148 (Procl.), ii. 42 (Archim.), ii. 566 (Papp.)

μαθηματικός, ὁ, ὁ, mathematical; μαθηματικός, ὁ, mathematician, ii. 2 (Ael.), ii. 61 (Papp.); ἡ μαθηματική (sc. ἐπιστήμη), mathematics, 4 (Archytas); τὰ μ. mathematics

μέγεθος, ὁ, ὁ, Ion. εἰς, τό, magnitude, 444 (Eucl.), ii. 50 (Archim.), ii. 412 (Ptol.)

μέθοδος, ἡ, following after, investigation, method, 90 (Procl.)

μεῖζον, ὁ, greater, more, 318 (Archim.); ἦτοι μ. ἔστιν ἡ ἐλάσσων, ii. 112 (Archim.); μ. ὀρθής, greater than a right angle, 438 (Eucl.); ἡ μ. (sc. εἰδεία), major in Euclid’s classification of irrationals, 458 (Eucl.)

μένειν, to remain, to remain stationary, 98 (Nicom.), 286 (Eutoc.)

μερίζειν, to divide, τι παρά τι, 50 (Theon Alex.)

μερισμός, ὁ, division, 16 (Schol. in Plat. Charm.), ii. 414 (Ptol.)

μέρος, ὁ, Ion. εἰς, τό, part; of a number, 66 (Eucl.); of a magnitude, 444 (Eucl.), ii. 584 (Papp.); τὰ μέρη, parts, directions, ἐσ’ ἔκατερα τὰ μ., in both directions, 438 (Eucl.)

μεσημβρανός, ὁ, ὁ, for μεσημβρός, of or for noon; μ. (sc. κύκλος), ὁ, meridian, ii. 268 (Cleom.)

μέσος, ἡ, ὁ, middle; ἡ μέση (sc. εἰδεία), mean (ἀριθμητική, γεωμετρικὴ, ἀρμονικὴ), ii. 568 (Papp.); μέση τῶν ΔΚ, ΚΓ, mean between ΔΚ, ΚΓ, 272 (Eutoc.); ἄκρος καὶ μ. λόγος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.); ἡ μέση (sc. εἰδεία), medial in Euclid’s classi-
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cification of irrationals, 458 (Eucl.); ἐκ δύο μέσων πρώτη, first bimedial, ἐκ δύο μέσων δεύτερα, second bimedial, etc., ibid.

μεσότης, ἴσος, ἴ, mean, ii. 566 (Papp.); μ. ἀριθμητική, geometric, γεωμετρική, άριθμητική (ὑπ-εωτία), 110-111 (Iamb.)

μετρεῖν, to measure, contain an integral number of times, 68 (Eucl.), ii. 54 (Archim.)

μέτρον, τό, measure, relation, ii. 294 (Prob. Bov.); κοινόν μ., common measure, ii. 210 (Archim.)

μέχρι, as far as, prep. with gen.; ἡ μέχρι τοῦ ἄξονος (ἐκ. γραμμῆς), ii. 256 (Archim.)

μήκος, Dor. μάκος, εσ., τό, length, 436 (Eucl.); distance of weight from fulcrum of a lever, ii. 206 (Archim.)

μῆνισκος, ὁ, crescent-shaped figure, lune, 238 (Eudemus ap. Simplic.)

μηχανή, ἴ, contrivance, machine, engine, ii. 26 (Plut.)

μηχανικός, ὁ, όν, of or for machines, mechanical, ii. 616 (Papp.); ἡ μηχανική (with or without τέχνη), mechanics, ii. 614 (Papp.); as subst., μηχανικός, ὁ, mechanician, ii. 616 (Papp.), ii. 496 (Damian.)

μηχανοσκόπος, ὁ, maker of engines, ii. 616 (Papp.)

μικρός, ὁ, ὁν., small, little; M. ἀστρονομοῦμενος (ἐκ. τόπος), Little Astronomy, ii. 409 n. b

μικτός, ὁ, ό, mixed; μ. γραμμή, ii. 360 (Procl.); μ. ἐπιστάσεως, ii. 470 (Heron)

μοῖρα, ας, ἴ, portion, part; in astron., degree, 50 (Theon Alex.); μ. τοιχία, χρονική, ii. 396 (Hypsicl.)

μοιάς, ἴδος, ἴ, unit, monad, 66 (Eucl.)

μοναχός, ὁ, όν, unique, singular; μ. λόγος, ii. 606 (Papp.)

μόριον, τό, part, 6 (Plat.)

μουσικός, ὁ, όν, Dor. μουσικός, ὁ, όν, musical; ἡ μουσική (ἐκ. τέχνης), poetry sung to music, music, 4 (Archytas)

μυρίας, ἴδος, ἴ, the number ten thousand, myriad, ii. 198 (Archim.); μ. ἀπλαί, διπλαί, τριαί, κτλ., a myriad raised to the first power, to the second power, and so on, ii. 355 n. α

μυρίων, ας, a, ten thousand, myriad; μ. μυριάδες, myriad myriads, ii. 198 (Archim.)

Nevev, to be in the direction of, ii. 6 (Aristarch.); of a straight line, to verge, i.e., to be so drawn as to pass through a given point and make a given intercept, 244 (Eudemus ap. Simplic.), 420 (Aristot.), ii. 188 (Archim.)
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νεῦσις, κος, ἰ, inclination, verging, problem in which a straight line has to be drawn through a point so as to make a given intercept, 245 n. a; στερέα ν., solid verging, 350 (Papp.); Νεῦσις, title of work by Apollonius, ii. 598 (Papp.)

'Oðós, ἰ, method, ii. 596 (Papp.)

οἴκειος, a, ov, proper to a thing; ὀ οἴκος, κύκλος, ii. 270 (Cleom.)

ὀκτάγωνος, ov, eight-cornered; as subst., ὀκτάγωνον, τó, regular plane figure with eight sides, octagon, ii. 196 (Archim.)

ὀκτάεδρος, ov, with eight faces; as subst., ὀκτάεδρον, τó, solid with eight faces, ii. 196 (Archim.)

ὀκταπλάσιος, a, ov, eightfold, ii. 584 (Papp.)

ὀκτωκαθεκαπλάσιος, ov, eighteen-fold, ii. 6 (Aristarch.)

ὀκτωκαττριακοστάεδρον, τó, solid with thirty-eight faces, ii. 196 (Archim.)

ὀλόκληρος, ov, complete, entire; as subst., ὀλόκληρον, τó, integer, ii. 534 (Dioph.)

ὀλός, η, ov, whole; τά ο, 444 (Eucl.)

ὀμαλός, η, ov, even, uniform, ii. 618 (Papp.)

ὀμαλός, adv., uniformly, 338 (Papp.)

ὀμιος, a, ov, like, similar; ὀμιόωνον, 288 (Eutoc.)

ὀμοιοι ἐπίπεδοι καὶ στερέοι ὀμιθμοί, 70 (Eucl.)

ὀμοιός, adv., similarly, ii. 176 (Archim.); τά ὀ. τε- γαμένα, the corresponding terms, ii. 166 (Archim.); ὀ. κείομαι, to be similarly situated, ii. 208 (Archim.)

ὀμολογεῖν, to agree with, admit; pass., to be allowed, admitted; τῷ ὀμολογούμενον, that which is admitted, premise, ii. 596 (Papp.)

ὀμολογος, ov, corresponding; ὀ. μεγέθεα, ii. 166 (Archim.); ὀ. πλευράς, ii. 208 (Archim.)

ὀμοταγής, ες, ranged in the same row or line, co-ordinate with, corresponding to, similar to, ii. 586 (Papp.)

ὁνομα, ατος, τό, name; ᾧ (ος, εἰθέλα), ἐκ δύο ὀνομάτων, binomial in Euclid's classification of irrationals, 458 (Eucl.)

ὀξυγώνιος, ov, acute-angled; ὀ. κώνος and ὀ. κώνου τομή, ii. 278 (Eutoc.)

ὀξύς, είς, ε, acute; ὀ. γωνία, acute angle, often with γωνία omitted, 438 (Eucl.)

ὀπτικός, ἦ, ov, of or for sight; ὀπτικά, τά, theory of laws of sight; as prop. name, title of work by Euclid, 156 (Procl.)

ὁραματικός, ἦ, ov, serving as instruments; ὀ. λήψις, mechanical solution, 260 (Eutoc.)
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ὁργανικῶς, adv., by means of instruments, 292 (Eutoc.)
ὁργανόν, τὸ, instrument, 294 (Eutoc.); dim. ὀργανίον, 294 (Eutoc.)
ὁρθός, a., ov, upright, erect; ἡ ὀρθὸς (τῶν εἴδους πλευρῶν), the erect side of the rectangle formed by the ordinate of a conic section applied to the parameter as base, latus rectum, an alternative name for the parameter, ii. 316 (Apollon.), ii. 322 (Apollon.)
ὁρθογώνιος, ov, having all its angles right, right-angled, orthogonal; ὁ τετράγωνον, 440 (Eucl.); ὁ παραβαλλόγραμμον, 268 (Eutoc.)
ὁρθὸς, ἡ, ov, right; ὁ γωνία, right angle, 438, 442 (Eucl.); ὁ κώνος, right cone, ii. 286 (Apollon.)
ὁρίζων, to separate, delimit, bound, define, 382 (Plat.); ἔδειξεν ὁρμημένη, finite straight line, 188 (Eucl.)
ὁρῶν, ὁ, boundary, 438 (Eucl.); term in a proportion, 112 (Archytas ap. Porph.), 114 (Nicom.)
oὖν, therefore, used of the steps in a geometrical proof, 326 (Eucl.)
ὀξεῖος, to be borne, to float in a liquid; ἐπὶ τῶν ὀξυμένων, On floating bodies, title of work by Archimedes, ii. 616 (Papp.)

Παρά, beside; παραβάλλειν π. to apply a figure to a straight line, 188 (Eucl.);
π. π. π. παραβάλλειν, to divide by, ii. 482 (Heron)
παραβάλλειν, to throw beside;
π. παρά, to apply a figure to a straight line, 188 (Eucl.); hence, since to apply a rectangle xy to a straight line x is to divide xy by x, π. =to divide, ii. 482 (Heron)
παραβολή, ἡ, juxtaposition; division (v. παραβάλλειν), hence quotient, ii. 530 (Dioph.); application of an area to a straight line, 186 (Eucl.); the conic section parabola, so called because the square on the ordinate is equal to a rectangle whose height is equal to the abscissa applied to the parameter as base, ii. 304 (Apollon.), 186 (Procl.), 280 (Eutoc.)
παράδοξος, ov, contrary to expectation, wonderful; ἡ π. γραμμή, the curve called paradoxical by Menelaus, 348 (Papp.); τὰ π., the paradoxes of Erycinus, ii. 573 (Papp.)
παρακείσθαι, to be adjacent, ii. 590 (Papp.), 282 (Eutoc.)
παραλληλεπίπεδον, τὸ, figure bounded by three pairs of parallel planes, parallelepiped, ii. 600 (Papp.)
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παράλληλόγραμμος, or, bounded by parallel lines; as subst., παράλληλόγραμμον, τό, parallelogram, 188 (Eucl.)

παράλληλος, or, beside one another, side by side, parallel, 270 (Eutoc.); π. εὐθεία, 440 (Eucl.)

παραμικῆς, es, Dor. παραμικῆς, es, oblong; σφαιροειδὲς π., ii. 164 (Archim.)

παραπλήρωμα, atos, τό, interstice, ii. 590 (Papp.); complement of a parallelogram, 190 (Eucl.)

παρασείας, to stretch out along, produce, 10 (Plat.)

παρατίθησις, εος, η, increase, ii. 412 (Ptol.)

παράπτωσις, or, hyper-supine; παράπτωσις, τό, a quadrilateral with a re-entrant angle, 482 (Papp.)

πᾶς, πᾶσα, πᾶ, all, the whole, every, any; π. σημεῖον, any point, 442 (Eucl.)

πεντάγωνος, or, pentagonal; π. ἄρθρος, 96 (Nicom.); as subst., πεντάγωνον, τό, pentagon, 222 (Iamb.)

περαίνειν, to bring to an end; πεπερασμένος, or, terminated, 280 (Eutoc.); γραμμαί πεπερασμέναι, finite lines, ii. 42 (Archim.)

πέρας, atos, τό, end, extremity; of a line, 436 (Eucl.); of a plane, 438 (Eucl.)

περατουθία, to limit, bound; εὐθεία περατουθία, 438 (Eucl.)

περιγράψεως, to circumscribe, ii. 48 (Archim.)

περιέχεσθαι, to contain, bound; τό περιεχόμενον ὑπὸ, the rectangle contained by, ii. 108 (Archim.); αἱ περιεχόμεσθαι τὴν γωνίαν γραμμαῖ, 438 (Eucl.); τό περιεχόμενον σχῆμα, 440 (Eucl.)

περιλαμβάνειν, to contain, include, ii. 104 (Archim.)

περίμετρος, or, very large, well-fitting; ἡ π. (sc. γραμμῆς) = περίμετρον, τό, perimeter, ii. 318 (Archim.), ii. 502 (Heron), ii. 386 (Theon Alex.)

περισσώς, Att. περιττάκις, adv., taken an odd number of times; π. ἄριτος ἄρθρος, odd-times even number, 68 (Eucl.); π. περισσός ἄρθρος, odd-times odd number, 68 (Eucl.)

περισσός, Att. περιττός, η, or, superfundus; subtle: ἄρθρος π., odd number, 66 (Eucl.)

περιτθέσαι, to place or put around, 94 (Aristot.)

περιφέρεια, ἡ, circumference or periphery of a circle, arc of a circle, 440 (Eucl.), ii. 412 (Ptol.)

περιφορά, ἡ, revolution, turn of a spiral, ii. 182 (Archim.)

περικύκλος, ὕπος, η, magnitude, size, ii. 412 (Ptol.)
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πίπτειν, to fall; of points, ii. 44 (Archim.); of a straight line, 286 (Eutoc.)

πλάγιος, στρ. oblique; π. διάμετρος, transverse diameter of a conic section, ii. 286 (Apollon.); π. πλευρά, transverse side of the rectangle formed by the ordinate of a conic section applied to the parameter as base, ii. 316 (Apollon.) and ii. 322 (Apollon.)

πλάσιον, to form; of numbers, ii. 528 (Dioph.)

πλάτος, ους, ιόν. eös, τό, breadth, 438 (Eucl.)

πλεόνενει, to widen, broaden, 88 (Nicom.)

πλευρά, άς, η, side; of a triangle, 440 (Eucl.); of a parallelogram, ii. 216 (Archim.); of a square, hence, square root, ii. 530 (Dioph.); of a number, 70 (Eucl.); πλαγία π., latus rectum of a conic section, ii. 322 (Apollon.); π. καὶ διαμέτρος, 132 (Theon Smyr.)

πλήθος, ους, Ιόν. eös, τό, number, multitude, 66 (Eucl.)

πνευματικός, ὁ, ὁ, of wind or air; Πνευματικά, τά, title of work by Heron, ii. 616 (Papp.)

ποιεῖν, to do, construct, ii. 566 (Papp.); to make, π. τομή, ii. 290 (Apollon.); to be equal to, to equal, ii. 526 (Dioph.)

πολλαπλασιάζειν, to multiply, 70 (Eucl.)

πολλαπλασιος, στρ. on, many times as large, multiple; of a number, 66 (Eucl.); of a magnitude, 444 (Eucl.); as subst., πολλαπλάσιον, τό, multiple; τά λοάκες π., equimultiples, 446 (Eucl.)

πολλαπλασιον, or = πολλαπλάσιος, ii. 212 (Archim.)

πόλος, στρ. pole; of a sphere, ii. 580 (Papp.); of a conchoid, 300 (Papp.)

πολύγωνος, στρ. having many angles, polygonal; comp., πολυγωνώτερος, ους, Ιόν. 592 (Papp.); as subst., πολύγωνον, τό, polygon, ii. 48 (Archim.)

πολύεδρος, στρ. having many bases; as subst., πολύεδρον, τό, polyhedron, ii. 572 (Papp.)

πολυπλασιασμός, στρ. multiplication, 16 (Schol. in Plat. Charm.), ii. 414 (Apollon.)

πολυπλευρός, στρ. many sided, multilateral, 440 (Eucl.)

πορίζειν, to bring about, find either by proof or by construction, ii. 598 (Papp.), 252 (Procl.)

πόρωμα, στρ. τό, corollary to a proposition, 480 (Procl.), ii. 294 (Apollon.); kind of proposition intermediate between a theorem and a problem, porism, 480 (Procl.)
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ποριστικὸς, ἢ, ὁ, able to supply or find; ποριστικὸν τὸν προταβέντος, ii. 598 (Papp.)
πραγματεία, ἢ, theory, investigation, 148 (Procl.), ii. 406 (Theon Alex.)
πρίσμα, ατος, τό, prism, ii. 470 (Heron)
πρόβλημα, ατος, τό, problem, 14 (Plat.), 258 (Eutoc.), ii. 566 (Papp.)
προβληματικός, ἢ, ὁ, of or for a problem; applied to species of analysis, ii. 598 (Papp.)
πρόβλημα, ὁ, clear, manifest, ii. 496 (Heron)
προψηλάθαι, to take the lead; προψηλατόν, τά, forward points, i.e., those lying on the same side of a radius vector of a spiral as the direction of its motion, ii. 184 (Archim.)
προκατασκευάζειν, to construct beforehand, 276 (Eutoc.)
προμήχας, ες, prolonged, oblong, 398 (Plat.)
πρός, Dor. πορί, prep., towards; ὁς ἢ ΔΚ πρὸς ΚΕ, the ratio ΔΚ : KE, 272 (Eutoc.)
προσπίπτειν, to fall, 300 (Eutoc.); αἱ προσπίπτοντας εὐθείας, 438 (Eucl.), ii. 594 (Papp.)
προσπέθεναι, to add, 444 (Eucl.)
πρότασις, εος, ἢ, proposition, enunciation, ii. 566 (Papp.)
πρωτεύειν, to propose, to enunciate a proposition, ii. 566 (Papp.); τὸ πρωτεϋν, that which was proposed, proposition, ii. 220 (Archim.)
πρώτιστος, ἢ, ὁ, also ὁς, ἢ, the very first, 90 (Nicom.)
πρώτος, ἢ, ὁ, first; π. ἀριθμός, prime number, 68 (Eucl.); but π. ἀριθμοί, numbers of the first order in Archimedes, ii. 198 (Archim.); in astron., π. έξηκοστόν, first sixtieth, minute, 50 (Theon Alex.); in geom., π. εὐθεία, first distance of a spiral, ii. 182 (Archim.)
πτώς, εος, ἢ, case of a theorem or problem, ii. 600 (Papp.)
πῦθην, ἓν, ὁ, base, basic number of a series, i.e., lowest number possessing a given property, 398 (Plat.); number of tens, hundreds, etc., contained in a number, ii. 354 (Papp.)
πυραμίς, ἱδος, ἢ, pyramid, 228 (Archim.)
Πέπευ, to incline; of the weights on a balance, ii. 208
ῥητός, ἢ, ὁ, rational, 398 (Plat.), 452 (Eucl.)
ῥίζα, ἴων, ῶς, ἢ, root; ἰχνευκωτάτη ῶς, 90 (Nicom.)
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ṇομβοκαίδης, ἐς, rhombus-shaped, rhomboidal: Ἀ. σχήμα, 440 (Eucl.)

ῥόμβος, ὁ, plane figure with four equal sides but with only the opposite angles equal, rhombus, 440 (Eucl.); ὁ στερεός, figure formed by two cones having the same base and their axes in a straight line, solid rhombus, ii. 44 (Archim.)

Σημαίνει, to signify, 188 (Procl.)

σημείον, Dor. σημεῖον, τό, point, 436 (Eucl.); σημ., ii. 522 (Dioph.)

σκαληνός, ὁ, ón, also ón, ὁν, oblique, scalene; κώνος, σ., ii. 286 (Apollon.)

σχῆς, oes, Ion. ὀπτ., τό, leg; σ. τῆς γωνίας, 264 (Eutoc.)

σφείδα, ἡ, surface traced by a circle revolving about a point not its centre, sphere, tore, torus, ii. 468 (Heron)

σφηρικός, ὁ, ὁν, pertaining to a sphere, sporic; σ. τομές, sporic sections, ii. 364 (Procl.); σ. γωνίαι, sporic curves, ii. 364 (Procl.)

στερεός, ὁ, ὁν, solid: σ. γωνία, solid angle, 293 (Plat.); σ. ἀρίθμος, cubic number; σ. τόπος, solid loci, ii. 600 (Apollon.); σ. πρόβλημα, solid problem, 348 (Papp.); σ. νόμος, solid verging, 350 (Papp.); as subst., στερεόν, τό, solid, 258 (Eutoc.)

στίγμή, ἡ, point, 80 (Nicom.)

στοιχείον, τό, element, ii. 596 (Papp.); elementary book, 150 (Procl.); Στοιχεία, τά, the Elements, especially Euclid’s

στοιχεῖωσις, eis, ἡ, elementary exposition, elements; Euclid’s Elements of geometry, 156 (Procl.); αἱ κατὰ μουσικὴν σ., the elements of music, 156 (Procl.); σ. τῶν κοινών, elements of conics, ii. 276 (Eutoc.)

στοιχειώτης, ὁ, ὁ, teacher of elements, writer of elements, esp. Euclid, ii. 596 (Papp.)

στρογγυλός, ὁ, ὁν, round, 392 (Plat.)

συγκεῖον, to lie together; as pass. of συνετέχεια, to be composed of, ii. 284 (Apollon.)

συγκρασία, ἡ, comparison; Τῶν πέντε σχημάτων ἡ, Comparison of the Five Figures, title of work by Aristaeus, ii. 348 (Hyps.)

συλλυγός, ἐς, yoked together, conjugate: σ. διάμετρος, σ. ἄξονας, ii. 288 (Apollon.)

σύμμετρος, ὁ, commensurate with, commensurable with; 380 (Plat.), 452 (Eucl.), ii. 208 (Archim.)

συμπαρατείναι, to stretch out alongside of, 188 (Procl.)
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σύμπας, σύμπαςα, σύμπας, all together, the sum of, ii. 314 (Dioph.)
συμπεράσμα, ατος, τό, conclusion, ii. 228 (Archim.)
συμπίπτειν, to meet, 190 (Eucl.): τὴν μὲν ἐν αὐτῇ ἀ., of curves which meet themselves, ii. 360 (Procl.)
συμπληρών, to fill, complete: α. παραλληλόγραμμον, 268 (Eutoc.)
σύμπτωμα, ατος, τό, property of a curve, 336 (Papp.)
σύμπτωμα, eως, ἕ̇̇, falling together, meeting, ii. 64 (Archim.), ii. 270 (Cleom.), 286 (Eutoc.)
συναγωγή, ἡ, collection: title of work by Pappus, ii. 568 (Papp.)
συναφής, αυτος, the sum of, 328 (Archim.)
συνάπτειν, to collect, gather: συνήμματος λόγας, compound ratio, ii. 602 (Papp.)
συνεχῆς, to approximate, ii. 414 (Ptol.), ii. 470 (Heron)
συνεχός, adv., near, approximately, ii. 488 (Heron)
συνεχῆς, ἐς, continuous, 442 (Eucl.): σ. ἀναλογία, continued proportion, ii. 566 (Papp.): σ. ἀνάλογος, 302 (Papp.): ἀνάλογος α., ii. 364 (Procl.)
συνέττει, v., συνεττεῖαι
συνέδως, eως, ἕ̇̇, putting together, composition: α.

λόγος, transformation of a ratio known as componendo, 448 (Eucl.): method of reasoning from assumptions to conclusions, in contrast with analysis, synthesis, ii. 596 (Papp.)
σύνθετος, αυτος, composite: α. ἀδιαμος, 68 (Eucl.): α. γραμμή, ii. 360 (Procl.)
συνστάσας, to set up, construct, 190 (Eucl.), 312 (Them.)
συνταχθεῖται, ὅ̇̇, putting together in order, systematic treatise, composite volume, collection: title of work by Ptolemy, ii. 408 (Suidas)
συνθεῖται, to place or put together, add together, used of the synthesis of a problem, ii. 160 (Archim.): συνθεῖται, lit. to one having compounded, the transformation of a ratio known as componendo, ii. 130 (Archim.)
συνσαζεῖται, eως, ἕ̇̇, grouping (of theorems), 150 (Procl.)
σφαιρά, ἡ, sphere, 466 (Eucl.), ii. 40 (Archim.), ii. 572 (Papp.)
σφαιρικός, ἡ, ἃ, spherical, ii. 584 (Papp.): Σφαιρικά, title of works by Menelaus and Theodosius
σφαιρόσκολα, ἡ, artificial sphere, making of the heavenly spheres, ii. 618 (Papp.)
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σχήμα, atos, τό, shape, figure, 438 (Eucl.), ii. 42 (Archim.)
sχηματοποιεῖν, to bring into a certain form or shape; σχηματοποιοῦσα γραμμή, curve forming a figure, ii. 360 (Procl.)

Tάλαντον, τό, weight known as the talent, ii. 490 (Heron)
τάξις, ὥσ, ἡ, order, arrangement, scheme, 112 (Iambl.)
ταρασσέω, to disturb; τεταραγμένη ἀναλογία, disturbed proportion, 450 (Eucl.)
tάσσεως, to draw up in order, ii. 598 (Papp.) ὁμοίως τεταραγμένα, ii. 166 (Archim.); perf. part. pass. used as adv., τεταραγμένος, ordinate-wise, ii. 286 (Apollon.); αἱ καταγόμεναι τεταραγμένος (sc. εὐθεία), the straight lines drawn ordinate-wise, i.e., the ordinates, of a conic section, ii. 308 (Apollon.); τεταραγμένος ἡ ΓΑ, ii. 224 (Archim.)

τόλμης, α, ov. perfect, complete, ii. 604 (Papp.); τ. ἀρθρός, 76 (Speusippus ap. Theol. Arith.), 70 (Eucl.), 398 (Plat.)
τέμνειν, to cut; of straight lines by a straight line, ii. 288 (Apollon.); of a curve by a straight line, 278 (Eutoc.); of a solid

by a plane, ii. 288 (Apollon.)
tευτεταγμένακαδεκαέδρον, τό, solid with fourteen faces, ii. 196 (Archim.)
tεταγμένος, τ. τάσσεως
tεταραγμένος, ov. τ. ταράσσεως
tεταρτημόριον, τό, fourth part, quadrant of a circle, ii. 582 (Papp.)
tετραγωνίζειν, to make square, ii. 494 (Heron); to square, 10 (Plat.); ἡ τετραγωνιζούσα (sc. γραμμή), quadratrix, 334 (Simpl.), 336 (Papp.)
tετραγωνικός, ἡ, ὁ, square; of numbers, ii. 526 (Dioph.); τ. πλευρά, square root, 60 (Theon Alex.)
tετραγωνισμός, ὁ, squaring, 310 (Aristot.); τοῦ κύκλου τ., 308 (Plut.); τοῦ μυρισκοῦ τ., 150 (Procl.)
tετράγωνον, ov. square, 308 (Aristophanes), ii. 504 (Heron); ἄρθρος τ., square number, ii. 514 (Dioph.); τ. κρίκον, square rings, ii. 470 (Heron); as subst., τετράγωνον, τό, square, 440 (Eucl.); square number, ii. 518 (Dioph.)
tετράκις, adv., four times, 326 (Archim.)
tετραπλάσιον, a, ov. four-fold, four times as much, 332 (Archim.)
tετραπλασιωτέρον, ov. later form of τετραπλάσιον

tετράπλευρος, ov. four-sided,
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*quadrilateral*: σχήμα τ., 440 (Eucl.)

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